THE NEWTON PROCEDURE FOR SEVERAL VARIABLES

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Abstract. Let us consider an equation of the form

$$P(\boldsymbol{x}, z) = z^m + w_1(\boldsymbol{x})z^{m-1} + \cdots + w_{m-1}(\boldsymbol{x})z + w_m(\boldsymbol{x}) = 0\,,$$

where $m > 1$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $n > 1$, is a vector of variables, k is an algebraically closed field of characteristic zero, $w_i(\boldsymbol{x}) \in k[\![\boldsymbol{x}]\!]$, and $w_m(\boldsymbol{x}) \neq 0$. The aim is to prove the Theorem of Newton-Puiseux, namely:

Theorem 1. *The roots of the above equation are formal power series with rational exponents of bounded denominators, whose Newton diagrams are contained in an S-cone.*

As an application, in Section 4, we deal with some topics of integral dependence of Puiseux power series. In particular, we construct a domain $k[\![\boldsymbol{x}]\!]^*$ containing $k[\![\boldsymbol{x}]\!]$, integrally closed in its quotioent field $k((\boldsymbol{x}))^*$ and this one is the algebraic closure of $k((\boldsymbol{x}))$.

1. Introduction

In the last decade, there have been several (successful) attempts to solve an equation of integral dependence

$$P(\boldsymbol{x}, z) = z^m + w_1(\boldsymbol{x})z^{m-1} + \cdots + w_{m-1}(\boldsymbol{x})z + w_m(\boldsymbol{x}) = 0\,,$$

where $m > 1$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $n > 1$, is a vector of variables, k is an algebraically closed field of characteristic zero, $w_i(\boldsymbol{x}) \in k[\![\boldsymbol{x}]\!]$, and $w_m(\boldsymbol{x}) \neq 0$. We will assume, in addition, that the equation has only simple roots in any algebraic closure of $k((\boldsymbol{x}))$, although this means no restriction. In our opinion, the the firt remarkable attempt is the one by McDonald (c.f. [5]). Later, González Pérez greatly extended MacDonald's results applying them to quasi-ordinary Puiseux power series (c.f. [3]). In all the cases we know, the production of the roots is the result of a non-easy combinatorial procedure based upon the Newton polyhedron of the whole equation.

We have taken completely different point of view, the simplest possible we could think. We single out a variable, say x_1 and solve the equation in (z, x_1) over the field $k((x_2, \ldots, x_n))$ using the elementary Newton procedure for two variables (cf. [7], chapter 4, §3). The gain in simplicity is enormous. The possible loss in generality is not so much, because an usual technique in geometry is to prepare the equations before solving them. Moreover, this simplicity makes our techniques suitable for applications in fields of Mathematics other than Algebra, since the tools we use belong to the common ground of the mathematical knowledge.

The key part of our work is to control where the monomials with negative exponents of the solutions lie. Surprisingly enough, the Jung-Abhyankar theorem (c.f. [1]) gives us the clue. In fact, in [6], theorem 13, we already proved theorem 1, based on the Jung-Abhyankar theorem (cited J-A from now on). Once we know how to control the monomials with negative exponents, we produce here a direct proof, i.e. a proof based only on a detailed analysis of the Newton procedure for two variables, without ressource to J-A.

This approach is the key step to give an elementary proof of J-A. In fact, we conjecture that one can do such a thing by elementary methods, based upon theorem 1. We will not deal here with such matter.

In section 2 we give a very simple description of the roots. In section 3 we prove theorem 1 in the way we said above. In section 4 we give the applications.

2. Puiseux power series and S-cones

In this section we introduce a special kind of Puiseux power series, which will be the roots of the equation $P(\boldsymbol{x}, z) = 0$. In other words, we are going to give meaning to the statement of Theorem 1. Let us fix an algebraically closed field k of characteristic zero, a vector of variables $\boldsymbol{x} = (x_1, \ldots, x_n)$, $n > 1$, and an integer $d > 0$. We will consistenly use the lexicographic order $\leq_{\text{lex}}$ on $\mathbb{R}^n$, and the corresponding group order on the group of monomials $M = \{\boldsymbol{x}^{\boldsymbol{a}}\}_{\boldsymbol{a}\in\mathbb{R}^n}$.

Notations 2. Let $\mathcal{F}_{n,d}$ be the set of all the functions $f\colon \frac{1}{d}\mathbb{Z}^n \to k$; then $\mathcal{F}_{n,d}$ is an abelian group with respect to the usual addition of functions. Let us write every $f \in \mathcal{F}_{n,d}$ as a formal sum $f = \sum_{\boldsymbol{a}\in\mathbb{Z}^n} f_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}/d}$ where $f_{\boldsymbol{a}} = f(\boldsymbol{a}/d) \in k$ and, if $\boldsymbol{a} = (a_1, \ldots, a_n)$, then $\boldsymbol{x}^{\boldsymbol{a}/d} = x_1^{a_1/d} \cdots x_n^{a_n/d}$. We call the *Newton diagram* of f the set

$$\mathcal{E}(f) = \left\{ \frac{\boldsymbol{a}}{d} \in \frac{1}{d}\mathbb{Z}^n \quad \middle| \quad \boldsymbol{a} \in \mathbb{Z}^n, \quad f_{\boldsymbol{a}} \neq \boldsymbol{0} \right\}.$$

Finally, let us denote by $K_{n,d}$ the subfield $K_{n,d} = k((x_n^{1/d}))\ldots((x_1^{1/d}))$ of $\mathcal{F}_{n,d}$, which is constructed by induction. If $n = 1$, then $K_{1,d} = k((x_1^{1/d}))$, the field of formal meromorphic functions in the variable $x_1^{1/d}$. Any element $\sum_{i\geq r} \alpha_i x_1^{i/d} \in k((x_1))$, $\alpha_i \in k$, gives the function $f\colon \frac{1}{d}\mathbb{Z} \to k$ defined by $f(i/d) = 0$ if $i < r$ and $f(i/d) = \alpha_i$ for $i \geq r$. Let us assume that $n > 1$ and that we have defined the subfield $L = k((x_n^{1/d}))\cdots((x_2^{1/d}))$ of $\mathcal{F}_{n-1,d}$; for each $\alpha \in L$, we denote by $f_\alpha\colon \frac{1}{d}\mathbb{Z}^{n-1} \to k$ the corresponding function. In this situation, $K_{n,d}$ is the field $L((x_1^{1/d}))$. Any element $\sum_{i\geq r} \alpha_i x_1^{i/d} \in L((x_1))$, $\alpha_i \in L$, gives the function $f\colon \frac{1}{d}\mathbb{Z}^n \to k$ defined by $f(i/d, a_2/d, \ldots, a_n/d) = 0$ if $i < r$ and $f(i/d, a_2/d, \ldots, a_n/d) = f_{\alpha_i}$ for $i \geq r$.

Proposition 3. *Let $0 \neq f \in \mathcal{F}_{n,d}$; then $f \in K_{n,d}$ if and only if $\mathcal{E}(f)$ is a well-ordered subset of $\frac{1}{d}\mathbb{Z}^n$.*

Proof. Let us assume that $f \in K_{n,d}$ and use induction on n. If $n = 1$, then $f \in k((x_1^{1/d}))$ and $\mathcal{E}(f) \subset \frac{1}{d}\mathbb{Z}$ is clearly well-ordered. Let us assume that $n > 1$ and the result true for $n - 1$. Let $\emptyset \neq \Omega \subset \mathcal{E}(f)$; since f is a power series in $x_1^{1/d}$, the set of the first components of the vectors in Ω must have a minimum a_1/d. Let $0 \neq u_1 \in k((x_n^{1/d}))\cdots((x_2^{1/d}))$ be the coefficient of $x_1^{a_1/d}$ in f and let us denote by E the subset of $\frac{1}{d}\mathbb{Z}^n$ consisting of all the vectors of $\mathcal{E}(u_1)$ with an added a_1/d at the beginning, as their first coordinate. By the induction assumption, $\emptyset \neq E \cap \Omega$ must have a minimum $(a_1/d, a_2/d, \ldots, a_n/d)$, which is the minimum of Ω, so $\mathcal{E}(f)$ is well-ordered.

Now, let us assume that $\mathcal{E}(f)$ is well-ordered and use again induction on n. If $n = 1$, then $\mathcal{E}(f)$ has a lower bound in $\frac{1}{d}\mathbb{Z}$, so $f \in k((x_1^{1/d}))$. Let us assume that $n > 1$ and the result true for $n-1$. Let a_1/d be the first component of the minimum of $\mathcal{E}(f)$. For a fixed $i \in \mathbb{Z}$, $i \geq a_1$, we define $u_i\colon \frac{1}{d}\mathbb{Z}^{n-1} \to k$ by the relation $u_i(b_2/d, \dots, b_n/d) = f(i/d, b_2/d, \dots, b_n/d)$. For any such i, the Newton diagram $\mathcal{E}(u_i)$ is either empty or well-ordered, so $u_i \in k((x_n^{1/d}))\cdots((x_2^{1/d}))$. Therefore, f can be written as $f = \sum_{i\geq a_1} u_i x_1^{i/d}$, which implies $f \in k((x_n^{1/d}))\cdots((x_2^{1/d}))((x_1^{1/d}))$. ▍

Definition 4. A monomial blowing-up is a $\mathbb{R}$-linear automorphism φ_{ij} of $\mathbb{R}^n$, $1 \leq i\,, j \leq n$, $i \neq j$ defined by

$$\varphi_{ij}(a_1, \dots, a_n) = (a_1, \dots, a_i \overset{j)}{+} a_j, \dots, a_n)\,.$$

A monomial blowing-down is the inverse automorphism of a monomial blowing-up.

Notations 5.

(1) The *product order* $\ll$ is defined by $(a_1, \dots, a_n) \ll (b_1, \dots, b_n)$ if and only if $a_i \leq b_i$, for all $i = 1, \dots, n$. Note that *the product order is preserved by any monomial blowing-up.*

(2) We will also consider φ_{ij}, or its inverse, as an automorphism of the multiplicative group M of the monomials $\boldsymbol{x}^{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathbb{R}$, sending $\boldsymbol{x}^{\boldsymbol{a}}$ onto $\boldsymbol{x}^{\varphi_{ij}(\boldsymbol{a})}$. This can be viewed as applying the substitutions $x_i \to x_i x_j$, $x_l \to x_l$, $l \neq i$. To apply φ_{ij} to $P(\boldsymbol{x}, z)$ means to apply it to all its monomials, leaving z fixed.

(3) A monomial blowing-up φ_{ij} preserves the lexicographic order if and only if $i < j$ (c.f. [6], proposition 5). If φ_{ij} is order-preserving, then so is φ_{ij}^{-1}. We will call them *order-preserving monomial blowing-ups* or *order-preserving monomial blowing-downs.*

Corollary 6. *Any order-preserving monomial blowing-up φ_{ij}, $i < j$, induces a field k-automorphism of $K_{n,d}$.*

We borrow from [6] (lemma 15) the following

Lemma 7. *Let $\emptyset \neq \Lambda \subset \mathbb{Z}^n_{\geq}$; then there exists a linear automorphism Φ of $\mathbb{R}^n$, which is a composition of a finite sequence of order-preserving monomial blowing-ups, and a vector with integer coordinates $\boldsymbol{a} \in \Phi(\Lambda)$ such that $\Phi(\Lambda) \subset \boldsymbol{a} + \mathbb{Z}^n_{\geq}$*

From this result we deduce an important consequence, namely

Corollary 8. *Let $\Lambda_1, \dots, \Lambda_r$ be a finite number of non-empty subsets of $\mathbb{Z}^n_{\geq}$. Then there exists a linear automorphism Φ of $\mathbb{R}^n$, which is a composition of a finite sequence of order-preserving monomial blowing-ups, and vectors $\boldsymbol{a}_i \in \Phi(\Lambda_i)$, $i = 1, \dots, r$, such that $\Phi(\Lambda_i) \subset \boldsymbol{a}_i + \mathbb{Z}^n_{\geq}$.*

Proof. Let us observe that, for every monomial blowing-up φ and any vector $\boldsymbol{b} \in \mathbb{Z}^n_{\geq}$, one has $\varphi(\boldsymbol{b} + \mathbb{Z}^n_{\geq}) \subset \varphi(\boldsymbol{b}) + \mathbb{Z}^n_{\geq}$. Let us prove the corollary by induction on r. If $r = 1$, this is lemma 7, so let us assume that $r > 1$ and the result true for $r-1$. There exist Φ' and $\boldsymbol{b}_i \in \Phi'(\Lambda_i)$, $i = 1, \dots, r-1$, such that $\Phi'(\Lambda_i) \subset \boldsymbol{b}_i + \mathbb{Z}^n_{\geq}$. On the other hand, $\Phi'(\Lambda_r) \subset \mathbb{Z}^n_{\geq}$ so, by lemma 7, there exist Φ'' and $\boldsymbol{a}_r \in \Phi''\Phi'(\Lambda_r)$ such that $\Phi''\Phi'(\Lambda_r) \subset \boldsymbol{a}_r + \mathbb{Z}^n_{\geq}$. If, for every $i = 1, \dots, r-1$, we write $\boldsymbol{a}_i = \Phi''(\boldsymbol{b}_i)$, $i = 1, \dots, r-1$, then, by the first observation,

$$\Phi''\Phi'(\Lambda_i) \subset \Phi''(\boldsymbol{b}_i + \mathbb{Z}^n_{\geq}) \subset \Phi''(\boldsymbol{b}_i) + \mathbb{Z}^n_{\geq} = \boldsymbol{a}_i + \mathbb{Z}^n_{\geq}\,.$$

If we set $\Phi = \Phi''\Phi'$, our result is proven. ▍

We introduce now the objects we are looking for, namely, the Puiseux power series in some $K_{n,d}$ whose Newton diagram is contained in an $\mathcal{S}$-cone. In a recent paper of ours (c.f. [6]), we dealt with a special case of polyhedral cones (see, for instance, [2], page 6), that will give rise to the S-cones here.

Definition 9. A *polyhedral cone* $\Gamma(\Delta)$ will be a subset of $\mathbb{R}^n$ defined as the projection, from the origin $\mathbf{0}$, of a compact polyhedron Δ contained in an affine hyperplane H, such that $\mathbf{0} \notin H$, and Δ has a non-empty interior in H. In other words, $\Gamma(\Delta) = \cup_{\boldsymbol{a}\in\Delta}\langle \boldsymbol{a}\rangle_+$, where $\langle \boldsymbol{a}\rangle_+$ is the half-line of the non-negative multiples of $\boldsymbol{a}$.

It is easy to see that the transform of a polyhedral cone by a monomial blowing-up, or a monomial blowing-down, is again a polyhedral cone. In [6], Theorem 6, we proved that a polyhedral cone $\Gamma(\Delta)$ can be brought to the first quadrant by a finite sequence of monomial blowing-ups (i.e., its transform is contained in $\mathbb{R}^n_{\geq}$) if and only if $\Gamma(\Delta) \cap (-\mathbb{R}_{\geq})^n = \{\mathbf{0}\}$.

Now we need to say more on polyhedral cones that can be brought to the first quadrant by a finite sequence of monomial blowing-ups, namely

Theorem 10. *Let $\Gamma(\Delta)$ be a polyhedral cone; the following conditions are equivalent:*

1. $\Gamma(\Delta)$ *can be brought to the first quadrant by a finite sequence of order-preserving monomial blowing-ups.*
2. *For every vector $\mathbf{0} \neq \boldsymbol{c} \in \Gamma(\Delta)$, its first non-zero component is positive.*

Proof. Let us observe that the first non-zero component of any vector is invariant by any order-preserving monomial blowing-up. Consequently, if there exists a vector $\mathbf{0} \neq \boldsymbol{c} \in \Gamma(\Delta)$ whose first non-zero component is negative, 1) cannot hold.

Conversely, let us asume that 2) holds and let $\{\boldsymbol{c}_1, \dots, \boldsymbol{c}_m\}$, $m \geq n$, be non-zero vectors such that the half-lines $\langle \boldsymbol{c}_i\rangle_+$ are the edges of $\Gamma(\Delta)$. Then there must exist a finite sequence of order-preserving monomial blowing-ups (call Φ their composition) such that $\Phi(\boldsymbol{c}_i) \in \mathbb{R}^n_{\geq}$, so $\Phi(\Gamma(\Delta)) \subset \mathbb{R}^n_{\geq}$. ∎

Definition 11. An S-cone is a polyhedral cone that can be brought to the first quadrant by a finite sequence of order-preserving monomial blowing-ups.

Corollary 12. *Let $\Gamma(\Delta) \subset \mathbb{R}^n$ be an S-cone and $0 \neq f \in \mathcal{F}_{n,d}$ be such that $\mathcal{E}(f) \subset \Gamma(\Delta) \cap \frac{1}{d}\mathbb{Z}^n$ then $f \in K_{n,d}$.*

Proof. (c.f. [6], proof of theorem 13). We know that $\frac{1}{d}\mathbb{Z}^n_{\geq}$ is well-ordered, so it is $\Gamma(\Delta)$, being the inverse image of some subset of $\frac{1}{d}\mathbb{Z}^n$ by a finite sequence of order-preserving monomial blowing-downs. This implies that $\mathcal{E}(f)$ is well-ordered and the lemma. ∎

3. The Newton Procedure for several variables

In this section, we construct the generalization to several variables of the classical Newton Procedure and prove Theorem 1. Therefore, we fix the equation $P(\boldsymbol{x}, z) = 0$ of the statement of this theorem.

Notations 13. Let us consider a polynomial

$$Q(\boldsymbol{x}^{1/r}, z) = v_0(\boldsymbol{x}^{1/r})z^m + v_1(\boldsymbol{x}^{1/r})z^{m-1} + \cdots + v_{m-1}(\boldsymbol{x}^{1/r})z + v_m(\boldsymbol{x}^{1/r})$$

where $m\,,n\,,r \in \mathbb{Z}_>$, $m > 1$, $\boldsymbol{x} = (x_1, \dots, x_n)$ is a vector of variables, $\boldsymbol{x}^{1/r} = (x_1^{1/r}, \dots, x_n^{1/r})$ and

$$v_i(\boldsymbol{x}^{1/r}) \in k((x_n^{1/r}))\cdots((x_2^{1/r}))[\![x_1^{1/r}]\!]\,, \quad k((x_n^{1/r}))\cdots((x_2^{1/r})) = k \text{ if } n = 1\,,$$

$v_0(\boldsymbol{x}^{1/r})v_m(\boldsymbol{x}^{1/r}) \neq 0$. We denote by $\mathcal{E}_1\big(Q(\boldsymbol{x}^{1/r}, z)\big)$ the *Newton diagram* of $Q(\boldsymbol{x}, z)$ as a polynomial only in (x_1, z), that is, we plot every monomial $x_n^{a_n/r} \cdots x_2^{a_2/r} x_1^{a_1/r} z^b$, $a_i \in \mathbb{Z}$, $i = 1, \dots, n$, occurring in $Q(\boldsymbol{x}^{1/r}, z)$ with a non-zero coefficient, onto the point $(a_1/r, b) \in \big(\frac{1}{r}\mathbb{Z}_\geq\big) \times \{0, 1, \dots, m\}$

Remark 14. Let $Q(\boldsymbol{x}^{1/r}, z) = 0$ be a polynomial as in Notations 13 with $n > 1$; then, for every order-preserving monomial blowing-up (or blowing-down) φ_{ij}, $i < j$, one has that $\varphi_{ij}\big(\mathcal{E}_1(P)\big) = \mathcal{E}_1(P)$. The reason is that, for any monomial $\boldsymbol{x}^{\boldsymbol{a}/r}$, the blowing-up $\varphi_{ij}(\boldsymbol{x}^{\boldsymbol{a}/r})$ has the same exponent of x_1 as $\boldsymbol{x}^{\boldsymbol{a}/r}$.

The proof of Theorem1 is achieved by induction on the number n of variables in the coefficients. We make the following induction assumption, which holds for $n = 1$ and Φ equal to the identity, by the classical Theorem of Newton-Puiseux:

Induction assumption (**IA**). For every Weierstraß polynomial with n variables in the coefficient ring,

$$\Pi(\boldsymbol{x}, z) = z^\mu + \omega_1(\boldsymbol{x}^{1/\varrho})z^{\mu-1} + \cdots + \omega_{\mu-1}(\boldsymbol{x}^{1/\varrho})z + \omega_\mu(\boldsymbol{x}^{1/\varrho}) \in k[\![x_1^{1/\varrho}, \dots, x_n^{1/\varrho}]\!][z],$$

with $\mu > 1$, and $\varrho \in \mathbb{Z}_>$, there exists a linear automorphism Φ of $\mathbb{R}^t$, which is a composition of a finite sequence of order-preserving monomial blowing-ups, and a positive integer π, such that all the roots of $\Phi\big(\Pi(\boldsymbol{x}, z)\big) = 0$ belong to $k[\![\boldsymbol{x}^{1/\pi}]\!]$.

Remark 15. As noted above, the case $n = 1$ is the very well known classical Newton-Puiseux Theorem. The suite requires the reader to know in some depth the proof, say as in [7].

For our purposes, it will suffice to show here a *very* brief sketch of the methods for $n = 1$ to fix some ideas and notations.

Suppose then an equation of the form

$$\Pi(x, z) = z^\mu + \omega_1(x^{1/\varrho})z^{\mu-1} + \cdots + \omega_\mu(x^{1/\varrho}) + \omega_\mu(x^{1/\varrho}),$$

where we will put $\omega_0 = 1$. If z_0 is to be a root of $\Pi(x, z)$, we can write

$$z_0 = \alpha_1 x^{\gamma_1} + \alpha_2 x^{\gamma_2} + \alpha_3 x^{\gamma_3} + \cdots\,, \qquad \gamma_1 < \gamma_2 < \cdots\,,$$

where $\gamma_i \in \mathbb{Q}_>$ for all i. Rewriting z_0 as $z_0 = x^\gamma(\alpha + z_0')$, with $\gamma = \gamma_1$ and $\alpha = \alpha_1$, and substituting back into $\Pi(x, z)$, we have

$$\begin{aligned} \Pi(x, z_0) &= \omega_\mu(x^{1/\varrho}) + \omega_{\mu-1}(x^{1/\varrho})\big[x^\gamma(\alpha + z_0')\big] + \cdots + \big[x^\gamma(\alpha + z_0')\big]^\mu \\ &= \omega_\mu(x^{1/\varrho}) + \omega_{\mu-1}(x^{1/\varrho})x^\gamma\alpha + \cdots + \omega_0 x^{\gamma\mu}\alpha^\mu + \Sigma(x, z_0'), \end{aligned} \tag{1}$$

where $\Sigma(x, z_0')$ contains all terms on z_0'. The idea behind the theorem is to solve for γ and α, and iterate the construction.

Since the order of z_0' is $\gamma_2 > 0$, each term in $\Sigma(x, z_0')$ has strictly greater order than some $\omega_{\mu-r}(x^{1/\varrho})x^{r\gamma}\alpha^r$. Now, a necessary condition for $\Pi(x, z_0')$ to vanish is that the lowest order terms cancel out, so there must be at least two values of r such that

$$\beta = \nu_{\mu-r_1} + r_1\gamma = \nu_{\mu-r_2} + r_2\gamma \leq \nu_{\mu-r} + r\gamma, \qquad \text{for } r = 0, \dots, \mu, \tag{2}$$

and where $\nu_{\mu-r}$ is the order of $\omega_{\mu-r}(x^{1/\varrho})$.

If we group the lowest order terms in Equation (1), we obtain an equation in α, called the *characteristic equation*, of the form

$$C(\alpha) = \sum_h \omega'_{\mu-h}\alpha^h, \qquad \omega'_{\mu-h} \in k, \tag{3}$$

and where h runs over all terms with $\nu_{\mu-h} + h\gamma = \beta$.

We need now to find possible values for γ, which we do by looking at the Newton diagram of $\Pi(x,z)$. Equation (2) implies that there exists a β such that all points of $\mathcal{E}\big(\Pi(x,z)\big)$ lie on or above the line $u + \gamma v = \beta$ and at least two lie exactly on it. The linear form $L(u,v) = u + \gamma v$ is called an *admissible linear form* for $\mathcal{E}\big(\Pi(x,z)\big)$. Bear in mind that the line $u + \gamma v = \beta$ might be vertical at the very first step.

The possible values of γ are then determined by the slopes of the Newton polygon, and once γ is fixed, we can solve for α in Equation (3). Once we have γ and α, we can write

$$\Pi_1(x, z'') = \Pi\big(x, x^{\gamma_1}\alpha_1 + z''\big),$$

and apply the previous procedure of computing the first term to $\Pi_1(x,z'')$, which is also monic in z''. The proof is completed in [7] by showing that (a) we can always solve for α in Equation (3), (b) after the very first step the Newton polygon has a segment of negative slope and that (c) after a finite number of steps, the γ_i have a common denominator (this is expressed by saying that the root z_0 has bounded denominators).

It is obvious that part (a) of the proof is trivial if we start from an algebraically closed field k, but throught the induction we will have $\omega_{\mu-h} \in k[\![x_2^{1/\varrho}, \dots, x_n^{1/\varrho}]\!]$. It should be noted that, since we will be applying this very procedure for the general case, considering $\mathcal{E}_1\big(P(\boldsymbol{x}, z)\big)$ and following the proof for $n = 1$, the only part we have to prove is (a). We do it in three lemmas.

Now, we start with our equation $P(\boldsymbol{x}, z) = 0$ and apply to it the classical Newton Procedure, taking x_1 and z as the independent and dependent variables, respectively. We consider the case in which the first admissible segment is vertical.

Lemma 16. *Let us assume that $\mathcal{E}_1(P)$ has a point on the vertical axis other than $(0,m)$ and that we choose as the first admissible segment the vertical one. Then there exists a linear automorphism Φ_1 of $\mathbb{R}^n$, which is a composition of a finite sequence of order-preserving monomial blowing-ups, leaving invariant the first coordinate of any vector, and a positive integer d_1 such that the roots of the corresponding $C(\alpha)$ belong to $k[\![x_2^{1/d_1}, \dots, x_n^{1/d_1}]\!]$.*

Proof. If we have chosen the vertical segment at the first step of the Newton Procedure, the corresponding characteristic equation is $C(\alpha) = 0$ (α is the unknown), where

$$C(\alpha) = \alpha^m + \widehat{w}_{i_1}(x_2, \dots, x_n)\alpha^{m-i_1} + \cdots + \widehat{w}_{i_s}(x_2, \dots, x_n)\alpha^{m-i_s}\,,$$

$s \geq 1$, $i_1 < \cdots < i_s$ and $\widehat{w}_{i_j}(x_2, \dots, x_n) = w_{i_j}(0, x_2, \dots, x_n) \neq 0$, $j = 1, \dots, s$. Then $C(\alpha) = 0$ is an equation of integral dependence over less than n variables and, by **IA**, there exist $d_1 \in \mathbb{Z}_>$ and a linear automorphism Φ_1 of $\mathbb{R}^n$, which is a composition of a finite sequence of order-preserving monomial blowing-ups, leaving invariant the first coordinate of any vector, such that the transforms by Φ_1 of all the roots of $\Phi_1\big(C(\alpha)\big) = 0$ belong to $k[\![x_2^{1/d_1}, \dots, x_n^{1/d_1}]\!]$. ∎

Lemma 17. *Let $P'(\boldsymbol{x}^{1/p}, z') = 0$ be an equation,*

$$P'(\boldsymbol{x}^{1/p}, z') = w'_0(\boldsymbol{x}^{1/p})(z')^m + w'_1(\boldsymbol{x}^{1/p})(z')^{m-1} + \cdots + w'_{m-1}(\boldsymbol{x}^{1/p})z' + w'_m(\boldsymbol{x}^{1/p})$$

with $m > 1$, $w'_i(\boldsymbol{x}^{1/p}) \in k[\![\boldsymbol{x}^{1/p}]\!]$, $p \in \mathbb{Z}_>$. Let us assume that the first step of the Newton Procedure, applied to $P'(\boldsymbol{x}^{1/p}, z')$, uses any admissible segment of negative slope in $\mathcal{E}_1(P')$. Under **IA***, there exist a linear automorphism Φ' of $\mathbb{R}^n$, which is a composition of a finite sequence of order-preserving monomial blowing-ups, and a positive integer p_1, such that the roots of the corresponding $C(\alpha)$ computed thorough the chosen segment, belong to $k[\![\boldsymbol{x}^{1/p_1}]\!]$.*

Proof. Let us write $\nu_t = \nu_{x_1}(w'_t(\boldsymbol{x}^{1/p}))$, $t = 0, 1, \ldots, m$. We have chosen an admissible linear form $L = u + \gamma v$, where u, v are the variables, with $\gamma \in \mathbb{Q}_>$, attaining a minimum $\mu \in \mathbb{Q}_>$ on $\mathcal{E}_1(P')$ at a finite set of points $(\nu_{m_1}, m-m_1), \ldots, (\nu_{m_s}, m-m_s)$, with $m - m_1 > \cdots > m - m_s$. Since $\mu = \nu_{m_t} + \gamma(m - m_t)$, $\forall t = 1, \ldots, s$, one must have $\nu_{m_1} < \cdots < \nu_{m_s}$.

Let us write $\mathcal{E}(w'_{m_t}(\boldsymbol{x}^{1/p})) = \frac{1}{p}\Lambda_{m_t}$ with $\Lambda_{m_t} \subset \mathbb{Z}^n_{\geq}$, for all $t = 1, \ldots, s$. By corollary 8, there exist a linear automorphism Φ'' of $\mathbb{R}^n$, which is a composition of a finite sequence of order-preserving monomial blowing-ups, and vectors $\boldsymbol{a}_{m_t} = (a_{m_t,1}, \ldots, a_{m_t,n}) \in \Phi''(\Lambda_{m_t})$, such that $\Phi''(\Lambda_{m_t}) \subset \boldsymbol{a}_{m_t} + \mathbb{Z}^n_{\geq}$, that is, $\Phi''(w'_{m_t}(\boldsymbol{x}^{1/p})) = \boldsymbol{x}^{\boldsymbol{a}_{m_t}/p}\widehat{w}'_{m_t}(\boldsymbol{x}^{1/p})$ where $\widehat{w}'_{m_t}(\boldsymbol{0}) \neq 0$, for all $t = 1, \ldots, s$. We now take the equation $\Phi''(P'(\boldsymbol{x}^{1/p}, z')) = 0$; one has $\mathcal{E}_1(\Phi''(P')) = \mathcal{E}_1(P')$ by remark 14.

In this situation, we have a set of positive rationals

$$\Omega_1 = \{\nu_{m_{t'}} - \nu_{m_t} = (a_{m_{t'},1} - a_{m_t,1})/p \mid 1 \leq t < t' \leq s\}$$

and a set of rationals

$$\Omega_2 = \{(a_{m_t,j} - a_{m_{t'},j})/p \mid 1 \leq t < t' \leq s\,, j = 2, \ldots, n\}\,.$$

We see that there exists a positive integer e such that each element of $e\Omega_1$ is greater than all the elements of Ω_2. In fact, it is enough to take the minimum ω_1 of Ω_1, the maximum ω_2 of Ω_2 and $e \in \mathbb{Z}_>$ such that $e\omega_1 > \omega_2$. For each t, t' such that $1 \leq t < t' \leq s$ and each $j = 2, \ldots, n$ we have that $e(a_{m_{t'},1}/p - a_{m_t,1}/p) > a_{m_t,j}/p - a_{m_{t'},j}/p$, so $a_{m_t,j}/p + ea_{m_t,1}/p < a_{m_{t'},j}/p + ea_{m_{t'},1}/p$. Let $\Phi''_1 = \varphi^e_{1n} \cdots \varphi^e_{12}$ (which clearly commute); then, if $\boldsymbol{b}_{m_t} = \Phi''_1(\boldsymbol{a}_{m_t}) = (b_{m_t,1}, \ldots, b_{m_t,n})$, one has $\nu_{m_t} = a_{m_t,1}/p = b_{m_t,1}/p$ and $\boldsymbol{b}_{m_1} \ll \boldsymbol{b}_{m_2} \ll \cdots \ll \boldsymbol{b}_{m_s}$. Moreover, $\Phi''_1\Phi''(w'_{m_t}(\boldsymbol{x}^{1/p})) = \boldsymbol{x}^{\boldsymbol{b}_{m_t}/p}\Phi''_1(\widehat{w}'_{m_t}(\boldsymbol{x}^{1/p}))$ and this last factor is a unit.

We now operate with the equation $\Phi''_1\Phi''(P'(\boldsymbol{x}^{1/p}, z')) = 0$ (recall that $\mathcal{E}_1(\Phi''_1\Phi''(P'(\boldsymbol{x}^{1/p}, z')) = \mathcal{E}_1(P'(\boldsymbol{x}^{1/p}, z')))$, with the same chosen linear form L. To compute the corresponding terms of the roots of $\Phi''_1\Phi''(P'(\boldsymbol{x}^{1/p}, z')) = 0$ we take the change of variable $z' = x_1^{\gamma}(\alpha + z'_1)$ (where α is the unknown) and solve the characteristic equation $C(\alpha) = 0$ with

$$C(\alpha) = \widehat{w}''_{m_1}(x_2^{1/p}, \ldots, x_n^{1/p})\alpha^{m-m_1} + \cdots + \widehat{w}''_{m_s}(x_2^{1/p}, \ldots, x_n^{1/p})\alpha^{m-m_s}\,,$$

where $\widehat{w}''_{m_t}(x_2^{1/p}, \ldots, x_n^{1/p}) = \Phi''_1\Phi''(w'_{m_t}(\boldsymbol{x}^{1/p}))x_1^{-\nu_{m_t}}$ evaluated at $x_1 = 0$, that is $\widehat{w}''_{m_t}(x_2^{1/p}, \ldots, x_n^{1/p}) = (\boldsymbol{x}^{\boldsymbol{b}_{m_t}/p}/x_1^{b_{m_t,1}})\ \Phi''_1(\widehat{w}'_t(0, x_2^{1/p}, \ldots, x_n^{1/p}))$ and the last factor is a unit. By the above arguments, $\widehat{w}''_{m_1}(x_2^{1/p}, \ldots, x_n^{1/p})$ divides all the other coefficients of the characteristic equation; dividing by it, $C(\alpha) = 0$ becomes an equation of integral dependence of α over $k[\![x_2^{1/p}, \ldots, x_n^{1/p}]\!]$.

By **IA**, there exist $p' \in \mathbb{Z}_>$ and a linear automorphism Φ_2'' of $\mathbb{R}^n$, which is a composition of a finite sequence of order-preserving monomial blowing-ups leaving invariant the first coordinate of every vector of $\mathbb{R}^n$, such that, if α_j, $j = 1, \ldots, r$ are the non-zero roots of $C(\alpha) = 0$, one has that $\Phi_2''(\alpha_j) \in k[\![x_2^{1/p'}, \ldots, x_n^{1/p'}]\!]$. Writing $\Phi' = \Phi_2''\Phi_1''\Phi''$ and taking a common denominator p_1, we have the lemma. ▍

Proposition 18. *Let $i \geq 1$ be any integer; there exists a linear automorphism Φ_i of $\mathbb{R}^n$, which is a composition of a finite sequence of order-preserving monomial blowing-ups, and a positive integer d_i such that, if we apply i steps of the classical Newton Procedure to the equation $\Phi_i\big(P(\boldsymbol{x}, z)\big) = 0$ in (x_1, z), in any possible way, the sum of the first i terms of any root we obtain belongs to $k[\![\boldsymbol{x}^{1/d_i}]\!]$.*

Proof. We remind that, by Remark 14, the evolution through the Newton Procedure of the Newton diagram $\mathcal{E}_1(P)$ of $P(\boldsymbol{x}, z)$ is the same as the evolution of the Newton diagram $\mathcal{E}_1\big(\Phi(P)\big)$, for any linear automorphism Φ of $\mathbb{R}^n$, which is a composition of a finite sequence of order-preserving monomial blowing-ups; only the coefficients of the characteristic equations change.

Lemmas 16 and 17 show that we can indeed solve the characteristic equation in each step of the Newton procedure, perhaps adding a composition of order-preserving blowing-ups for every negative slope of the corresponding Newton diagram: since all Φ_i are a composition of order-preserving monomial blowing-ups, we have that the characteristic equation of $\Phi_i\big(P(\boldsymbol{x}, z)\big)$ is exactly $\Phi_i\big(C(\alpha)\big)$. With this in mind, and the fact that any order-preserving monomial blowing-up preserves the first quadrant of $\mathbb{R}^n$, the proposition is an obvious consequence of lemmas 16 and 17. ▍

Lemma 19. *Let $P'(\boldsymbol{x}, z') = 0$ be an equation,*

$$P'(\boldsymbol{x}, z') = w_0'(\boldsymbol{x}^{1/p})(z')^m + w_1'(\boldsymbol{x}^{1/p})(z')^{m-1} + \cdots + w_{m-1}'(\boldsymbol{x}^{1/p})z' + w_m'(\boldsymbol{x}^{1/p})$$

with $m > 1$, $w_i'(\boldsymbol{x}^{1/p}) \in k[\![\boldsymbol{x}^{1/p}]\!]$, $i = 0, 1, \ldots, m$, $p \in \mathbb{Z}_>$. Let us assume that $w_i'(0, x_2^{1/p}, \ldots, x_n^{1/p}) = 0$, for all $i \in \{0, \ldots m-2, m\}$ and that $\beta = w_{m-1}'(0, x_2^{1/p}, \ldots, x_n^{1/p}) \neq 0$. Then $P'(\boldsymbol{x}, z') = 0$ has only one root with positive x_1-order and there exist a linear automorphism Φ' of $\mathbb{R}^n$, which is a composition of a finite sequence of order-preserving monomial blowing-ups, such that the transform of this root by Φ' belongs to $k[\![\boldsymbol{x}^{1/p}]\!]$.

Proof. The only admissible segment of $\mathcal{E}_1(P')$ with negative slope consists just of the two points $(0, 1)$ and $\big(\nu_{x_1}\big(w_m'(\boldsymbol{x}^{1/p})\big), 0\big)$, so the admissible linear form is $L = u + \gamma v$ with $\gamma = \nu_{x_1}\big(w_m'(\boldsymbol{x}^{1/p})\big)$, and the minimum it attains on $\mathcal{E}_1(P')$ is γ. The characteristic equation is $0 = C(\alpha) = \beta\alpha + \alpha'$, where α' is the result of making $x_1 = 0$ in $w_m'(\boldsymbol{x}^{1/p})/x_1^{\nu_{x_1}(w_m'(\boldsymbol{x}^{1/p}))}$, so $\alpha = -\alpha'/\beta$. This yields αx_1^γ as the only possible first term of any root of $P'(\boldsymbol{x}, z') = 0$ with positive x_1-order.

Now, we must perform the change of variables $z' = x_1^\gamma(\alpha + z_1')$ and divide the result by x_1^γ. The transform of the monomial $\beta z'$ is $\beta\alpha x_1^\gamma + \beta x_1^\gamma z_1'$. The first summand of this expression cancels with the initial form in x_1 of $w_m'(\boldsymbol{x}^{1/p})$. After this cancellation and division by x_1^γ it remains the monomial $\beta z_1'$, which cannot be cancelled with any other coming from $x_1^a(z')^b$ because all of them contain a power of x_1 with exponent of the form $L(a, b) > \gamma$. This shows that the transform equation is of the same form as $P'(\boldsymbol{x}, z') = 0$, with the same β. This implies the uniqueness of the root with positive x_1-order.

Let us write $\mathcal{E}(\beta) = \frac{1}{p}\Lambda$, with $\emptyset \neq \Lambda \subset \mathbb{Z}^{n-1}_{\geq}$. By lemma 7, there exists a linear automorphism Φ' of $\mathbb{R}^{n-1}$, which is a composition of a finite sequence of order-preserving monomial blowing-ups, and a vector with integer coordinates $(a_2, \ldots, a_n) \in \Phi'(\Lambda)$ such that $\Phi'(\Lambda) \subset (a_2, \ldots, a_n) + \mathbb{Z}^{n-1}_{\geq}$, that is $\Phi'(\beta) = x_2^{a_2/p} \cdots x_n^{a_n/p}\beta'$, where β' is a unit in $k[\![x_2^{1/p}, \ldots, x_n^{1/p}]\!]$. Since all the monomials occurring in $P'(\boldsymbol{x}, z')$ contain x_1 raised to a power of the form a/p, $a \in \mathbb{Z}_>$, except those in $\beta z'$, the same happens with $\Phi'(P'(\boldsymbol{x}, z'))$ and the exception is $x_2^{a_2/p} \cdots x_n^{a_n/p}\beta'$. Let $\Phi'' = \varphi_{1n}^{a_n} \cdots \varphi_{12}^{a_2}$; then all the monomials in $\Phi''\Phi'(P'(\boldsymbol{x}, z'))$ are divisible by $x_2^{a_2/p} \cdots x_n^{a_n/p}$ and only those occurring in $x_2^{a_2/p} \cdots x_n^{a_n/p}\beta'$ are not divisible by x_1. Applying the Newton Procedure to $\Phi''\Phi'\big(P'(\boldsymbol{x}, z')\big)$, as we did before to $P'(\boldsymbol{x}, z')$, it is now clear that the only root with positive x_1-order of $\Phi''\Phi'\big(P'(\boldsymbol{x}, z')\big) = 0$ belongs to $k[\![\boldsymbol{x}^{1/p}]\!]$. ▮

Theorem 20. *There exists a positive integer d and a linear automorphism Φ of $\mathbb{R}^n$, which is a composition of a finite sequence of order-preserving monomial blowing-ups, such that all the roots of $\Phi\big(P(\boldsymbol{x}, z)\big) = 0$ belong to $k[\![\boldsymbol{x}^{1/d}]\!]$.*

Proof. For any $i \geq 1$, proposition 18 tells us that our theorem is true if we consider, not the whole roots, but the truncation of them to the first i terms. In fact, this proposition tells us this result only for some roots of the equation. Taking all the automorphisms, composing them, and taking a common denominator, we have the result proven for all the roots because $\mathbb{R}^n_0$ is stable by any monomial blowing-up. We know that the classical Newton Procedure, followed with all the necessary choices to compute all the roots of $P(\boldsymbol{x}, z) = 0$ arrives at a step in which all the equations are of the type of the one in lemma 19. Composing with the new order-preserving monomial blowing-ups given by this lemma, we have our result. ▮

We finally arrive to the

Proof of Theorem 1. By theorem 20, the roots $\zeta_1, \ldots, \zeta_m$ of $\Phi\big(P(\boldsymbol{x}, z)\big) = 0$ belong to $k[\![\boldsymbol{x}^{1/d}]\!]$. Taking into account that $\Phi\big(P(\boldsymbol{x}, z)\big) = \prod_{i=1}^m (z - \zeta_i)$ and the fact that every monomial blowing-down is a field k-automorphism of $K_{n,d}$ (c.f. corollary 6), we have that

$$P(\boldsymbol{x}, z) = \Phi^{-1}\Phi(P(\boldsymbol{x}, z)) = \prod_{i=1}^{m} \big(z - \Phi^{-1}(\zeta_i)\big),$$

so the roots of $P(\boldsymbol{x}, z)$ are the $\Phi^{-1}(\zeta_i)$, $i = 1, \ldots, m$ and $\mathcal{E}\big(\Phi^{-1}(\zeta_i)\big) \subseteq \Phi^{-1}(\mathbb{R}^n_{\geq})$, which is an $\mathcal{S}$-cone. ▮

4. Applications: integral and algebraic closures

Throughout this section, we will denote by $\mathcal{S}$ the set of the finite compositions of order-preserving monomial blowing-downs of $\mathbb{R}^n$ and define $\Lambda = \{\Phi(\mathbb{R}^n_{\geq}) \mid \Phi \in \mathcal{S}\}$. To shorten the sentences, we will simply say "blowing-up" (res. "blowing-down") instead of order-preserving monomial blowing-ups (resp. blowing-downs).

Remark 21. Let $1 \leq i < j \leq n$ be two indices and φ_{ij} (resp. φ_{ij}^{-1}) be the corresponding blowing-up (resp. blowing-down); we write φ_{ij} (resp. φ_{ij}^{-1}) in matrix form as $\boldsymbol{a} \to \boldsymbol{a}B$. Then $B = E_{ij}(1)$ (resp. $B = E_{ij}(-1)$), the elementary matrix equal to the $n \times n$ identity matrix I_n except for the fact that it has a 1 (resp. -1)

at the (i, j) position. Then the matrix of the composition of a finite sequence of blowing-ups (resp. blowing-downs) has always 1's at the main diagonal and it is upper-triangular.

A $n \times n$ matrix A with 1's at the main diagonal and upper-triangular is the matrix of the composition of a finite sequence of blowing-ups if and only if it has non-negative integer entries. In fact, the condition is obviously necessary. If A has non-negative integer entries, then a suitable right-multiplication by a finite number of matrices of the form $E_{ij}(-1)$, $i < j$, gives the identity matrix I_n, so A is the matrix of the composition of a finite sequence of blowing-ups.

In the case of blowing-downs, we can say nothing about entries. It is clear that, if a matrix A with 1's at the main diagonal has non-positive entries outside it, then A is the matrix of the composition of a finite sequence of blowing-downs, for it can be right-multiplied by a finite sequence of matrices $E_{ij}(1)$, $i < j$, to obtain I_n. However, this condition being sufficient, it is not necessary:

$$\begin{pmatrix} 1 & -6 & -8 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & -6 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -9 & 28 \\ 0 & 1 & -12 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4}$$

On the other hand, let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be the canonical base of $\mathbb{R}^n_{\geq}$ and let Φ be a composition of a finite sequence of blowig-ups (resp. blowing-downs); then

$$\Phi(\mathbb{R}^n_{\geq}) = \left\{ \sum_{i=1}^{n} \lambda_i \Phi(\boldsymbol{e}_i) \,\middle|\, \lambda_i \in \mathbb{R}_{\geq} \, , \, i = 1, \ldots, n \right\} ,$$

the set of non-negative linear combinations of $\{\Phi(\boldsymbol{e}_1), \ldots, \Phi(\boldsymbol{e}_n)\}$. If A is the matrix corresponding to Φ according to the above notations, then the row vectors of A are just $\{\Phi(\boldsymbol{e}_1), \ldots, \Phi(\boldsymbol{e}_n)\}$.

Lemma 22. *For every $\Phi \in \mathcal{S}$, all the elements of $\Phi(\mathbb{R}^n_{\geq}) \setminus \{\mathbf{0}\}$ are lexicographically greater than $\mathbf{0}$. Moreover, $\mathbb{R}^n_{\geq} \subset \Phi(\mathbb{R}^n_{\geq})$.*

Proof. The first assertion is trivial; let us show the second. Let A be the matrix of Φ; then A^{-1} is the matrix of a composition of blowing-ups and $I_n = A^{-1}A$, which means that the vectors of the canonical basis of $\mathbb{R}^n$ belong to the semigroup generated by the rows of A, so $\mathbb{R}^n_{\geq} \subset \Phi(\mathbb{R}^n_{\geq})$. ▌

Remark 23. It is not true in general that, if $\Gamma \in \Lambda$ and $\Phi \in \mathcal{S}$ then $\Gamma \subset \Phi(\Gamma)$ or $\Phi(\Gamma) \subset \Gamma$. For instance, if Γ is given by the row vectors of the matrix A, Φ is given by the matrix B then $\Phi(\Gamma)$ is given by the row vectors of the matrix $Q = AB$, where

$$A = \begin{pmatrix} 1 & -4 & -1 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} 1 & -4 & 19 \\ 0 & 1 & -14 \\ 0 & 0 & 1 \end{pmatrix} ,$$

then

$$AQ^{-1} = \begin{pmatrix} 1 & 0 & -20 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} , \quad QA^{-1} = \begin{pmatrix} 1 & 0 & 20 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix} ,$$

and both have a negative entry.

Lemma 24. *Let $\Gamma_1 = \Phi_1(\mathbb{R}^n_{\geq})$, $\Gamma_2 = \Phi_2(\mathbb{R}^n_{\geq})$, with Φ_1, $\Phi_2 \in \mathcal{S}$ be two S-cones. Then there exists $\Phi \in \mathcal{S}$ such that $\Gamma_1 \subseteq \Phi(\mathbb{R}^n_{\geq})$ and $\Gamma_2 \subset \Phi(\mathbb{R}^n_{\geq})$.*

Proof. Let A_1, A_2 be the respective matrices of Φ_1 and Φ_2; it is easy to see that there exists a matrix B, corresponding to a finite composition of blowing-ups, such that $X = A_1 B$, $Y = A_2 B$ have their row vectors in $\mathbb{R}^n_{\geq}$. The matrix B^{-1} corresponds to a $\Phi \in \mathcal{S}$ and $A_1 = XB^{-1}$, $A_2 = YB^{-1}$, which means that the row vectors of A_1, A_2 belong to the semigroup generated by the row vectors of B^{-1}. This shows that $\Gamma_1 \subseteq \Phi(\mathbb{R}^n_{\geq})$ and $\Gamma_2 \subseteq \Phi(\mathbb{R}^n_{\geq})$. ∎

Definition 25. If $\Gamma \in \Lambda$ and $d \in \mathbb{Z}_>$, we will write $k[\![\Gamma, d]\!]$ for the subring of $K_{n,d}$ consisting of the Puiseux power series whose Newton diagram is contained in Γ.

Lemma 26. *For $\Gamma, \Gamma' \in \Lambda$, $d, d' \in \mathbb{Z}_>$ one has that*

$$k[\![\Gamma, d]\!] \subset k[\![\Gamma', d']\!] \iff \Gamma \subseteq \Gamma' \text{ and } d | d' .$$

Therefore, the set of rings $k[\![\Gamma, d]\!]$, together with the inclusions, is a direct system of k-algebras. Also the set of their quotient fields $Q(k[\![\Gamma, d]\!])$, together with the inclusions, is a direct system of fields.

The proof is straightforward in view of lemma 24.

Definition 27. The k-algebra $\bigcup_{(\Gamma,d)\in\Lambda\times\mathbb{Z}_>} k[\![\Gamma, d]\!]$ will be denoted by $k[\![x_1, \ldots, x_n]\!]^* = k[\![\boldsymbol{x}]\!]^*$. If $Q(-)$ denotes quotient fields, field $\bigcup_{(\Gamma,d)\in\Lambda\times\mathbb{Z}_>} Q(k[\![\Gamma, d]\!])$ will be denoted by $k((x_1, \ldots, x_n))^* = k((\boldsymbol{x}))^*$. Note that $k((\boldsymbol{x}))^*$ is the quotient field of $k[\![\Gamma, d]\!]^*$.

We take again the Newton arguments. From theorem 1 we derive an easy consequence, namely the following

Corollary 28. *The roots of a polynomial*

$$P(\boldsymbol{x}^{1/d}, z) = z^m + \omega_1(\boldsymbol{x}^{1/d}) z^{m-1} + \cdots + \omega_m(\boldsymbol{x}^{1/d}) ,$$

where $\omega_i(\boldsymbol{x}^{1/d}) \in k[\![\boldsymbol{x}^{1/d}]\!]$, $i = 1, \ldots, m$, are Puiseux power series in some $K_{n,dd'}$, such that their Newton diagrams are contained in an S-cone.

Remark 29. Note that Theorem 1 does not guarantee that all series with exponents in S-cones are integral over $k[\![\boldsymbol{x}]\!]$. For instance, the power series $f = x^{1/2}\sqrt{1 - x/y}$ is algebraic over $k((x, y))$, its minimal polynomial is $z^2 - x(1 - x/y) \notin k[\![x, y]\!]$, so it cannot be integral over $k[\![x, y]\!]$ because this ring is integrally closed (c.f. [8], theorem 4, page 260).

Finally we arrive at the results we wanted, namely

Theorem 30. *The ring $k[\![\boldsymbol{x}]\!]^*$ is integrally closed (in its quotient field).*

Proof. It suffices to show that every polynomial

$$P = z^m + v_1(\boldsymbol{x}^{1/d_1}) z^{m-1} + \cdots + v_m(\boldsymbol{x}^{1/d_m}),$$

where $v_i \in k[\![\Gamma_i, d_i]\!]$, has all its roots in $k[\![\boldsymbol{x}]\!]^*$. Note that, by taking common denominators and common cones, we can suppose that there exists a single pair (Γ, d) such that $v_i(\boldsymbol{x}^{1/d_i}) = v_i(\boldsymbol{x}^{1/d}) \in k[\![\Gamma, d]\!]$. Now, $\Gamma = \Phi(\mathbb{R}^n_{\geq})$ for some $\Phi \in \mathcal{S}$; write

$$P' = \Phi^{-1}(P) = z^m + \omega_1(\boldsymbol{x}^{1/d}) z^{m-1} + \cdots + \omega_m(\boldsymbol{x}^{1/d}),$$

with $\omega_i(\boldsymbol{x}^{1/d}) \in k[\![\boldsymbol{x}^{1/d}]\!]$, for $i = 1, \ldots, m$.

We now apply Corollary 28 to this polynomial; then all the roots of P' belong to some $k[\![\Gamma', dd']\!]$, where $\Gamma' \in \Lambda$; call them $g_1, \ldots, g_r$. But this means that $\Phi(g_i) \in k[\![\Phi(\Gamma'), dd']\!]$ are the roots of P. ∎

Corollary 31. *$k[\![\boldsymbol{x}]\!]^*$ is the integral closure of $R = \bigcup_{\Gamma\in\Lambda} k[\![\Gamma, 1]\!]$.*

Proof. This proof uses well-known facts on Galois theory of Puiseux power series (c.f., for instance, [4], Chapter V, §1.2 and §1.3). Any $f \in k[\![\Gamma, d]\!]$ gives raise to an algebraic extension of $Q(R)$ generated by monomials with exponents in Γ. By ordinary Puiseux power series computations, the minimal polynomial of f over $Q(R)$ is and equation of integral dependence over R (c.f. Kiyek, loc.cit.). This ends our proof ▌

Theorem 32. *The field $k((\boldsymbol{x}))^*$ is algebraically closed.*

Proof. For any algebraic extension $L = k((\boldsymbol{x}))^*[\alpha]$, there exists $c \in k[\![\boldsymbol{x}]\!]^*$ such that $\alpha' = c\alpha$ is integral over $k[\![\boldsymbol{x}]\!]^*$, and $L = k((\boldsymbol{x}))^*[\alpha']$ (take a common denominator $c \in k[\![\boldsymbol{x}]\!]^*$ of the coefficients of the minimal polynomial of α over $k((\boldsymbol{x}))^*$). This means that α' satisfies an equation of the form

$$P = z^m + v_1(\boldsymbol{x}^{1/d_1})z^{m-1} + \cdots + v_m(\boldsymbol{x}^{1/d_m}).$$

By Theorem 30, the roots of such an equation lie in $k[\![\boldsymbol{x}]\!]^*$, and thus, $\alpha \in k((x))^*$. ▌

Corollary 33. *$k((x))^*$ is the algberaic closure of $Q(R)$.*

The proof is straightforward from theorem 32 and corollary 33.

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References

[1] S.S. Abhyankar, *On the ramification of algebraic functions.* Amer. J. Math., **36** (1955) 575–592.

[2] G. Ewald. *Combinatorial convexity and algebraic geometry.* Graduate Texts in Mathematics. Vol. **168**, Springer-Verlag, 1996.

[3] P.D. González Pérez: *Singularités quasi-ordinaires toriques et polyhèdre de Newton du discriminant*, Canad. J. Math. Vol. **52**(2), 2000. pp. 348–368

[4] K. Kiyek; J.L. Vicente: *Resolution of curve and surface singularities in Characteristc Zero*, Vol **4**. Algebras and Applications. Kluwer Academic Publishers. ISBN 1-4020-2028-7, Dordrecht, The Netherlands, 2004.

[5] John McDonald, *Fiber polytopes and fractional power series.* Journal of Pure Appl. Algebra, Vol. **104**, 1995 213–233.

[6] M. J. Soto and J. L. Vicente. *Polyhedral cones and monomial blowing-ups.* Linear Algebra and its Applications. Vol. **412**, 2006, pp. 362–372

[7] Robert J. Walker. *Algebraic curves.* Springer-Verlag. New York. 1978.

[8] O. Zariski and P. Samuel. *Commutative Algebra I.* Van Nostrand, 1958.

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