S0022-247X(03)00845-X/FLA AID:9036 Vol.●●●(●●●) yjmaa9036 P.1 (1-11)
ELSGMLTM(YJMAA):m1 2003/11/25 Prn:27/11/2003; 13:15 by:ML p. 1

Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

ELSEVIER

J. Math. Anal. Appl. ●●● (●●●●) ●●●–●●●

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Averaging the k largest distances among n: k-centra in Banach spaces

Pier Luigi Papini [1] and Justo Puerto [*,2]

Received 4 November 2002

Submitted by J.B. Conway

Abstract

Given a Banach space X let $A \subset X$ containing at least k points. In location theory, reliability analysis, and theoretical computer science, it is useful to minimize the sum of distances from the k furthest points of A: this problem has received some attention for X a finite metric space (a network), see, e.g., [Discrete Appl. Math. 109 (2001) 293]; in the case $X = E^n$, $k = 2$ or 3, and A compact some results have been given in [Math. Notes 59 (1996) 507]; also, in the field of theoretical computer science it has been considered in [T. Tokuyama, Minimax parametric optimization problems in multidimensional parametric searching, in: Proc. 33rd Annu. ACM Symp. on Theory of Computing, 2001, pp. 75–84]. Here we study the above problem for a finite set $A \subset X$, generalizing—among others things—the results in [Math. Notes 59 (1996) 507].

1. Introduction

Let X be a Banach space; let $A = \{a_1, \dots, a_n\} \subset X$, $n \geqslant 3$, $a_i \neq a_j$ for $i \neq j$, a finite set whose cardinality will be denoted by $\#A$. Also, we denote by $\delta(A)$ the diameter of A.

Given $x \in X$, let $\sigma(x) = (\sigma_1(x), \dots, \sigma_n(x))$ be an ordering of the elements of $\{1, 2, \dots, n\}$ such that $\|x - a_{\sigma_1(x)}\| \geqslant \|x - a_{\sigma_2(x)}\| \geqslant \cdots \geqslant \|x - a_{\sigma_n(x)}\|$.

Given an integer k, $1 \leqslant k \leqslant n$, we set:

$$r_k(A, x) = \frac{1}{k}\sum_{i=1}^{k} \|x - a_{\sigma_i(x)}\| \quad \text{and} \quad r_k(A) = \inf_{x \in X} r_k(A, x).$$

* Corresponding author.
E-mail address: puerto@us.es (J. Puerto).

[1] The research of the first author was partially supported by the Italian national group G.N.A.M.P.A.

[2] The author thanks Spanish ministry of Science and Technology through grant number BFM2001-2378.

0022-247X/$ – see front matter © 2003 Published by Elsevier Inc.
doi:10.1016/j.jmaa.2003.11.011

S0022-247X(03)00845-X/FLA AID:9036 Vol.●●●(●●●) yjmaa9036 P.2 (1-11)
ELSGMLTM(YJMAA):m1 2003/11/25 Prn:27/11/2003; 13:15 by:ML p. 2

Clearly, $r_1(A)$ is the Chebyshev radius of A, that we shall also denote by $r(A)$, while $r_n(A)$ is the minimum average of distances from the points of A, usually denoted by $\mu(A)$. (We also use this notation when referring to others' results.) A point x (when it exists) such that $r_k(A,x) = r_k(A)$ will be called a k-*centrum* of A.

In particular, a 1-centrum of A is a (Chebyshev) center; an n-centrum of A is a median (or Fermat point). The term k-centrum was coined in the early seventies [15] to refer to the minimization of the function $r_k(A,x)$ when X is a finite metric space. The reader should notice that this term (k-centrum) differs from n-*center* as it is used in recent papers. In the latter, n-center means center or median for n-point sets or n-flat of a given finite set.

In this paper, we study the functions $r_k(A,x)$ and the k-centra; these problems, apart from some results given in [23], have been also considered in [11,15,16] from an algorithmic point of view. The interested reader can also find different applications of these functions in different areas of applied mathematics as reliability: optimization of systems k-out-of-n [1]; location analysis [13] or in decision theory [22], among others.

2. Preliminary results

We start with a simple remark; clearly, given a finite set $A = \{a_1,\dots,a_n\}$, for any $x \in X$ we have

$$r_1(A,x) \geqslant r_2(A,x) \geqslant \cdots \geqslant r_n(A,x).$$

From this we have the following remark.

Remark 2.1. For any A we have

$$r(A) \geqslant r_2(A) \geqslant \cdots \geqslant r_{n-1}(A) \geqslant \mu(A). \tag{1}$$

Remark 2.2. We can also give estimates in the "opposite" sense. Let $1 \leqslant k \leqslant j \leqslant n$. Given any $A = \{a_1,\dots,a_n\}$, for every $x \in X$ we have $kr_k(A,x) = \sum_{i=1}^{k} \|x - a_{\sigma_i(x)}\| \leqslant \sum_{i=1}^{j} \|x - a_{\sigma_i(x)}\| = jr_j(A,x)$; taking infimum on x, we obtain

$$kr_k(A) \leqslant jr_j(A). \tag{2}$$

A better estimate is the following (whose proof is almost trivial) proposition.

Proposition 2.1. *Given $A = \{a_1,\dots,a_n\}$, let $n \geqslant 2h$ with h an integer $1 \leqslant h \leqslant n/2$. If i, j is a pair of indexes such that $\|a_i - a_j\| = \delta(A)$, set $A_1 = A \setminus \{a_i, a_j\}$; then let i_1, j_1 be indexes such that $a_{i_1}, a_{j_1} \in A_1$ and $\|a_{i_1} - a_{j_1}\| = \delta(A_1)$; then define $A_2 = A_1 \setminus \{a_{i_1}, a_{j_1}\}$. Proceeding in this way, we obtain*

$$2hr_{2h}(A) \geqslant \delta(A) + \delta(A_1) + \delta(A_2) + \cdots + \delta(A_{h-1}). \tag{3}$$

The next result gives us some structural properties of the $r_k(A,x)$ function. They are direct consequences of basic properties of the norm in X and thus, its proof is left out.

S0022-247X(03)00845-X/FLA AID:9036 Vol.•••(•••) yjmaa9036 P.3 (1-11)
ELSGMLTM(YJMAA):m1 2003/11/25 Prn:27/11/2003; 13:15 by:ML p. 3

Proposition 2.2. *Let $A = \{a_1, \ldots, a_n\}$ and let k be an integer $1 \leqslant k \leqslant n$; then the function $r_k(A, x)$ $(x \in X)$ is 1-Lipschitz continuous and convex. Moreover, if X is strictly convex, $r_k(A, x)$ is strictly convex outside lines containing at least k points of A.*

Given A, let for $\varepsilon \geqslant 0$ and $1 \leqslant k \leqslant n = \#A$,

$$s_k(A, \varepsilon) = \left\{x \in X \colon r_k(A, x) \leqslant r_k(A) + \varepsilon\right\}. \tag{4}$$

According to Proposition 2.2, the sets $s_k(A, \varepsilon)$ are always closed and convex. Also, in a dual space, the functions $x \to \|x - a\|$ are weak*-lower semicontinuous, so the sets $s_k(A, \varepsilon)$ are bounded, w*-closed, and w*-compact. Therefore, the (possibly empty) set

$$s_k(A) = \bigcap_{\varepsilon > 0} s_k(A, \varepsilon) \tag{5}$$

is always closed, bounded, and convex, and its elements are the k-centra of A, i.e., the points x such that $r_k(A, x) = r_k(A)$.

By standard w*-compactness arguments we obtain the following proposition.

Proposition 2.3. *If X is a dual space (in particular, if X is reflexive), then $s_k(A) \neq \emptyset$ for any finite set A and any k between 1 and $\#A$.*

Remark 2.3. The above result is true, for example, if $X = l_\infty$. Also, the same result holds if X is norm-one complemented in X^{**}. The proof in the case of existence of norm-one projection is simple (and obtains following the line of proofs in [19]). General results of this type have been given in [19].

Next result shows that also other spaces have the same properties.

Theorem 2.1. *If $X = c_0$, then for every $A = \{a_1, \ldots, a_n\}$ and $1 \leqslant k \leqslant n$ we have $s_k(A) \neq \emptyset$.*

Proof. We may consider A as a subset of l^∞. Since l^∞ is a dual space, there exists $x = (x^{(1)}, x^{(2)}, \ldots, x^{(n)}, \ldots) \in l^\infty$ such that $r_k(A, x) = \inf\{r_k(A, y) \colon y \in l^\infty\}$. Since A is in c_0 there exists an index h such that $|a_i^{(j)}| \leqslant \|x - a_{\sigma_k(x)}\|$, for all $j > h$ and $i = 1, \ldots, n$. Then, $x_0 = (x^{(1)}, \ldots, x^{(h)}, 0, \ldots, 0, \ldots) \in c_0$ and

$$\|x_0 - a_i\| \leqslant \sup\left\{\sup\left\{\left|a_i^{(j)}\right| \colon j > h\right\}, \sup\left\{\left|x^{(j)} - a_i^{(j)}\right| \colon j \leqslant h\right\}\right\} \leqslant \|x - a_{\sigma_k(x)}\|,$$

for $i = 1, \ldots, n$. Hence, $r_k(A, x_0) \leqslant \|x - a_{\sigma_k(x)}\| \leqslant r_k(A, x) = r_k(A)$ and so $r_k(A, x_0) = r_k(A)$. $\square$

Remark 2.4. There are spaces where for some finite sets, centers and/or medians do not always exist; one of these spaces is a hyperplane of c_0 considered in [12]. (This does not contradict Theorem 2.1.) Examples of four-point sets with a center but without median, or with a median but without a center are indicated in [12,20]. Examples of three-point sets without k-centra for any k are shown at the end of this paper.

S0022-247X(03)00845-X/FLA AID:9036 Vol.●●●(●●●) yjmaa9036 P.4 (1-11)
ELSGMLTM(YJMAA):m1 2003/11/25 Prn:27/11/2003; 13:15 by:ML p. 4

Remark 2.5. Let $A \subset F$, A containing at least k points, F finite. Then $r_k(A,x) \leqslant r_k(F,x)$ for all $x \in X$, and so $r_k(A) \leqslant r_k(F)$ $(1 \leqslant k \leqslant \#A)$. Also, if $r_k(A) = r_k(F)$, then $s_k(A) \subset s_k(F)$.

Remark 2.6. If $m_k \in s_k(A)$ and c is a center of F, then we have the almost trivial estimate

$$\|m_k - c\| \leqslant d(A, m_k) + r(A), \tag{6}$$

where $d(A, m_k) = \inf_{x \in A} \|x - m_k\|$ denotes the distance of m_k from the set A. In fact, if $\|m_k - a_i\| = d(A, m_k)$, then we have

$$\|m_k - c\| \leqslant \|m_k - a_i\| + \|a_i - c\| \leqslant d(A, m_k) + r(A).$$

Remark 2.7. It is clear that $x \in s_n(A)$ and $\|x - a_i\| =$ constant $i = 1, 2, \ldots, n$, implies $x \in s_1(A)$. (See, for example, [3] for results of this type.) More generally, if $c_k \in s_k(A)$ and the k farthest points to c_k in A are at the same distance r_k from c_k, then we have $r(A) \leqslant r(A, c_k) = r_k(A)$; so for $i = 1, \ldots, k$, $r_i(A) = r_k(A)$, and then $c_k \in s_i(A)$.

3. General results on k-centra

We start with a general result concerning k-centra, which generalizes results contained in [23], well-known for $k = \#A$.

Theorem 3.1. *Let X be a strictly convex space and $A \subset X$; if k is odd, then $s_k(A)$ $(1 \leqslant k \leqslant n)$ contains at most one point; if k is even and $s_k(A)$ contains x' and x'', $x' \neq x''$, then there exist (at least) k points of A on the line passing through x' and x''.*

Proof. Given $A = \{a_1, \ldots, a_n\}$ and k, $1 \leqslant k \leqslant n$, if x', x'' belong to $s_k(A)$, then according to the convexity of $s_k(A)$ also $x = (x' + x'')/2$ belongs to $s_k(A)$. Let $a_1, \ldots, a_k$ be the k points of A furthest away to x' and x'', so that $\sum_{i=1}^{k} \|x - a_i\| = kr_k(A)$. Then, we have

$$\begin{aligned} kr_k(A) &= \sum_{i=1}^{k} \left\| \frac{x' + x''}{2} - a_i \right\| \leqslant \sum_{i=1}^{k} \left(\frac{\|x' - a_i\|}{2} + \frac{\|x'' - a_i\|}{2} \right) \\ &\leqslant \frac{kr_k(A, x')}{2} + \frac{kr_k(A, x'')}{2} = kr_k(A), \end{aligned}$$

so all these inequalities are equalities. This means two facts: (1) $a_1, \ldots, a_k$ are also the k points in A furthest to x; and (2) $x' - a_i = \lambda_i(x'' - a_i)$ for some non-negative λ_i, $i = 1, \ldots, k$; therefore $x', x'', a_1, \ldots, a_k$ are all collinear. This is impossible for k odd because in this case the unique median of $A' = \{a_1, \ldots, a_k\}$ is the only point of A' leaving $(k-1)/2$ points of $a_1, \ldots, a_k$ to each side ("central point"); for k even, all points letting $k/2$ on each side are medians of A'. □

Remark 3.1. The proof of the above theorem shows that if X is a strictly convex space and $A \subset X$, if $\#A$ is odd, or $\#A$ is even and does not contain k collinear points, then $s_k(A)$ $(1 \leqslant k \leqslant n)$ contains at most one point. (The last result follows also from Proposition 2.2.)

S0022-247X(03)00845-X/FLA AID:9036 Vol.●●●(●●●) yjmaa9036 P.5 (1-11)
ELSGMLTM(YJMAA):m1 2003/11/25 Prn:27/11/2003; 13:15 by:ML p. 5

When $k = 2$ we have no uniqueness result. (See Remark 3.3 below.)

Theorem 3.2. *For any $A \subset X$ we have $r(A) = r_2(A)$.*

Proof. Assume by contradiction, that $r_2(A) < r(A)$ for some $A = \{a_1, \dots, a_n\}$. Take $x \in X$ such that $r_2(A, x) = r(A) - \sigma$ for some $\sigma > 0$; we have $r(A, x) \geqslant r(A)$ (by definition) so there exists $a_i \in A$ such that $\|x - a_i\| \geqslant r(A)$.

For any $a_j \in A$, $j \neq i$, we have

$$\frac{\|x - a_i\| + \|x - a_j\|}{2} \leqslant r_2(A, x) = r(A) - \sigma,$$

so

$$\|x - a_j\| \leqslant 2r(A) - 2\sigma - \|x - a_i\| \leqslant 2r(A) - 2\sigma - r(A) = r(A) - 2\sigma.$$

If $x_\lambda = \lambda a_i + (1 - \lambda)x$, $0 \leqslant \lambda \leqslant 1$, then we have $\|x_\lambda - x\| = \lambda\|a_i - x\|$;

$$\begin{aligned}\frac{1}{2}\big(\|a_i - x_\lambda\| + \|x_\lambda - a_j\|\big) &\leqslant \frac{1}{2}\big(\|a_i - x\| - \|x - x_\lambda\| + \|x_\lambda - x\| + \|x - a_j\|\big)\\ &\leqslant r(A) - \sigma \quad \text{for all } j \neq i.\end{aligned}$$

Choose $\lambda \in (0, 1)$ so that $\|x_\lambda - a_i\| = r(A) - \sigma$; we obtain, for all $j \neq i$

$$\|x_\lambda - a_j\| \leqslant 2\big(r(A) - \sigma\big) - \|a_i - x_\lambda\| = 2r(A) - 2\sigma - \big(r(A) - \sigma\big) = r(A) - \sigma;$$

therefore $r(A, x_\lambda) \leqslant r(A) - \sigma$, a contradiction. □

Remark 3.2. In general, in any space, we have $r_3(A) < r_2(A)$ for some A: for example, also in the Euclidean plane E^2, there are three-point sets where the center and the median do not coincide.

We have proved (Theorem 3.2) that $r_1(A) = r_2(A)$ always. On the contrary, the equality $r_k(A) = r_{k+1}(A)$ for $k \geqslant 2$ does not happen frequently and it has some strong implications. We shall discuss now this fact, giving a converse of Remark 2.7.

Theorem 3.3. *Let $r_k(A) = r_{k+1}(A)$ for some $k \geqslant 1$ and $A = \{a_1, \dots, a_n\}$; $n > k$. Then $s_k(A) \subset s_{k+1}(A)$. (In particular, by Theorem 3.2, if c is a center of A, then $c \in s_2(A)$.) Moreover, if $c_k \in s_k(A)$, then (at least) the $k + 1$ points of A which are farthest to c_k have the same distance $r_k(A)$ from it; in addition, for $i = 1, \dots, k$, $r_i(A) = r_k(A)$; $c_k \in s_i(A)$; $s_i(A) \subset s_{i+1}(A)$. (Note that if X is strictly convex, then $s_{k+1}(A)$ is a singleton for $k \geqslant 2$ since the $k + 1$ points farthest to c_k are not collinear.)*

Proof. Let $r_k(A) = r_{k+1}(A)$; $c_k \in s_k(A)$. Order the elements of A so that $\|c_k - a_1\| \geqslant \|c_k - a_2\| \geqslant \dots \geqslant \|c_k - a_n\|$; we have

$$r_k(A) = \frac{1}{k}\sum_{i=1}^{k}\|c_k - a_i\| \geqslant \frac{1}{k+1}\sum_{i=1}^{k+1}\|c_k - a_i\| = r_{k+1}(A, c_k) \geqslant r_{k+1}(A).$$

S0022-247X(03)00845-X/FLA AID:9036 Vol.●●●(●●●) yjmaa9036 P.6 (1-11)
ELSGMLTM(YJMAA):m1 2003/11/25 Prn:27/11/2003; 13:15 by:ML p. 6

Therefore, our assumption implies that $c_k \in s_{k+1}(A)$; moreover,

$$\frac{1}{k+1}\left(\sum_{i=1}^{k}\|c_k - a_i\| + \|c_k - a_{k+1}\|\right) = \frac{1}{k}\sum_{i=1}^{k}\|c_k - a_i\|$$

implies

$$\frac{\|c_k - a_{k+1}\|}{k+1} = \left(\frac{1}{k} - \frac{1}{k+1}\right)\sum_{i=1}^{k}\|c_k - a_i\| = \frac{r_k}{k+1},$$

so $\|c_k - a_{k+1}\| = r_k(A)$; but then, since

$$\|c_k - a_{k+1}\| \leqslant \min_{1\leqslant i\leqslant k}\|c_k - a_i\| \leqslant \frac{1}{k}\sum_{i=1}^{k}\|c_k - a_i\| = r_k(A),$$

$\|c_k - a_1\| = \cdots = \|c_k - a_k\| = \|c_k - a_{k+1}\|$. By recalling Remark 2.7, we obtain the conclusion. □

Remark 3.3. In general, also if X is the Euclidean plane, a 2-centrum of A is not a center: for example, if $A = \{(0,1); (0,-1); (\varepsilon,0)\}$, $0 \leqslant \varepsilon \leqslant 1$, then the unique center of A is the origin, while all points $(0,\alpha)$; $|\alpha| \leqslant (1-\varepsilon^2)/2$, are 2-centra.

Remark 3.4. If A has at most one $(k+1)$-centrum and $r_k(A) = r_{k+1}(A)$, then $x \in s_{k+1}(A) \Rightarrow s_k(A) \subseteq \{x\}$. Without the assumption of uniqueness on $s_{k+1}(A)$ this is not true, as the following example shows. Let X be the plane with the max norm, and $A = \{(-\frac{9}{10},0); (\frac{11}{10},1); (-\frac{9}{10},-1)\}$; we have $r_2(A) = r_3(A) = 1$; $P = (\frac{1}{10},0)$ belongs to $s_2(A) \subset s_3(A)$; the origin belongs to $s_3(A)$ but not to $s_2(A)$.

Our next result, whose proof follows from the definition of $r_k(A)$, extends [3, Proposition 2.7].

Theorem 3.4. *Let $m_k \in s_k(A)$, $m_j \in s_j(A)$, $\max\{k,j\} \leqslant n = \#A$. Then we have*

$$\|m_k - m_j\| \leqslant r_k(A) + r_j(A). \tag{7}$$

In particular, if $j = k$ and $\{m_k, m'_k\} \subset s_k(A)$, then

$$\|m_k - m'_k\| \leqslant 2r_k(A). \tag{8}$$

Remark 3.5. The estimates (7) and (8) are sharp. (See [3, Example 2.9].) But if we assume that X is strictly convex, then we have better estimates. In fact, according to Remark 3.1, in this case (for $k \neq 2$) we have uniqueness of solutions in many cases. But for $k \neq j$ we cannot give better inequalities (see [4, §4]) apart from the fact that strict inequality holds in both (7) and (8).

Now assume that we have equality in (7). Looking at the proof of Theorem 3.4, we obtain subsequently; for the j farthest points to m_j, a_i, $i = 1,2,\ldots,j$, we have $\|m_j - a_i\| + \|a_i - m_k\| = \|m_j - m_k\|$; the j farthest points to m_k, all have distance $r_k(A,m_k)$ from it; therefore, if $j > k$ then $r_k(A) = r_j(A)$ and both m_k and m_j belong

S0022-247X(03)00845-X/FLA AID:9036 Vol.●●●(●●●) yjmaa9036 P.7 (1-11)
ELSGMLTM(YJMAA):m1 2003/11/25 Prn:27/11/2003; 13:15 by:ML p. 7

to $s_j(A)$. If $j=k$, then the k farthest points to m_j $[m_k]$ are on the sphere of radius r_k centered at m_j [respectively at m_k]; moreover the distance between the centers of the two balls is twice the radius r_k.

In the following we consider a localization property of the k-centra with respect to $\mathrm{co}(A)$, the convex hull of the set A.

Theorem 3.5. *If X is a two-dimensional space, or if X is a Hilbert space, then for any A and any k $(1 \leqslant k \leqslant \#A)$, it holds $s_k(A) \cap \mathrm{co}(A) \neq \emptyset$. Moreover, if X is a Hilbert space, or if $\dim(X)=2$ and X is strictly convex, then $s_k(A) \subset \mathrm{co}(A)$.*

Proof. The assumptions imply that $s_k(A) \neq \emptyset$. If $\dim(X)=2$ then (see [21]) for every $x \in X$ there exists $x^* \in \mathrm{co}(A)$ such that $\|x^*-a\| \leqslant \|x-a\|$ for any $a \in A$; i.e., $\|x^*-a_i\| \leqslant \|x-a_i\|$ for $i=1,\dots,n=\#A$, so $r_k(A,x^*) \leqslant r_k(A,x)$: if we take $x \in s_k(A)$, this shows that there also exists $x^* \in s_k(A) \cap \mathrm{co}(A)$.

Now let X be Hilbert or if $\dim(X)=2$, X strictly convex; if $x \notin \mathrm{co}(A)$, let x^* be the best approximation to x from $\mathrm{co}(A)$: we have $\|x^*-a_i\| < \|x-a_i\|$ for $i=1,\dots,n$, so $r_k(A,x^*) < r_k(A,x)$, thus an element of $s_k(A)$ must belong to $\mathrm{co}(A)$. □

Corollary 3.1. *Let X be Hilbert or if $\dim(X)=2$, X strictly convex; given $A \subset X$ with no subset of k points being collinear, if $m_k \in s_k(A)$ and $c \in s_1(A)$, then $\|m_k-c\|=r(A)$ implies that $m_k \in A$.*

Proof. Follow the line of the proof of [4, Proposition 5.1]. □

Another interesting property of k-centra of a set A is that they allow to characterize inner product spaces in terms of their intersection with the convex hull of A. Characterizations of this type are known from the sixties. (See [8,9].) The same property concerning medians was considered in the nineties by Durier [7], where partial answers were given. It has been proved only recently for medians of three-point sets, this result can be found in [6].

Theorem 3.6. *If $\dim(X) \geqslant 3$ and the norm of X is not hilbertian, then there exists a three-point set A such that $s_3(A) \cap \mathrm{co}(A) = \emptyset$.*

By using such theorem, it is not difficult to obtain the following proposition.

Proposition 3.1. *If $\dim(X) \geqslant 3$ and the norm of X is not hilbertian, then for every $n \geqslant 3$ there exists an n-point set F such that $s_3(F) \cap \mathrm{co}(F) = \emptyset$.*

Proof. We prove the result for $n=4$, the extension to $n \geqslant 4$ being similar.

Under the assumptions done, according to Proposition 2.2, $\inf_{x \in \mathrm{co}(A)} r_3(A,x)$ is always attained; now take $A=\{a_1,a_2,a_3\}$ as given by Theorem 3.6: for some $\sigma>0$ we have

$$\inf_{x \in \mathrm{co}(A)} r_3(A,x) = r_3(A) + 4\sigma > r_3(A).$$

S0022-247X(03)00845-X/FLA AID:9036 Vol.●●●(●●●) yjmaa9036 P.8 (1-11)
ELSGMLTM(YJMAA):m1 2003/11/25 Prn:27/11/2003; 13:15 by:ML p. 8

Take $\bar{x} \in X$ such that $r_3(A, \bar{x}) < r_3(A) + \sigma$; it is not a restriction to assume that $\|\bar{x} - a_3\| \leqslant \min\{\|\bar{x} - a_1\|, \|\bar{x} - a_2\|\}$. Now take $a_4 \notin A$ such that $\|a_3 - a_4\| \leqslant \sigma$ and let $F = A \cup \{a_4\}$. We have $r_3(F, \bar{x}) \leqslant r_3(A, \bar{x}) + \sigma \leqslant r_3(A) + 2\sigma$. Now take $y \in \mathrm{co}(F)$: there is $x \in \mathrm{co}(A)$ such that $\|x - y\| \leqslant \sigma$; therefore $|r_3(F, y) - r_3(F, x)| \leqslant \sigma$, so $r_3(F, y) \geqslant r_3(F, x) - \sigma \geqslant r_3(A, x) - \sigma \geqslant r_3(A) + 3\sigma$; thus

$$\inf_{y \in \mathrm{co}(F)} r_3(F, y) \geqslant r_3(A) + 3\sigma \geqslant r_3(F, \bar{x}) + \sigma \geqslant r_3(F) + \sigma,$$

this proves the thesis. □

Given a set A with n points and $k < n$, we can divide the space X into $\binom{n}{k}$ regions R_j, so that when x is taken in one of these regions, the same k points of A are the farthest to x; of course, inside each of these regions there are $k!$ different possible orderings $\sigma_1, \ldots, \sigma_k$. It is possible to have $R_i \cap R_j \neq \emptyset$ (the values of the kth distance can be equal to the $(k+1)$th one); also, if R_j is determined by $a_1, \ldots, a_k$ then $a_i \notin R_j$ for $i = 1, \ldots, k$. Also in general the medians of $a_1, \ldots, a_k$ (if they exist) do not belong to R_j. Note that these regions are not in general convex: for example, if X if the plane with the max norm, given $a_1 = (1, 0)$ and $a_2 = (-1, 0)$, the set $\|x - a_1\| \geqslant \|x - a_2\|$ is not convex. But the same is true, for some pair, in any space with a non-hilbertian norm.

If X is a Hilbert space, then the regions R_j are convex: in fact, consider, e.g., the region R determined by the points $a_1, \ldots, a_k$, $k < \#A$: then

$$R = \bigcap_{i=1}^{k} \bigl\{x \in X\colon \|x - a_h\| \leqslant \|x - a_i\| \text{ for } h = k+1, \ldots, n\bigr\}.$$

R is the intersection of $k(n-k)$-convex regions, therefore it is convex. A detailed analysis of these sets can be found in [13]. (Not only for Hilbert spaces.) Also in the particular case of two-dimensional spaces some geometrical properties as well as the complexity analysis are given in [14].

Minimizing $r_k(A)$ is equivalent to solve $\binom{n}{k}$ constrained Fermat problems; then looking for the minimum of the values obtained: for each R_j, determined by k given points, say $\{a_1, \ldots, a_k\}$, look for a median of these points, restricted to the "feasible region" R_j. Algorithms for the solution of this kind of problems in two-dimensional spaces can be found in [14]; also, in networks (finite metric spaces) algorithms are given in [10,16].

Given X, consider for $k \in \mathbb{N}$ the parameter

$$J_k(X) = \sup\left\{\frac{2r_k(A)}{\delta(A)}\colon A \subset X \text{ finite}, \ \max\{2, k\} \leqslant \#A\right\}. \tag{9}$$

For $k = 1$, the number $J_1(X) = J(X)$ is called the finite Jung constant and has been studied intensively; in general, $1 \leqslant J(X) \leqslant 2$, while the value of $J(X)$ gives information on the structure of X. As shown partially in [5] and later completely in [18], we always have

$$J(X) = \sup\left\{\frac{2\mu(A)}{\delta(A)}\colon A \subset X \text{ finite}, 2 \leqslant n = \#A\right\}.$$

Since $\mu(A) \leqslant r_k(A) \leqslant r(A)$ always (see (1)), we obtain the following result.

S0022-247X(03)00845-X/FLA AID:9036 Vol.●●●(●●●) yjmaa9036 P.9 (1-11)
ELSGMLTM(YJMAA):m1 2003/11/25 Prn:27/11/2003; 13:15 by:ML p. 9

Theorem 3.7. *In every space X, for every positive integer k, we have*

$$J_k(X) = J(X). \tag{10}$$

Our last result in this section was already known for medians (see [4]) but it can be extended to general k-centra.

Proposition 3.2. *Let $m_k \in s_k(A)$ for some set A. Assume that $A_k \subset A$, $\#A_k = k$ and $r_k(A) = \frac{1}{k}\sum_{a\in A_k}\|m_k - a\|$. If $\|m_k - \frac{1}{k}\sum_{a\in A_k} a\| = r_k(A)$ then X is not strictly convex.*

Proof. By the triangular inequality we have

$$r_k(A) = \left\| m_k - \frac{1}{k}\sum_{a\in A_k} a \right\| \leqslant \frac{1}{k}\sum_{a\in A_k}\|m_k - a\| = r_k(A).$$

Thus, m_k is also a center of A_k and $r_k(A) = r(A_k)$. Now, we apply first claim in [4, Proposition 3.1] to the set A_k to get the result. □

4. Concluding remarks

To conclude our analysis of k-centra, we study several properties of these points regarding equilateral sets. Recall that A is called *equilateral* if $\|a_i - a_j\| =$ constant for $i \neq j$, $1 \leqslant i, j \leqslant n = \#A$. Also, recall that the centroid of a finite set A is given by the point $\frac{1}{\#A}\sum_{a\in A} a$. For equilateral sets there are several nice properties connecting centers, medians and centroids (see [2]). Some of them can be extended further to k-centra.

Proposition 4.1. *Let A be an equilateral set in an inner product space X and let $k \geqslant 3$; then the centroid of A belongs to $s_k(A)$.*

Proof. Assume that 0 is the center of A; then $\langle a_i, a_j\rangle =$ constant for $i \neq j$, $1 \leqslant i, j \leqslant n = \#A$. Let $y = \sum_{j=1}^{n}\lambda_j a_j$; then the function $f(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^{k}\|y - a_i\|$ is symmetric.

In Hilbert spaces it always exists $m_k \in s_k(A) \cap \operatorname{co}(A)$. Moreover, under the hypothesis of the proposition $s_k(A)$ is a singleton, then m_k is the unique minimizer of f and $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1/n$; thus m_k is the centroid of A. □

Remark 4.1. Let $A = \{a_1, \dots, a_n\}$ be an equilateral set with $\|a_i - a_j\| = d$, $\forall i \neq j$; then it is easy to see that

$$r_k(A, x) \geqslant \frac{d}{2} \quad \text{for any } x \in X.$$

Indeed, for any $x \in X$, $kr_k(A, x)$ is attained as a sum of distances from x to k points of A. Let us denote by $A_k(x)$ the subset of A containing the points that define $r_k(A, x)$. $A_k(x)$ itself is an equilateral set with $\|a_i - a_j\| = d$, $\forall i \neq j$, $a_i, a_j \in A_k(x)$; then

$$kr_k(A, x) = \sum_{a\in A_k(x)}\|a - x\| \geqslant \frac{kd}{2},$$

S0022-247X(03)00845-X/FLA AID:9036 Vol.●●●(●●●) yjmaa9036 P.10 (1-11)
ELSGMLTM(YJMAA):m1 2003/11/25 Prn:27/11/2003; 13:15 by:ML p. 10

where inequality comes from [2, Lemma 4.1] applied to the set $A_k(x)$. (Also if k is even, it follows from (3).)

Proposition 4.2. *For any equilateral set in the hypothesis of Remark* 4.1, *the conditions* $r(A) = d/2$ *and* $r_k(A) = d/2$ *are equivalent, for any* $k = 2, 3, \ldots, n$. *In these cases* k*-centra for any* $k = 1, 2, \ldots, n = \#A$ *coincide.*

Proof. Runs parallel to [2, Proposition 5.1] except for the details of considering partial sums of k-largest distances. □

From this last result we can present an example of set without k-centra for any k. [2, Example 5.2] is an equilateral three-point set without median. Now, we apply Proposition 4.2 to conclude that the set in that example cannot have k-centra for any $k = 1, 2, 3$.

References

[1] R.E. Barlow, F. Proschan, Statistical Theory of Reliability and Life Testing, in: Int. Ser. Decision Process., Ser. Quant. Methods Decision Making, Holt, Rinehart & Winston, New York, 1975.
[2] M. Baronti, E. Casini, P.L. Papini, Equilateral sets and their central points, Rend. Mat. Appl. (7) 13 (1993) 133–148.
[3] M. Baronti, E. Casini, P.L. Papini, Central points of finite sets, Rend. Circ. Mat. Palermo (2) Suppl. 40 (1996) 63–72.
[4] M. Baronti, E. Casini, P.L. Papini, Centroids, centers, medians: what is the difference?, Geom. Dedicata 68 (1997) 157–168.
[5] M. Baronti, G. Lewicki, On some constants in Banach spaces, Arch. Math. 62 (1994) 438–444.
[6] C. Benítez, M. Fernández, M.L. Soriano, Location of the Fermat–Torricelli medians of three points, Trans. Amer. Math. Soc. 354 (2002) 5027–5038.
[7] R. Durier, Constrained and unconstrained problems in location theory and inner product spaces, Numer. Funct. Anal. Optim. 18 (1997) 297–310.
[8] A.L. Garkavi, On the Chebyshev center and the convex hull of a set, Uspekhi Mat. Nauk. USSR 19 (1964) 139–145.
[9] V. Klee, Circumspheres and inner products, Math. Scand. 8 (1960) 363–370.
[10] S. Nickel, J. Puerto, A unified approach to network location problems, Networks 34 (1999) 283–290.
[11] W. Ogryczak, A. Tamir, Minimizing the sum of the k largest functions in linear time, Inform. Process. Lett. 85 (2003) 117–122.
[12] P.L. Papini, Existence of centers and medians, Preprint, 2001.
[13] J. Puerto, F.R. Fernández, Geometrical properties of the symmetrical single facility location problem, J. Nonlinear Convex Anal. 1 (2000) 321–342.
[14] A.M. Rodríguez-Chía, S. Nickel, J. Puerto, F.R. Fernández, A flexible approach to location problems, Math. Methods Oper. Res. 51 (2000) 69–89.
[15] P. Slater, Centers to centroids in graphs, J. Graph Theory 2 (1978) 209–222.
[16] A. Tamir, The k-centrum multi-facility location problem, Discrete Appl. Math. 109 (2001) 293–307.
■Uncited■ [17] T. Tokuyama, Minimax parametric optimization problems in multi-dimensional parametric searching, in: Proc. 33-rd Annu. ACM Symp. on Theory of Computing, 2001, pp. 75–84.
[18] L. Veselý, Equality of two Banach space constants related to different notions of radius of a set, Boll. Un. Mat. Ital. A (7) 9 (1995) 515–520.
[19] L. Veselý, Generalized centers of finite sets in Banach spaces, Acta Math. Univ. Comenian. (N.S.) 66 (1997) 83–95.
[20] L. Veselý, A Banach space in which all compact sets, but all not bounded sets, admit Chebyshev centers, Arch. Math., in press.

S0022-247X(03)00845-X/FLA AID:9036 Vol.●●●(●●●) yjmaa9036 P.11 (1-11)
ELSGMLTM(YJMAA):m1 2003/11/25 Prn:27/11/2003; 13:15 by:ML p. 11

[21] R.E. Wendell, A.P. Hurter Jr., Location theory, dominance and convexity, Oper. Res. 21 (1973) 314–320.
[22] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Syst. Man Cybern. 18 (1988) 183–190.
[23] V.A. Zalgaller, Distances to the two and three most distant points, Math. Notes 59 (1996) 507–510.