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A SUM OPERATOR WITH APPLICATIONS TO SELF–IMPROVING PROPERTIES OF POINCARÉ INEQUALITIES IN METRIC SPACES

BRUNO FRANCHI, CARLOS PÉREZ, AND RICHARD L. WHEEDEN

Abstract. We define a class of summation operators with applications to the self-improving nature of Poincaré–Sobolev estimates, in fairly general quasimetric spaces of homogeneous type. We show that these sum operators play the familiar role of integral operators of potential type (e.g., Riesz fractional integrals) in deriving Poincaré–Sobolev estimates in cases when representations of functions by such integral operators are not readily available. In particular, we derive norm estimates for sum operators and use these estimates to obtain improved Poincaré–Sobolev results.

1. Introduction.

It is well-known that Poincaré–Sobolev estimates in Euclidean space can be derived as corollaries of norm inequalities for Riesz fractional integral operators. For example, the classical estimate

$$\left(\int_B |f(x)-f_B|^q\,dx\right)^{1/q} \le c\left(\int_B |\nabla f(x)|^p\,dx\right)^{1/p}, \quad \frac{1}{q}=\frac{1}{p}-\frac{1}{n}, \quad 1<p<n, \tag{1}$$

where B is a Euclidean ball in $\mathbb{R}^n$ and $f_B = \frac{1}{|B|}\int_B f(x)\,dx$, can be derived from the norm inequality

$$\left(\int_{\mathbb{R}^n} |I_1 f(x)|^q\,dx\right)^{1/q} \le c\left(\int_{\mathbb{R}^n} |f(x)|^p\,dx\right)^{1/p} \tag{2}$$

for the same values of p and q, where c is independent of f and

$$I_1 f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}}\,dy$$

is the Riesz transform of f of order 1. Similarly, although (2) is false in case $p=1$ and $q=n/(n-1)$, the case $p=1$ of (1) can be derived from the following weak-type analogue of (2):

$$\big|\{x\in\mathbb{R}^n : |I_1 f(x)|>\lambda\}\big|^{(n-1)/n} \le \frac{c}{\lambda}\,||f||_{L^1(\mathbb{R}^n)},\ \lambda>0,$$

with c independent of λ and f, where $|E|$ denotes the Lebesgue measure of a set E.

The well-known pointwise representation inequality

$$|f(x)-f_B| \le c\,I_1(|\nabla f|\,\chi_B)(x), \quad x\in B,$$

with c independent of x, B and f, makes it clear how (1) follows from (2) in case $p>1$, and a far less obvious argument based on truncation can be used when $p=1$ (see [23], [18], [28]).

In fact, norm estimates for more general integral transforms have been used recently to derive Poincaré–Sobolev estimates for vector fields in fairly general settings, such as on manifolds and groups, and even on abstract metric spaces in the sense of [6]. For example, let $\rho(x,y)$ be a metric on $\mathbb{R}^n$ that is induced by a collection X of Carnot–Carathéodory vector fields, and suppose that Lebesgue measure is a doubling measure for ρ-balls, i.e., that $|B(x,2r)|\le C|B(x,r)|$ with C

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independent of x and r, where $B(x,r)$ denotes the ρ-ball with center x and radius r. Then the operator

$$I(f)(x) = \int_{\mathbb{R}^n} f(y) \frac{\rho(x,y)}{|B(x,\rho(x,y))|}\, dy$$

has known mapping properties from L^p to L^q with p,q related naturally in terms of the doubling property, and these mapping properties lead to Poincaré–Sobolev estimates of the form

$$\left(\frac{1}{|B|}\int_B |f(x)-f_B|^q\, dx\right)^{1/q} \le c\, r(B) \left(\frac{1}{|B|}\int_B |Xf(x)|^p\, dx\right)^{1/p},$$

where $r(B)$ is the radius of the ρ-ball B. The reason why Poincaré–Sobolev estimates follow is that there is a representation inequality of the form

$$|f(x)-f_B| \le c\, I(|Xf|\,\chi_B)(x), \quad x \in B, \tag{3}$$

with c independent of x, B and f; see e.g. [11], [15], [20] for precise statements of this representation, and see e.g. [28] for the mapping properties of the operator I.

On the other hand, starting with work of Saloff-Coste [27], it is known that Poincaré–Sobolev estimates have a self-improving nature, in the sense that it is possible to derive estimates for general p,q from particular special cases such as

$$\frac{1}{|B|}\int_B |f(x)-f_B|\, dx \le c\, r(B) \left(\frac{1}{|B|}\int_B |Xf|^{p_0}\, dx\right)^{1/p_0} \tag{4}$$

for some p_0, without explicit mention of any integral operator at all.

A partial explanation for the apparent mystery about the role of integral operators in the self-improving technique was given in [11] and with sharp constants in [21] (see also [15], [20], [22], [25]). It was shown there that in case $p_0 = 1$, (4) is in fact equivalent to (3). In particular, by assuming (4) with $p_0 = 1$, we also have (3), and the more general Poincaré–Sobolev estimates then follow from the corresponding norm estimates for the integral operator I.

However, in case $p_0 > 1$, no sharp representation analogous to (3) is known to follow from (4). When $p_0 > 1$, the difficulty that one encounters in trying to adapt the arguments which lead from (4) to (3) in case $p_0 = 1$ is related to the presence of the exponent $1/p_0$: the functional $a(B)$ defined by

$$a(B) = r(B) \left(\frac{1}{|B|}\int_B |g|^{p_0}\, dx\right)^{1/p_0} \qquad (g \text{ and } p_0 \text{ fixed})$$

is not easy to add over a class of non-overlapping (or even disjoint) balls B if $p_0 > 1$. Thus, starting from an estimate of the type

$$\frac{1}{|B|}\int_B |f - f_B|\, dx \le c\, a(B)$$

for all balls B with $a(B)$ as above and $p_0 > 1$, or with an even more general functional $a(B)$, it is not clear how to build an integral operator whose norm estimates imply improved Poincaré–Sobolev estimates like

$$\left(\frac{1}{|B|}\int_B |f - f_B|^q\, dx\right)^{1/q} \le C\, a(B) \quad \text{for some } q > 1.$$

The main purpose of this paper is to show that the familiar role of integral operators is instead played by a sum operator $T(x)$ which is formed by adding $a(B)$ over an appropriate chain of balls associated with a point x:

$$T(x) = \sum_{B \text{ in a chain for } x} a(B).$$

In case $p_0 = 1$, the sum operator becomes an integral operator, but in any case, the L^p to L^q mapping properties of the sum operator can be derived in much the same ways as those for integral transforms of potential type, and these norm estimates for T lead to correspondingly more general Poincaré estimates. We will be able to obtain such results for a fairly general class of functionals $a(B)$ which includes the special choice

$$a(B) = r(B) \left(\frac{1}{|B|} \int_B |Xf|^{p_0} \, dx \right)^{1/p_0}.$$

See also [29], [17] and [8] for other types of operators which involve adding integral averages; the sums in [17] involve integral averages over annuli, while those in [29] involve averages over portions of dyadic "cubes". On the other hand, the sums in [8] involve concentric balls centered at x that hence have countable overlapping. In particular we improve some results obtained in [17].

We will use the sum operator to strengthen several of the self-improving results obtained in [16], and also to derive results for the weighted BV spaces defined in [2]. For example, we will prove the following result, in which we use the notation $|B|_\omega = \int_B d\omega$ for the ω-measure of B. Let $p_0 > 0$ and X be a differential operator on $\mathbb{R}^n$ for which

$$\frac{1}{|B|} \int_B |f - f_B| \, dx \leq c \, r(B) \left(\frac{1}{|B|} \int_B |Xf|^{p_0} \, dx \right)^{1/p_0}$$

for all ρ-balls B and all Lipschitz functions f. If ω is a measure which satisfies the doubling condition

$$|B|_\omega \leq c \left(\frac{r(B)}{r(\tilde{B})} \right)^N |B|_\omega, \quad \tilde{B} \subset B,$$

for all ρ-balls $\tilde{B}, B$, then we have

$$\left(\frac{1}{|B|_\omega} \int_B |f - f_B|^q \, d\omega \right)^{1/q} \leq C \, r(B) \left(\frac{1}{|B|} \int_B |Xf|^p \, dx \right)^{1/p} \tag{5}$$

with p, q related by

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{N}, \quad p_0 \leq p < q < \infty,$$

and with C independent of f and B. It was proved in [16] that such a result holds under the stronger assumption that $\omega \in A_\infty(dx)$ (the definition of $A_\infty(dx)$ is given after Corollary 2.15), but we will be able to deduce it by assuming only the doubling condition. In fact, a more general result is proved in Corollary 2.16 below, replacing (5) by

$$\left(\frac{1}{|B|_\omega} \int_B |f - f_B|^q \, d\omega \right) \leq C \, r(B) \left(\frac{1}{|B|_{vdx}} \int_B |Xf|^p \, v \, dx \right)^{1/p}. \tag{6}$$

In this version, Lebesgue measure dx on the right side of the conclusion is replaced by a more general measure vdx, provided that $p_0 \leq p < q < \infty$, that we replace our assumption about the doubling condition of order N by the balance condition

$$\frac{r(\tilde{B})}{r(B)} \left(\frac{|\tilde{B}|_\omega}{|B|_\omega} \right)^{1/q} \leq C \left(\frac{|\tilde{B}|_{vdx}}{|B|_{vdx}} \right)^{1/p}, \quad \tilde{B} \subset B, \tag{7}$$

and provided $v \in A_{p/p_0}(dx)$ (again, the definition of $A_p(dx)$ is given after Corollary 2.15).

Note that the possibility of choosing $q = p$ is not addressed in the result just mentioned. However, in §3, we will show that if ω is absolutely continuous with respect to Lebesgue measure, it is possible to treat the case $q = p > p_0 \geq 1$ by assuming a stronger version of the balance condition, a version which we shall refer to as a Fefferman–Phong type strengthening of the condition; if $d\omega = w\,dx$, this strengthening involves replacing $|\tilde{B}|_\omega$ in the numerator on the left side of (7) by the larger quantity

$$\mathcal{A}_r(\omega, \tilde{B}) = \left(\int_{\tilde{B}} w^r \, dx \right)^{1/r} |\tilde{B}|^{1/r'}$$

for some $r > 1$, $1/r + 1/r' = 1$. See §3 for the exact statements.

We shall refer to inequalities like (6) as *two-measure (or two-weight) inequalities*. To illustrate the general interest of two-weight inequalities in applications, consider the paper [5], where the authors prove a Harnack inequality for anisotropic degenerate/singular elliptic equations of the form $\mathrm{div}\,(A(x)Du) = 0$ in an open set Ω, when

$$\lambda(x)|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \Lambda(x)|\xi|^2$$

for $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$. There, a two-weight inequality for the pair of measures $\Lambda\,dx$ and $\lambda\,dx$ is a key tool used in the proof, and it is obtained directly from a balance condition akin to (7). Clearly, enlarging the class of weights for which (6) holds would yield Harnack inequalities for more general classes of pde's. Following the spirit of [5], the same condition is used in [14] to prove a compensated compactness theorem and then a homogenization result for nonlinear degenerate elliptic pde's with oscillating coefficients. Analogously, compact imbedding of weighted Sobolev and BV spaces can be deduced from two-weight Sobolev-Poincaré inequalities (see, e.g., [13]).

But two-weight inequalities also arise when dealing with isotropic equations of the form $\mathrm{div}\,(w(x)Du) = 0$ in case w does not belong to the class A_2 (the situation for $w \in A_2$ is well-understood due to [9]), but when nevertheless ω can be *estimated* from below and from above by weights satisying a two-weight Sobolev-Poincaré inequality. An assumption of this type is much weaker than the A_2-condition which requires more delicate control of the weight on every ball.

Finally, we note that our motivation for deriving results in rather general quasimetric spaces is that the theory then works in important non-Euclidean settings like Carnot–Carathéodory metric spaces associated with subelliptic differential operators, graphs and fractal sets (see e.g. [17] for references).

2. Main results and proofs.

Throughout the paper, we shall consider a fixed quasimetric space $(\mathcal{S}, \rho)$ endowed with a doubling Borel measure μ that makes $(\mathcal{S}, \rho, \mu)$ a *quasimetric space of homogeneous type* in the sense that the following properties hold:

(i) $\rho(x, y) \ge 0$ for all $x, y \in \mathcal{S}$, and $\rho(x, y) = 0$ iff $x = y$;
(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in \mathcal{S}$;
(iii) $\rho(x, y) \le K\big[\rho(x, z) + \rho(z, y)\big]$ for all $x, y, z \in \mathcal{S}$.

If $x \in \mathcal{S}$ and $r > 0$, let $B(x, r)$ denote the ρ-ball centred at x of radius r, i.e., $B(x, r) = \{y \in \mathcal{S} : \rho(x, y) < r\}$. If B is a ρ-ball, we will often call B simply 'a ball', and we will denote its radius by $r(B)$ and its μ-measure by $|B|_\mu$. Moreover, if $c > 0$, we shall denote by cB the ball with the same center as B and such that $r(cB) = cr(B)$. Whenever we speak of a "measure", we mean a nonnegative Borel measure.

We always assume that the following *doubling property* holds for μ:

(iv) There exists $A > 0$ such that

$$|B(x, 2r)|_\mu \le A\,|B(x, r)|_\mu$$

for all $x \in \mathcal{S}$ and r.

Definition 2.1. We say that a locally finite Borel measure ω belongs to the class $D = D(\mathcal{S}, \rho)$ if there is a constant $A_\omega > 1$ so that ω satisfies the doubling condition

$$|B(x, 2r)|_\omega \le A_\omega \, |B(x, r)|_\omega \tag{8}$$

for all $x \in \mathcal{S}$ and $r > 0$, where we denote $|E|_\omega = \int_E d\omega$ for any measurable set E. In case ω is absolutely continuous with respect to μ, i.e., if $d\omega = w \, d\mu$ for a nonnegative function $w \in L_{loc}(d\mu)$, we write $|E|_\omega = |E|_{w d\mu}$ and call w a weight function.

Remark 2.2. It is easy to see that (8) implies

$$|B(x, tr)|_\omega \le A_\omega \, t^{\log_2 A_\omega} \, |B(x, r)|_\omega \tag{9}$$

for $t > 1$, $r > 0$ and $x \in \mathcal{S}$. We shall say that ω satisfies the doubling condition of order N and write $\omega \in D_N = D_N(\mathcal{S}, \rho)$ if

$$|B(x, tr)|_\omega \le C \, t^N \, |B(x, r)|_\omega$$

for $t > 1$, $r > 0$ and $x \in \mathcal{S}$. If $\omega \in D$, then by [30], p. 269, assuming as we shall that all annuli $B(x, R) \setminus B(x, r)$ with $0 < r < R$ are nonempty, ω also satisfies a reverse doubling condition: there exist $\alpha, \beta > 1$ depending on A_ω such that

$$|B(x, \alpha r)|_\omega \ge \beta \, |B(x, r)|_\omega \tag{10}$$

for all $r > 0$, and hence

$$|B(x, tr)|_\omega \ge c \, t^\epsilon \, |B(x, r)|_\omega \tag{11}$$

for all $x \in \mathcal{S}$ and $t > 1$, where ϵ and c are positive constants depending on α and β. We will usually be dealing only with the class of subballs of some fixed ball B_0, and then we only need the conditions above for such balls.

GEOMETRIC HYPOTHESES: Let B_0 be a fixed ball in $(\mathcal{S}, \rho)$. We suppose that for each $x \in B_0$, there exists a chain of balls $\{B_j\} = \{B_j(x)\}_{j=1}^\infty$ satisfying

- (H1) $B_j \subset B_0$ for all $j \ge 0$;
- (H2) $r(B_j) \approx 2^{-j} r(B_0)$ for all $j \ge 0$;
- (H3) $\rho(B_j, x) \le c r(B_j)$ for all $j \ge 0$,

where $\rho(B_j, x)$ denotes the distance from x to B_j, and we assume that the constants in (H2) and (H3) are independent of x and j. Note that the balls $B_j(x)$ may or may not contain x, but the sequence $\{B_j(x)\}$ depends on x.

From now on, any positive constant that depends at most on K, A and the constants in (H2) and (H3) will be called a *geometric constant.*

It follows from (H2), (H3) and (iii) that

- (H4) If $j < k$ then $B_k \subset C B_j$, where C is a geometric constant.

Remark 2.3. We know from [20] and [15] that a chain of balls satisfying (H1)–(H3), and so also (H4), exists in metric spaces satisfying the *segment (or geodesic) property*, i.e., in metric spaces such that for every pair of points $x, y \in \mathcal{S}$ there is a continuous curve $\gamma : [0, T] \to \mathcal{S}$ connecting x and y such that $\rho(\gamma(t), \gamma(s)) = |t - s|$ for all $s, t \in [0, T]$. In fact, we then also have the extra properties

- (H5) For all $j \ge 0$, $B_j \cap B_{j+1}$ contains a ball S_j with $r(S_j) \approx r(B_j)$;
- (H6) $\rho(B_j, x) \approx r(B_j)$ for all $j \ge 0$;
- (H7) $\{B_j\}$ has bounded overlaps.

Moreover, the constants in (H5)–(H7) are geometric constants.

Typically, Carnot–Carathéodory and Riemannian metrics satisfy the segment property (see Remark 2.6 of [16] for references).

Definition 2.4. Let $a : B \to a(B)$ be a nonnegative functional defined on balls $B \subset B_0$. If $x \in B_0$, let

$$T(x) = \sum_{j=0}^{\infty} a(B_j(x)), \tag{12}$$

where $\{B_j(x)\}_{j=1}^{\infty}$ is a sequence of balls satisfying (H1), (H2), and (H3), and $B_0(x) = B_0$ for all $x \in B_0$.

We call $T(x)$ a sum operator associated with the functional $a(B)$.

The significance of $T(x)$ lies in the following simple pointwise representation formula.

Theorem 2.5. *Suppose (H1)–(H3) and (H5) hold. Let $f \in L^1(B_0, \mu)$ be such that for any ball $B \subset B_0$,*

$$\frac{1}{|B|_\mu} \int_B |f - f_B| \, d\mu \le c\, a(B), \tag{13}$$

where $f_B = \frac{1}{|B|_\mu} \int_B f \, d\mu$. Then for μ-a.e. $x \in B_0$,

$$|f(x) - f_{B_0}| \le C\, T(x), \tag{14}$$

where C is a geometric constant which also depends on the constant in (13).

Remark 2.6. We thank Professor G. Lu for pointing out that the conclusion of Theorem 2.5 holds with a weaker hypothesis. In fact, by using the methods of [19], the left-hand side of (13) can be replaced by

$$\left(\frac{1}{|B|_\mu} \int_B |f - f_B|^\epsilon \, d\mu \right)^{1/\epsilon}$$

for any $\epsilon > 0$. In particular, it can also be replaced by the weak $L^1(B, \mu)$ norm of $f - f_B$, i.e., by

$$\|f - f_B\|_{L^{1,\infty}(B,\mu)} = \sup_{\lambda > 0} \frac{\lambda}{|B|_\mu} |\{x \in B : |f - f_B| > \lambda\}|_\mu.$$

The fact that the weak norm can be substituted follows from Kolmogorov's inequality: if $0 < q < r$, then for nonnegative measurable functions g,

$$\left(\frac{1}{|B|_\mu} \int_B g(x)^q \, d\mu \right)^{1/q} \le \left(\frac{r}{r - q} \right)^{1/q} \|g\|_{L^{r,\infty}(B,\mu)}, \tag{15}$$

where the norm on the right is the weak $L^r(B, \mu)$ norm.

Moreover, the role of the constants f_B and f_{B_0} in (13) and (14) can instead be played by appropriate polynomials, and in this way, our main results have analogues for high order Poincaré–Sobolev estimates. We refer to [20] for the definition and necessary properties of polynomials in quasimetric spaces. Remark 2.6 also applies here.

We can now state one of our main results, a weak type estimate for the operator T.

Theorem 2.7. *Let $0 < q < \infty$ and $\omega \in D$. Suppose (H1)–(H3) hold, and that there exist positive constants θ and c so that $\theta < 1$ and*

$$\sum_j \{a(Q_j)^q \, |Q_j|_\omega\}^\theta \le c \, \{a(B_0)^q \, |B_0|_\omega\}^\theta \tag{16}$$

for all collections $\{Q_j\}$ of pairwise disjoint subballs of B_0. Then

$$\sup_{\lambda > 0} \lambda \, |\{x \in B_0 : \ T(x) > \lambda\}|_\omega^{1/q} \le C\, a(B_0) \, |B_0|_\omega^{1/q}, \tag{17}$$

where C is a geometric constant which also depends on the constant in (16).

Remark 2.8. Since $\theta < 1$, condition (16) implies that a and ω also satisfy the condition (called $\mathcal{D}_q$ in [16])

$$\sum_j a(Q_j)^q\, |Q_j|_\omega \le c\, a(B_0)^q\, |B_0|_\omega$$

for any family $\{Q_j\}$ of pairwise disjoint subballs of B_0. In fact, this condition is weaker than (16) and corresponds to the limit case $\theta = 1$ (not allowed here). However, we stress again that the present results, unlike those in [16], do not require that $\omega \in A_\infty(\mu)$.

Remark 2.9. In Theorems 2.5 and 2.7, the chain $\{B_j(x)\}$ is not required to satisfy either (H6) or (H7).

In Proposition 2.13 below, we will give important examples of functionals $a(B)$ which satisfy (16). Here we mention the simple special case when

$$a(B) = r(B)\left(\frac{1}{|B|_\mu}\int_B g^p\, d\mu\right)^{1/p}$$

for $1 \le p < \infty$ and a fixed function $g \ge 0$ (e.g., $g = |Xf|$ for some f, where X is a differential operator). In fact, for this choice of $a(B)$, we shall see that if $\mu \in D_N$ then (16) is valid for $\omega = \mu, 1 \le p < N, 1/q = 1/p - 1/N$ and $\theta = p/q$, i.e.,

$$\sum_j a(Q_j)^p\, |Q_j|_\mu^{p/q} \le c\, a(B_0)^p\, |B_0|_\mu^{p/q}, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{N}, \quad 1 \le p < N$$

for any family of pairwise disjoint subballs of B_0.

As a first consequence of Theorems 2.5 and 2.7, we shall derive the following weak self-improving property of Poincaré's inequality in B_0.

Theorem 2.10. *Let (H1)–(H3) and (H5) hold. Suppose also that $\omega \in D$ and (16) holds for some $\theta < 1$ and some $1 < q < \infty$. If f is a real-valued Borel function on B_0 that satisfies*

$$\frac{1}{|B|_\mu}\int_B |f - c_B|\, d\mu \le c\, a(B) \tag{18}$$

for every ball $B \subset B_0$, where c_B is a real number depending on B and f, then

$$\sup_{\lambda > 0} \lambda\, |\{x \in B_0 :\ |f(x) - f_{B_0}| > \lambda\}|_\omega^{1/q} \le C\, a(B_0)\, |B_0|_\omega^{1/q}, \tag{19}$$

where $f_{B_0} = \frac{1}{|B_0|_\mu}\int_{B_0} f\, d\mu$ and C is a geometric constant which also depends on the constants in (16) and (18).

We now prove Theorems 2.5, 2.7 and 2.10, beginning with Theorem 2.7. Throughout the proofs, we shall denote by c, C different positive constants which may change from place to place.

Proof of Theorem 2.7. For J to be chosen and $x \in B_0$, write

$$T(x) = \sum_{j=0}^{\infty} a(B_j(x)) = \sum_{j=0}^{J} + \sum_{j=J+1}^{\infty} = I + II.$$

Then

$$\begin{aligned} I &= \sum_{j=0}^{J} a(B_j(x))\, |B_j(x)|_\omega^{1/q} \cdot |B_j(x)|_\omega^{-1/q} \\ &\le c\, a(B_0)\, |B_0|_\omega^{1/q} \sum_{j=0}^{J} |B_j(x)|_\omega^{-1/q}, \end{aligned}$$

by the one-term version of (16), namely

$$a(B)\,|B|_\omega^{1/q} \le c\,a(B_0)\,|B_0|_\omega^{1/q}, \quad B \subset B_0.$$

By (H4), $B_J(x) \subset CB_j(x)$ if $j \le J$, and then by reverse doubling (11), (H2) and (9),

$$\begin{aligned} |B_J(x)|_\omega &\le c \left(\frac{r(B_J(x))}{r(B_j(x))}\right)^\epsilon |CB_j(x)|_\omega \qquad \text{if } j \le J \\ &\le c\, 2^{(j-J)\epsilon} |B_j(x)|_\omega . \end{aligned}$$

Thus

$$\sum_{j=0}^{J} |B_j(x)|_\omega^{-1/q} \le c \sum_{j=0}^{J} 2^{(j-J)\epsilon/q} |B_J(x)|_\omega^{-1/q} \le c\,|B_J(x)|_\omega^{-1/q},$$

and so

$$I \le c\,a(B_0)\,|B_0|_\omega^{1/q}\,|B_J(x)|_\omega^{-1/q}.$$

Notice now by (H3) that there exists a geometric constant $\alpha > 1$ such that $x \in \alpha B_j(x)$ for all $j \ge 0$. Thus, we can write

$$\begin{aligned} II &= \sum_{j=J+1}^{\infty} a(B_j(x)) = \sum_{j=J+1}^{\infty} \left[a(B_j(x))\,|B_j(x)|_\omega^{\frac{1}{q}-\frac{1}{\theta q}} \right] |B_j(x)|_\omega^{\frac{1}{\theta q}-\frac{1}{q}} \\ &\le \left[\sup_{B: B \subset B_0, x \in \alpha B} a(B)\,|B|_\omega^{\frac{1}{q}-\frac{1}{\theta q}} \right] \sum_{j=J+1}^{\infty} |B_j(x)|_\omega^{\frac{1}{\theta q}-\frac{1}{q}} . \end{aligned}$$

If $j \ge J+1$, then $B_j(x) \subset CB_J(x)$ by (H4), and consequently by (11),

$$|B_j(x)|_\omega \le c \left(\frac{r(B_j(x))}{r(B_J(x))}\right)^\epsilon |CB_J(x)|_\omega \qquad \text{if } j \ge J+1.$$

Since $1/(\theta q) - 1/q > 0$, it follows by using (H2) that

$$\sum_{j=J+1}^{\infty} |B_j(x)|_\omega^{\frac{1}{\theta q}-\frac{1}{q}} \le c\,|B_J(x)|_\omega^{\frac{1}{\theta q}-\frac{1}{q}} .$$

Letting $S(x)$ be defined by

(20) $$S(x) = \sup_{B: B \subset B_0, x \in \alpha B} a(B)\,|B|_\omega^{\frac{1}{q}-\frac{1}{\theta q}},$$

we obtain

$$II \le c\,S(x)\,|B_J(x)|_\omega^{\frac{1}{\theta q}-\frac{1}{q}} .$$

Hence

(21) $$T(x) = I + II \le c \left\{ a(B_0)\,|B_0|_\omega^{1/q}\,|B_J(x)|_\omega^{-1/q} + S(x)\,|B_J(x)|_\omega^{\frac{1}{\theta q}-\frac{1}{q}} \right\}.$$

We claim that

(22) $$T(x) \le c\,S(x)^\theta \left[a(B_0)\,|B_0|_\omega^{1/q} \right]^{1-\theta} .$$

If $S(x)$ is infinite, (22) is obvious. If $S(x)$ is finite, pick J such that the two terms on the right side of (21) are comparable, i.e., so that

$$|B_J(x)|_\omega^{1/(\theta q)} \approx \frac{a(B_0)\,|B_0|_\omega^{1/q}}{S(x)} .$$

Indeed, to see that this choice is possible, let $M = a(B_0)\,|B_0|_\omega^{1/q}/S(x)$. Then

$$0 < M \leq \frac{a(B_0)\,|B_0|_\omega^{1/q}}{a(B_0)\,|B_0|_\omega^{\frac{1}{q}-\frac{1}{\theta q}}} = |B_0|_\omega^{1/(\theta q)},$$

or equivalently $|B_0|_\omega \geq M^{\theta q}$. By reverse doubling, $|B_j(x)|_\omega \to 0$ as $j \to \infty$. Hence, there exists J such that

$$|B_{J+1}(x)|_\omega < M^{\theta q} \qquad \text{and} \qquad |B_J(x)|_\omega \geq M^{\theta q}.$$

Thus $|B_{J+1}(x)|_\omega < |B_J(x)|_\omega$. On the other hand, $|B_{J+1}(x)|_\omega \approx |B_J(x)|_\omega$ since by (H4),

$$B_{J+1}(x) \subset C B_J(x), \tag{23}$$

and then we have

$$|B_J(x)|_\omega \leq |CB_J(x)|_\omega \leq \left(\frac{r(CB_J(x))}{r(B_{J+1})}\right)^N |B_{J+1}(x)|_\omega \leq c\,|B_{J+1}(x)|_\omega,$$

where the next-to-last inequality follows from (23) and doubling applied to the balls $B_{J+1}(x)$ and $CB_J(x)$, and the last inequality follows from (H2). Hence $|B_J(x)|_\omega \approx M^{\theta q}$, as desired, and we then obtain (22) by direct computation.

If $T(x) > \lambda$, then (22) implies that

$$\lambda < c\,S(x)^\theta \left[a(B_0)\,|B_0|_\omega^{1/q}\right]^{1-\theta}.$$

Hence, by definition of $S(x)$, there exists a ball B_x with $B_x \subset B_0$, $x \in \alpha B_x$ and

$$\lambda < c\,\left[a(B_x)\,|B_x|_\omega^{\frac{1}{q}-\frac{1}{\theta q}}\right]^\theta \left[a(B_0)\,|B_0|_\omega^{1/q}\right]^{1-\theta},$$

so that

$$\lambda^q\,|B_x|_\omega \leq c\,a(B_x)^{\theta q}|B_x|_\omega^\theta \left[a(B_0)\,|B_0|_\omega^{1/q}\right]^{q(1-\theta)}. \tag{24}$$

Since the collection of balls $\mathcal{C} = \{\alpha B_x \,:\, x \in B_0 \text{ and } T(x) > \lambda\}$ covers $\{x \in B_0 \,:\, T(x) > \lambda\}$, an argument of Vitali type shows that there is a disjoint countable subfamily $\{\alpha B_k\}_{k=1}^\infty$ of $\mathcal{C}$ (thus (24) holds for each B_k) and a geometric constant $\alpha_1 > 1$ such that

$$\{x \in B_0 \,:\, T(x) > \lambda\} \subset \bigcup_{k=1}^\infty \alpha_1 \alpha B_k.$$

Hence

$$\begin{aligned}
|\{x \in B_0 \,:\, T(x) > \lambda\}|_\omega &\leq \sum_{k=1}^\infty |\alpha_1\alpha B_k|_\omega \\
&\leq c\sum_{k=1}^\infty |B_k|_\omega \qquad \text{by doubling} \\
&\leq \frac{c}{\lambda^q}\sum_{k=1}^\infty a(B_k)^{\theta q}|B_k|_\omega^\theta \left[a(B_0)\,|B_0|_\omega^{1/q}\right]^{q(1-\theta)},
\end{aligned}$$

by (24). Since the balls B_k are disjoint and lie in B_0, we obtain from (16) that

$$\begin{aligned}
|\{x \in B_0 \,:\, T(x) > \lambda\}|_\omega &\leq \frac{c}{\lambda^q}\,a(B_0)^{\theta q}|B_0|_\omega^\theta \left[a(B_0)\,|B_0|_\omega^{1/q}\right]^{q(1-\theta)} \\
&= \frac{c}{\lambda^q}\,a(B_0)^q\,|B_0|_\omega.
\end{aligned}$$

Thus (17) is proved, and so the proof of Theorem 2.7 is complete. We note in passing that the last part of the argument can easily be adapted to show that $S(x)$ itself satisfies a weak type estimate, and so is finite almost everywhere. □

Proof of Theorems 2.5 and 2.10. Let us first prove Theorem 2.5. By (H2) and (H3), we have that for μ-a.e. $x \in B_0$ (e.g., for every Lebesgue point x of f),

$$f_{B_j(x)} \to f(x) \quad \text{as} \quad j \to \infty. \tag{25}$$

Now let x be a point of B_0 such that (25) holds. Then

$$|f(x) - f_{B_0}| \le \sum_{j=0}^{\infty} |f_{B_j} - f_{B_{j+1}}|,$$

where $B_j = B_j(x)$ (recall that $B_0(x) = B_0$ by definition). But by doubling, keeping (H5) in mind,

$$\begin{aligned} |f_{B_j} - f_{B_{j+1}}| &\le |f_{B_j} - f_{S_j}| + |f_{S_j} - f_{B_{j+1}}| \\ &\le \frac{1}{|S_j|_\mu} \int_{S_j} |f - f_{B_j}|\, d\mu + \frac{1}{|S_j|_\mu} \int_{S_j} |f - f_{B_{j+1}}|\, d\mu \\ &\le \frac{c}{|B_j|_\mu} \int_{B_j} |f - f_{B_j}|\, d\mu + \frac{c}{|B_{j+1}|_\mu} \int_{B_{j+1}} |f - f_{B_{j+1}}|\, d\mu \\ &\le c\,[a(B_j) + a(B_{j+1})]\,, \end{aligned}$$

by (13). Hence,

$$\begin{aligned} |f(x) - f_{B_0}| &\le c \sum_{j=0}^{\infty} [a(B_j) + a(B_{j+1})] \\ &\le c \sum_{j=0}^{\infty} a(B_j) = c\,T(x). \end{aligned}$$

This proves (14) and so also Theorem 2.5.

To prove Theorem 2.10, notice first that the function f there satisfies $f \in L^1(B_0, \mu)$ and that the constant c_B in (18) can be replaced by $f_B = \frac{1}{|B|_\mu} \int_B f\, d\mu$, by a standard argument. Notice also that the set $\{x \in B_0 : \ |f(x) - f_{B_0}| > \lambda\}$ is a Borel set, and hence it is ω-measurable by definition. Theorem 2.10 then follows from Theorems 2.5 and 2.7. □

Remark 2.11. The proofs of Theorems 2.7 and 2.10 do not require the mild monotonicity condition

$$a(B_1) \le c\, a(B_2) \qquad \text{when} \qquad B_1 \subset B_2 \subset CB_1,$$

but this follows from the one–term version of (16) if we assume (16) holds with B_0 replaced by any subball of B_0, since $\omega \in D$.

The following strong type Poincaré result is a corollary of Theorem 2.10 and an interpolation argument.

Corollary 2.12. *Let the assumptions of Theorem 2.10 hold, including (16) for some $\theta < 1$ and $\omega \in D$. Then if $0 < r < q$,*

$$\left(\frac{1}{|B_0|_\omega} \int_{B_0} |f - f_{B_0}|^r\, d\omega \right)^{1/r} \le c\, a(B_0), \tag{26}$$

where c is a geometric constant c which depends also on r, q.

Proof. By Hölder's inequality, we only need to prove (26) when $\theta q < r < q$. To do this, note that if $\theta q < \tilde{q} < q$, then (16) still holds when q is replaced by $\tilde{q}$ and θ is replaced by $\theta_1 = (q\theta)/\tilde{q}$. Indeed

$$\begin{aligned}\sum_j a(Q_j)^{\tilde{q}\theta_1} |Q_j|_\omega^{\theta_1} &= \sum_j a(Q_j)^{q\theta} |Q_j|_\omega^{\theta} \cdot |Q_j|_\omega^{\theta_1-\theta} \\ &\le a(B_0)^{q\theta} |B_0|_\omega^{\theta} |B_0|_\omega^{\theta_1-\theta} = a(B_0)^{\tilde{q}\theta_1} |B_0|_\omega^{\theta_1},\end{aligned}$$

by (16) and the facts that $\theta_1 - \theta > 0$ and $|Q_j|_\omega \le |B_0|_\omega$. Since $\theta_1 < 1$, we can apply (19) with $\tilde{q}$ in place of q, and we conclude by the Marcinkiewicz interpolation theorem. □

Let us now give some applications of the previous results. We begin by giving an example of a functional $a(B)$ which satisfies (16).

Proposition 2.13. *Let ν and ω be Borel measures on B_0. Given p and a function g which satisfy $0 < p < \infty$, $g \in L^p(B_0, d\nu)$ and $g \ge 0$, define*

$$a(B) = r(B) \left(\frac{1}{|B|_\nu} \int_B g^p \, d\nu \right)^{1/p}. \tag{27}$$

If the balance condition

$$\frac{r(B)}{r(B_0)} \left(\frac{|B|_\omega}{|B_0|_\omega} \right)^{1/q} \le c \left(\frac{|B|_\nu}{|B_0|_\nu} \right)^{1/p} \tag{28}$$

is valid for some q and all $B \subset B_0$, then condition (16) with $\theta = p/q$ holds for (27).

In particular, in case both $\omega = \mu$ and $\nu = \mu$, the balance condition (28) amounts to the doubling condition $\mu \in D_N, N = (1/p - 1/q)^{-1}$, which explains the example that we mentioned earlier before Theorem 2.10.

Proof. If $\{Q_j\}$ are disjoint subballs of B_0, then

$$\begin{aligned}\sum_j a(Q_j)^p \, |Q_j|_\omega^{p/q} &= \sum_j \left(\frac{r(Q_j)^p \, |Q_j|_\omega^{p/q}}{|Q_j|_\nu} \int_{Q_j} g^p \, d\nu \right) \\ &\le C \, \frac{r(B_0)^p \, |B_0|_\omega^{p/q}}{|B_0|_\nu} \sum_j \int_{Q_j} g^p \, d\nu \qquad \text{(by (28))} \\ &\le C \, \frac{r(B_0)^p \, |B_0|_\omega^{p/q}}{|B_0|_\nu} \int_{B_0} g^p \, d\nu = C \, a(B_0)^p |B_0|_\omega^{p/q}. \quad \square\end{aligned}$$

By combining Corollary 2.12 and Proposition 2.13, we immediately obtain the following self-improving result.

Corollary 2.14. *Suppose that (H1)–(H3) and (H5) hold, $\omega \in D(\mathcal{S}, \rho)$, and f is a real-valued Borel function in B_0. Let ν be a Borel measure in $\mathcal{S}$ such that the balance condition (28) holds for some pair p, q with $0 < p < q$, and let r satisfy $0 < r < q$. If there exists $g \ge 0$ such that for every ball $B \subset B_0$ there is a constant $c_B \in \mathbb{R}$ with*

$$\frac{1}{|B|_\mu} \int_B |f - c_B| \, d\mu \le c \, r(B) \left(\frac{1}{|B|_\nu} \int_B g^p \, d\nu \right)^{1/p}, \tag{29}$$

then

$$\left(\frac{1}{|B_0|_\omega} \int_{B_0} |f - f_{B_0}|^r \, d\omega \right)^{1/r} \le c \, r(B_0) \left(\frac{1}{|B_0|_\nu} \int_{B_0} g^p \, d\nu \right)^{1/p}. \tag{30}$$

As in [16], we are not able in general to prove the end-point result with $r = q$. However, again as in [16], the sharp result with $r = q$ is true when we are dealing with a right-hand side which acts on truncated functions like a differential operator. Let us state some typical results of this kind including a new application to weighted Poincaré inequalities for generalized BV-functions; we refer to [16] for further examples and references.

Consider a functional $b(B, f)$ of two variables of the form

$$b : \mathcal{B} \times \mathcal{F} \to (0, \infty),$$

where $\mathcal{F}$ is an appropriate set of functions contained in $L^1_{loc}(\mathcal{S}, \mu)$ and $\mathcal{B}$ denotes the family of all balls in $(\mathcal{S}, \rho)$. Given a nonnegative function h and a positive real number λ, the truncation $\tau_\lambda(h)$ is defined by

$$\tau_\lambda(h)(x) = \min\{h(x), 2\,\lambda\} - \min\{h(x), \lambda\} = \begin{cases} 0 & \text{if } h(x) \le \lambda \\ h(x) - \lambda & \text{if } \lambda < h(x) \le 2\,\lambda \\ \lambda & \text{if } h(x) > 2\,\lambda. \end{cases}$$

In the Euclidean case, $\mathcal{F}$ can be chosen to be the class of Lipschitz continuous functions if $b(B, f)$ is defined by

$$b(B, f) = r(B) \left(\frac{1}{|B|_\nu} \int_B |\nabla f|^p \, d\nu \right)^{\frac{1}{p}}$$

for some Borel measure ν and some $p > 0$. More generally, ∇ could be replaced in this example by any first order differential operator X with $X1 = 0$, i.e., with no zero order term. In the general case, we shall assume that $\mathcal{F}$ has the properties

(H8) $f \in \mathcal{F} \Rightarrow f + \lambda, \lambda f \in \mathcal{F}$ for $\lambda \in \mathbb{R}$
(H9) $f \in \mathcal{F} \Rightarrow |f| \in \mathcal{F}$
(H10) $f \in \mathcal{F} \Rightarrow \tau_\lambda(|f|) \in \mathcal{F}$ for $\lambda \ge 0$,

and that the following natural relationships between the functional b and $\mathcal{F}$ hold:

(H11) $b(B, f) = b(B, f + \lambda)$ for all $f \in \mathcal{F}$ and $\lambda \in \mathbb{R}$
(H12) $b(B, |f|) \le b(B, f)$ for all $f \in \mathcal{F}$
(H13) There exist $q > 0$ and a constant C such that for any nonnegative $f \in \mathcal{F}$, any ball B and any sequence λ_k of the form $\{\lambda_k = 2^k \lambda\}, k = 1, 2, \ldots, \lambda > 0$,

$$\sum_{k=1}^{\infty} b(B, \tau_{\lambda_k}(f))^q \le C\, b(B, f)^q. \tag{31}$$

If $b(B, f)$ is the functional mentioned above as an example, then (31) is valid whenever $q \ge p$; this is proved in [16], p. 118, in case $p = 1$, and the general case is similar.

We assume that b and $\mathcal{F}$ have all the properties listed above and also that b satisfies the following condition (a condition like (16) but for every $B \in \mathcal{B}$ and $f \in \mathcal{F}$, uniformly in B and f): there exist positive constants θ and c with $\theta < 1$ so that

$$\sum_j \{b(Q_j, f)^q \, |Q_j|_\omega\}^\theta \le c\,\{b(B, f)^q \, |B|_\omega\}^\theta \tag{32}$$

for all collections $\{Q_j\}$ of disjoint subballs of B. The value of q here is the same as in (31).

Once again, if $b(B, f)$ is the functional given above as an example, then (32) holds with $\theta = p/q$ provided the balance condition (28) is valid for the pair ω, ν; the proof is similar to that of Proposition 2.13.

The proof of the next result is analogous to that of [16], Theorem 3.1, and relies on a truncation argument which by-passes the interpolation argument.

Corollary 2.15. *Suppose (H1)–(H3) and (H5) hold and that the functional* b *and the class* $\mathcal{F}$ *satisfy the conditions above, including (31) and (32) for some* $\omega \in D$ *and some* q, θ *with* $q > 0$ *and* $0 < \theta < 1$*. Suppose also that*

$$\frac{1}{|B|_\mu} \int_B |f - c_B|\, d\mu \le c\, b(B, f) \tag{33}$$

with c_B *depending on* B *and* f*, uniformly for* $f \in \mathcal{F}$ *and* $B \subset B_0$*. Then for all* $f \in \mathcal{F}$*,*

$$\left(\frac{1}{|B_0|_\omega} \int_{B_0} |f - f_{B_0}|^q\, d\omega \right)^{1/q} \le C\, b(B_0, f), \tag{34}$$

where C *is a geometric constant which also depends on the constants in (31) and (32).*

In particular, Corollary 3.2 of [16] still holds (at least when $p < q$) with the weaker assumption that $\omega \in D$ instead of $\omega \in A_\infty(\mu)$. More precisely, we have the following result, in which we say that a nonnegative function $v \in A_p(\nu), 1 \le p < \infty$, if

$$\left(\frac{1}{|B|_\nu} \int_B v\, d\nu \right) \left(\frac{1}{|B|_\nu} \int_B v^{-p'/p}\, d\nu \right)^{p/p'} \le C, \quad 1 < p < \infty, \quad p' = \frac{p}{p-1},$$
$$\frac{1}{|B|_\nu} \int_B v\, d\nu \le C\, ess\, inf_B\, v, \quad p = 1,$$

for all balls B, with C independent of B. Moreover, we say that $v \in A_\infty(\nu)$ if $v \in A_p(\nu)$ for some p, $1 \le p < \infty$.

Corollary 2.16. *Let* μ *and* ν *be doubling Borel measures in* $(\mathbb{R}^n, \rho)$*,* $p_0 > 0$ *and* X *be a differential operator for which*

$$\frac{1}{|B|_\mu} \int_B |f - f_B|\, d\mu \le C\, r(B) \left(\frac{1}{|B|_\nu} \int_B |Xf|^{p_0}\, d\nu \right)^{1/p_0} \tag{35}$$

for all balls B *and all Lipschitz functions* f*. Let* $p_0 \le p < q < \infty$*, and assume that* $\omega \in D$*,* $v \in A_{p/p_0}(\nu)$*, and the following balance condition holds:*

$$\frac{r(\tilde{B})}{r(B)} \left(\frac{|\tilde{B}|_\omega}{|B|_\omega} \right)^{1/q} \le C \left(\frac{|\tilde{B}|_{vd\nu}}{|B|_{vd\nu}} \right)^{1/p} \tag{36}$$

for all balls $\tilde{B}, B$ *such that* $\tilde{B} \subset B$*. Then*

$$\left(\frac{1}{|B|_\omega} \int_B |f - f_B|^q\, d\omega \right)^{1/q} \le C\, r(B) \left(\frac{1}{|B|_{vd\nu}} \int_B |Xf|^p\, v\, d\nu \right)^{1/p} \tag{37}$$

with C *independent of* f *and* B*.*

Remark 2.17. Typically, the above result applies to the Carnot–Carathéodory metric space associated with a family of Lipschitz continuous vector fields.

Let us now state a result for generalized BV functions. In the classical Euclidean setting we refer for instance to [31], Ch. 5. In particular, if Ω is a bounded open subset of $\mathbb{R}^n$, we say that $f \in BV(\Omega)$ if

$$\|Df\|(\Omega) := \sup \left\{ \int_\Omega f \operatorname{div} \phi\, dx : |\phi| \le 1, \phi \in \operatorname{Lip}_0(\Omega) \right\} < \infty.$$

It is known ([31], Remark 5.1.2) that $\|Df\|$ is a Radon measure, i.e., a regular Borel measure that is finite on compact sets.

In [2], the author introduces a class $BV_\sigma(\Omega) \subset BV(\Omega)$ of weighted BV-functions with respect to a weight function $\sigma \in A_1^*$, i.e., a weight function $\sigma \in A_1(dx)$ with respect to Euclidean balls and which is also lower semicontinuous. More precisely, assuming that σ is defined and satisfies the A_1^*-condition in a neighborhood of $\bar{\Omega}$, a function $f \in L^1(\Omega, \sigma dx)$ is said to belong to $BV_\sigma(\Omega)$ if

$$\sup\left\{\int_\Omega f \operatorname{div}\phi\, dx : |\phi| \le \sigma, \phi \in \mathrm{Lip}_0(\Omega)\right\} < \infty.$$

By using the Riesz representation theorem, given any $f \in BV_\sigma(\Omega)$, we can canonically associate a Radon measure $\mathrm{var}_\sigma f$ such that $\mathrm{var}_\sigma f(\mathcal{U}) = \int_\mathcal{U} \|Df\|\, \sigma\, dx$ for any Borel set $\mathcal{U} \subset \Omega$. In particular, Theorem 3.2 in [2] states that the following Poincaré inequality holds:

$$\int_B |f(x) - f_B|\, \sigma(x)\, dx \le C\, r(B)\, \mathrm{var}_\sigma f(B) \tag{38}$$

for any Euclidean ball B in Ω with $r(B) \le 1$ and for any $f \in BV_\sigma(\Omega)$.

It is well-known that if $\sigma \in A_1(dx)$, then the measure $\sigma\, dx = d\mu$ satisfies the doubling condition (iv), and the segment property obviously holds for Euclidean distance. If w is a doubling weight with respect to Lebesgue measure and Euclidean balls, we want to find suitable conditions on w so that we can use Corollary 2.15 to improve (38) to the two-weight inequality

$$\left(\frac{1}{|B|_{wdx}}\int_B |f(x) - f_B|^q\, w(x)\, dx\right)^{1/q} \le C\, \frac{r(B)}{|B|_{\sigma dx}}\, \mathrm{var}_\sigma f(B) \tag{39}$$

for a suitable $q = q(\sigma, w) > 1$. To this end, set

$$\mathcal{F} = BV_\sigma(\Omega) \qquad \text{and} \qquad b(B, f) = \frac{r(B)}{|B|_{\sigma dx}}\, \mathrm{var}_\sigma f(B)$$

for any Euclidean ball B. Let us first show that assumptions (H8)–(H13) are satisfied. In doing so, we shall use the co-area formula for BV_σ-functions proved in [3], Theorem 2.4.5. If $E \subset \mathbb{R}^n$ is any Borel set and $\mathcal{U}$ is an open set, we say that E has finite σ-perimeter in $\mathcal{U}$ if $\chi_E \in BV_\sigma(\mathcal{U})$, and we put $\|\partial E\|_\sigma(\mathcal{U}) = \mathrm{var}_\sigma \chi_E(\mathcal{U})$ (see [31], 5.4.1, for the analogous statement in the unweighted setting $\sigma \equiv 1$). Then Theorem 2.4.5 in [3] reads as follows (we refer for instance to [31], Theorem 5.4.4 for the classical unweighted results):

Theorem 2.18 (Co-area formula for BV_σ-functions)**.** *Let $f \in L^1_{\mathrm{loc}}(d\mu)$ where $d\mu = \sigma\, dx$ and $\sigma \in A_1^*$. Set $E_t = E_t(f) = \{x \in \Omega : f(x) > t\}$ for $t \in \mathbb{R}$. If $\mathcal{U}$ is an open set in Ω, then $f \in BV_\sigma(\mathcal{U})$ if and only if the map $t \to \|\partial E_t\|_\sigma(\mathcal{U})$ belongs to $L^1(\mathbb{R})$. In addition, if $f \in BV_\sigma(\mathcal{U})$, then*

$$\mathrm{var}_\sigma f(\mathcal{U}) = \int_\mathbb{R} \|\partial E_t\|_\sigma(\mathcal{U})\, dt. \tag{40}$$

Note that (H8) and (H11) are straightforward consequences of the definition of BV_σ. To prove (H9), (H10), (H12) and (H13), first note that if $-\infty \le a < b \le +\infty$ and we put

$$\tau_{a,b}(f)(x) = \begin{cases} a & \text{if } f(x) \le a \\ f(x) & \text{if } a < f(x) \le b \\ b & \text{if } f(x) > b, \end{cases}$$

then

$$\{x \in \Omega : \tau_{a,b}(f)(x) > t\} = \begin{cases} \Omega & \text{if } t < a \\ E_t(f) & \text{if } a \le t < b \\ \emptyset & \text{if } t \ge b. \end{cases}$$

Therefore, $\tau_\lambda(f) = \tau_{\lambda,2\lambda}(f) - \lambda$, and hence (40) and (H11) imply that

$$\mathrm{var}_\sigma \tau_\lambda(f)(\mathcal{U}) = \int_\lambda^{2\lambda} \|\partial E_t\|_\sigma(\mathcal{U})\, dt, \tag{41}$$

which in turn implies (H10) for $f \ge 0$ and (H13). Eventually, (H9) and (H12), as well as (H10) (whether $f \ge 0$ or not) follow since $|f| = \tau_{0,+\infty}(f) - \tau_{-\infty,0}(f)$.

With $\mathcal{F}$ and $b(B,f)$ as above, we want to apply Corollary 2.15 with ω and μ replaced there by $w\,dx$ and $\sigma\,dx$, respectively. Note that $w\sigma^{-1}$ is clearly a doubling weight with respect to $\sigma\,dx$ and Euclidean balls, and that (38) implies (33). Thus, in order to apply Corollary 2.15, we only need to show that (32) holds for a suitable choice of θ and q. Thus, suppose there exists $q \geq 1$ such that (cf. (28) with $p = 1$)

$$\frac{r(B)}{r(B_0)}\left(\frac{|B|_{wdx}}{|B_0|_{wdx}}\right)^{1/q} \leq C\,\frac{|B|_{\sigma dx}}{|B_0|_{\sigma dx}} \tag{42}$$

for all balls $B \subset B_0$. Then, if $\{B_j\}$ is a family of disjoint subballs of B_0,

$$\begin{aligned}
&\sum_j b(B_j,f)\,|B_j|_{wdx}^{1/q}\\
&\quad= \sum_j \frac{r(B_j)}{|B_j|_{\sigma dx}}\,\mathrm{var}_\sigma f(B_j)\,|B_j|_{wdx}^{1/q} \leq C\,\frac{r(B_0)}{|B_0|_{\sigma dx}}\,|B_0|_{wdx}^{1/q}\sum_j \mathrm{var}_\sigma f(Q_j)\\
&\quad\leq C\,\frac{r(B_0)}{|B_0|_{\sigma dx}}\,|B_0|_{wdx}^{1/q}\cdot\mathrm{var}_\sigma f(B_0) = C\,b(B_0,f)|B_0|_{wdx}^{1/q},
\end{aligned}$$

since $\mathrm{var}_\sigma f$ is a Borel measure. Thus (42) implies (32) for $\theta = 1/q$, and our two-weight self-improving result can be stated as follows.

Theorem 2.19. *Let $\sigma \in A_1^*$ be fixed, and let w be a doubling weight with respect to Euclidean distance and Lebesgue measure. If there exists $q > 1$ such that (42) is satisfied, then (39) holds for any $f \in BV_\sigma(\Omega)$.*

More generally, let $(\mathcal{S},\rho)$ be a metric space endowed with a doubling Borel measure μ, so that $(\mathcal{S},\rho,\mu)$ is a metric space of homogeneous type. Assume in addition that $(\mathcal{S},\rho,\mu)$ is *of Poincaré type*, i.e., that for any $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{S},\mathbb{R})$,

$$\frac{1}{|B|_\mu}\int_B |f - f_B|\,d\mu \leq C\,r(B)\frac{1}{|B|_\mu}\int_B \|\nabla f\|\,d\mu \tag{43}$$

for all balls B, where

$$\|\nabla f\| = \liminf_{t\to 0^+}\frac{1}{t}\sup_{\rho(x,y)\leq t}|f(x) - f(y)|.$$

From now on, we assume that ρ and μ are fixed, and hence $\mathcal{S}$ will stand for $(\mathcal{S},\rho,\mu)$. Following [1] and [24], if $\Omega \subset \mathcal{S}$ is an open set, we can define the class $BV_{\mathcal{S}}(\Omega) = BV(\Omega,\rho,\mu)$ of bounded variation functions in $\mathcal{S}$ by a relaxation argument starting from ρ-Lipschitz continuous functions f as follows.

Definition 2.20. We say that $f \in L^1(\Omega)$ belongs to $BV_{\mathcal{S}}(\Omega)$ if there exists a sequence $(f_h)_{h\in\mathbb{N}}$ in $\mathrm{Lip}_{\mathrm{loc}}(\Omega,\mathbb{R}) \cap L^1(\Omega)$ such that $f_h \to f$ in $L^1(\Omega)$ as $h \to \infty$ and

$$\begin{aligned}
&\|Df\|_{\mathcal{S}}(\Omega)\\
&:= \inf\left\{\liminf_{h\to\infty}\int_\Omega \|\nabla f_h\|\,d\mu\,;\ f_h \in \mathrm{Lip}_{\mathrm{loc}}(\Omega,\mathbb{R})\cap L^1(\Omega)\,;\ f_h \to f \text{ in } L^1(\Omega) \text{ as } h\to\infty\right\} < \infty.
\end{aligned}$$

Remark 2.21. This abstract theory applies for instance to the BV spaces associated with Lipschitz continuous vector fields in [12] provided a Poincaré inequality holds, and to BV spaces associated with the strong-A_∞ weights of G. David and S. Semmes (see [4] for details).

If $f \in BV_{\mathcal{S}}(\Omega)$ for any bounded open set $\Omega \subset \mathcal{S}$, we say that $f \in BV_{\mathcal{S},\mathrm{loc}}$. In this case, such a procedure yields a variation $\|Df\|_{\mathcal{S}}$ coinciding with a positive measure on open subsets of Ω ([1], Theorem 3.3, or [24], Theorem 3.4). Since a co-area formula still holds in this setting ([24], Proposition 4.2), we can repeat our previous arguments yielding identity (41) to apply Corollary 2.15 and then eventually to prove the following two-weight result.

Theorem 2.22. *Let $(\mathcal{S}, \rho, \mu)$ be a metric space of homogeneous type which is also of Poincaré type (i.e., (43) holds for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{S}, \mathbb{R})$), and suppose that (H1)–(H3) and (H5) hold. Let $w\, d\mu \in D$ be such that there exists $q > 1$ so that*

$$\frac{r(B)}{r(B_0)} \left(\frac{|B|_{\omega d\mu}}{|B_0|_{\omega d\mu}} \right)^{1/q} \le C\, \frac{|Q|_\mu}{|B_0|_\mu}$$

for all balls $B \subset B_0$. Then

$$\left(\frac{1}{|B|_{\omega d\mu}} \int_B |f - f_B|^q\, \omega\, d\mu \right)^{1/q} \le C\, \frac{r(B)}{|B|_\mu}\, \|Df\|_{\mathcal{S}}(B) \tag{44}$$

for all $f \in BV_{\mathcal{S}}(B_0)$ and $B \subset B_0$.

Remark 2.23. If we assume in addition that $(\mathcal{S}, \rho)$ enjoys the segment property, then both Theorems 2.19 and 2.22 could be proved alternatively through a representation formula and an L^p, L^q continuity result for integral operators of potential type in spaces of homogeneous type. See [20] for the form of this representation, and see e.g. [10] for the L^p, L^q continuity result. Similar representation formulas were introduced earlier in [11] and [15] in case a stronger assumption is satisfied by the measures involved. In the case of Theorem 2.19 for example, the assumption requires the existence of $c > 0$ such that for all balls B, $\tilde{B}$ with $\tilde{B} \subset B \subset B_0$,

$$\frac{|B|_{wdx}}{|\tilde{B}|_{wdx}} \ge c\, \frac{r(B)}{r(\tilde{B})},$$

which fails to hold for general A_1^* weights (think for instance of $w(x) = |x|^{-n+\epsilon}$ for $0 < \epsilon < 1$). However, by [20], this stronger condition is not required if the representation formula in [11], [15] is altered slightly by adding an innocuous constant term to the right-hand side.

3. The case $p = q$.

In this section, we shall consider the special case when the functional $a(B)$ is given by

$$a(B) = r(B) \left(\frac{1}{|B|_\mu} \int_B g^{p_0}\, d\mu \right)^{1/p_0}, \quad B \subset B_0, \tag{45}$$

where $p_0 \ge 1$, $\mu \in D$, and $g \ge 0$. We will not need to assume that g is a derivative, but we will assume that ω is absolutely continuous with respect to μ:

$$d\omega = w\, d\mu.$$

Our goal is to derive an analogue of Corollary 2.16 in which q is allowed to equal p if $p > p_0$, i.e., to prove that for appropriate w and v, the estimate

$$\left(\frac{1}{|B_0|_{wd\mu}} \int_{B_0} |f - f_{B_0}|^p\, w\, d\mu \right)^{1/p} \le C\, r(B_0) \left(\frac{1}{|B_0|_{vd\mu}} \int_{B_0} g^p\, v\, d\mu \right)^{1/p}$$

with $p > p_0$ can be deduced from an initial assumption of the form

$$\frac{1}{|B|_\mu} \int_B |f - f_B|\, d\mu \le C\, r(B) \left(\frac{1}{|B|_\mu} \int_B g^{p_0}\, d\mu \right)^{1/p_0}$$

for all balls $B \subset B_0$. The exact statement is given in Theorem 3.1 below. It will be convenient to assume as we may that *supp* $g \subset B_0$, and then to define $a(B)$ by the same formula for all $B \subset \mathcal{S}$.

Let

$$\mathcal{A}_r(w, B) = \left(\int_B w^r\, d\mu \right)^{1/r} |B|_\mu^{1/r'}, \quad r > 1, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Note that $|B|_{wd\mu} \le \mathcal{A}_r(w, B)$ for any w by Hölder's inequality, and that if $w \in A_\infty(d\mu)$, then $|B|_{wd\mu} \approx \mathcal{A}_r(w, B)$ uniformly in B if r is sufficiently close to 1. In this section, in order to prove a direct strong type estimate for the sum operator $T(x)$, we will assume a different form of the balance condition. We will assume that for a given $p > p_0$, there exists $r > 1$ so that for all $B \subset cB_0$ ($c > 1$ is an appropriate geometric constant),

$$\left(\frac{r(B)}{r(B_0)}\right)^{p_0}\left(\frac{\mathcal{A}_r(w,B)}{|cB_0|_{wd\mu}}\right)^{\frac{p_0}{p}}\left(\frac{\mathcal{A}_r(\sigma,B)}{|cB_0|_{\sigma d\mu}}\right)^{1-\frac{p_0}{p}} \le C\,\frac{|B|_\mu}{|B_0|_\mu}, \quad \sigma = v^{-\frac{1}{(p/p_0)-1}}. \tag{46}$$

We refer to this condition as a strengthened balance condition in the Fefferman–Phong sense. In case $v \in A_{p/p_0}(d\mu)$, it is easy to check that (46) amounts to the balance condition

$$\frac{r(B)}{r(B_0)}\left(\frac{\mathcal{A}_r(w,B)}{|cB_0|_{wd\mu}}\right)^{1/p} \le C\left(\frac{|B|_{vd\mu}}{|B_0|_{vd\mu}}\right)^{1/p}, \quad B \subset cB_0. \tag{47}$$

Moreover, if $w \in A_\infty(d\mu)$, (47) is equivalent to (36) in case $p = q$ (and $\nu = \mu$):

$$\frac{r(B)}{r(B_0)}\left(\frac{|B|_{\omega d\mu}}{|B_0|_{\omega d\mu}}\right)^{1/p} \le c\left(\frac{|B|_{vd\mu}}{|B_0|_{vd\mu}}\right)^{1/p}, \quad B \subset B_0.$$

We will prove the analogue of Corollary 2.16 given in the next theorem.

Theorem 3.1. *Assume that (H1)–(H3) and (H5) hold for a ball B_0 in a space $(\mathcal{S},\rho,\mu)$ of homogeneous type. Let f be a function which satisfies*

$$\frac{1}{|B|_\mu}\int_B |f - f_B|\,d\mu \le C\,r(B)\left(\frac{1}{|B|_\mu}\int_B g^{p_0}\,d\mu\right)^{1/p_0}, \quad B \subset B_0,$$

for some $p_0 \ge 1$ and some function $g \ge 0$. If w and v are a pair of weights so that the balance condition (47) holds for some $p > p_0$, $r > 1$ and all $B \subset cB_0$, and if $v \in A_{p/p_0}(d\mu)$, then

$$\left(\int_{B_0} |f - f_{B_0}|^p\,w\,d\mu\right)^{1/p} \le C\,|cB_0|_{wd\mu}^{1/p}\,r(B_0)\left(\frac{1}{|B_0|_{vd\mu}}\int_{B_0} g^p\,v\,d\mu\right)^{1/p}.$$

Note that the function g above is not assumed to be a derivative. Note also that $w\,d\mu$ is not assumed to be a doubling measure; if $w\,d\mu$ is doubling then we may take $c = 1$ in the conclusion.

We will use the following result about sum operators as a basis for deriving Theorem 3.1.

Theorem 3.2. *Assume that (H1)–(H3) hold for a ball B_0 in a space $(\mathcal{S},\rho,\mu)$ of homogeneous type. Let T be the sum operator formed by using the functional $a(B)$ in (45) for some $p_0 \ge 1$. Let w and v be weights which satisfy (46) for some $p > p_0$ and all $B \subset cB_0$. Then*

$$\left(\int_{B_0} T^p\,w\,d\mu\right)^{1/p} \le C_{B_0}\left(\int_{B_0} g^p\,v\,d\mu\right)^{1/p} \tag{48}$$

with

$$C_{B_0} = C\,\frac{r(B_0)\,|cB_0|_{wd\mu}^{\frac{1}{p}}|cB_0|_{\sigma d\mu}^{\frac{1}{p_0}-\frac{1}{p}}}{|B_0|_\mu^{\frac{1}{p_0}}}.$$

Remark 3.3. As always, μ is assumed to be a doubling measure but none of w, v or σ is assumed to be a doubling weight. If $v \in A_{p/p_0}(d\mu)$, then (48) means simply that

$$\|T\|_{L^p_{wd\mu}(B_0)} \le C\,r(B_0)\,|cB_0|_{wd\mu}^{1/p}\left(\frac{1}{|B_0|_{vd\mu}}\int_{B_0} g^p\,v\,d\mu\right)^{1/p},$$

since if $v \in A_{p/p_0}(d\mu)$ then

$$\frac{|cB_0|_{\sigma d\mu}^{\frac{1}{p_0}-\frac{1}{p}}}{|B_0|_\mu^{\frac{1}{p_0}}} \le \frac{C}{|B_0|_{vd\mu}^{\frac{1}{p}}}.$$

Finally, note that Theorem 3.2 is a strong type result, as opposed to our earlier weak type result about $T(x)$.

Proof of Theorem 3.2. To prove the theorem, we will use a grid of dyadic sets in $\mathcal{S}$ which are "almost balls", as constructed in [28]. In fact, the following has been proved there:

> If $\tau = 8K^5$ (where K is the quasimetric constant for ρ), then for any (large negative) integer m, there are points $\{x_j^k\}$ and a family $\mathcal{D}_m = \{D_j^k\}$ of sets for $k = m, m+1, \cdots$ and $j = 1, 2, \cdots$ such that
> - $B(x_j^k, \tau^k) \subset D_j^k \subset B(x_j^k, \tau^{k+1})$
> - For each $k = m, m+1, \cdots$, the family $\{D_j^k\}$ is pairwise disjoint in j, and $\mathcal{S} = \cup_j D_j^k$.
> - If $m \le k < l$, then either $D_j^k \cap D_i^l = \emptyset$ or $D_j^k \subset D_i^l$.

We call the family $\mathcal{D} = \cup_{m\in\mathbb{Z}} \mathcal{D}_m$ a dyadic cube decomposition of $\mathcal{S}$ and refer to the sets in $\mathcal{D}$ as dyadic cubes. A dyadic cube will usually be denoted by Q, and $B(Q)$ will denote the containing ball described above with $\frac{1}{\tau}B(Q) \subset Q \subset B(Q)$; thus, if $Q = D_j^k$ then $B(Q) = B(x_j^k, \tau^{k+1})$. We set $\ell(Q) = r(B(Q))/\tau$ and call $\ell(Q)$ the "sidelength" of Q. We note that while the cubes in each $\mathcal{D}_m$ have the dyadic properties listed above, there may be no nestedness properties of the cubes in $\mathcal{D}_{m_1}$ relative to the cubes in $\mathcal{D}_{m_2}$ if m_1, m_2 are different.

Since $supp\, g \subset B_0$, the theorem will follow by proving (48) with integration on the right-hand side extended over $\mathcal{S}$. Let $x \in B_0$. By definition,

$$T(x) = \sum a(B),$$

where the sum is over all balls B in a chain for x. Define

$$T_m(x) = \sum_{B: r(B) \ge \tau^m} a(B),$$

where the sum is only over those balls in the same chain whose radius is at least τ^m. Since $T_m(x)$ increases to $T(x)$ as $m \to -\infty$, it is enough to prove (48) with T replaced by T_m for the same constant C_{B_0} (independent of m).

Fix m. If B belongs to the chain for x and $r(B) \ge \tau^m$, then if $r(B) \approx 2^{-n} r(B_0)$, $n \ge 0$, we can choose pairwise disjoint dyadic cubes $Q_\ell^n \in \mathcal{D}_m$, $\ell = 1, \dots, N$, of comparable size to B (i.e., with $\ell(Q_\ell^n) \approx r(B)$) such that $B \subset \bigcup_{\ell=1}^N Q_\ell^n$. In fact, N can be chosen to be independent of B. If Q is a dyadic cube, let

$$a(Q) = \ell(Q) \left(\frac{1}{|Q|_\mu} \int_Q g^{p_0}\, d\mu \right)^{1/p_0}.$$

Since $r(B) \approx \ell(Q_\ell^n)$, it follows from doubling that $|B|_\mu \approx |Q_\ell^n|_\mu$. Thus, since the Q_ℓ^n are disjoint in ℓ, there is a geometric constant c depending possibly also on N and p_0 so that

$$a(B) = r(B) \left(\frac{1}{|B|_\mu} \int_B g^{p_0}\, d\mu \right)^{1/p_0}$$

$$\le c \sum_\ell \ell(Q_\ell^n) \left(\frac{1}{|Q_\ell^n|_\mu} \int_{Q_\ell^n} g^{p_0}\, d\mu \right)^{1/p_0} = c \sum_\ell a(Q_\ell^n).$$

Since $\rho(x, B) \le c r(B)$ (by (H3)), then $\rho(x, Q_\ell^n) \le c\, \ell(Q_\ell^n)$ for all ℓ. Hence,

$$T_m(x) \le c \sum_{\substack{\ell, n \,: Q_\ell^n \in \mathcal{D}_m \\ \rho(x, Q_\ell^n) \le c\, \ell(Q_\ell^n)}} a(Q_\ell^n). \tag{49}$$

By duality,

$$\|T_m\|_{L^p_{wd\mu}(B_0)} = \sup_{\substack{h\geq 0,\, supp\, h\subset B_0 \\ \|h\|_{L^{p'}_{d\mu}(B_0)}=1}} \int T_m\, h\, w^{\frac{1}{p}}\, d\mu, \quad \frac{1}{p}+\frac{1}{p'}=1.$$

Since $B\subset B_0$, it is easy to see that each Q^n_ℓ is contained in cB_0, and consequently we obtain

$$\begin{aligned}\int T_m\, h\, w^{\frac{1}{p}}\, d\mu &\leq c \sum_{Q\in\mathcal{D}_m;Q\subset cB_0} a(Q) \int_{cB(Q)} h\, w^{\frac{1}{p}}\, d\mu \\ &= c \sum_{Q\in\mathcal{D}_m;Q\subset cB_0} \ell(Q) \left(\frac{1}{|Q|_\mu}\int_Q g^{p_0}\, d\mu\right)^{\frac{1}{p_0}} \int_{cB(Q)} h\, w^{\frac{1}{p}}\, d\mu := c\, S.\end{aligned}$$

To prove Theorem 3.2, it is enough to show that

$$S \leq C_{B_0} \left(\int g^p\, v\, d\mu\right)^{\frac{1}{p}} \left(\int h^{p'}\, d\mu\right)^{\frac{1}{p'}}. \tag{50}$$

We may assume without loss of generality that

$$\left(\frac{1}{|B_0|_\mu}\int_{B_0} g^{p_0}\, d\mu\right)^{\frac{1}{p_0}} = 1.$$

For $\gamma>1$ to be chosen and $k\in\mathbb{Z}$, let

$$\mathcal{C}_k = \{Q\in\mathcal{D}_m : Q\subset cB_0\, ; \gamma^k < \left(\frac{1}{|Q|_\mu}\int_Q g^{p_0}\, d\mu\right)^{\frac{1}{p_0}} \leq \gamma^{k+1}\}. \tag{51}$$

Then

$$\begin{aligned} S &= \sum_{Q\in\mathcal{D}_m;Q\subset cB_0} \ell(Q) \left(\frac{1}{|Q|_\mu}\int_Q g^{p_0}\, d\mu\right)^{\frac{1}{p_0}} \int_{cB(Q)} h\, w^{\frac{1}{p}}\, d\mu \\ &= \sum_k \sum_{Q\in\mathcal{C}_k} \ell(Q) \left(\frac{1}{|Q|_\mu}\int_Q g^{p_0}\, d\mu\right)^{\frac{1}{p_0}} \int_{cB(Q)} h\, w^{\frac{1}{p}}\, d\mu \\ &= \sum_{k\leq 0} + \sum_{k\geq 1} \quad := S_1 + S_2. \end{aligned}$$

Let us first estimate S_1. We have

$$\begin{aligned} S_1 &\leq \sum_{k\leq 0}\sum_{Q\in\mathcal{C}_k} \ell(Q)\gamma^{k+1} \int_{cB(Q)} h\, w^{\frac{1}{p}}\, d\mu \\ &\leq \sum_{k\leq 0}\gamma^{k+1} \sum_{Q\in\mathcal{D}_m:Q\subset cB_0} \ell(Q) \int_{cB(Q)} h\, w^{\frac{1}{p}}\, d\mu. \end{aligned}$$

We claim that if B is any ball, then

$$\sum_{Q\in\mathcal{D}_m;Q\subset cB} \ell(Q) \int_{cB(Q)} h\, w^{\frac{1}{p}}\, d\mu \leq c\, r(B) \int_{cB} h\, w^{\frac{1}{p}}\, d\mu. \tag{52}$$

To prove (52), note that the left-hand side of (52) is at most

$$\sum_{\ell:\tau^\ell\leq cr(B)} \sum_{\substack{Q\in\mathcal{D}_m,\, Q\subset cB \\ \ell(Q)=\tau^\ell}} \tau^\ell \int_{cB(Q)} h\, w^{\frac{1}{p}}\, d\mu$$

$$\leq \sum_{\ell:\tau^\ell \leq cr(B)} \tau^\ell \int_{cB} \left[\sum_{Q\in\mathcal{D}_m:\ell(Q)=\tau^\ell} \chi_{cB(Q)} \right] h\, w^{\frac{1}{p}}\, d\mu \quad := I.$$

But

$$\sum_{Q\in\mathcal{D}_m:\ell(Q)=\tau^\ell} \chi_{cB(Q)}(y) \leq C$$

uniformly with respect to ℓ (cf. (59) of [26]). Thus since

$$\sum_{\ell:\tau^\ell \leq cr(B)} \tau^\ell \leq c\, r(B),$$

we obtain

$$I \leq Cr(B) \int_{cB} h\, w^{\frac{1}{p}}\, d\mu.$$

which proves (52).

Going back to S_1, we obtain from (52) that

$$S_1 \leq c\, r(B_0) \int_{cB_0} h\, w^{\frac{1}{p}}\, d\mu.$$

Thus, since $\frac{1}{|B_0|_\mu} \int_{B_0} g^{p_0}\, d\mu = 1$, we may write

$$S_1 \leq c\, r(B_0) \left(\frac{1}{|B_0|_\mu} \int_{B_0} g^{p_0}\, d\mu \right)^{\frac{1}{p_0}} \int_{cB_0} h\, w^{\frac{1}{p}}\, d\mu$$

$$= c\, r(B_0) \left(\frac{1}{|B_0|_\mu} \int_{B_0} g^{p_0}\, v^{\frac{1}{s}} v^{-\frac{1}{s}}\, d\mu \right)^{\frac{1}{p_0}} \int_{cB_0} h\, w^{\frac{1}{p}}\, d\mu$$

with $s = p/p_0$. By Hölder's inequality,

$$S_1 \leq c\, r(B_0) \frac{1}{|B_0|_\mu^{\frac{1}{p_0}}} \left(\int_{B_0} g^p\, v\, d\mu \right)^{\frac{1}{p}} \left(\int_{B_0} v^{-\frac{s'}{s}}\, d\mu \right)^{\frac{1}{p_0 s'}} \left(\int_{cB_0} h^{p'}\, d\mu \right)^{\frac{1}{p'}} |cB_0|_{w d\mu}^{\frac{1}{p}}$$

$$\leq C_{B_0} \left(\int_{B_0} g^p\, v\, d\mu \right)^{\frac{1}{p}} \left(\int_{cB_0} h^{p'} d\mu \right)^{\frac{1}{p'}}.$$

This completes our estimation of S_1.

To estimate S_2, let $\{Q_j^k\}_j$ be the maximal dyadic cubes in $\mathcal{D}_m$ with

$$\left(\frac{1}{|Q|_\mu} \int_Q g^{p_0}\, d\mu \right)^{\frac{1}{p_0}} > \gamma^k.$$

The Q_j^k are disjoint in j by maximality. We do not assume $Q_j^k \subset cB_0$, but if $k \geq 1$, this must be so for a suitably large geometric constant c provided γ is large, as we now show. In fact, if Q_j^k is not contained in cB_0 and c is sufficiently large depending on the quasimetric constant K, then $\ell(Q_j^k)$ is at least comparable to $r(B_0)$ since Q_j^k must intersect B_0 (due to the support of g). Consequently, we must have $|B_0|_\mu \leq c_1 |Q_j^k|_\mu$ by doubling, with c_1 depending on c, and then

$$1 = \left(\frac{1}{|B_0|_\mu} \int_{B_0} g^{p_0}\, d\mu \right)^{\frac{1}{p_0}} = \left(\frac{1}{|B_0|_\mu} \int_{B_0} g^{p_0}\, d\mu \right)^{\frac{1}{p_0}}$$

$$\geq c_1^{-\frac{1}{p_0}} \left(\frac{1}{|Q_j^k|_\mu} \int_{Q_j^k} g^{p_0}\, d\mu \right)^{\frac{1}{p_0}} \geq c_1^{-\frac{1}{p_0}} \gamma^k,$$

which is impossible for $k \geq 1$ if γ is sufficiently large.

Thus $Q_j^k \subset cB_0$ if $k \geq 1$. By maximality and since μ is doubling we have

$$\gamma^k < \left(\frac{1}{|Q_j^k|_\mu} \int_{Q_j^k} g^{p_0}\, d\mu \right)^{\frac{1}{p_0}} < c\gamma^k \leq \gamma^{k+1}, \tag{53}$$

if γ is large, so that $Q_j^k \in \mathcal{C}_k$ when $k \geq 1$. On the other hand, again by maximality, any cube $Q \in \mathcal{C}_k$ is contained in a cube Q_j^k for some j. Then

$$S_2 \leq c \sum_k \gamma^{k+1} \sum_j \sum_{Q \in \mathcal{D}_m : Q \subset Q_j^k} \ell(Q) \int_{cB(Q)} h\, w^{\frac{1}{p}}\, d\mu.$$

If we write $B_j^k = B(Q_j^k)$ and apply (52), we obtain that S_2 is bounded by

$$\begin{aligned} & c \sum_k \gamma^{k+1} \sum_j \ell(Q_j^k) \int_{cB_j^k} h\, w^{\frac{1}{p}}\, d\mu \\ & \leq c\gamma \sum_{k,j} \ell(Q_j^k) \left(\frac{1}{|Q_j^k|_\mu} \int_{Q_j^k} g^{p_0}\, d\mu \right)^{\frac{1}{p_0}} \int_{cB_j^k} h\, w^{\frac{1}{p}}\, d\mu \\ & = c\gamma \sum_{k,j} a(Q_j^k) \int_{cB_j^k} h\, w^{\frac{1}{p}}\, d\mu. \end{aligned} \tag{54}$$

By Hölder inequality with exponents $(pr)'$, pr,

$$\begin{aligned} \int_{cB_j^k} h\, w^{\frac{1}{p}} d\mu & \leq \left(\int_{cB_j^k} h^{(pr)'} d\mu \right)^{\frac{1}{(pr)'}} \left(\int_{cB_j^k} w^r d\mu \right)^{\frac{1}{pr}} \\ & \leq \left(\int_{cB_j^k} h^{(pr)'} d\mu \right)^{\frac{1}{(pr)'}} \mathcal{A}_r(w, cB_j^k)^{\frac{1}{p}}\, |cB_j^k|_\mu^{-\frac{1}{pr'}}. \end{aligned}$$

Then, by Hölder's inequality for p, p', (54) and so also S_2 is bounded by

$$c\gamma \left[\sum_{k,j} a(Q_j^k)^p \mathcal{A}_r(w, cB_j^k) \right]^{\frac{1}{p}} \left[\sum_{k,j} \left(\int_{cB_j^k} h^{(pr)'} d\mu \right)^{\frac{p'}{(pr)'}} |Q_j^k|_\mu^{-\frac{p'}{pr'}} \right]^{\frac{1}{p'}}. \tag{55}$$

We stress the fact that $Q_j^k, B_j^k \subset cB_0$, as we proved above. Note that $-p'/pr' = -p'/(pr)' + 1$, so the second factor in (55) is

$$\left[\sum_{k,j} \left(\frac{1}{|Q_j^k|_\mu} \int_{cB_j^k} h^{(pr)'} d\mu \right)^{\frac{p'}{(pr)'}} |Q_j^k|_\mu \right]^{\frac{1}{p'}}.$$

Let

$$\Omega_k = \{x : \sup_{Q \in \mathcal{D}_m : x \in Q} \left(\frac{1}{|Q|_\mu} \int_Q g^{p_0}\, d\mu \right)^{\frac{1}{p_0}} > \gamma^k \}.$$

Then $\Omega_k = \bigcup_j Q_j^k$. Let

$$E_j^k = Q_j^k \setminus \Omega_{k+1}.$$

Note $E_j^k \subset \Omega_k \setminus \Omega_{k+1}$, and therefore the sets $\{E_j^k\}$ are disjoint in both k and j. We claim that

$$|Q_j^k|_\mu \leq 2\, |E_j^k|_\mu. \tag{56}$$

If so, the second factor in (55) is bounded by

$$\left[\sum_{k,j}\left(\frac{1}{|Q_j^k|_\mu}\int_{cB_j^k} h^{(pr)'}\,d\mu\right)^{p'/(pr)'} 2\,|E_j^k|_\mu\right]^{1/p'}.$$

On the other hand, if $x \in E_j^k$ then $x \in Q_j^k$, so that if we denote by M the Hardy-Littlewood maximal function defined by

$$M(f)(x) = \sup_{B:x\in B}\frac{1}{|B|_\mu}\int_B |f|\,d\mu,$$

we obtain from the doubling of μ that

$$\frac{1}{|Q_j^k|_\mu}\int_{cB_j^k} h^{(pr)'}\,d\mu \le cM\left(h^{(pr)'}\right)(x) \quad \text{if } x \in E_j^k.$$

Hence, the second factor in (55) is bounded by

$$c\left[\sum_{j,k}\int_{E_j^k} M\left(h^{(pr)'}\right)^{\frac{p'}{(pr)'}}\,d\mu\right]^{\frac{1}{p'}}$$

$$\le c\left[\int M\left(h^{(pr)'}\right)^{\frac{p'}{(pr)'}}\,d\mu\right]^{\frac{1}{p'}} \le c\left[\int h^{p'}\,d\mu\right]^{\frac{1}{p'}} \tag{57}$$

since $p'/(pr)' > 1$.

To prove (56), it is enough to show that

$$|Q_j^k \cap \Omega_{k+1}|_\mu \le \frac{1}{2}\,\mu(Q_j^k).$$

Write

$$\begin{aligned}|Q_j^k \cap \Omega_{k+1}|_\mu &= |Q_j^k \cap \bigcup_i Q_i^{k+1}|_\mu \\ &= \sum_i |Q_j^k \cap Q_i^{k+1}|_\mu.\end{aligned}$$

If $Q_j^k \cap Q_i^{k+1} \ne \emptyset$, then either $Q_i^{k+1} \subset Q_j^k$ or $Q_j^k \subset Q_i^{k+1}$ and $Q_j^k \ne Q_i^{k+1}$. But the last is impossible, since by maximality of Q_j^k it would imply that

$$\gamma^k \ge \left(\frac{1}{|Q_i^{k+1}|_\mu}\int_{Q_i^{k+1}} g^{p_0}\,d\mu\right)^{\frac{1}{p_0}},$$

which is false since the right-hand side exceeds γ^{k+1}. Thus $Q_i^{k+1} \subset Q_j^k$ if the two intersect, so that

$$\begin{aligned}
|Q_j^k \cap \Omega_{k+1}|_\mu &= \sum_{i:Q_i^{k+1}\subset Q_j^k} |Q_i^{k+1}|_\mu \\
&\le \sum_{i:Q_i^{k+1}\subset Q_j^k} \frac{1}{\gamma^{(k+1)p_0}} \int_{Q_i^{k+1}} g^{p_0}\, d\mu \\
&\le \frac{1}{\gamma^{(k+1)p_0}} \int_{Q_j^k} g^{p_0}\, d\mu \quad \text{since the } Q_i^{k+1} \text{ are disjoint in } i \\
&\le \frac{1}{\gamma^{(k+1)p_0}} (c\gamma^k)^{p_0} |Q_j^k|_\mu \quad \text{by (57)} \\
&= \left(\frac{c}{\gamma}\right)^{p_0} |Q_j^k|_\mu \le \frac{1}{2}\mu(Q_j^k)
\end{aligned}$$

if γ is chosen sufficiently large. Thus, our claim (56) is proved.

We now want to estimate the first factor in (55). Recall that

$$a(Q) = \ell(Q)\left(\frac{1}{|Q|_\mu}\int_Q g^{p_0}\, d\mu\right)^{\frac{1}{p_0}}$$

and $p > p_0 \ge 1$. Then, writing again $B_j^k = B(Q_j^k)$ and setting $s = p/p_0$, we have

$$\sum_{k,j} a(Q_j^k)^p \mathcal{A}_r(w, cB_j^k) \le c \sum_{k,j} r(B_j^k)^p \left(\frac{1}{|B_j^k|_\mu} \int_{B_j^k} g^{p_0}\, v^{\frac{1}{s}}\, v^{-\frac{1}{s}}\, d\mu\right)^s \mathcal{A}_r(w, cB_j^k).$$

By Hölder's inequality with exponents $(s'r)', s'r$, the last sum is bounded by

$$c\sum_{k,j} r(B_j^k)^p |B_j^k|_\mu^{-s} \left(\int_{B_j^k} g^{p_0(s'r)'}\, v^{\frac{(s'r)'}{s}}\, d\mu\right)^{\frac{s}{(s'r)'}} \left(\int_{B_j^k} v^{-\frac{s'r}{s}}\, d\mu\right)^{\frac{s}{s'r}} \mathcal{A}_r(w, cB_j^k). \tag{58}$$

Remember that by definition of $\mathcal{A}_r$,

$$\left(\int_{B_j^k} v^{-\frac{s'r}{s}}\, d\mu\right)^{\frac{s}{s'r}} = \mathcal{A}_r\left(v^{-\frac{s'}{s}}, B_j^k\right)^{\frac{s}{s'}} |B_j^k|_\mu^{-\frac{s}{s'r'}}.$$

In addition, since $s/s' = s - 1 = p/p_0 - 1$, we have by (46) that if B is any subball of cB_0, then

$$\left(\frac{r(B)}{r(B_0)}\right)^p \left[\frac{\mathcal{A}_r(w,B)}{|cB_0|_{wd\mu}}\right] \left[\frac{\mathcal{A}_r(v^{-\frac{s'}{s}}, B)}{\int_{cB_0} v^{-\frac{s'}{s}}\, d\mu}\right]^{\frac{s}{s'}} \le\ c\left[\frac{|B|_\mu}{|B_0|_\mu}\right]^{\frac{p}{p_0}}.$$

Applying this with $B = cB_j^k$, recalling that μ is doubling, and writing c in place of c^2 as necessary, we obtain that (58) is bounded by

$$\begin{aligned}
&\frac{cr(B_0)^p |cB_0|_{wd\mu} \left(\int_{cB_0} v^{-\frac{s'}{s}}\, d\mu\right)^{\frac{s}{s'}}}{|B_0|_\mu^{\frac{p}{p_0}}} \\
&\qquad\cdot \sum_{j,k} |B_j^k|_\mu^{\frac{p}{p_0} - s - \frac{s}{s'r'}} \left(\int_{B_j^k} g^{p_0(s'r)'}\, v^{\frac{(s'r)'}{s}}\, d\mu\right)^{\frac{s}{(s'r)'}}.
\end{aligned} \tag{59}$$

The first factor in (59) is precisely the scaling factor $C_{B_0}^p$ appearing in (48). To estimate the second factor in (59) (i.e., the sum), note that

$$\frac{p}{p_0} - s - \frac{s}{s'r'} = -\frac{s}{s'r'} = 1 - \frac{s}{(s'r)'}$$

since

$$\begin{aligned}\frac{s}{(s'r)'} - \frac{s}{s'r'} &= s\left(\frac{1}{(s'r)'} - \frac{1}{s'r'}\right)\\ &= s\left(1 - \frac{1}{s'r} - \frac{1}{s'r'}\right) = s\left(1 - \frac{1}{s'}\right) = s\,\frac{1}{s} = 1.\end{aligned}$$

Therefore, the sum in (59) equals

$$\sum_{j,k}\left(\frac{1}{|B_j^k|_\mu}\int_{B_j^k} g^{p_0(s'r)'}\, v^{\frac{(s'r)'}{s}}\, d\mu\right)^{\frac{s}{(s'r)'}} |B_j^k|_\mu,$$

which as before (using $|B_j^k|_\mu \approx |Q_j^k|_\mu \le c|E_j^k|_\mu$) is bounded by

$$c\int M\left(g^{p_0(s'r)'} v^{\frac{(s'r)'}{s}}\right)^{\frac{s}{(s'r)'}} d\mu.$$

Since $\frac{s}{(s'r)'} > 1$, the last integral is at most

$$c\int\left[g^{p_0(s'r)'} v^{\frac{(s'r)'}{s}}\right]^{\frac{s}{(s'r)'}} d\mu = c\int g^{p_0 s}\, v\, d\mu = c\int g^p\, v\, d\mu.$$

Combining estimates and taking the p-th root shows that the first factor in (55) is bounded by $C_{B_0}\left(\int g^p\, v\, d\mu\right)^{1/p}$. Using this together with the estimate (57) for the second factor in (55), we see that that (55), and so also S_2, is bounded by

$$C_{B_0}\left(\int g^p\, v\, d\mu\right)^{1/p}\left(\int h^{p'}\, d\mu\right)^{1/p'}.$$

We have already shown that S_1 has the same bound, and therefore so does S, i.e., (50) holds, and the proof is complete. □

Proof of Theorem 3.1.

The hypothesis of Theorem 3.1 together with Theorem 2.5 gives $|f(x) - f_{B_0}| \le cT(x)$ for μ-a.e. $x \in B_0$, where T is the sum operator formed by using the functional

$$a(B) = r(B)\left(\frac{1}{|B|_\mu}\int_B g^{p_0}\, d\mu\right)^{1/p_0}.$$

By hypothesis, $v \in A_{p/p_0}(d\mu)$ and the balance condition (47) holds. Applying Theorem 3.2 (see Remark 3.3 in particular), Theorem 3.1 follows immediately. □

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Bruno Franchi: Dipartimento di Matematica, Università di Bologna, Piazza di porta San Donato, 5, 40126 Bologna, Italy.

E-mail address: `franchib@dm.unibo.it`

Carlos Pérez: Departmento de Análisis Matemático, Facultad de Matemáticas, Universidad de Sevilla, 41080 Sevilla, Spain.

E-mail address: `carlosperez@us.es`

Richard. L. Wheeden: Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903, USA.

E-mail address: `wheeden@math.rutgers.edu`