Estimates with A_∞ Weights
for Various Singular Integral Operators.

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Sunto. – *Si studia la limitatezza di certe classi di integrali singolari sullo spazio $L^p(w)$, $w \in A_\infty$.*

1. – Introduction.

Most of the operators in Harmonic Analysis are known to be bounded on $L^p(\mathbf{R}^n)$ for certain values of p, and also bounded on $L^p(w)$, $w \in A_p$, for the same range of p. Sometimes, stronger forms of these boundedness properties hold. For instance, let $Tf(x) = $ p.v. $\int k(x-y) f(y) \, dy$ be a classical singular integral operator, and let M be the Hardy-Littlewood maximal operator. A remarkable theorem due to R. Coifman states that for each $0 < p < \infty$, and for each $w \in A_\infty$, there exists $C = C_{w,p} > 0$ such that

$$(1) \qquad \int_{\mathbf{R}^n} |Tf(y)|^p \, w(y) \, dy \leqslant C \int_{\mathbf{R}^n} Mf(y)^p \, w(y) \, dy \,,$$

for each smooth f.

Coifman's proof of (1) is based on a difficult good-λ inequality involving the maximal singular integral operator T^* and M, which uses estimates for T^* due to M. Cotlar.

The aim of this paper is to give a different proof of (1), under conditions that allow for consideration of a wider class of operators. Our approach combines the following two ingredients. First, we prove a pointwise estimate. Namely, given $0 < s < 1$, there exists $C = C_s > 0$ such that

$$(2) \qquad M_s^{\#}(Tf)(x_0) \leqslant CMf(x_0) \quad f \in \mathcal{D}(\mathbf{R}^n), \ x_0 \in \mathbf{R}^n$$

where $M_s^{\#}$ is the s-sharp maximal operator defined as

$$M_s^{\#}(g) = M^{\#}(|g|^s)^{1/s} \,,$$

$M^{\#}$ being the sharp maximal operator of C. Feffermn and E. Stein. The second basic ingredient is the following estimate of C. Fefferman and E. Stein. For each $0 < p < \infty$, and for each $w \in A_{\infty}$, there exists $C = C_{w,p} > 0$ such that

$$(3) \qquad \int_{R^n} Mf(y)^p w(y)\, dy \leq C \int_{R^n} M^{\#} f(y)^p w(y)\, dy \qquad f \in \mathcal{O}(\boldsymbol{R}^n).$$

A very simple proof of (3) can be found in [17], p. 42.

Once (2) is proved, the Lebesgue differentiation theorem, together with (3), yield

$$\int_{R^n} |Tf(y)|^p w(y)\, dy \leq \int_{R^n} M(|Tf|^s)(y)^{p/2} w(y)\, dy \leq$$

$$C \int_{R^n} M^{\#}(|Tf|^s)(y)^{p/s} w(y)\, dy = C \int_{R^n} M_s^{\#}(Tf)(y)^p w(y)\, dy \leq$$

$$C \int_{R^n} Mf(y)^p w(y)\, dy .$$

REMARK 1.1. – Our result extends substantially the one obtained by J. Bruna and B, Koremblum (cf. [3]). They obtained an inequality similar to (2) with $M_s^{\#}$ replaced by the maximal operator M_w defined as

$$M_w f(x) = \sup_{x \in Q} \frac{1}{|Q|} \|f\|_{L^{1,\infty}(Q)}.$$

However, their estimate does not provide weighted inequalities.

REMARK 1.2. – K. Yabuta has shown (cf. [22]), that it is possible to enlarge the class of weights for which (3) holds. Indeed, Yabuta shows that this is the case for the class of weights C_q, $q > p > 1$, which contains the class A_{∞}. Therefore, a corresponding result holds for T and M.

REMARK 1.3. – It is well known that the estimate (2) is false for $s = 1$, although for each $r > 1$ there exists $C = C_r > 0$, such that

$M^{\#}(Tf)(x_0) \leqslant C M_r f(x_0)$, $f \in \mathcal{D}(\mathbf{R}^n)$, $x_0 \in \mathbf{R}^n$. On the other hand it is possible to show that a local version of (3) yields the following sharp form

$$M^{\#}(Tf)(x_0) \leqslant C M(Mf)(x_0) \quad f \in \mathcal{D}(\mathbf{R}^n), \ x_0 \in \mathbf{R}^n.$$

The class of Calderón-Zygmund operators in the sense of R. Coifman and Y. Meyer (cf. [6]), will be our model case, and we will prove (2) for these operators in Section 2.1 below. However, with similar techniques it is possible to extend (2) to weakly strongly singular Calderón-Zygmund operators (cf. [11] p. 21, [1], [2]), as well as to some pseudo-differential operators in the Hörmander class (cf. [14]) and to a class of oscillatory integral operators, related to those introduced by D. H. Phong and E. Stein (cf. [20]). A very interesting class of operators for which our approach does not seem to work, is the class of rough singular integrals. It is known, however, that these operators are bounded on $L^p(w)$ for $w \in A_p$ (cf. [10]).

In Section 2.2, we obtain weighted estimates for commutators with BMO functions in the context of L^p and Morrey spaces. Finally, Sections 3 and 4 are dedicated to study the vector-valued case. Our applications include Littlewood-Paley square functions and general Hardy-Littlewood maximal operators.

2. – Singular integrals.

2.1. – *Calderón-Zygmund operators, the model case.*

We start by recalling the definition of the motivating example. A kernel on $\mathbf{R}^n \times \mathbf{R}^n$ will be a locally integrable complex-valued function k, defined on $\Omega = \mathbf{R}^n \times \mathbf{R}^n \setminus diagonal$.

We will impose on the kernel k an integral smoothness condition, very close to the one used by B. Jawerth and A. Torchinsky (cf. [16] p. 256), to prove various estimates for local sharp maximal functions.

For each ball $B = B(x_0 r)$, let

$$D_B k(y) = \frac{1}{|B|} \frac{1}{|B|} \int_B \int_B |k(z, y) - k(x, y)| \, dx \, dz.$$

Then, the smoothness condition reads as follows,

(D)   There are constants $C, N > 0$ such that

$$\sup_{r>0} \int_{|y-x_0|>Nr} |f(y)| |D_B k(y)| \, dy \leq CMf(x_0),$$

for all $f \in \mathcal{D}(\mathbf{R}^n)$ and $x_0 \in \mathbf{R}^n$.

Every standard kernel in the sense of R. Coifman and Y. Meyer (cf. [6]), satisfies condition (D), as well convolution kernels of Dini type.

We say that a linear and continuous operator $T \colon \mathcal{D}(\mathbf{R}^n) \to \mathcal{D}'(\mathbf{R}^n)$ is associated with a kernel k, if

$$\langle Tf, g \rangle = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} k(x,y) \, g(x) \, f(y) \, dx \, dy \,,$$

whenever $f, g \in \mathcal{D}(\mathbf{R}^n)$ with $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$.

THEOREM 2.1. – *Let T be an operator associated with a kernel k satisfying condition* (D). *Let us suppose that T extends to a bounded operator from $L^1(\mathbf{R}^n)$ into $L^{1,\infty}(\mathbf{R}^n)$. Then, for each $0 < s < 1$, there exists $C = C_s > 0$ such that*

(4) $M_s^{\#}(Tf)(x_0) \leq CMf(x_0) \quad f \in \mathcal{D}(\mathbf{R}^n), \; x_0 \in \mathbf{R}^n$.

PROOF. – To prove (4), we will show that for each $0 < s < 1$, each ball $B = B(x_0, r)$ and for some complex constant $c = c_B$, there exists $C = C_s > 0$ such that,

$$\left(\frac{1}{|B|} \int_B \left| |Tf|^s - |c|^s \right| dx \right)^{1/s} \leq CMf(x_0).$$

Let $f = f_1 + f_2$, where $f_1 = f\chi_{B(x_0,Nr)}$, N being the same constant as in condition (D). Using the notation $g_Q = 1/|Q| \int_Q g(y) \, dy$, we pick

$c = (Tf_2)_B$. Now, since $\left| \, |a|^s - |b|^s \right| \leqslant |a - b|^s$ for $0 < s < 1$, we obtain

$$\left(\frac{1}{|B|} \int_B \left| \, |Tf|^s - |(T_2)_B|^s \right| dx \right)^{1/s} \leqslant C \left(\frac{1}{|B|} \int_B |Tf_1|^s \, dx \right)^{1/s} +$$

$$C \left(\frac{1}{|B|} \int_B |Tf_2 - (Tf_2)_B|^s \, dx \right)^{1/s} = C(I + II).$$

Since $T: L^1(\mathbf{R}^n) \to L^{1,\infty}(\mathbf{R}^n)$ and $0 < s < 1$, Kolmogorov's inequality (cf. [13] p. 485 for instance), yields

$$I \leqslant \frac{1}{|B|} \int_B |f_1| \, dx = \frac{C}{|B(x_0, Nr)|} \int_{B(x_0, Nr)} |f| \, dx \leqslant CMf(x_0).$$

To take care of II we observe that Jensen's inequality together with Fubini's theorem yield,

$$II \leqslant \frac{1}{|B|} \int_B |Tf_2 - (Tf_2)_B| \, dx \leqslant \int_{|y - x_0| > Nr} |f(y)| D_B k(y) \, dy,$$

which is bounded by a multiple of $Mf(x_0)$, according to condition (D). This completes the proof of Theorem 2.1. ∎

As we mentioned in the introduction, condition (D) and Theorem 2.1 can be adapted to include the so called weakly strongly singular Calderón-Zygmund operators. These are singular integral operators which enjoy properties similar to those expected in the Calderón-Zygmund operator theory, while the kernel are more singular in the diagonal than the standard ones.

2.2. – Commutators with BMO functions.

The next result is a sharp version of the sufficient condition in [7], Theorem I.

THEOREM 2.2. – *Let T be an operator satisfying the hypothesis of Theorem 2.1 and let $b \in BMO$, fixed. Then, for each $w \in A_\infty$, $r > 1$, $0 < p < \infty$, there exists $C = C_{w,r,p} > 0$ such that*

$$(5) \quad \int_{\mathbf{R}^n} |[b, T] f(y)|^p w(y) \, dy \leqslant C \int_{\mathbf{R}^n} M_r f(y)^p w(y) \, dy \quad f \in \mathcal{Q}(\mathbf{R}^n),$$

where $[b, T]$ denotes the commutator of T with the operator of multi-plication by b.

The proof of Theorem 2.2 is based on the following a priori estimate.

For each $b \in BMO$, $0 < s < 1$, $1 < r < \infty$, there exists $C = C_{b,s,r} > 0$ such that,

$$(6) \quad M_s^{\#}([b, T] f)(x) \leqslant C\|b\|_{BMO}\, (M_{sr}(Tf)(x) + M_r f(x)) \qquad f \in \mathcal{D}(\boldsymbol{R}^n).$$

Selecting s small enough, it is not hard to see that standard properties of the A_∞ class of weights together with Theorem 2.1, imply the conclusion of Theorem 2.2. The idea of relating commutators with the sharp maximal operator of C. Fefferman and E. Stein is due to J. O. Strömberg (cf. [15]).

REMARK 2.3. – Inequality (5) is false for $r = 1$, and in fact one can give the following sharper result

$$\int_{R^n} |[b, T] f(y)|^p w(y)\, dy \leqslant C \int_{R^n} M(Mf)^p w(y)\, dy \,,$$

where b and w satisfy the same conditions as in (5). These results can be found in [19].

The a priori estimate (6) was proved for $s = 1$ in [8], p. 326. Following this paper we will use (6) to obtain weighted estimates on Morrey spaces.

More specifically, let us consider a nonincreasing function $\varphi\colon (0, \infty) \to (0, \infty)$. Give, $0 < p < \infty$, and given a weight w, that is to say a nonnegative and locally integrable function on $\boldsymbol{R}^n$, we define the class $M_{\varphi,w}^p(\boldsymbol{R}^n)$ as those functions f in L_{loc}^p such that

$$\sup_B \frac{1}{\varphi(|B|)} \left(\frac{1}{|B|} \int_B |f(y)|^p w(y)\, dy \right)^{1/p} = \|f\|_{M_{\varphi,w}^p} < \infty \,,$$

where the supremum is taken over all balls B. This definition coincides with the one in [8], p. 324 when $1 < p < \infty$, $w = 1$, $\varphi(t) = t^{(\lambda - n)/np}$, $0 < \lambda < n$.

We first observe that for each ball B, and for each $0 < p < \infty$

$$M^p_{\varphi,w}(\mathbf{R}^n) \subset L^p(w(M\chi_B)^\gamma),$$

provided that

(i) there is a positive constant $0 < D < 1$ for which

$$\varphi(2t) \leq D\varphi(t)$$

and

(ii) $\max\{0, 1 + (p \log D)/(n \log 2)\} < \gamma$.

The proof is similar to the one in [4] p. 275, and we shall omit it. We are ready now to extend Lemma 3 in [8] p. 326.

LEMMA 2.4. – *Let φ and γ satisfy conditions* i) *and* ii) *above. Suppose also that w is a weight such that $w(M\chi_B)^\gamma$ belongs to A_∞ uniformly on each ball B. Then for each $0 < p < \infty$, there exists a positive constant C such that*

(7) $$\|Mf\|_{M^p_{\varphi,w}} \leq C\|M^\# f\|_{M^p_{\varphi,w}}.$$

The proof in [8] applies with obvious changes and will be omitted.

REMARK 2.5. – There are interesting nontrivial weights satisfying the assumption of the lemma. Recall that a weight w satisfies the RH_∞ condition if there is a constant c such that for each ball B

(8) $$\sup_{y \in B} w(y) \leq \frac{1}{|B|} \int_B w(y)\, dy.$$

Let w be one of these weights, and let $(M\mu)^\gamma$ be finite a.e. with $0 < \gamma < 1$, where $\mu = \chi_{B_0}$, for a fixed ball B_0. We claim that $w(M\mu)^\gamma$ satisfies the A_∞ condition. Indeed, let $1 < r < 1/\gamma$. Then,

$$\left(\frac{1}{|B|}\int_B (w(y)(M\mu(y))^\gamma)^r dy\right)^{1/r} \leq \sup_{y \in B} w(y)\left(\frac{1}{|B|}\int_B (M\mu(y))^{r\gamma} dy\right)^{1/r} \leq$$

$$\frac{c}{|B|}\int_B w(y)\, dy \inf_{y \in B}(M\mu(y))^\gamma \leq \frac{c}{|B|}\int_B w(y)(M\mu(y))^\gamma dy,$$

since $(M\mu(y))^\gamma$ satisfies the A_1 condition (cf. [13]).

Some important examples of functions satisfying the RH_∞ condition are the polynomials. If π is a polynomial, the $|\pi| \in RH_\infty$, (cf. [9] p. 16).

REMARK 2.6. – The A_∞ class can be replaced in Lemma 2.4 by the larger class weak-A_∞ which was introduced in [21]. A weight w satisfies the weak-A_∞ condition if there are positive constants c and δ such that

$$w(E) \leqslant c \left(\frac{|E|}{|Q|} \right)^\delta w(2Q)$$

for each cube Q and any measurable set $E \subset Q$. It is not hard to see as in [21] that inequality (3) still holds with w satisfying the weak-A_∞ condition. Hence, the proof of (7) follows with obvious modifications.

Estimates (6) and (7) imply the a priori estimate

$$\|[T,b]f\|_{M^p_{\tilde{p},w}} \leqslant C\|f\|_{M^p_{\tilde{p},w}},$$

for each smooth function f. Analogously, from (4) and (7) we can deduce

$$\|Tf\|_{M^p_{\tilde{p},w}} \leqslant C\|f\|_{M^p_{\tilde{p},w}},$$

for each smooth function f.

3. – Vector-valued extension and applications.

It is well known that Littlewood-Paley type estimates may be viewed as an extension of the Calderón-Zygmund theory (cf. [13] Chapter V). In the same way, it is possible to extend the previous results within this context. We will describe some consequences of this extension.

Let us consider one of the most typical square functions. Namely, let $\psi \in \mathcal{S}(\boldsymbol{R}^n)$ be a nontrivial radial function satisfying $\int_{\boldsymbol{R}^n} \psi = 0$. We

set $\psi_t(x) = (1/t^n)\psi(x/t)$, $Q_t f = f * \psi_t$, and we consider the nonlinear operator

$$Sf(x) = \left(\int\limits_0^\infty |Q_t f(x)|^2 \frac{dt}{t}\right)^{1/2}.$$

THEOREM 3.1. – *For each $0 < p < \infty$ and for each $w \in A_\infty$, there exists $C = C_{w,p} > 0$ such that*

$$(9) \qquad \int\limits_{R^n} Sf(y)^p w(y)\, dy \leq C \int\limits_{R^n} Mf(y)^p w(y)\, dy \qquad f \in \mathcal{D}(R^n),$$

An analogous result holds for the discrete version of S, denoted as $\tilde{S}$ and defined by

$$\tilde{S}f(x) = \left(\sum_k |Q_{2^k} f(x)|^2\right)^{1/2}.$$

We will close this section considering the following generalization of the Hardy-Littlewood maximal operator. Given $\phi \in L^1(R^n)$, we define the maximal operator

$$M_\phi f(x) = \sup_{\delta > 0} |f * \phi_\delta(x)|,$$

where, as usual, $\phi_\delta = (1/\delta^n)\phi(x/\delta)$. This operator can be viewed as a vector-valued singular integral, $M_\phi f(x) = \|Tf(x)\|_B$, where $B = l^\infty(R^+)$ and $Tf(x) = (f * \phi_\delta(x))_{\delta > 0}$.

Furthermore, if we assume that ϕ satisfies $|\phi(x - y) - \phi(x)| \leq C(|y|/|x|^{n+1})$, for $|x| > 2|y| > 0$, we have the following result

THEOREM 3.2. – *For each $0 < p < \infty$ and for each $w \in A_\infty$, there exists $C = C_{w,p} > 0$ such that*

$$(10) \qquad \int\limits_{R^n} M_\phi f(y)^p w(y)\, dy \leq C \int\limits_{R^n} Mf(y)^p w(y)\, dy \qquad f \in \mathcal{D}(R^n).$$

4. – A further consideration.

In this section we will state a further extension of Coifman's estimate (1) which is a consequence of extrapolation methods due to C. Pérez (cf. [18]).

THEOREM 4.1. – *Let* $0 < p, q < \infty$, *and let* P *be any of the opera-tors* $T, S, \tilde{S}, M_{\ddagger}$ *described in Sections 2 and 3. Then, for each weight* $w \in A_\infty$ *there exists* $C = C_{w,p,q} > 0$ *such that*

$$(11) \qquad \left\| \left(\sum_{i=0}^{\infty} |Pf_i|^q \right)^{1/q} \right\|_{L^p(w)} \leq C \left\| \left(\sum_{i=0}^{\infty} (Mf_i)^q \right)^{1/q} \right\|_{L^p(w)},$$

for all functions $f_i \in \mathcal{D}(\mathbf{R}^n)$.

Furthermore, if $b \in BMO$ *for each* $w \in A_\infty$, $r > 1$, *there exists* $C = C_{w,r,p,q} > 0$ *such that*

$$(12) \qquad \left\| \left(\sum_{i=0}^{\infty} |[b,T]f_i|^q \right)^{1/q} \right\|_{L^p(w)} \leq C \left\| \left(\sum_{i=0}^{\infty} (M_r f_i)^q \right)^{1/q} \right\|_{L^p(w)},$$

for all functions $f_i \in \mathcal{D}(\mathbf{R}^n)$.

The roots of this result can be found in the celebrated vector-valued extension of the Hardy-Littlewood maximal theorem, due to C. Fefferman and E. Stein (cf. [12]). However, we have been mostly influenced by the work of J. Garcia-Cuerva and J. L. Rubio de Fran-cia on extrapolation for weights (cf. [13] Chapter IV).

REFERENCES

[1] J. ALVAREZ - M. MILMAN, H^p *continuity properties of Calderón-Zyg-mund operators*, J. Math. An. Appl., **118** (1986), 67-79.

[2] J. ALVAREZ - M. MILMAN, *Vector-valued inequalities for strongly singu-lar Calderón-Zygmund operators*, Rev. Mat. Iberoamericana, **2** (1986), 405-425.

[3] J. BRUNA - B. KORENBLUM, *A note on Calderón-Zygmund singular in-tegral operators*, Bull. Amer. Math. Soc., **16** (1987), 271-273.

[4] S. CHIARENZA - M. FRASCA, *Morrey spaces and the Hardy-Littlewood maximal function*, Rend. Mat. (7), **7** (1987), 273-279.

[5] R. COIFMAN, *Distribution function inequalities for singular integrals*, Proc. Acad. Sci. U.S.A., **69** (1972), 2838-2839.

[6] R. COIFMAN - Y. MEYER, *Au delà des opérateurs pseudo-différentiels*, Astérisque, **57** (1979), 2nd edition.

[7] R. COIFMAN - R. ROCHBERG - G. WEISS, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103** (1976), 611-635.

[8] G. DiFAZIO - M. A. RAGUSA, *Commutators and Morrey spaces*, Boll. Un. Mat. Ital. (7), **5-A** (1991), 323-332.

[9] R. A. DeVore - R. C. Sharpley, *Maximal functions measuring smoothness*, Mem. Amer. Math. Soc., 47 (1984), 1-115.

[10] J. Duoandikotxea - J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math., 84 (1986), 541-561.

[11] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math., 124 (1970), 9-36.

[12] C. Fefferman - E. M. Stein, *Some maximal inequalities*, Amer. J. Math., 93 (1971), 107-115.

[13] J. Garcia-Cuerva - J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Math. Stud., 116 (1985).

[14] L. Hörmander, *Pseudo-differential operators and hypoelliptic equations*, Proc. Symp. Pure Math., 10 (1967), 138-183.

[15] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Mat., 16 (1978), 263-270.

[16] B. Jawert - A. Torchinsky, *Local sharp maximal functions*, J. Approx. Theory, 43 (1985), 231-270.

[17] J. L. Journé, *Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón*, Lect. Notes in Math., 994 (1983).

[18] C. Pérez, *Extrapolation results for operators*, preprint (1992).

[19] C. Pérez, *Two weighted norm inequalities for commutators of singular integral operators with BMO functions*, in preparation.

[20] D. H. Phong - E. M. Stein, *Hilbert integrals, singular integrals and Radon transforms*, Acta Math., 157 (1985), 99-157.

[21] E. T. Sawyer, *Two weight norm inequalities for certain maximal and integral operators*, in *Harmonic analysis* (F. Ricci and G. Weiss eds.), Lect. Notes in Math., 908 (1982), 102-127.

[22] K. Yabuta, *Sharp maximal function and C_p condition*, Arch. Math., 55 (1990), 151-155.

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