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A REMARK ON WEIGHTED INEQUALITIES FOR GENERAL MAXIMAL OPERATORS

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Abstract

Let $1 < p < \infty$, and let w, v be two non–negative functions. We give a sufficient condition on w, v for which the general maximal operator $M_{\mathcal{B}}$ is bounded from $L^p(v)$ into $L^p(w)$. Our condition is stronger but closely related to the $A_{p,\mathcal{B}}$ condition for two weights.

1 Introduction

Let $\mathcal{Q}$ be the family of all open cubes in $\mathbb{R}^n$ with sides parallel to the axes and let $M_{\mathcal{Q}} = M$ denote the Hardy–Littlewood maximal operator. According to a fundamental theorem of E. Sawyer [?], M is a bounded operator from $L^p(v)$ into $L^p(w)$, $1 < p < \infty$, if and only if $(w, v) \in S_p$, that is there is a positive constant c such that

$$\int_Q M(v^{1-p'}\chi_Q)(y)^p\, w(y)dy \leq c \int_Q v(y)^{1-p'}\, dy \qquad Q \in \mathcal{Q}. \tag{1}$$

On the other hand, it is well known that Muckenhoupt's A_p condition for two weights,

$$\left(\frac{1}{|Q|}\int_Q w(y)\, dy\right)^{1/p} \left(\frac{1}{|Q|}\int_Q v(y)^{1-p'}\, dy\right)^{1/p'} \leq c \qquad Q \in \mathcal{Q}, \tag{2}$$

is not equivalent to S_p unless $v = w$ (cf. [?] also [?] p. 433). One problem with E. Sawyer's condition is that it is very difficult to test in practice since it involves the operator M on it. It would be interesting to obtain sufficient conditions close in form to (??). In [?] we initiated this program and showed that it is enough to consider conditions such as (??) but replacing the local $L^{p'}$ average norm involved on the weight v by appropiate stronger norms. An antecedent of these results can be found in C. Neugebauer's paper [?].

This note is devoted to study the corresponding two weight problem

$$\int_{\mathbb{R}^n} M_{\mathcal{B}}f(y)^p\, w(y)dy \le c \int_{\mathbb{R}^n} |f(y)|^p v(y)dy, \tag{3}$$

where $\mathcal{B}$ is a general basis. By a basis $\mathcal{B}$ in $\mathbb{R}^n$ we mean a collection of open sets in $\mathbb{R}^n$. The study of general maximal operators $M_{\mathcal{B}}$ arise in many situations in Fourier Analysis where the geometry involved is other than the one given by the cubes or balls. Of course, in one–parameter Fourier Analysis the basis $\mathcal{Q}$ plays a central role together with the basis $\mathcal{D}$ of all dyadic cubes. A corresponding role in multiparameter Fourier Analysis is played by the basis $\mathcal{R}$ of all rectangles with sides parallel to the axes. An example of "exotic" but interesting basis is given by the Córdoba–Zygmund basis $\Re$ in $\mathbb{R}^3$ of all rectangles with sidelengths of the form $\{s, t, st\}$.

For a general basis $\mathcal{B}$ $M_{\mathcal{B}}$ denotes the maximal operator associated to $\mathcal{B}$ defined by

$$M_{\mathcal{B}}f(x) = \sup_{x\in B} \frac{1}{|B|}\int_B |f(y)|\, dy$$

if $x \in \cup_{B\in\mathcal{B}}$ and $M_{\mathcal{B}}f(x) = 0$ otherwise. We say that w is a weight associated to the basis $\mathcal{B}$ if w is a non–negative measurable function in $\mathbb{R}^n$ such that $w(B) = \int_B w(y)\, dy < \infty$ for each B in $\mathcal{B}$. The weight w belongs to the class $A_{p,\mathcal{B}}$, $1 < p < \infty$, if there is a constant c such that

$$\left(\frac{1}{|B|}\int_B w(y)\, dy\right)\left(\frac{1}{|B|}\int_B w(y)^{1-p'}\, dy\right)^{p-1} \le c \tag{4}$$

for all $B \in \mathcal{B}$. For $p = \infty$, we set $A_{\infty,\mathcal{B}} = \cup_{p>1} A_{p,\mathcal{B}}$.

In [?] we introduced the following class of bases that we shall use later.

DEFINITION 1.1 A basis $\mathcal{B}$ is a *Muckenhoupt* basis if for each $1 < p < \infty$, and every $w \in A_{p,\mathcal{B}}$

$$M_{\mathcal{B}} : L^p(w) \to L^p(w).$$

It is shown in [?] that this definition is equivalent to saying that for each $1 < p < \infty$

$$M_{\mathcal{B},w} : L^p(w) \to L^p(w) \qquad w \in A_{\infty,\mathcal{B}}. \tag{5}$$

Here $M_{\mathcal{B},w}$ denotes the weighted maximal operator defined by

$$M_{\mathcal{B},w}f(x) = \sup_{x\in B} \frac{1}{w(B)} \int_B |f(y)|\, w(y)dy.$$

Most of the important bases are Muckenhoupt bases, and in particular those mentioned above: $\mathcal{Q}, \mathcal{D}, \mathcal{R}$ and $\Re$.

Let now (w, v) be a couple of weights associated to the basis $\mathcal{B}$. Extending a previous result of E. Sawyer in [**?**] for the strong maximal operator, B. Jawerth [**?**] gave a necessary and sufficient condition for $M_{\mathcal{B}}$ to be bounded from $L^p(v)$ into $L^p(w)$ under no restriction on $\mathcal{B}$. Let $\mathcal{F}$ be the family of all finite unions $G = \cup_{j=1}^N B_j$ of sets in $\mathcal{B}$; Jawerth's condition is that for some positive constant c

$$\int_G M_{\mathcal{B}} \left(v^{1-p'} \chi_G \right) (y)^p\, w(y)dy \le c \int_G v(y)^{1-p'}\, dy \qquad G \in \mathcal{F}. \tag{6}$$

This condition is even harder to verify than Sawyer's condition S_p. In Theorem **??** below we shall provide a simpler sufficient condition which is not necessary.

We shall use the following class of weights.

DEFINITION 1.2 *We say that a weight w associated to the basis $\mathcal{B}$ satisfies condition (A) if there are constants $0 < \lambda < 1$, $0 < c = c(\lambda) < \infty$ such that for all measurable set E*

$$(A) \qquad w\left(\{ x \in \mathbb{R}^n : M_{\mathcal{B}}(\chi_E)(x) > \lambda \} \right) \le c\, w(E).$$

Before stating our main theorem we shall make some remarks concerning condition (A).

This class of weights was considered by Jawerth in [**?**], although the unweighted version goes back to the work of A. Córdoba in [**?**]. One reason which makes condition (A) interesting is the fact that it is weaker than the $A_{\infty,\mathcal{B}}$ condition whenever the basis $\mathcal{B}$ is a Muckenhoupt basis. To see this let $w \in A_{\infty,\mathcal{B}}$; then $w \in A_{r,\mathcal{B}}$, for some $1 < r < \infty$, and by standard properties of the $A_{p,\mathcal{B}}$ weights

$$\frac{|E|}{|B|} \le c \left(\frac{w(E)}{w(B)} \right)^{1/r},$$

for each measurable subset $E \subset B \in \mathcal{B}$. It follows then that $M_{\mathcal{B}}(\chi_E)(x) \leq c\,(M_{\mathcal{B},w}(\chi_E)(x))^{1/r}$, and therefore, if $\mathcal{B}$ is a Muckenhoupt basis, (**??**) yields for all $\lambda > 0$

$$w\left(\{x \in \mathbb{R}^n : M_{\mathcal{B}}(\chi_E)(x) > \lambda\}\right) \leq w(\{x \in \mathbb{R}^n : M_{\mathcal{B},w}(\chi_E)(x) > \frac{\lambda^r}{c^r}\}) \leq$$

$$\leq \frac{c^r}{\lambda^r} \int_{\mathbb{R}^n} \chi_E(x)^r\, w(x)dx = c(\lambda)\, w(E),$$

which is condition (A).

In fact B. Jawerth and A. Torchinsky have shown in [**?**] that the $A_{\infty,\mathcal{R}}$ condition is strictly stonger than condition (A). As an example they show (cf. p. 270 in [**?**]) that if the weight w in $\mathbb{R}^n$, $n > 1$, is A_∞ in each variable except in one where is merely doubling, then w satisfies condition (A) while w does not belong to $A_{\infty,\mathcal{R}}$ as is well known.

A first result on the two weight problem was remarked by the author in [**?**]. We pointed out that the following generalized Fefferman–Stein's type inequality

$$\int_{\mathbb{R}^n} M_{\mathcal{B}}f(y)^p\, w(y)dy \leq c \int_{\mathbb{R}^n} |f(y)|^p M_{\mathcal{B}}\, w(y)dy, \tag{7}$$

holds assuming that $\mathcal{B}$ is a Muckenhoupt basis, and that the weight w satisfies condition (A). We recall that the classical Fefferman–Stein inequality for the Hardy–Littlewood maximal operator M has no restriction on w.

At this point we mention that a particular instance of this result was previously obtained by K. C. Lin in [**?**]. His result is for the strong maximal operator $M_{\mathcal{R}}$ in dimension $n = 2$, and with the weight w satisfying the $A_{\infty,\mathcal{R}}$ condition.

The main result of this paper is the following.

THEOREM 1.3 *Let $1 < p < \infty$, and let $\mathcal{B}$ a general basis satistying $M_{\mathcal{B}} : L^s(\mathbb{R}^n) \to L^s(\mathbb{R}^n)$ for all $1 < s < \infty$. Suppose that (w, v) is a couple of weights such that w satisfies (A), and that for some $1 < r < \infty$ there is a contant c such that*

$$\frac{1}{|B|} \int_B w(y)\, dy \left(\frac{1}{|B|} \int_B v(y)^{(1-p')r}\, dy \right)^{(p-1)/r} \leq c, \tag{8}$$

for all $B \in \mathcal{B}$. Then

$$M_{\mathcal{B}} : L^p(v) \to L^p(w). \tag{9}$$

It is easy to check that $(w, M_{\mathcal{B}}w)$ satisfies (**??**) and then we have the following corollary.

COROLLARY 1.4 *Let $1 < p < \infty$, and let $\mathcal{B}$ a general basis satistying $M_{\mathcal{B}} : L^s(\mathbb{R}^n) \to L^s(\mathbb{R}^n)$ for all $1 < s < \infty$. Suppose that w is a weight which satisfies condition (A). Then*

$$\int_{\mathbb{R}^n} M_{\mathcal{B}}f(y)^p\, w(y)dy \leq c \int_{\mathbb{R}^n} f(y)^p\, M_{\mathcal{B}}w(y)dy.$$

In particular if $\mathcal{B}$ is a Muckenhoupt basis we always have that $M_{\mathcal{B}} : L^s(\mathbb{R}^n) \to L^s(\mathbb{R}^n)$, $1 < s < \infty$ yielding (**??**) as a is corollary.

COROLLARY 1.5 *Let $1 < p < \infty$, and let $\mathcal{B}$ a Muckenhoupt basis. Suppose that w is a weight which satisfies condition (A). Then*

$$\int_{\mathbb{R}^n} M_{\mathcal{B}}f(y)^p\, w(y)dy \leq c \int_{\mathbb{R}^n} f(y)^p\, M_{\mathcal{B}}w(y)dy.$$

2 Proof of the Theorem

First we claim that (**??**) implies that

$$M_{\mathcal{B}} : L^p(v) \to L^{p,\infty}(w), \tag{10}$$

where $L^{p,\infty}(w)$ is the weighted Lorentz space defined by all the functions f such that $\sup_{\lambda>0} \left(\lambda^p\, w(\{x \in \mathbb{R}^n : M_{\mathcal{B}}f(x) > \lambda\})\right) < \infty$.

To prove (**??**) we just need to prove that there is a constant c such that

$$w(K) \leq \frac{c}{t^p} \int_{\mathbb{R}^n} f(y)^p\, v(y)dy,$$

for each $t > 0, f \geq 0$, and for any compact subset K of $\{x \in \mathbb{R}^n : M_{\mathcal{B}}(f)(x) > t\}$. By the compactness of K and the definition of $M_{\mathcal{B}}f$ we can find a finite collection $B_1, \ldots, B_N$ such that

$$K \subset \cup_{j=1}^N B_j \qquad \textit{and} \qquad t < \frac{1}{|B_j|} \int_{B_j} f(y)\, dy, \tag{11}$$

for each $j = 1, \ldots, N$. We follow now a well known selecting procedure argument (cf. [**?**] p. 463 for instance). Let $\tilde{B}_1 = B_1$ and, once $\tilde{B}_1, \ldots, \tilde{B}_{k-1}$

have been selected, we choose $\tilde{B}_k$ to be the first set in the given sequence (if any) such that

$$\left|\tilde{B}_k \cap (\cup_{j=1}^{k-1} \tilde{B}_j)\right| < \lambda \left|\tilde{B}_k\right|.$$

Now, we claim that

$$\cup_{j=1}^{N} B_j \subset \{x \in \mathbb{R}^n : M_{\mathcal{B}}(\chi_{\cup_{j=1}^{M} \tilde{B}_j})(x) \geq \lambda\}. \tag{12}$$

Let $x \in \cup_{j=1}^{N} B_j$; if x belongs to some $\tilde{B}_k$ it is of course obvious that it is contained on the set to the right since $\lambda < 1$. If on the other hand $x \in B_j$ for some B_j which has been discarded in the selection process, we must have $\left|B_j \cap (\cup_{j=1}^{M} \tilde{B}_j)\right| \geq \lambda |B_j|$, and therefore $M_{\mathcal{B}}(\chi_{\cup_{j=1}^{M} \tilde{B}_j})(x) \geq \lambda$. Now, since w satisfies condition (A), (**??**) yields

$$w(\cup_{j=1}^{N} B_j) \leq c\, w(\cup_{j=1}^{M} \tilde{B}_j).$$

This together with (**??**), (**??**) and Hölder's inequality yields the following estimate

$$w(K) \leq c\, w(\cup_{j=1}^{M} \tilde{B}_j) \leq c \sum_j \left(\frac{1}{t\left|\tilde{B}_j\right|} \int_{\tilde{B}_j} f(y)\, dy \right)^p w(\tilde{B}_j) =$$

$$c\, \frac{1}{t^p} \sum_j \left(\frac{1}{\left|\tilde{B}_j\right|} \int_{\tilde{B}_j} f(y) v(y)^{1/p} v(y)^{-1/p}\, dy \right)^p w(\tilde{B}_j) \leq$$

$$c\, \frac{1}{t^p} \sum_j \left(\frac{1}{\left|\tilde{B}_j\right|} \int_{\tilde{B}_j} f(y)^{(p'r)'} v(y)^{(p'r)'/p}\, dy \right)^{\frac{p}{(p'r)'}} \left(\frac{1}{\left|\tilde{B}_j\right|} \int_{\tilde{B}_j} v(y)^{-p'r/p}\, dy \right)^{\frac{p}{p'r}} \frac{w(\tilde{B}_j)}{\left|\tilde{B}_j\right|} \left|\tilde{B}_j\right| \leq$$

$$c\, K\, \frac{1}{t^p} \sum_j \left(\frac{1}{\left|\tilde{B}_j\right|} \int_{\tilde{B}_j} f(y)^{(p'r)'} v(y)^{(p'r)'/p}\, dy \right)^{\frac{p}{(p'r)'}} \left|\tilde{B}_j\right| \leq .$$

Denote $E_j = \tilde{B}_j \backslash \cup_{i=1}^{j-1} \tilde{B}_i$, so that $\{E_j\}$ is a disjoint family with $E_j \subset \tilde{B}_j$ and $\left|\tilde{B}_j\right| < \frac{1}{1-\lambda} |E_j|$. Then

$$w(K) \leq c\, \frac{1}{t^p} \sum_j \left(\frac{1}{\left|\tilde{B}_j\right|} \int_{\tilde{B}_j} f(y)^{(p'r)'} v(y)^{(p'r)'/p}\, dy \right)^{\frac{p}{(p'r)'}} |E_j| \leq$$

$$\frac{c}{t^p}\sum_j \int_{E_j} M_{\mathcal{B}}(f^{(p'r)'}v^{(p'r)'/p})(y)^{\frac{p}{(p'r)'}}\,dy \leq \frac{c}{t^p}\int_{\mathbb{R}^n} M_{\mathcal{B}}(f^{(p'r)'}v^{(p'r)'/p})(y)^{\frac{p}{(p'r)'}}\,dy \leq$$

$$\frac{c}{t^p}\int_{\mathbb{R}^n} f(y)^p\,v(y)dy,$$

since by hypothesis $M_{\mathcal{B}} : L^s(\mathbb{R}^n) \to L^s(\mathbb{R}^n)$, $1 < s < \infty$.

To conclude the proof of the theorem we observe first that we always have that

$$M_{\mathcal{B}} : L^\infty(v) \to L^\infty(w). \tag{13}$$

Now, denoting condition (**??**) by $A_{p,r}$ we see that (w, v) satisfies $A_{\bar{p},\bar{r}}$ for some $1 < \bar{p} < p$, $1 < \bar{r} < \infty$; in fact $\frac{p-1}{r} + 1 < \bar{p} < p$, $1 < \bar{r} < \frac{\bar{p}-1}{p-1}r$ will do it:

$$\frac{1}{|B|}\int_B w(y)\,dy\left(\frac{1}{|B|}\int_B (v(y)^{-1})^{(\frac{\bar{r}}{\bar{p}-1})}\,dy\right)^{1/\frac{\bar{r}}{\bar{p}-1}} \leq$$

$$\frac{1}{|B|}\int_B w(y)\,dy\left(\frac{1}{|B|}\int_B (v(y)^{-1})^{(\frac{r}{p-1})}\,dy\right)^{1/\frac{r}{p-1}} \leq K.$$

By the above argument $A_{\bar{p},\bar{r}}$ and $M_{\mathcal{B}} : L^s(\mathbb{R}^n) \to L^s(\mathbb{R}^n)$, $1 < s < \infty$, yield

$$M_{\mathcal{B}} : L^{\bar{p}}(v) \to L^{\bar{p},\infty}(w).$$

This together with (**??**) implies $M_{\mathcal{B}} : L^p(v) \to L^p(w)$ by the Marcinkiewicz's interpolation theorem. This concludes the proof of the theorem. □

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