

SUBMANIFOLDS OF GENERALIZED SASAKIAN SPACE FORMS

Pablo Alegre and Alfonso Carriazo

Abstract. In the present paper submanifolds of generalized Sasakian-space-forms are studied. We focus on almost semi-invariant submanifolds, these generalize invariant, anti-invariant, and slant submanifolds. Sectional curvatures, Ricci tensor and scalar curvature are also studied. The paper finishes with some results about totally umbilical submanifolds.

1. INTRODUCTION

Recently, in [1] D. E. Blair and the authors introduced the notion of a *generalized Sasakian space form* as that almost contact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$ whose curvature tensor satisfies

$$\begin{aligned} \widetilde{R}(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &+ f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \end{aligned} \quad (1.1)$$

for all vector fields X, Y, Z and certain differentiable functions f_1, f_2, f_3 on \widetilde{M} . This generalizes the concept of Sasakian space form as well as generalized complex space form did with complex space form. A generalized Sasakian space form with functions f_1, f_2, f_3 will be denoted by $\widetilde{M}(f_1, f_2, f_3)$. If it is a Sasakian manifold, then the functions are constant and $f_1 - 1 = f_2 = f_3$, as it was proved in [1]. The theory of generalized Sasakian space forms was continued by the authors in [2], which is mainly devoted to the study of the structures on these spaces.

Received May 21, 2007, accepted October 4, 2007.

Communicated by Bang-Yen Chen.

2000 *Mathematics Subject Classification*: 53C25, 53C40.

Key words and phrases: Almost contact metric manifold, Generalized Sasakian space form, Almost semi-invariant submanifold.

Both authors are partially supported by the PAI group FQM-327 (Junta de Andalucía, Spain, 2007). The second author is also partially supported by the MEC-FEDER grant MTM2004-04934-C04-04.

On the other hand, in [4] A. Bejancu initiated the study of CR-submanifolds of an almost Hermitian manifold generalizing invariant and anti-invariant submanifolds. The extension of this concept to submanifolds of almost contact metric manifolds was made in [5] by A. Bejancu and N. Papaghiuc; they called them semi-invariant submanifolds. Later, the study of almost semi-invariant submanifolds of framed metric manifolds, as a generalization of both CR-submanifolds and semi-invariant submanifolds, was initiated by M. M. Tripathi and K. D. Singh in [18], and followed by Tripathi and I. Mihai in [17].

Finally, in [16] almost semi-invariant submanifolds of generalized complex space forms were studied. We now present an analysis of almost semi-invariant submanifolds of generalized Sasakian space forms. Some results about these submanifolds on contact geometry have been already given in [15].

After a section containing some background and many new examples of almost semi-invariant submanifolds, we initiate the study of submanifolds of a generalized Sasakian space form by characterizing invariant and anti-invariant submanifolds by means of the action of the curvature tensor. Next, for an almost semi-invariant submanifold of a generalized Sasakian space form, we introduce the notions of \mathcal{D}^λ -sectional curvature and $(\mathcal{D}^\lambda, \mathcal{D}^\mu)$ -sectional curvature. Moreover, we also obtain some results about the Ricci tensor and the scalar curvature. The last part is devoted to totally umbilical submanifolds.

2. PRELIMINARIES AND EXAMPLES

At this stage, we recall some definitions and basic formulas which we will use later. We focus on submanifold theory's concepts. For general background on almost contact Riemann geometry we refer to [6]. We just recall the usual Sasakian structure on \mathbf{R}^{2m+1} , (ϕ_0, ξ, η, g) , given by

$$\begin{aligned}\eta &= \frac{1}{2} \left(dz - \sum_{i=1}^m y^i dx^i \right), & \xi &= 2 \frac{\partial}{\partial z}, \\ g &= \eta \otimes \eta + \frac{1}{4} \sum_{i=1}^m (dx^i \otimes dx^i + dy^i \otimes dy^i), \\ \phi_0 \left(X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i} + Z \frac{\partial}{\partial z} \right) &= \sum_{i=1}^m \left(Y_i \frac{\partial}{\partial x^i} - X_i \frac{\partial}{\partial y^i} \right) + \sum_{i=1}^m Y_i y^i \frac{\partial}{\partial z},\end{aligned}$$

where $\{x^i, y^i, z\}$, $i = 1 \dots m$ are the cartesian coordinates.

Given a submanifold M of an almost contact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$, we also use g for the induced Riemannian metric on M . We denote by $\widetilde{\nabla}$ the Levi-Civita connection on \widetilde{M} and by ∇ the induced Levi-Civita connection on M .

Thus the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V,$$

for vector fields X, Y tangent to M and a vector field V normal to M , where h denotes the second fundamental form, ∇^\perp the normal connection and A_V the shape operator in the direction of V . The second fundamental form and the shape operator are related by

$$g(h(X, Y), V) = g(A_V X, Y).$$

M is called a *totally geodesic submanifold* if h vanishes identically.

We denote by \tilde{R} and R the curvature tensors of \tilde{M} and M in the same way. They are related by Gauss and Codazzi's equations

$$\begin{aligned} &\tilde{R}(X, Y, Z, W) \\ &= R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(X, Z), h(Y, W)), \end{aligned} \tag{2.1}$$

$$(\tilde{R}(X, Y)Z)^\perp = (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z), \tag{2.2}$$

respectively, where $\tilde{R}(X, Y, Z)^\perp$ denotes the normal component of $\tilde{R}(X, Y)Z$ and

$$(\tilde{\nabla}_X h)(Y, Z) = \nabla_X^\perp (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.3}$$

The mean curvature vector H is defined by $H = (1/\dim M)\text{trace } h$. M is said to be *minimal* if H vanishes identically. Moreover it is called a *totally umbilical submanifold* if

$$h(X, Y) = g(X, Y)H, \tag{2.4}$$

for any X, Y tangent vector fields on M .

Let \mathcal{V} and \mathcal{W} be differentiable distributions on M . Then, M is said to be

- (i) a $(\mathcal{V}, \mathcal{W})$ -mixed totally geodesic submanifold if $h(X, Y) = 0$ for all $X \in \mathcal{V}$ and $Y \in \mathcal{W}$,
- (ii) a \mathcal{V} -totally geodesic submanifold if it is $(\mathcal{V}, \mathcal{V})$ -mixed totally geodesic,
- (iii) \mathcal{V} -minimal if $H_{\mathcal{V}} = (1/\dim \mathcal{V})\text{trace } h|_{\mathcal{V}} = 0$.

Let us consider now a submanifold M of an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$, tangent to the structure vector field ξ . Put $\phi X = TX + NX$ for any tangent vector field X , where TX (resp. NX) denotes the tangential (resp. normal) component of ϕX . Similarly, $\phi V = tV + nV$ for any normal vector field V with tV tangent and nV normal to M .

Two well-known classes of submanifolds of an almost contact metric manifold are invariant and anti-invariant submanifolds. In the first case, the tangent space

of the submanifold is invariant under the action of the almost contact structure ϕ , where as in the second case it is mapped into the normal space. We describe below a sort of submanifolds that generalize these two classes:

Definition 2.1. [18]. A submanifold M of an almost contact metric manifold \widetilde{M} , tangent to the structure vector field ξ , is said to be an almost semi-invariant submanifold if there exists k functions $\lambda_1, \dots, \lambda_k$, defined on M with values in $(0, 1)$, such that

- (i) $-\lambda_1^2(p), \dots, -\lambda_k^2(p)$ are distinct eigenvalues of $T^2|_{\mathcal{D}}$ at $p \in M$, with

$$T_p M = \mathcal{D}_p^1 \oplus \mathcal{D}_p^0 \oplus \mathcal{D}_p^{\lambda_1} \oplus \dots \oplus \mathcal{D}_p^{\lambda_k} \oplus \text{Span}\{\xi_p\},$$

where \mathcal{D}_p^λ , $\lambda \in \{1, 0, \lambda_1(p), \dots, \lambda_k(p)\}$, denotes the eigenspace associated to the eigenvalue $-\lambda^2$.

- (ii) The dimensions of $\mathcal{D}_p^1, \mathcal{D}_p^0, \mathcal{D}_p^{\lambda_1}, \dots, \mathcal{D}_p^{\lambda_k}$ are independent of $p \in M$.

If in addition, each λ_i is constant, then M is called a skew semi-invariant submanifold.

Consequently, it is easy to verify the following equalities:

$$\mathcal{D}_p^1 = \text{Ker}(N|_{\mathcal{D}})_p, \quad \mathcal{D}_p^0 = \text{Ker}(T|_{\mathcal{D}})_p \quad \text{and} \quad \mathcal{D}_p^{\lambda_i} = \text{Ker}(T^2|_{\mathcal{D}} + \lambda_i^2(p)I)_p,$$

$i = 1, \dots, k$, for all $p \in M$.

The definition enables us to consider T -invariant mutually orthogonal distributions

$$\mathcal{D}^\lambda = \bigcup_{x \in M} \mathcal{D}_x^\lambda, \quad \lambda \in \{1, 0, \lambda_1, \dots, \lambda_k\},$$

on M such that $TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_1} \oplus \dots \oplus \mathcal{D}^{\lambda_k} \oplus \text{Span}\{\xi\}$,

For an almost semi-invariant submanifold M of \widetilde{M} we have

$$T^\perp M = \underline{\mathcal{D}}^1 \oplus \underline{\mathcal{D}}^0 \oplus \underline{\mathcal{D}}^{\lambda_1} \oplus \dots \oplus \underline{\mathcal{D}}^{\lambda_k},$$

where $\underline{\mathcal{D}}^1 = \text{Ker}(t)$, $\underline{\mathcal{D}}^0 = \text{Ker}(n)$, $\underline{\mathcal{D}}^\lambda = N\mathcal{D}^\lambda$, $t\underline{\mathcal{D}}^\lambda = \mathcal{D}^\lambda$ and $\underline{\mathcal{D}}^\lambda = \text{Ker}(n^2 + \lambda^2 I)$ with $\lambda \in \{\lambda_1, \dots, \lambda_k\}$. The distributions $\underline{\mathcal{D}}^\lambda$, $\lambda \neq 0$, are n -invariant.

We denote by U^λ (resp. \underline{U}^λ) the orthogonal projection from TM on \mathcal{D}^λ (resp. $\underline{\mathcal{D}}^\lambda$). Then we have:

$$g(TX, TY) = \sum_{\lambda} \lambda^2 g(U^\lambda X, U^\lambda Y), \quad g(nV, nW) = \sum_{\lambda} \lambda^2 g(\underline{U}^\lambda V, \underline{U}^\lambda W). \quad (2.5)$$

Now we present some different examples of almost semi invariant submanifolds. They are based on the fine relation between them and slant submanifolds. First we

remember that given $X \in T_xM$ a tangent vector at the point x , the Wirtinger angle $\theta(X)$ is the angle between ϕX and T_xM . A differentiable distribution is called *slant* if $\theta(X)$ is the same for all $X \neq 0$ in the distribution at every point of M . A submanifold of an almost contact metric manifold is called *slant* if for any $x \in M$ and any $X \in T_xM$, linearly independent of ξ_x , the Wirtinger angle is a constant. The following proposition characterizes slant submanifolds.

Proposition 2.2. [8, pg. 128] *Let M be a submanifold of an almost contact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$ tangent to ξ . Then, M is a slant submanifold if and only if there is a constant $\mu \in [0, 1]$ such that $T^2 = -\mu^2(I - \eta \otimes \xi)$. In such a case, if θ is the slant angle, $\mu^2 = \cos^2 \theta$.*

From this characterization we deduce:

Proposition 2.3. *Every slant submanifold M of an almost contact metric manifold with slant angle θ is a skew semi-invariant submanifold with $TM = \mathcal{D}^\lambda \oplus \text{Span}\{\xi\}$ such that $\lambda = \cos \theta$.*

If the Wirtinger angle does not depend on the choice of $X \in T_xM$ but it does on the choice of the point $x \in M$, then M is called a *quasi-slant submanifold*. They were introduced in almost Hermitian geometry by F. Etayo, [13], and the same notion can be considered in almost contact geometry just by taking X independent of ξ_x . The characterization of quasi-slant submanifolds is similar to that of slant ones, but now $T^2 = -\mu^2(I - \eta \otimes \xi)$, for μ a certain function. So every quasi-slant submanifold is an almost semi-invariant one, and every almost semi-invariant submanifold with $TM = \mathcal{D}^\lambda \oplus \text{Span}\{\xi\}$ is quasi-slant.

We can construct some specific examples of quasi-slant submanifolds from the examples of slant submanifolds in complex and contact geometry given in [11] and [8], respectively. First, we present the following result:

Theorem 2.4. *Given a quasi-slant submanifold S of \mathbf{C}^2*

$$x(u', v') = (f_1(u', v') + if_3(u', v'), f_2(u', v') + if_4(u', v'))$$

whose Wirtinger angle, θ , is different from 0 and $\pi/2$ everywhere, with $\partial/\partial u'$ and $\partial/\partial v'$ non-zero and orthogonal. Then

$$y(u, v, t) = 2(f_1(u, v), f_2(u, v), f_3(u, v), f_4(u, v), t), \tag{2.6}$$

defines an almost semi-invariant submanifold M in \mathbf{R}^5 with its usual Sasakian structure. Furthermore $\{e_1, e_2, \xi\}$, with

$$e_1 = \frac{\partial}{\partial u} + \left(2f_3 \frac{\partial f_1}{\partial u} + 2f_4 \frac{\partial f_2}{\partial u} \right) \frac{\partial}{\partial t}, \quad e_2 = \frac{\partial}{\partial v} + \left(2f_3 \frac{\partial f_1}{\partial v} + 2f_4 \frac{\partial f_2}{\partial v} \right) \frac{\partial}{\partial t},$$

is an orthogonal basis of TM , and $TM = \mathcal{D}^\lambda \oplus \text{Span}\{\xi\}$ with $\lambda = \cos \theta$.

Proof. It is just a simple computation to prove that the basis is orthogonal, and that $\mathcal{D}^\lambda = \text{Span}\{e_1, e_2\}$ is the eigenspace of $T^2|_{\mathcal{D}}$ associated to $-\lambda^2$. ■

Using this theorem we can give some examples of almost semi-invariant submanifolds, by considering some different quasi-slant submanifolds of \mathbf{C}^2 . All these examples can be verified by straightforward computations.

Example 2.5. Let $f(u, v)$ be a differentiable function, and let us consider the submanifold M of \mathbf{R}^5 given by:

$$y(u, v, t) = 2(u \cos f(u, v), u \sin f(u, v), v, 0, t).$$

We ask f to be $f = f(u)$ or $f = f(v)$ in order for $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ to be orthogonal. Then $TM = \mathcal{D}^\lambda \oplus \text{Span}\{\xi\}$ with $\lambda = \frac{\cos f - u \sin f f'}{\sqrt{1 + u^2 f'^2}}$.

Example 2.6. Given a differentiable function $k(u) \neq c/u$, for every constant c , the submanifold given by

$$y(u, v, t) = 2(e^{k(u)u} \cos u \cos v, e^{k(u)u} \sin u \cos v, e^{k(u)u} \cos u \sin v, e^{k(u)u} \sin u \sin v, t)$$

is an almost semi-invariant submanifold with $TM = \mathcal{D}^\lambda \oplus \text{Span}\{\xi\}$ and $\lambda = \frac{k'u + k}{\sqrt{(k'u + k)^2 + 1}}$.

Example 2.7. We want the equation

$$y(u, v, t) = 2(u, k \cos v, v, k \sin v, t)$$

to define an almost semi-invariant submanifold M in \mathbf{R}^5 with $TM = \mathcal{D}^\lambda \oplus \text{Span}\{\xi\}$. In order to apply Theorem , we choose either $k = k(u)$ or $k = k(v)$. In the first case,

$$\lambda = \frac{kk' + 1}{\sqrt{1 + k'^2} \sqrt{k^2 + 1}},$$

and $k(u)$ must be different from $\sqrt{-2u + c}$ and ce^u , for any constant c , so that λ be different from 0, 1. In the second case, $\lambda = 1/\sqrt{1 + k^2 + k'^2}$.

Now we give some examples with two distributions in the decomposition of the tangent space. They are based on bi-slant submanifolds (see [9]), but with Wirtinger angles changing from one point to other.

Example 2.8. Let us consider \mathbf{R}^9 with its usual Sasakian structure, and its submanifold M given by the following equation

$$x(u, v, w, s, t) = 2(u, 0, w, 0, v \cos \theta_1, v \sin \theta_1, s \cos \theta_2, s \sin \theta_2, t),$$

where $\theta_1 = \sqrt{3} \log(v)$ and $\theta_2 = \sqrt{3} \log(s)$, $v, s \in (1, +\infty)$. Then M is a five dimensional almost semi-invariant submanifold with two non-constant functions, λ_1, λ_2 . To prove this fact, we just take the orthogonal basis

$$\begin{aligned} e_1 &= 2 \left(\frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial z} \right), & e_2 &= (\cos \theta_1 - \sqrt{3} \sin \theta_1) \frac{\partial}{\partial y_1} + (\sin \theta_1 + \sqrt{3} \cos \theta_1) \frac{\partial}{\partial y_2}, \\ e_3 &= 2 \left(\frac{\partial}{\partial x_3} + y_3 \frac{\partial}{\partial z} \right), & e_4 &= (\cos \theta_2 - \sqrt{3} \sin \theta_2) \frac{\partial}{\partial y_3} + (\sin \theta_2 + \sqrt{3} \cos \theta_2) \frac{\partial}{\partial y_4}, \\ e_5 &= 2 \frac{\partial}{\partial z} = \xi, \end{aligned}$$

and we define the distributions $\mathcal{D}^{\lambda_1} = \text{Span}\{e_1, e_2\}$, and $\mathcal{D}^{\lambda_2} = \text{Span}\{e_3, e_4\}$. It is clear that $TM = \mathcal{D}^{\lambda_1} \oplus \mathcal{D}^{\lambda_2} \oplus \text{Span}\{\xi\}$, and, for $i = 1, 2$, \mathcal{D}^{λ_i} is a slant distribution with angle θ_i , being $\mathcal{D}_p^{\lambda_i}$ the eigenspace of $T^2|_{\mathcal{D}}$ at $p \in M$ associated to the eigenvalue $-\cos^2 \theta_i$.

Example 2.9. The equation $x(u, v, s, t) = 2(u, v, s, 0, t)$, defines an almost semi-invariant submanifold in \mathbf{R}^5 with its usual almost contact structure. In this case, $TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \text{Span}\{\xi\}$, just taking $\mathcal{D}^1 = \text{Span} \left\{ 2 \left(\frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial z} \right), 2 \frac{\partial}{\partial y_1} \right\}$ and $\mathcal{D}^0 = \text{Span} \left\{ 2 \left(\frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial z} \right) \right\}$.

Example 2.10. Let us consider \mathbf{R}^7 with its usual Sasakian structure, and its submanifold M given by

$$x(u, v, s, t) = 2(u, 0, 0, v, s \cos \theta, s \sin \theta, t),$$

with $\theta = \sqrt{3} \log s$, where $s \in (1, +\infty)$. Then M is an almost semi-invariant submanifold whose tangent space admits the following decomposition

$$TM = \mathcal{D}^0 \oplus \mathcal{D}^\lambda \oplus \text{Span}\{\xi\},$$

where $\mathcal{D}^0 = \text{Span}\{e_3\}$, $\mathcal{D}^\lambda = \text{Span}\{e_1, e_2\}$, with $\lambda = \cos \theta$, and

$$\begin{aligned} e_1 &= 2 \left(\frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial z} \right), & e_2 &= 2 \frac{\partial}{\partial y_1}, \\ e_3 &= (\cos \theta - \sqrt{3} \sin \theta) \frac{\partial}{\partial y_2} + (\sin \theta + \sqrt{3} \cos \theta) \frac{\partial}{\partial y_3}, & e_4 &= 2 \frac{\partial}{\partial z} = \xi. \end{aligned}$$

Since sections 3 and 6 will be more concerned with the invariant and anti-invariant cases of semi-invariance, it seems to be helpful to include illustrative examples of such kinds of submanifolds. Actually, we can get examples of invariant submanifolds just by taking $f = 0$ in Example 2.5 or $k = ce^u$ in Example 2.7. Similarly, we obtain anti-invariant submanifolds with $f = \pi/2$ or $f = \arccos(1/u)$ in Example 2.5, $k = 0$ or $k = c/u$ in Example 2.6 and $k = \sqrt{-2u + c}$ in Example 2.7.

3. SUBMANIFOLDS OF A GENERALIZED SASAKIAN SPACE FORM

In this section, we first state some results characterizing invariant and anti-invariant submanifolds of a generalized Sasakian space form by means of the curvature tensor.

Lemma 3.1. *Let M be a submanifold of a generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$. If M is either invariant or anti-invariant, then $\tilde{R}(X, Y)Z$ is tangent and $\tilde{R}(X, Y)V$ is normal to M , for any X, Y, Z tangent to M and any V normal to M .*

Proof. Let us first notice that both statements above are equivalent because $\tilde{R}(X, Y, Z, V) = -\tilde{R}(X, Y, V, Z)$. Therefore, we must only prove the first one.

On the one hand, if M is invariant, then

$$\begin{aligned} \tilde{R}(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &+ f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \end{aligned}$$

is tangent to M because so are $X, Y, \phi X$ and ϕY .

And on the other hand, if M is anti-invariant, then $g(X, \phi Z) = g(Y, \phi Z) = g(X, \phi Y) = 0$, so

$$\begin{aligned} \tilde{R}(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \end{aligned}$$

which is tangent, concluding the proof. ■

Even more, if X, Y are orthogonal to ξ then $\tilde{R}(X, Y)Z$ also is, for all Z tangent to M .

Since both conditions of the proposition above are equivalent, from now on we will only refer to the first one.

With some additional conditions we can prove a kind of converse:

Lemma 3.2. *Let M be a connected submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. If $f_2(p) \neq 0$, for each $p \in M$, and TM is invariant under the action of $\widetilde{R}(X, Y)$, X, Y tangent to M , then M is either invariant or anti-invariant.*

Proof. For X, Y tangent to M ,

$$\begin{aligned} \widetilde{R}(X, Y)X &= f_1\{g(Y, X)X - g(X, X)Y\} - 3f_2g(Y, \phi X)\phi X \\ &+ f_3\{\eta(X)^2Y - \eta(Y)\eta(X)X + g(X, X)\eta(Y)\xi - g(Y, X)\eta(X)\xi\} \end{aligned}$$

should be tangent, so $f_2g(Y, \phi X)\phi X$ is tangent. As $f_2 \neq 0$ at any point, either ϕX is tangent or $g(Y, \phi X) = 0$, for all Y tangent to M , and then ϕX is normal to M . Taking into account that M is a connected submanifold, for each point $p \in M$, one of the two conditions holds: either $N_p = 0$ or $T_p = 0$. Finally, using an argument of continuity, either $N \equiv 0$ or $T \equiv 0$, i.e. M is either invariant or anti-invariant. ■

Actually, it is enough for TM to be invariant by the action of $\widetilde{R}(X, Y)$ when X, Y are orthogonal to ξ .

Joining both lemmas we arrive to:

Theorem 3.3. *Let M be a connected submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with $f_2 \neq 0$ everywhere. Then, M is either invariant or anti-invariant if and only if TM is invariant under the action of $\widetilde{R}(X, Y)$, for any X, Y tangent to M .*

Hence, for an almost semi-invariant submanifold we deduce the following result:

Corollary 3.4. *Let M be a connected almost semi-invariant submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$, such that $\mathcal{D}^0 \neq \{0\} \neq \mathcal{D}^1 \oplus \mathcal{D}^{\lambda_1} \oplus \dots \oplus \mathcal{D}^{\lambda_k}$. If TM is invariant under the action of $\widetilde{R}(X, Y)$, then f_2 must vanish along the submanifold.*

We can prove a similar result to Lemma 3.1 for $\widetilde{R}(U, V)$, $U, V \in T^\perp M$.

Proposition 3.5. *Let M be a submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. If M is invariant, then TM and $T^\perp M$ are invariant under the action of $\widetilde{R}(U, V)$ for any U, V normal to M .*

Proof. Let us first notice that both statements are equivalent because $\widetilde{R}(U, V, W, X) = -\widetilde{R}(U, V, X, W)$, so we only prove the first one.

As M is invariant, given U normal to M , $g(X, \phi U) = -g(\phi X, U) = 0$, for any tangent vector field X . Hence, ϕU is normal to M .

Therefore,

$$\begin{aligned}\tilde{R}(U, V)X &= f_1\{g(V, X)U - g(U, X)V\} \\ &\quad + f_2\{g(U, \phi X)\phi V - g(V, \phi X)\phi U + 2g(U, \phi V)\phi X\} \\ &\quad + f_3\{\eta(U)\eta(X)V - \eta(V)\eta(X)U + g(U, X)\eta(V)\xi - g(V, X)\eta(U)\xi\} \\ &= 2f_2g(U, \phi V)\phi X,\end{aligned}$$

which is tangent. ■

Since from being anti-invariant we can not deduce that ϕV is normal, we do not give a similar result for anti-invariant submanifolds. But we can prove a converse.

Proposition 3.6. *Let M be a connected submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. If $f_2(p) \neq 0$, for each $p \in M$, and $T^\perp M$ is invariant under the action of $\tilde{R}(U, V)$, U, V normal to M , then M is either invariant or anti-invariant.*

Proof. It is similar to Lemma 3.2's proof, just imposing that $\tilde{R}(U, V)U$ must be normal for any U, V normal to M . ■

4. SECTIONAL CURVATURES

Let M be an almost semi-invariant submanifold of an almost contact metric manifold \widetilde{M} . For a unit vector $X \in \mathcal{D}^\lambda$, $\lambda \neq 0$, we define the \mathcal{D}^λ -sectional curvature of X as:

$$H_\lambda(X) = K_M \left(X, \frac{1}{\lambda}TX \right). \quad (4.1)$$

Let $\{E_1, \dots, E_{n(\lambda)}\}$ and $\{F_1, \dots, F_{n(\mu)}\}$ be local orthonormal references of \mathcal{D}^λ and \mathcal{D}^μ respectively. We define the $(\mathcal{D}^\lambda, \mathcal{D}^\mu)$ -sectional curvature as:

$$\varrho_{\lambda\mu} = \sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\mu)} K_M(E_i \wedge F_j). \quad (4.2)$$

The analogous definitions in complex geometry were introduced by Tripathi in [16] and Ronsse in [14]. Obviously the definition above does not depend on the basis.

Now we study the \mathcal{D}^λ -sectional curvatures for submanifolds of a generalized Sasakian space form. First, we remind the reader that T is called \mathcal{W} -commutative, for a certain distribution \mathcal{W} , if $h(X, TY) = h(TX, Y)$, for any $X, Y \in \mathcal{W}$. Secondly,

let us notice that, in such a case, the sectional curvature of the plane described by two orthogonal unit vector fields, X, Y tangent to M , is given by

$$K_M(X \wedge Y) = R(X, Y, Y, X) = f_1 + 3f_2g^2(X, TY) - f_3\{\eta^2(X) + \eta^2(Y)\} - \|h(X, Y)\|^2 + g(h(X, X), h(Y, Y)), \tag{4.3}$$

where we have used (1.1) and Gauss' equation (2.1).

Theorem 4.1. *Let \widetilde{M} be an almost semi-invariant submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. If T is \mathcal{D}^λ -commutative, $\lambda \neq 0$, then for all unit $X \in \mathcal{D}^\lambda$:*

$$H_\lambda(X) = f_1 + 3\lambda^2 f_2 - \|h(X, X)\|^2 - \frac{1}{\lambda^2} \|h(X, TX)\|^2. \tag{4.4}$$

Therefore

$$H_\lambda(X) \leq f_1 + 3\lambda^2 f_2, \tag{4.5}$$

and the equality holds if M is \mathcal{D}^λ -totally geodesic.

Proof. By using (4.3), we see that

$$H_\lambda(X) = f_1 + 3f_2g^2(X, \phi(1/\lambda TX)) - f_3\{\eta^2(X) + \eta^2(1/\lambda TX)\} - \|h(X, 1/\lambda TX)\|^2 + g(h(X, X), h(1/\lambda TX, 1/\lambda TX)),$$

for any vector field $X \in \mathcal{D}^\lambda$.

Moreover, $g(X, \phi(1/\lambda TX)) = g(X, 1/\lambda T^2X) = g(X, -\lambda^2/\lambda X) = -\lambda$, so

$$H_\lambda(X) = f_1 + 3f_2\lambda^2 - \frac{1}{\lambda^2}\{\|h(X, TX)\|^2 - g(h(X, X), h(TX, TX))\}. \tag{4.6}$$

Finally, as T is \mathcal{D}^λ -commutative

$$h(TX, TX) = h(X, T^2X) = h(X, -\lambda^2X) = -\lambda^2h(X, X), \tag{4.7}$$

which used in (4.6) gives (4.4). ■

In particular, as every Sasakian space form is a generalized Sasakian space form we can prove the following corollary. This is related to one given by M. Barros and F. Urbano in [3]: for a nearly Kaehler generalized complex space form $M^{2n}(\mu, \alpha)$, given \mathcal{W} an integrable distribution and $X \in \mathcal{W}$, we have $H(X) \leq \mu$.

Corollary 4.2. *Let M be an almost semi-invariant submanifold of a Sasakian space form $\widetilde{M}(c)$. Then, the ϕ -sectional curvature $H(X)$ of M satisfies $H(X) \leq c$, for any X tangent to M .*

Proof. From the definition of \mathcal{D}^1 -sectional curvature, for an unit $X \in \mathcal{D}^1$ we have $H(X) = K_M(X, \phi X) = K_M(X, TX) = H_1(X)$, and

$$H(X) \leq f_1 + 3f_2 = \frac{c+3}{4} + 3\left(\frac{c-1}{4}\right) = c,$$

follows from (4.5). ■

Now we are interested on the $(\mathcal{D}^\lambda, \mathcal{D}^\mu)$ -sectional curvature in a generalized Sasakian space form, depending on the values of λ and μ .

Theorem 4.3. *Let M be an almost semi-invariant submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. Let $\{E_1, \dots, E_{n(\lambda)}\}$ and $\{F_1, \dots, F_{n(\mu)}\}$ be local orthonormal references of \mathcal{D}^λ and \mathcal{D}^μ respectively. The following equations hold:*

(1) *If $\lambda \neq \mu$, then:*

$$\varrho_{\lambda\mu} = n(\lambda)n(\mu)f_1 + g(H_\lambda, H_\mu) - \sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\mu)} \|h(E_i, F_j)\|^2.$$

(2) *If $\lambda = \mu = 0$, then:*

$$\varrho_{00} = n(0)^2 f_1 + \|H_0\|^2 - \sum_{i=1}^{n(0)} \sum_{j=1}^{n(0)} \|h(E_i, E_j)\|^2.$$

(3) *If $\lambda \neq 0$, then:*

$$\varrho_{\lambda\lambda} = n(\lambda)^2 f_1 + 3f_2 n(\lambda) \lambda^2 + \|H_\lambda\|^2 - \sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\lambda)} \|h(E_i, E_j)\|^2.$$

Proof.

(1) Since \mathcal{D}^μ is T -invariant, $E_i \in \mathcal{D}^\lambda$ and $TF_j \in \mathcal{D}^\mu$, and $g(E_i, \phi F_j) = g(E_i, TF_j) = 0$. Then, computing $\varrho_{\lambda\mu}$ by (4.3) we have:

$$\begin{aligned} \varrho_{\lambda\mu} &= \sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\mu)} \{f_1 - \|h(F_j, E_i)\|^2 + g(h(E_i, E_i), h(F_j, F_j))\} \\ &= n(\lambda)n(\mu)f_1 + g(H_\lambda, H_\mu) - \sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\mu)} \|h(E_i, F_j)\|^2. \end{aligned}$$

(2) It can be proved by using the same method.

- (3) Now we choose an adapted local orthonormal basis, $\{E_1, \dots, E_{\frac{n(\lambda)}{2}}, \dots, E_{n(\lambda)}\}$, for \mathcal{D}^λ with $E_{\frac{n(\lambda)}{2}+i} = \frac{1}{\lambda}TE_i$, $1 \leq i \leq n(\lambda)/2$. By using this basis,

$$\sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\lambda)} g(E_i, TE_j)^2 = n(\lambda)\lambda^2,$$

because $g(E_i, TE_j) = 0$ always but for $j = \frac{n(\lambda)}{2} + i$ and $j = \frac{n(\lambda)}{2} - i$. For such values we have

$$g(E_i, TE_j) = g\left(E_i, \frac{1}{\lambda}T^2E_i\right) = g\left(E_i, \frac{-\lambda^2}{\lambda}E_i\right) = -\lambda.$$

Thus, by virtue of (4.3), we conclude the proof. ■

We finish this section by connecting these curvatures with the notions of $(\mathcal{D}^\lambda, \mathcal{D}^\mu)$ -totally geodesic and \mathcal{D}^λ -totally geodesic submanifolds. From the above theorem we deduce immediately this one:

Theorem 4.4. *Let M be an almost semi-invariant submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$.*

- (1) *If H_λ is orthogonal to H_μ , $\lambda \neq \mu$, then $\varrho_{\lambda\mu} \leq n(\lambda)n(\mu)f_1$ and the equality holds if and only if M is $(\mathcal{D}^\lambda, \mathcal{D}^\mu)$ -mixed totally geodesic.*
- (2) *If M is \mathcal{D}^λ -minimal then $\varrho_{\lambda\lambda} \leq n(\lambda)^2f_1 + 3f_2n(\lambda)\lambda^2$, and the equality holds if and only if M is \mathcal{D}^λ -totally geodesic.*

There are examples satisfying each of the above theorem's items. In Example 2.8, H_{λ_1} and H_{λ_2} are orthogonal, and M is $(\mathcal{D}^{\lambda_1}, \mathcal{D}^{\lambda_2})$ -mixed totally geodesic. Example 2.9 is totally geodesic, so it satisfies both items. Finally, Example 2.10 is \mathcal{D}^λ -totally geodesic and $(\mathcal{D}^0, \mathcal{D}^\lambda)$ -mixed totally geodesic but not \mathcal{D}^0 -totally geodesic.

5. RICCI TENSOR AND SCALAR CURVATURE

Let M be a $(m + 1)$ -dimensional submanifold of a $(2m + 1)$ -dimensional generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$, with ξ tangent to M . We consider $\{E_1, \dots, E_m, E_{m+1} = \xi\}$ and $\{N_1, \dots, N_{2n-m}\}$ local orthonormal basis of TM and $T^\perp M$ respectively, and denote $A_{N_\nu} = A_\nu$.

The curvature tensors of M and \widetilde{M} are related by means of Gauss' equation, (2.1). Therefore, from (1.1) we obtain the following result:

Lemma 5.1. *The Ricci tensor S of a $(m + 1)$ -dimensional submanifold M of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ is given by*

$$S(X, Y) = mf_1g(X, Y) + 3f_2g(TX, TY) + f_3\{(1 - m)\eta(X)\eta(Y) - g(X, Y)\} + \sum_{\nu=1}^{2n-m} \{(m + 1)(\text{trace } A_\nu)g(A_\nu X, Y) - g(A_\nu X, A_\nu Y)\},$$

for any X, Y vector fields on M .

Hence, for an almost semi-invariant submanifold, using the decomposition (2.5), we are able to state:

Lemma 5.2. *The Ricci tensor S of a $(m + 1)$ -dimensional almost semi-invariant submanifold M of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ is given by*

$$S(X, Y) = \sum_{\lambda} (mf_1 + 3f_2\lambda^2 - f_3)g(U^\lambda X, U^\lambda Y) + m(f_1 - f_3)\eta(X)\eta(Y) + \sum_{\nu=1}^{2n-m} \{(m + 1)(\text{trace } A_\nu)g(A_\nu X, Y) - g(A_\nu X, A_\nu Y)\}, \tag{5.1}$$

for any X, Y vector fields on M .

For an almost semi-invariant submanifold of a Sasakian space form $\widetilde{M}^{2n+1}(c)$, we get

$$S(X, Y) = \sum_{\lambda} \frac{(m - 1 + 3\lambda^2)c + 3(m - \lambda^2) + 1}{4}g(U^\lambda X, U^\lambda Y) + m\eta(X)\eta(Y) + \sum_{\nu=1}^{2n-m} \{(m + 1)(\text{trace } A_\nu)g(A_\nu X, Y) - g(A_\nu X, A_\nu Y)\},$$

just by putting $f_1 = \frac{c + 3}{4}$ and $f_2 = f_3 = \frac{c - 1}{4}$ in (5.1).

Now we will study the scalar curvature.

Lemma 5.3. *The scalar curvature τ of a $(m + 1)$ -dimensional almost semi-invariant submanifold M of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ is given by*

$$\tau = f_1 + \frac{1}{(m + 1)m} \{3f_2 \sum_{\lambda} n(\lambda)\lambda^2 - 2mf_3 + (m + 1)^2\|H\|^2 - \|h\|^2\}, \tag{5.2}$$

with $\lambda \in \{0, 1, \lambda_1, \dots, \lambda_k\}$.

Proof. Considering an adapted local orthonormal basis in \mathcal{D}^λ , as in Lemma 4.3, a direct computation of

$$\tau = \frac{1}{(m+1)m} \sum_{i,j=1}^{m+1} R(E_i, E_j, E_j, E_i)$$

concludes the proof. ■

Once again, we can settle this result for a Sasakian space form:

$$\tau = \frac{c+3}{4} + \frac{1}{(m+1)m} \left\{ \frac{3c-3}{4} \sum_{\lambda} n(\lambda)\lambda^2 - 2m\frac{c-1}{4} + (m+1)^2 \|H\|^2 - \|h\|^2 \right\}.$$

From the results above the following statements can be argued.

Theorem 5.4. *Let M be a $(m+1)$ -dimensional, minimal almost semi-invariant submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. Then:*

- (1) $S - \sum_{\lambda} \{mf_1 + 3f_2\lambda^2 - f_3\}g \circ (U^\lambda \otimes U^\lambda) - m(f_1 - f_3)\eta \otimes \eta$,
with $\lambda \in \{0, 1, \lambda_1, \dots, \lambda_k\}$, is negative semi-definite in TM .
- (2) $\tau \leq f_1 + \frac{1}{(m+1)m} \{3f_2 \sum_{\lambda} n(\lambda)\lambda^2 - 2mf_3\}$, $\lambda \in \{0, 1, \lambda_1, \dots, \lambda_k\}$.

Proof. By virtue of Lemma 5.2,

$$\begin{aligned} S(X, X) - \sum_{\lambda} (mf_1 + 3f_2\lambda^2 - f_3)g(U^\lambda X, U^\lambda X) - m(f_1 - f_3)\eta(X)\eta(X) \\ = - \sum_{\nu=1}^{2n-m} g(A_\nu X, A_\nu X) \leq 0, \end{aligned}$$

where we take into account that

$$\sum_{\nu=1}^{2n-m} (\text{trace } A_\nu)g(A_\nu X, X) = \sum_{i=1}^{m+1} g(h(X, X), h(E_i, E_i)) = (m+1)g(h(X, X), H),$$

and that $H = 0$, because M is minimal.

The second part comes directly from Lemma 5.3. ■

Totally geodesic, minimal almost semi-invariant submanifolds can be characterized by means of either the Ricci tensor or the scalar curvature.

Theorem 5.5. *Let M be a $(m+1)$ -dimensional, minimal almost semi-invariant submanifold of generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. The following conditions are equivalent:*

- (i) M is totally geodesic,
- (ii) $\tau = f_1 + \frac{1}{(m+1)m} \{3f_2 \sum_{\lambda} n(\lambda)\lambda^2 - 2mf_3\}$,
- (iii) $S(X, Y) = \sum_{\lambda} (mf_1 + 3f_2\lambda^2 - f_3)g(U^\lambda X, U^\lambda Y) + m(f_1 - f_3)\eta(X)\eta(Y)$,
for all X, Y tangent to M .

Proof. Conditions (i) and (ii) are equivalent because of Lemma 5.3 and the assumption of M being a minimal submanifold.

Condition (i) implies (iii). Conversely if (iii) holds, from Lemma 5.2 it follows that $A_V X = 0$ for any normal vector V and therefore it is totally geodesic. ■

Similar results to Theorems 5.4 and 5.5 could be given for almost semi-invariant submanifolds of a Sasakian space form, $\widetilde{M}(c)$, just by putting $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$.

6. TOTALLY UMBILICAL SUBMANIFOLDS

In [10], B.-Y Chen studied umbilical submanifolds in the case of spaces of constant curvature. Also Chen and K.Ogiue, [12], considered such immersions in complex space forms, and D. E. Blair and L. Vanhecke, [7], in Sasakian space forms. Finally, Vanhecke, in [19], dealt with those submanifolds in generalized complex space forms. Our purpose is to give similar results in almost contact geometry, this is, for totally umbilical submanifolds of generalized Sasakian space forms.

We first need the following lemma.

Lemma 6.1. *Let M be an m -dimensional, connected, totally umbilical submanifold ($m > 2$) of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with $f_2 \neq 0$. Then M is either an invariant or an anti-invariant submanifold.*

Proof. From (2.3) and (2.4) we arrive at

$$(\widetilde{\nabla}_X h)(Y, Z) = g(Y, Z)\nabla_X^\perp H.$$

Hence Codazzi's equation (2.2) becomes

$$(\widetilde{R}(X, Y)Z)^\perp = g(Y, Z)\nabla_X^\perp H - g(X, Z)\nabla_Y^\perp H. \tag{6.1}$$

We can choose unit vector fields, X, Y with Y orthogonal to X and ϕX . Therefore, from (6.1) we obtain:

$$(\tilde{R}(X, Y)Y)^\perp = \nabla_X^\perp H. \tag{6.2}$$

In fact,

$$\tilde{R}(X, Y)Y = f_1g(Y, Y)X + f_3(\eta(X)\eta(Y)Y - \eta(Y)\eta(Y)X - g(Y, Y)\eta(X)\xi)$$

is tangent to M and then $(\tilde{R}(X, Y)Y)^\perp = 0$. Thus from (6.2) we get $\nabla_X^\perp H = 0$, for any X . From (6.1), $(\tilde{R}(X, Y)Z)^\perp = 0$, for any tangent fields X, Y, Z , which means that TM is invariant under the action of $R(X, Y)$. Therefore, M is either an invariant or an anti-invariant submanifold by virtue of Lemma 3.2. ■

Theorem 6.2. *Let M be a connected, totally umbilical submanifold of a generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$ with $f_2 \neq 0$. Then, M is one of the following submanifolds:*

(i) *a space with sectional curvatures given by the function*

$$K_M(X \wedge Y) = f_1 - f_3\{\eta^2(X) + \eta^2(Y)\} + g(H, H),$$

immersed in \tilde{M} as an anti-invariant submanifold,

(ii) *a generalized Sasakian space form $M(f_1 + g(H, H), f_2, f_3)$, immersed in \tilde{M} as an invariant submanifold.*

Proof. For a totally umbilical submanifold the equation of Gauss (2.1) becomes:

$$R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}g(H, H), \tag{6.3}$$

for any vector fields X, Y, Z, W tangent to M . Hence, if the ambient space is a generalized Sasakian space form, and X, Y are unit vector fields in M , the sectional curvature of the plane section spanned by X and Y is given by:

$$K_M(X \wedge Y) = f_1 + 3f_2g^2(X, TY) - f_3\{\eta^2(X) + \eta^2(Y)\} + g(H, H).$$

Thus, as we work under the hypothesis of Lemma 6.1, M is either anti-invariant or invariant. In the first case, the above sectional curvature is given by:

$$K_M(X \wedge Y) = f_1 - f_3\{\eta^2(X) + \eta^2(Y)\} + g(H, H).$$

Otherwise, if M is an invariant submanifold, an almost contact structure (ϕ, ξ, η, g) , could be given to M . And from (6.3), it follows

$$\begin{aligned} R(X, Y)Z &= (f_1 + g(H, H))\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ &\quad + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \end{aligned}$$

for any X, Y, Z tangent to M , so M is a generalized Sasakian space form $M(f_1 + g(H, H), f_2, f_3)$. ■

Finally, from Theorem 6.2 we directly obtain

Corollary 6.3. *Let M be a connected totally geodesic submanifold of a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with $f_2 \neq 0$. Then, M is one of the following submanifolds:*

(i) *a space with sectional curvatures given by the function*

$$K_M(X \wedge Y) = f_1 - f_3\{\eta^2(X) + \eta^2(Y)\},$$

immersed in \widetilde{M} as an anti-invariant submanifold,

(ii) *a generalized Sasakian space form $M(f_1, f_2, f_3)$, immersed in \widetilde{M} as an invariant submanifold.*

ACKNOWLEDGMENT

The authors are thankful to the referee of this paper, for some very pertinent suggestions which have improved it.

REFERENCES

1. P. Alegre, D. E. Blair and A. Carriazo, Generalized Sasakian space forms, *Israel J. Math.*, **141** (2004), 157-183.
2. P. Alegre and A. Carriazo, Structures on generalized Sasakian-space-forms, *Differential Geom Appl.*, **26** (2008), 656-666.
3. M. Barros and F. Urbano, CR-submanifolds of generalized complex space forms. *An. Stiint. Univ. Al. I. Cuza Iasi.*, **25** (1979), 355-363.
4. A. Bejancu, CR-submanifolds of a Kaehler manifold. I, *Proc. Amer. Math. Soc.*, **69(1)** (1978), 135-142.
5. A. Bejancu and N. Papaghiuc, Almost semi-invariant submanifolds of a Sasakian manifold, *Bull. Math. Soc. Sci. Math. R. S. Roumanie.*, **28** (1984), 13-40.

6. D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser, Boston, 2002.
7. D. E. Blair and Lieven Vanhecke, Umbilical submanifolds of Sasakian space forms, *J. Differential Geometry*, **13(2)** (1978), 273-278.
8. J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández, Slant submanifolds in Sasakian manifolds, *Glasg. Math. J.*, **42** (2000), 125-138.
9. A. Carriazo, Bi-slant immersions, *Proceedings ICRAMS*, (2000), 88-97.
10. B.-Y. Chen, *Geometry of Submanifolds*, Marcel Dekker, New York, 1973.
11. B.-Y. Chen, Slant immersions, *Bull. Austral. Math. Soc.*, **41** (1990), 135-147.
12. B.-Y. Chen and K. Ogiue, Two theorems on Kaehler manifolds, *Michigan Math. J.*, **21** (1975), 225-229.
13. F. Etayo, On Quasi-slant submanifolds of an almost Hermitian manifold, *Publ. Math. Debrecen*, **53(1-2)** (1998), 217-223.
14. G. S. Ronsse, Generic and skew CR submanifolds of a Kaehler manifold, *Bull. Inst. Math. Acad. Sinica*, **18** (1990), 127-141.
15. M. M. Tripathi, Almost semi-invariant submanifolds of trans-Sasakian manifolds, *J. Indian Math. Soc.*, **62** (1996), 225-245.
16. M. M. Tripathi, Generic submanifolds of generalized complex space form, *Publ. Math. Debrecen*, **50** (1997), 373-392.
17. M. M. Tripathi and Ion Mihai, Submanifolds of Framed Metric manifolds and S-manifolds, *Note Mat.*, **20(2)** (2001), 135-164.
18. M. M. Tripathi and K. D. Singh, Almost semi-invariant submanifolds of an ξ -framed metric manifold, *Demonstratio Math.*, **29** (1996), 413-426.
19. L. Vanhecke, Umbilical Submanifolds of Generalized Complex Space Forms, *Bull. Inst. Math. Acad. Sinica*, **4** (1976), 183-190.

Pablo Alegre and Alfonso Carriazo
Department of Geometry and Topology,
Faculty of Mathematics,
University of Sevilla,
Apdo, Correos 1160,
41080 Sevilla,
Spain
E-mail: psalerue@upo.es
carriazo@us.es