Quantitative asymptotic regularity results for the composition of two mappings

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Abstract

In this paper, we use techniques which originate from proof mining to give rates of asymptotic regularity and metastability for a sequence associated to the composition of two firmly nonexpansive mappings.

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1 Introduction

This paper continues the work initiated in [1] where the asymptotic behavior of compositions of finitely many firmly nonexpansive mappings was studied with the main focus on asymptotic regularity and convergence results. Since the subdifferential of a proper, convex and lower semi-continuous function is a maximal monotone operator and the resolvent of a monotone operator is firmly nonexpansive, certain splitting methods applied to convex minimization problems are a very relevant instance where compositions of firmly nonexpansive mappings appear. In this line, Bauschke, Combettes and Reich [5] proved that if f and g are proper, convex and lower semicontinuous functions defined on a Hilbert space H, composing alternatively the resolvents of f and g, one obtains weak convergence to a solution (if it exists) of the following minimization problem associated to f and g:

$$\operatorname*{argmin}_{(x,y)\in H\times H}\left(f(x)+g(y)+\frac{1}{2\lambda}\|x-y\|^2\right),\tag{1}$$

where $\lambda > 0$. This problem covers, among others, the convex feasibility problem for two sets both in the consistent and inconsistent case.

More recently, see [4], problem (1) was considered in the setting of CAT(0) spaces where it can be approached as in Hilbert spaces by applying alternatively the two resolvents of f and g, J_λ^f and J_λ^g, respectively (which are well-defined in this context). When evaluating the values of the resolvents, errors may also be

taken into account. Thus, given a starting point x_0, one can construct the sequences (x_n) and (y_n) defined by

$$d\left(y_n, J_\lambda^g x_n\right) \le \varepsilon_n \quad \text{and} \quad d\left(x_{n+1}, J_\lambda^f y_n\right) \le \delta_n, \quad \text{for each } n \in \mathbb{N}, \tag{2}$$

where $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\sum_{n=0}^{\infty} \delta_n < \infty$. Note that, if (x^*, y^*) is a solution of (1), then $y^* = J_\lambda^g(x^*)$ and $x^* = J_\lambda^f(y^*)$, so $\text{Fix}(J_\lambda^f \circ J_\lambda^g) \neq \emptyset$. At the same time, if $x^* \in \text{Fix}(J_\lambda^f \circ J_\lambda^g)$, then $(x^*, J_\lambda^g(x^*))$ is a solution of (1). Since J_λ^f and J_λ^g are firmly nonexpansive, by [1, Theorem 3.3], it follows that the sequences (x_n) and (y_n) are asymptotically regular (that is, $\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$ and $\lim_{n \to \infty} d(y_n, y_{n+1}) = 0$), provided problem (1) has a solution. If the range of one of the resolvents is boundedly compact, by [1, Theorem 4.2], there exists $u \in \text{Fix}(J_\lambda^f \circ J_\lambda^g)$ such that (x_n) and (y_n) converge to u and $J_\lambda^g u$, respectively.

In this paper, we use techniques which originate from proof mining (see section 2.3 below and [10] for more details) to give explicit quantitative forms of these results. Section 3 contains our main result that provides a rate of asymptotic regularity for the sequences (x_n) and (y_n) obtained by composing alternatively two general firmly nonexpansive mappings (with or without errors) in CAT(0) spaces. Section 4 focuses on rates of metastability for these two sequences: based on general facts from computability theory one can rule out the existence of computable rates of convergence for (x_n) (or for (y_n)). However, metastability

$$\forall k \in \mathbb{N} \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists n \in \mathbb{N} \, \forall i, j \in [n, n + g(n)] \, (d(x_i, x_j) < \frac{1}{k+1}),$$

though noneffectively equivalent to the full Cauchy property of (x_n), does admit (on general logical grounds) effective bounds $\Phi(k, g)$ on '$\exists n \in \mathbb{N}$'. We call such a bound Φ a rate of metastability. This concept has been known in logic as the Kreisel 'no-counterexample interpretation' of which it is a special instance, and for the case at hand also coincides with the Gödel functional interpretation (see [10]). In 2007, the concept was rediscovered by T. Tao ([14]) who introduced the name 'metastability' for it. Disregarding error terms for the moment, in our situation a rate of metastability might be seen as a far reaching generalization of a rate of asymptotic regularity as the latter results as the special case of the former where $g \equiv 1$:
From

$$\forall k \in \mathbb{N} \, \exists n \le \Phi(k, 1) \, (d(x_n, x_{n+1}) < \frac{1}{k+1}),$$

the fact that $d(x_n, x_{n+1})$ is nonincreasing immediately gives

$$\forall k \in \mathbb{N} \, \forall n \ge \Phi(k, 1) \, (d(x_n, x_{n+1}) < \frac{1}{k+1}).$$

2   Preliminaries

2.1   CAT(0) spaces

Let (X, d) be a metric space. A *geodesic path* that joins two points $x, y \in X$ is a mapping $\gamma : [0, l] \subseteq \mathbb{R} \to X$ such that $\gamma(0) = x$, $\gamma(l) = y$ and $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [0, l]$. The image of γ is called a *geodesic segment* from x to y. A point $z \in X$ belongs to such a geodesic segment if there exists $t \in [0, 1]$ such that $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t)d(x, y)$ and we write $z = (1 - t)x + ty$. (X, d) is a *(uniquely) geodesic space* if every two points in X are joined by a (unique) geodesic path. A subset C of X is *convex* if it contains all geodesic segments that join any two points in C. For more details on geodesic metric spaces, see [6].

There are several equivalent conditions for a geodesic metric space (X, d) to be CAT(0), one of them being the following inequality (see, for example, [3, Theorem 1.3.3]) which is to be satisfied for any four points $x, y, u, v \in X$

$$d(x, y)^2 + d(u, v)^2 \le d(x, v)^2 + d(y, u)^2 + 2d(x, u)d(y, v). \tag{3}$$

CAT(0) spaces include Hilbert spaces, $\mathbb{R}$-trees, Euclidean buildings, complete simply connected Riemannian manifolds of nonpositive sectional curvature, and many other important spaces.

2.2 Firmly nonexpansive mappings

Firmly nonexpansive mappings were introduced in Banach spaces by Bruck [8] (in the context of Hilbert spaces, these mappings are precisely the firmly contractive ones considered earlier by Browder [7]). Recently, Bruck's definition was extended to a nonlinear setting in [2] (see also [13]; in the case of the Hilbert ball this is already due to [9]).

Definition 2.1. *Let C be a nonempty subset of a $CAT(0)$ space (X, d). A mapping $T : C \to X$ is firmly nonexpansive if*

$$d(Tx, Ty) \leq d((1 - \lambda)x + \lambda Tx, (1 - \lambda)y + \lambda Ty),$$

for all $x, y \in C$ and $\lambda \in [0, 1]$.

Let X be a complete $CAT(0)$ space. The metric projection onto closed and convex subsets of X is firmly nonexpansive. Another important example of a firmly nonexpansive mapping is the *resolvent* of a convex, lower semi-continuous and proper function $f : X \to (-\infty, +\infty]$,

$$J_\lambda^f(x) := \operatorname*{argmin}_{z \in X} \left(f(z) + \frac{1}{2\lambda} d(x, z)^2 \right),$$

where $\lambda > 0$.

In $CAT(0)$ spaces, every firmly nonexpansive mapping satisfies the condition below which was called property (P_2) in [1]. Moreover, in this setting, every mapping with property (P_2) is nonexpansive. Note also that in Hilbert spaces, this notion coincides with firm nonexpansivity.

Definition 2.2. *Let C be a nonempty subset of a metric space (X, d). A mapping $T : C \to X$ satisfies property (P_2) if*

$$2d(Tx, Ty)^2 \leq d(x, Ty)^2 + d(y, Tx)^2 - d(x, Tx)^2 - d(y, Ty)^2,$$

for all $x, y \in C$.

2.3 Proof mining

During the last two decades a systematic program of 'proof mining' has emerged as a new applied form of proof theory and has successfully been applied to a number of areas of core mathematics (see [10] for a comprehensive treatment up to 2008). This logic-based program has its roots in Georg Kreisel's pioneering ideas of 'unwinding of proofs' going back to the 1950's and is concerned with the extraction of explicit effective bounds from prima facie noneffective proofs. General logical metatheorems guarantee such extractions for large classes of proofs and provide algorithms (based on so-called proof interpretations) for the actual extraction from a given proof. This approach has been applied with particular success in the context of nonlinear analysis including fixed point theory, ergodic theory, topological dynamics, continuous optimization and abstract Cauchy problems.

One condition that guarantees such results is that the statement proven has (if written in the appropriate formal framework) the form

$$\forall k \in \mathbb{N} \, \forall g \in \mathbb{N}^{\mathbb{N}} \, \forall \underline{x} \in \underline{X}^{(X)} \, \exists n \in \mathbb{N} \, A_\exists(k, g, \underline{x}, n),$$

where $\underline{x}$ is a tuple of parameters ranging over various metric, hyperbolic or normed spaces $\underline{X}$ or suitable classes of mappings between such spaces and $A_\exists$ is purely existential. This is not the case for the usual formulation of the Cauchy property which is of the form $\forall \exists \forall$ but is satisfied for the (equivalent) metastable formulation since the bounded quantifier $\forall i, j \in [n, n + g(n)]$ can be disregarded.

However, for asymptotic regularity results $d(x_n, x_{n+1}) \to 0$ (rather than the convergence of (x_n) itself) one usually can obtain full rates of convergence. One reason for this is that the sequence $(d(x_n, x_{n+1}))_{n \in \mathbb{N}}$ often is nondecreasing (as is the case for the sequences $(x_n), (y_n)$ defined by (4) below). Then $d(x_n, x_{n+1}) \to 0$ is equivalent to

$$\forall k \in \mathbb{N} \, \exists n \in \mathbb{N} \, (d(x_n, x_{n+1}) < \frac{1}{k+1}),$$

3

which has the right logical form and any bound $\Phi(k)$ on n is a rate of asymptotic regularity.

In the next section we will present rates of asymptotic regularity that have been extracted using this methodology from a noneffective asymptotic regularity proof given in [1]. The noneffectivity of that proofs comes from the (repeated) use of the convergence of bounded monotone sequences which is known to fail in a computable reading. The analysis of the proof was obtained by first replacing the use of the limits of such sequences by sufficiently good Cauchy-points instead and then applying a well-known and effective rate of metastability for the Cauchy property of bounded monotone sequences to suitably chosen parameters g and ε (see the end of the proof of Theorem 3.1 below).

In the final section we will apply an effective proof mining result from [12] to convert the rates of asymptotic regularity into rates of metastability provided that the underlying space is compact.

As usual in applications of the proof mining methodology, the final bounds and the proofs of their correctness can be stated in ordinary mathematical terms without any reference to tools or concepts from logic.

3 Rate of asymptotic regularity

Let X be a metric space, $T_1, T_2 : X \to X$ and (x_n) and (y_n) be defined by

$$y_n := T_1 x_n \quad \text{and} \quad x_{n+1} := T_2 y_n, \quad \text{for each } n \in \mathbb{N}. \tag{4}$$

Theorem 3.1. *Let (X, d) be a CAT(0) space and let $T_1, T_2 : X \to X$ satisfy property (P_2). Denote $S := T_2 \circ T_1$ and suppose that $\mathrm{Fix}(S) \neq \emptyset$. Let $x_0 \in X$ and $b > 0$ such that there exists $u \in \mathrm{Fix}(S)$ with $2d(x_0, u) \leq b$. Define the sequences (x_n) and (y_n) by (4). Then*

$$\forall \varepsilon > 0 \, \forall n \geq \Phi(\varepsilon, b) \, (d(y_n, y_{n+1}) \leq d(x_n, x_{n+1}) \leq \varepsilon),$$

where

$$\Phi(\varepsilon, b) := k \left\lceil \frac{2b(1 + 2^k)}{\varepsilon} \right\rceil + 1, \quad \text{with } k := \left\lceil \frac{2b}{\varepsilon} \right\rceil.$$

Proof. Let $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$. Since T_1 and T_2 satisfy property (P_2) we have that

$$2d(y_{n+k+1}, y_{n+1})^2 \leq d(x_{n+k+1}, y_{n+1})^2 + d(x_{n+1}, y_{n+k+1})^2 - d(x_{n+k+1}, y_{n+k+1})^2 - d(x_{n+1}, y_{n+1})^2$$

and

$$2d(x_{n+k+1}, x_{n+1})^2 \leq d(y_{n+k}, x_{n+1})^2 + d(y_n, x_{n+k+1})^2 - d(y_{n+k}, x_{n+k+1})^2 - d(y_n, x_{n+1})^2.$$

These inequalities together with (3) yield

$$d(y_{n+k+1}, y_{n+1})^2 + d(x_{n+k+1}, x_{n+1})^2 \leq d(x_{n+1}, x_{n+k+1})d(y_{n+1}, y_{n+k}) + d(x_{n+1}, x_{n+k+1})d(y_n, y_{n+k+1}).$$

Because $d(y_{n+k+1}, y_{n+1})^2 + d(x_{n+k+1}, x_{n+1})^2 \geq 2d(y_{n+k+1}, y_{n+1})d(x_{n+k+1}, x_{n+1})$ we obtain that

$$2d(y_{n+k+1}, y_{n+1})d(x_{n+k+1}, x_{n+1}) \leq d(x_{n+1}, x_{n+k+1})d(y_{n+1}, y_{n+k}) + d(x_{n+1}, x_{n+k+1})d(y_n, y_{n+k+1}).$$

Hence,

$$2d(y_{n+k+1}, y_{n+1}) - d(y_{n+1}, y_{n+k}) \leq d(y_n, y_{n+k+1})$$

(note that the above relation is also true when $d(x_{n+k+1}, x_{n+1}) = 0$). Since

$$d(y_{n+1}, y_{n+k}) \leq d(y_n, y_{n+k-1}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+k-1}),$$

it follows that $d(y_{n+1}, y_{n+k}) \leq (k-1)d(y_n, y_{n+1})$. Denote $r_{n,k} := d(y_n, y_{n+k}) \leq d(x_n, x_{n+k}) \leq 2d(x_0, u) \leq b$. Then

$$2r_{n+1,k} - (k-1)r_{n,1} \leq r_{n,k+1}.$$

We show next that for any $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$,

$$r_{n,k} \geq kr_{n+k,1} - k2^k(r_{n,1} - r_{n+k,1}). \tag{5}$$

4

For $k = 1$ this relation obviously holds for every $n \in \mathbb{N}$. Suppose that (5) holds for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$,

$$
\begin{aligned}
r_{n,k+1} &\geq 2r_{n+1,k} - (k-1)r_{n,1} \geq 2\left(kr_{n+1+k,1} - k2^k(r_{n+1,1} - r_{n+1+k,1})\right) - (k-1)r_{n,1} \\
&\geq 2\left(kr_{n+1+k,1} - k2^k(r_{n,1} - r_{n+1+k,1})\right) - (k-1)r_{n,1} \\
&= (k+1)r_{n+1+k,1} - k2^{k+1}(r_{n,1} - r_{n+1+k,1}) - (k-1)r_{n,1} + (k-1)r_{n+1+k,1} \\
&= (k+1)r_{n+1+k,1} - (k2^{k+1} + k - 1)(r_{n,1} - r_{n+1+k,1}) \\
&\geq (k+1)r_{n+1+k,1} - (k+1)2^{k+1}(r_{n,1} - r_{n+1+k,1}).
\end{aligned}
$$

Thus (5) holds for every $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$, from where

$$
r_{n,1} - (1 + 2^k)(r_{n,1} - r_{n+k,1}) \leq \frac{r_{n,k}}{k} \leq \frac{b}{k}.
$$

Let $\varepsilon > 0$ and $k := \left\lceil \dfrac{2b}{\varepsilon} \right\rceil$. Then for all $n \in \mathbb{N}$,

$$
r_{n,1} - (1 + 2^k)(r_{n,1} - r_{n+k,1}) \leq \frac{\varepsilon}{2}.
$$

Note that $(r_{n,1})$ is a nonincreasing sequence bounded by b. Using [10, Proposition 2.27] for $\varepsilon' := \dfrac{\varepsilon}{2(1 + 2^k)}$ and $g \equiv k$, there exists $N \leq k \left\lceil \dfrac{2b(1 + 2^k)}{\varepsilon} \right\rceil$ such that $r_{N,1} - r_{N+k,1} \leq \dfrac{\varepsilon}{2(1 + 2^k)}$ and so $r_{N,1} \leq \varepsilon$. Thus, for $n \geq \Phi(\varepsilon, b)$,

$$
d(y_n, y_{n+1}) \leq d(x_n, x_{n+1}) \leq d(y_{n-1}, y_n) \leq r_{N,1} \leq \varepsilon.
$$

$\square$

Suppose now that (x_n) and (y_n) are defined by

$$
d(y_n, T_1 x_n) \leq \varepsilon_n \quad \text{and} \quad d(x_{n+1}, T_2 y_n) \leq \delta_n, \quad \text{for each } n \in \mathbb{N}, \tag{6}
$$

where $\displaystyle\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\displaystyle\sum_{n=0}^{\infty} \delta_n < \infty$. For $n \in \mathbb{N}$, denote $\gamma_n := \varepsilon_n + \delta_n$.

Lemma 3.2. *Let (X, d) be a CAT(0) space and let $T_1, T_2 : X \to X$ satisfy property (P_2). Denote $S := T_2 \circ T_1$. Then for every $n \in \mathbb{N}$,*

(i) $d(x_{n+1}, Sx_n) \leq \gamma_n$.

(ii) If $u \in \mathrm{Fix}(S)$, $d(x_{n+1}, u) \leq \gamma_n + d(x_n, u)$.

Proof. (i) $d(x_{n+1}, Sx_n) \leq d(x_{n+1}, T_2 y_n) + d(T_2 y_n, T_2(T_1 x_n)) \leq \delta_n + d(y_n, T_1 x_n) \leq \gamma_n$.
(ii) Let $u \in \mathrm{Fix}(S)$. Then

$$
\begin{aligned}
d(x_{n+1}, u) &\leq d(x_{n+1}, T_2 y_n) + d(T_2 y_n, u) \leq \delta_n + d(y_n, T_1 u) \\
&\leq \delta_n + d(y_n, T_1 x_n) + d(T_1 x_n, T_1 u) \leq \gamma_n + d(x_n, u).
\end{aligned}
$$

$\square$

Theorem 3.3. *Let (X, d) be a CAT(0) space and let $T_1, T_2 : X \to X$ satisfy property (P_2) with $\mathrm{Fix}(S) \neq \emptyset$, where $S := T_2 \circ T_1$. Let $x_0 \in X$. Define the sequences (x_n) and (y_n) by (6). Suppose that $\displaystyle\sum_{n=0}^{\infty} \gamma_n$ converges with Cauchy modulus α, i.e., $\alpha : (0, \infty) \to \mathbb{N}$,*

$$
\forall \varepsilon > 0 \, \forall k \in \mathbb{N} \left(\sum_{i=\alpha(\varepsilon)}^{\alpha(\varepsilon)+k} \gamma_i \leq \varepsilon \right).
$$

5

Let $B \geq 0$ such that $\sum\limits_{n \geq 0} \gamma_n \leq B$ and $b > 0$ such that there exists $u \in \mathrm{Fix}(S)$ with $d(x_0, u) \leq b$. Then

$$\forall \varepsilon > 0 \, \forall n \geq \Phi'(\varepsilon, b, B, \alpha) \, (d(x_n, x_{n+1}) \leq \varepsilon),$$

where

$$\Phi'(\varepsilon, b, B, \alpha) := \alpha(\varepsilon/3) + \Phi(\varepsilon/3, 2(b + B)),$$

and Φ is the rate of asymptotic regularity from Theorem 3.1. For $d(y_n, y_{n+1})$ the same result holds with rate $\Phi''(\varepsilon, b, B, \alpha) := \Phi'(\varepsilon/2, b, B, \alpha)$.

Proof. Let $(x_n), (y_n)$ be defined as in (6) and consider for $S := T_2 \circ T_1$,

$$z_n := S^n(x_{\alpha(\varepsilon/3)}).$$

Note that (z_n) is the sequence (x_n) defined by (4) with the starting point $z_0 = x_{\alpha(\varepsilon/3)}$.
By induction, Lemma 3.2.(ii) gives

$$d(x_n, u) \leq d(x_0, u) + \sum_{i=0}^{n-1} \gamma_i.$$

Thus,

$$d(z_0, u) = d(x_{\alpha(\varepsilon/3)}, u) \leq d(x_0, u) + \sum_{i=0}^{\alpha(\varepsilon/3)-1} \gamma_i \leq b + B$$

and so one can apply Theorem 3.1 to the sequence (z_n) to get that for every $n \geq \Phi(\varepsilon/3, 2(b + B))$,

$$d(z_n, z_{n+1}) \leq \frac{\varepsilon}{3}.$$

One easily shows by induction on n that for all $n \in \mathbb{N}$,

$$d(z_n, x_{\alpha(\varepsilon/3)+n}) \leq \sum_{i=\alpha(\varepsilon/3)}^{\alpha(\varepsilon/3)+n-1} \gamma_i \leq \frac{\varepsilon}{3}.$$

For $n = 0$ this is obvious and for the induction step we argue (using Lemma 3.2(i))

$$d(z_{n+1}, x_{\alpha(\varepsilon/3)+n+1}) \leq d(Sz_n, Sx_{\alpha(\varepsilon/3)+n}) + d(Sx_{\alpha(\varepsilon/3)+n}, x_{\alpha(\varepsilon/3)+n+1})$$
$$\leq d(z_n, x_{\alpha(\varepsilon/3)+n}) + \gamma_{\alpha(\varepsilon/3)+n}.$$

Hence for all $n \geq \alpha(\varepsilon/3) + \Phi(\varepsilon/3, 2(b + B))$

$$d(x_n, x_{n+1}) \leq d(x_n, z_{n-\alpha(\varepsilon/3)}) + d(z_{n-\alpha(\varepsilon/3)}, z_{n-\alpha(\varepsilon/3)+1}) + d(z_{n-\alpha(\varepsilon/3)+1}, x_{n+1}) \leq \varepsilon.$$

The claim for (y_n) follows from

$$d(y_n, y_{n+1}) \leq d(T_1 x_n, T_1 x_{n+1}) + \varepsilon_n + \varepsilon_{n+1} \leq d(x_n, x_{n+1}) + \varepsilon_n + \varepsilon_{n+1}$$

and the fact that for $n \geq \Phi'(\frac{\varepsilon}{2}, b, B, \alpha) \geq \alpha(\varepsilon/6)$ one has $\varepsilon_n + \varepsilon_{n+1} \leq \frac{\varepsilon}{6}$ and so

$$d(y_n, y_{n+1}) \leq d(x_n, x_{n+1}) + \frac{\varepsilon}{6} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{6} < \varepsilon.$$

$\square$

Remark 3.4. *In the situation of Corollary 3.3 from [1] we can take the quadratic rate θ from that corollary instead of the exponential rate Φ in Theorem 3.3 above.*

4 Rate of metastability

Consider the sequences $(x_n), (y_n)$ from (6).

Lemma 4.1. $\chi(n, m, r) := m(r + 1)$ *is a modulus of uniform quasi-Fejér monotonicity (in the sense of* [12]) *of* (x_n) *w.r.t.* $F := \mathrm{Fix}(S)$, *where* $S := T_2 \circ T_1$, *and the error terms* $\gamma_n := \delta_n + \varepsilon_n$, *i.e.*

$$\forall r, n, m \in \mathbb{N} \, \forall p \in X \, \left(d(p, Sp) \leq \frac{1}{\chi(n,m,r)+1} \to \right.$$
$$\left. \forall l \leq m \, \left(d(x_{n+l}, p) < d(x_n, p) + \sum_{i=n}^{n+l-1} \gamma_i + \frac{1}{r+1} \right) \right).$$

Analogously for (y_n) *with* $\gamma_n' := \varepsilon_{n+1} + \delta_n$ *and* $S' := T_1 \circ T_2$.

Proof. An easy calculation (see (14) in [1]) gives that for all $p \in X$ and $n \in \mathbb{N}$

$$d(x_{n+1}, Sp) \leq d(x_n, p) + \gamma_n$$

and so
$$d(x_{n+1}, p) \leq d(x_n, p) + \gamma_n + d(p, Sp).$$

Hence,
$$d(x_{n+l}, p) \leq d(x_n, p) + \sum_{i=n}^{n+l-1} \gamma_i + l \cdot d(p, Sp).$$

The claim is now immediate.
The second claim for (y_n) is proved analogously using

$$d(y_{n+1}, S'p) \leq d(y_n, p) + \gamma_n'.$$

$\square$

Lemma 4.2. *Under the assumptions of Theorem 3.3, let* β *be a rate of convergence for* $\gamma_n \to 0$ *and* $\Phi'(\varepsilon) := \Phi'(\varepsilon, b, B, \alpha)$ *be the rate of convergence for* $d(x_n, x_{n+1}) \to 0$ *from Theorem 3.3. Then* $\Phi_\beta(\varepsilon) := \max\{\beta(\varepsilon/2), \Phi'(\varepsilon/2)\}$ *is a rate of asymptotic regularity for* $d(x_n, Sx_n) \to 0$, *i.e.*

$$\forall \varepsilon > 0 \, \forall n \geq \Phi_\beta(\varepsilon) \, (d(x_n, Sx_n) \leq \varepsilon).$$

Analogously for (y_n) *and* $S' := T_1 \circ T_2$ *with* β *being replaced by a rate of convergence* β' *for* $\gamma_n' \to 0$ *and* $\Phi_{\beta'}'(\varepsilon) := \max\{\beta'(\varepsilon/2), \Phi''(\varepsilon/2)\}$.

Proof. Let $n \geq \Phi_\beta(\varepsilon)$. Then (using Lemma 3.2.(i))

$$d(x_n, Sx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Sx_n) \leq \frac{\varepsilon}{2} + \gamma_n \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For (y_n) one reasons analogously. Thus, for all $n \geq \Phi_{\beta'}'(\varepsilon) = \max\{\beta'(\varepsilon/2), \Phi'(\varepsilon/4)\}$,

$$d(y_n, S'y_n) \leq d(y_n, y_{n+1}) + d(y_{n+1}, S'y_n) \leq \frac{\varepsilon}{2} + \gamma_n' \leq \varepsilon.$$

$\square$

Corollary 4.3. $\widehat{\Phi}_\beta(k, N) := \max\{N, \max\{\Phi_\beta(1/(i+1)) : i \leq k\}\}$ *is a monotone* $\liminf$*-bound for* (x_n) *w.r.t.* $\mathrm{Fix}(S)$ *in the sense of* [12]. *Analogously for* (y_n) *with* $\Phi_{\beta'}'$ *and* $\mathrm{Fix}(S')$.

Definition 4.4 (see [12]). *Let* (X, d) *be a totally bounded metric space. We call a function* $\gamma : \mathbb{N} \to \mathbb{N}$ *a modulus of total boundedness for* X *if for any sequence* (a_n) *in* X

$$\exists 0 \leq i < j \leq \gamma(k) \, \left(d(a_i, a_j) \leq \frac{1}{k+1} \right).$$

Theorem 4.5. *Under the assumptions of Theorem 3.3, let X additionally be totally bounded with a modulus of total boundedness γ. Then (x_n) is Cauchy with rate of metastability $\widehat{\Psi}(k, g, \gamma, \alpha, \beta, b, B)$, i.e.*

$$\forall k \in \mathbb{N}, \ \forall g : \mathbb{N} \to \mathbb{N} \, \exists n \leq \widehat{\Psi}(k, g, \gamma, \alpha, \beta, b, B) \, \forall i, j \in [n, n + g(n)] \ \left(d(x_i, x_j) \leq \frac{1}{k+1} \right),$$

where

$$\widehat{\Psi}(k, g, \gamma, \alpha, \beta, b, B) := \widehat{\Psi}_0(P), \ P := \gamma(8k + 7) + 1, \ \xi(k) := \alpha(1/(k+1)),$$
$$\chi_g^M(n, k) := (\max_{i \leq n} g(i)) \cdot (k + 1),$$

and $\widehat{\Psi}_0$ is defined recursively

$$\widehat{\Psi}_0(0) := 0, \ \widehat{\Psi}_0(n + 1) := \widehat{\Phi}_\beta \left(\chi_g^M \left(\widehat{\Psi}_0(n), 8k + 7 \right), \xi(8k + 7) \right).$$

Analogously for (y_n) with $\Phi'_{\beta'}$ and $\xi(k) := \alpha'(1/(k+1))$, where α' is a Cauchy modulus for $\sum_{n=0}^{\infty} \gamma'_n$.

Proof. The result follows from Theorem 6.4 in [12] together with Lemma 4.1 and Corollary 4.3. Note that in our case $G = H = id$ and so we can take $\alpha_G := \beta_H := id$ as well. $\square$

References

[1] Ariza-Ruiz, D., López-Acedo, G., Nicolae, A.: The asymptotic behavior of the composition of firmly nonexpansive mappings. J. Optim. Theory Appl. **167**, 409–429 (2015)

[2] Ariza, D., Leuştean, L., López-Acedo, G.: Firmly nonexpansive mappings in classes of geodesic spaces. Trans. Amer. Math. Soc. **366**, 4299–4322 (2014)

[3] Bačák, M.: Convex analysis and optimization in Hadamard spaces. De Gruyter Series in Nonlinear Analysis and Applications, vol. 22. Walter de Gruyter & Co., Berlin (2014)

[4] Banert, S.: Backward-backward splitting in Hadamard spaces. J. Math. Anal. Appl. **414**, 656–665 (2014)

[5] Bauschke, H.H., Combettes, P.L., Reich, S.: The asymptotic behavior of the composition of two resolvents. Nonlinear Anal. **60**, 283–301 (2005)

[6] Bridson, M., Haefliger, A.: Metric Spaces of Non-positive Curvature. Grundlehren der Mathematischen Wissenschaften, vol. 319. Springer, Berlin (1999)

[7] Browder, F.E.: Convergence theorems for sequences of nonlinear operators in Banach spaces. Math. Z. **100**, 201–225 (1967)

[8] Bruck, R.E.: Nonexpansive projections on subsets of Banach spaces. Pacific J. Math. **47**, 341–355 (1973)

[9] Goebel, K., Reich, S.: Uniform convexity, hyperbolic geometry, and nonexpansive mappings. Monographs and Textbooks in Pure and Applied Mathematics, 83. Marcel Dekker, Inc., New York (1984)

[10] Kohlenbach, U.: Applied Proof Theory: Proof Interpretations and their Use in Mathematics. Springer Monographs in Mathematics. Springer, Berlin-Heidelberg (2008)

[11] Kohlenbach, U.: On the quantitative asymptotic behavior of strongly nonexpansive mappings in Banach and geodesic spaces. Israel J. Math. (in press)

[12] Kohlenbach, U., Leuştean, L., Nicolae, A.: Quantitative results on Fejér monotone sequences. Preprint 2015. arXiv:1412.5563.

[13] Nicolae, A.: Asymptotic behavior of averaged and firmly nonexpansive mapping in geodesic spaces. Nonlinear Anal. **87**, 102–115 (2013)

[14] Tao, T.: Soft analysis, hard analysis, and the finite convergence principle, Essay posted May 23, 2007, appeared in: Tao, T., Structure and Randomness: Pages from Year One of a Mathematical Blog. Amer. Math. Soc., Providence, RI (2008)