The structure of attractors in non-autonomous perturbations of gradient-like systems

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Abstract

We study the stability of attractors under non-autonomous perturbations that are uniformly small in time. While in general the pullback attractors for the non-autonomous problems converge towards the autonomous attractor only in the Hausdorff semi-distance (upper semicontinuity), the assumption that the autonomous attractor has a 'gradient-like' structure (the union of the unstable manifolds of a finite number of hyperbolic equilibria) implies convergence (i.e. also lower semicontinuity) provided that the local unstable manifolds perturb continuously.

We go further when the underlying autonomous system is itself gradient-like, and show that all trajectories converge to one of the hyperbolic trajectories as $t \to \infty$. In finite-dimensional systems, in which we can reverse time and apply similar arguments to deduce that all bounded orbits converge to a hyperbolic trajectory as $t \to -\infty$, this implies that the 'gradient-like' structure of the attractor is also preserved under small non-autonomous perturbations: the pullback attractor is given as the union of the unstable manifolds of a finite number of hyperbolic trajectories.

Introduction

In autonomous systems the theory of global attractors is well-developed for both ordinary and partial differential equations (e.g. Chepyzhov & Vishik, 2002; Hale, 1988; Stuart & Humphries, 1996; Temam, 1988). Nevertheless, there are only detailed results on the structure of such attractors for gradient-like systems: in this case the attractors are formed from the union of the unstable manifolds of the equilibrium points. While results on the upper semi-

continuity of attractors under perturbation (no 'explosion') hold for a wide class of equations, these gradient-like examples are the only systems for which full continuity results are available.

Here we show that similar results hold even when the perturbations are non-autonomous. The key assumption is that the local stable and unstable manifolds of hyperbolic equilibria perturb in a smooth way, and we present our main results in an abstract form which we believe serves to keep both the hypotheses and the arguments clearer than they would be in particular examples.

In gradient-like systems every trajectory tends to one of the equilibria. Ball & Peletier (1977) showed that a similar result holds for systems that are asymptotically autonomous, with a limit system that is gradient-like. Here we show a similar result for small non-autonomous perturbations of gradient like systems, namely that all solutions tend to distinguished hyperbolic trajectories corresponding to the equilibria of the unperturbed system. Ball & Peletier's result is then a corollary of ours.

In finite-dimensional systems one can reverse the sense of time. It follows in this case that every trajectory defined for all time also tends to one of these hyperbolic trajectories as $t \to -\infty$. In this situation we can show that the structure of the autonomous attractor is also preserved under small non-autonomous perturbations: the pullback attractor is the union of the unstable manifolds of the hyperbolic trajectories.

To end the paper we discuss the application of our results to finite and infinite-dimensional semilinear equations on Banach spaces, making use of recent results on the stability on local stable and unstable manifolds due to Carvalho & Langa (2006).

Standing assumptions

Throughout the paper we will assume that all of the conditions in this section are satisfied.

Let $\mathscr{B}$ be a Banach space with norm $\|\cdot\|$. Suppose that we have an underlying autonomous dynamical system $\{S_0(t)\}_{t \geq 0}$ defined on $\mathscr{B}$, where

$$\lim_{t \downarrow 0} S_0(t)x = S_0(0)x = x, \ x \in \mathscr{B} \qquad S_0(t+s) = S_0(t)S_0(s) \text{ for all } t, s \geq 0,$$

and for each $t \geq 0$ the operator $S_0(t)$ is continuous from $\mathscr{B}$ into $\mathscr{B}$. We assume that this system has a global attractor $\mathcal{A}_0$, i.e. a compact invariant set that

attracts the orbit of all bounded subsets X of $\mathscr{B}$,

$$\operatorname{dist}(S_0(t)X, \mathcal{A}_0) \to 0 \qquad \text{as} \qquad t \to \infty,$$

where

$$\operatorname{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|.$$

Our main object of study will be a family of non-autonomous dynamical systems on $\mathscr{B}$ with solution operators $S_\eta(t, s)$ satisfying

$$\lim_{t \downarrow s} S_\eta(t, s)x = S_\eta(s, s)x = x, \qquad x \in \mathscr{B}$$

and

$$S_\eta(t, s) = S_\eta(t, r)S_\eta(r, s) \qquad \text{for all} \qquad t \geq r \geq s,$$

that converge to $S_0(t)$ in the strongly uniform sense that

$$\sup_{s \in \mathbb{R}} \|S_\eta(t + s, s)u_0 - S_0(t)u_0\| \to 0 \qquad (1)$$

as $\eta \to 0$ uniformly for $t \in [0, T]$ and $u_0 \in X$ (with X any bounded subset of $\mathscr{B}$). We will write '$S_\eta \Rightarrow S_0$' as a shorthand for this convergence.

We assume that for each S_η with η small enough there exists a pullback attractor $\mathcal{A}_\eta(\cdot)$: this is a family of compact sets $\mathcal{A}_\eta(t)$ that is invariant in the sense that

$$S_\eta(t, s)\mathcal{A}_\eta(s) = \mathcal{A}_\eta(t) \qquad \text{for all} \qquad t \geq s,$$

and attracts all bounded sets *in the pullback sense*, i.e.

$$\operatorname{dist}(S_\eta(t, s)X, \mathcal{A}_\eta(t)) \to 0 \qquad \text{as} \qquad s \to -\infty.$$

See, for example, Chepyzhov & Vishik (2002), Crauel, Debussche, & Flandoli (1997), Schmalfuss (2000), Kloeden & Schmalfuss (1998).

Outline of results

Under the condition that the attractors $A_\eta(t)$ are uniformly bounded (in both η and t) we first prove (Theorem 1.1) that the attractor of $S_0(\cdot)$ is upper semicontinuous under non-autonomous perturbations, i.e. that

$$\sup_{t \in \mathbb{R}} \operatorname{dist}(\mathcal{A}_\eta(t), \mathcal{A}_0) \to 0 \qquad \text{as} \qquad \eta \to 0.$$

All of our remaining results are for perturbations of systems with what we

term *gradient-like attractors*, i.e. in which

$$\mathcal{A}_0 = \overline{\bigcup_{j=1}^{n} W^{\mathrm{u}}(e_j)},$$

where $W^{\mathrm{u}}(e_j)$ denotes the unstable manifold of a hyperbolic equilibrium point e_j.

We also make a key additional assumption that the local unstable manifolds of hyperbolic equilibria perturb continuously. In particular, the hyperbolic equilibria e_j of the autonomous system are replaced by bounded complete (i.e. defined for all $t \in \mathbb{R}$) trajectories $e_j^{\eta}(t)$, the linearisation around which enjoys an exponential dichotomoy.

Under these conditions we can also prove (Theorem 2.3) that the attractor is lower semicontinuous, and so

$$\sup_{t \in \mathbb{R}} \mathrm{dist}_{\mathrm{H}}(\mathcal{A}_\eta(t), \mathcal{A}_0) \to 0 \qquad \text{as} \qquad \eta \to 0,$$

where $\mathrm{dist}_{\mathrm{H}}$ is the Hausdorff metric,

$$\mathrm{dist}_{\mathrm{H}}(A, B) = \max(\mathrm{dist}(A, B), \mathrm{dist}(B, A)). \tag{2}$$

Next we consider in more detail the possible limits of trajectories in small non-autonomous perturbations of gradient systems (which, of course, have gradient-like attractors). For the underlying autonomous system it is known that all trajectories converge to an equilibrium as $t \to +\infty$, and that all bounded orbits defined for all $t \in \mathbb{R}$ (which in fact form the elements of the attractor) also converge to an equilibrium as $t \to -\infty$.

In Theorem 3.7 we obtain similar behaviour of the perturbed systems, showing that every trajectory is asymptotic as $t \to +\infty$ to one of the complete trajectories $e_j^{\eta}(t)$.

In finite-dimensional systems one can simply reverse the sense of time, and deduce that every bounded complete trajectory is also asymptotic to one of the $e_j^{\eta}(t)$ as $t \to -\infty$. It follows that in this case the pullback attractors for the perturbed systems have the same 'gradient-like' form as for the underlying autonomous equation, namely

$$\mathcal{A}_\eta(t) = \overline{\bigcup_{j=1}^{n} W^{\mathrm{u}}(e_j^{\eta})(t)}.$$

In the final section we show how our results apply to semilinear equations on Banach spaces, also discussing the case of equations that are asymptotically

autonomous with a gradient-like limit. Here we recover a result due to Ball & Peletier (1977), showing that every trajectory tends to one of the equilibria of the limit system.

While this paper leaves open the significant problem of proving a similar 'structure theorem' for infinite-dimensional systems, it does give the first examples of pullback attractors in non-autonomous equations that have a non-trivial but well-understood structure. A proof of the infinite-dimensional version of the structure theorem is given in Carvalho et al. (2006).

1 Upper semicontinuity

The argument showing upper semicontinuity of attractors is very simple, and follows that in the autonomous case (e.g. Theorem 10.16 in Robinson, 2001; Theorem I.1.2 in Temam, 1988). We note that the main assumption in the theorem, that the pullback attractors are uniformly bounded, is a strong one. There are, however, interesting non-autonomous systems for which the pullback attractor is unbounded as $|t| \to \infty$, see for example Langa, Robinson, & Suárez (2002). However, restricting to such uniformly bounded attractors does eliminate the possibility of certain pathologies (see the comment after Lemma 2.1).

Theorem 1.1 *Suppose that $S_\eta \Rightarrow S_0$ as $\eta \to 0$ (in the sense of (1)). Then the following two statements are equivalent: (i) there exists a bounded subset B of $\mathscr{B}$ such that*

$$\mathcal{A}_\eta(t) \subseteq B \qquad for\ all \qquad t \in \mathbb{R} \quad and\ all\ 0 \leq \eta \leq \eta_0 \tag{3}$$

and (ii) the attractor $\mathcal{A}_0$ perturbs 'upper semicontinuously':

$$\sup_{t \in \mathbb{R}} \operatorname{dist}(\mathcal{A}_\eta(t), \mathcal{A}_0) \to 0 \qquad as \qquad \eta \to 0. \tag{4}$$

In the proof we denote by $N(U, \epsilon)$ the ϵ neighbourhood of U, i.e.

$$N(U, \epsilon) = \{x \in \mathscr{B} : \operatorname{dist}(x, U) < \epsilon\}.$$

Proof. It is clear that (4) implies (3), so we concentrate on showing that (3) implies (4). Given an $\epsilon > 0$ there exists a $T > 0$ such that

$$S_0(T)B \subseteq N(\mathcal{A}_0, \epsilon/2).$$

Now choose η_0 such that

$$\sup_{s \in \mathbb{R}} \|S_\eta(T+s,s)u_0 - S_0(T)u_0\| < \epsilon/2.$$

Then given any $u \in \mathcal{A}_\eta(t)$ we have $u = S_\eta(t,t-T)v$, where $v \in \mathcal{A}_\eta(t-T)$. Since $v \in B$, it follows that

$$S_0(T)v \in N(\mathcal{A}_0, \epsilon/2) \qquad \text{and} \qquad \|S_\eta(t,t-T)v - S_0(T)v\| < \epsilon/2,$$

and so $u \in N(\mathcal{A}_0, \epsilon)$ and the result follows. $\square$

A related result, weakening the requirement that $A_\eta(t)$ is uniformly bounded in t and obtaining convergence uniformly for t in bounded subsets of $\mathbb{R}$ is given in Caraballo & Langa (2003); see also Cheban (2004).

2   Gradient-like attractors and lower semicontinuity

In order to proceed further, we consider the case of a gradient-like attractor: We assume that the attractor $\mathcal{A}_0$ is given as the closure of the union of the unstable manifolds of a finite number of hyperbolic stationary points $\{e_j\}_{j=1}^n$,

$$\mathcal{A}_0 = \overline{\bigcup_{j=1}^n W^{\mathrm{u}}(e_j)}.$$

Central to our argument is the persistence of hyperbolic fixed points and the continuity of their stable and unstable manifolds under small non-autonomous perturbations.

We say that $x(\cdot) : \mathbb{R} \to \mathscr{B}$ is a *complete trajectory* of $S_\eta(\cdot, \cdot)$ if

$$S_\eta(t,s)x(s) = x(t) \qquad \text{for all} \qquad t \geq s.$$

The unstable manifold of such a complete trajectory $x(t)$, $W^{\mathrm{u}}(x(\cdot))(t)$, is defined as

$$W^{\mathrm{u}}(x(\cdot))(s) = \{v \in \mathscr{B} : \ S_\eta(t,s)v \text{ is defined for all } t \leq s$$
$$\text{and } \|S_\eta(t,s)v - x(t)\| \to 0 \text{ as } t \to -\infty\}.$$

We will require the following simple result, guaranteeing that the unstable manifold of any complete trajectory that is bounded as $t \to -\infty$ must be contained in the pullback attractor.

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Lemma 2.1 *Suppose that $x(t)$ is a complete trajectory of S_η, such that for some $M > 0$, $\|x(t)\| \le M$ for all $t \le 0$. Then $W^u(x(\cdot))(t) \in \mathcal{A}_\eta(t)$ for all $t \in \mathbb{R}$.*

Proof.   Take $v \in W^u(x(\cdot))(s)$. Then by definition we have

$$\|S_\eta(t,s)v - x(t)\| \le M \qquad \text{for all} \qquad t \le t^*(s,v)$$

and hence $\|S_\eta(t,s)v\| \le 2M$. It follows that for every $t \le t^*$, $S_\eta(t,s)v$ is contained in the fixed bounded set

$$B = \{x : \|x\| \le 2M\}.$$

Since

$$\text{dist}(S(s,t)B, \mathcal{A}_\eta(s)) \to 0 \qquad \text{as} \qquad t \to -\infty,$$

it follows that $v \in \mathcal{A}_\eta(s)$ as claimed. $\square$

We note that the counterexample in Section 5 of Langa et al. (2002) shows that the requirement that $x(t)$ is bounded in the past is necessary.

2.1   Lower semicontinuity

We now show that gradient-like attractors are lower semicontinuous under non-autonomous perturbations. The argument is based on the autonomous proof of Humphries (see Stuart & Humphries, 1996; and see also Hale & Raugel, 1989), for which the main additional ingredient is an assumption on the behaviour of the local unstable manifolds of the original equation under perturbation.

Since this assumption is key to all that follows, we give it a formal status. We use $B(x, \delta)$ to denote the open ball in $\mathscr{B}$ of radius δ centred at x.

Definition 2.2 *Let the standing assumptions hold. If e is an equilibrium point of S_0 we say that* the manifold structure near e is stable under perturbation *if there exists a $\delta > 0$ such that for any ϵ with $0 < \epsilon < \delta$ there exists an η_0 such that for all $0 < \eta < \eta_0$: (i) there is a complete trajectory $e^\eta(\cdot)$ of S_η with*

$$\|e^\eta(t) - e\| < \epsilon \qquad \text{for all} \qquad t \in \mathbb{R},$$

and this is the unique complete bounded trajectory lying entirely within $B(e, \delta)$, (ii) the local unstable manifold of e perturbs continuously:

$$\text{dist}_H(W^u(e) \cap B(e,\delta), W^u(e^\eta(\cdot))(t) \cap B(e,\delta)) < \epsilon \qquad \text{for all} \qquad t \in \mathbb{R},$$

and (iii) if for some t_* we have

$$S_\eta(t, s)u_s \in B(e, \delta) \tag{5}$$

for all $t \leq t^*$ then

$$\|S_\eta(t, s)u_s - e^\eta(t)\| \to 0 \qquad as \qquad t \to -\infty,$$

while if (5) holds for all $t \geq t^*$ then

$$\|S_\eta(t, s)u_s - e^\eta(t)\| \to 0 \qquad as \qquad t \to +\infty.$$

We are now in a position to prove a general lower semicontinuity result.

Theorem 2.3 *Suppose that $\mathcal{A}_0$ is gradient-like and $S_\eta \Rightarrow S_0$. Assume further that the manifold structure near each equilibrium e_j, $j = 1, \ldots, n$, is stable under perturbation. Then*

$$\sup_{t \in \mathbb{R}} \operatorname{dist}(\mathcal{A}_0, \mathcal{A}_\eta(t)) \to 0 \qquad as \qquad \eta \to 0.$$

Note that in the proof we in fact only use parts (i) and (ii) of the regularity assumption.

Proof.   We have to show that there exists an η_0 such that

$$\sup_{t \in \mathbb{R}} \operatorname{dist}(\mathcal{A}_0, \mathcal{A}_\eta(t)) < \epsilon \qquad \text{for all} \qquad \eta < \eta_0. \tag{6}$$

Since $\mathcal{A}_0$ is compact we can find a finite set of points $\{x_j\}_{j=1}^M \in \mathcal{A}_0$ such that

$$\mathcal{A}_0 \subset \bigcup_{j=1}^M B(x_j, \epsilon/4).$$

To prove (6) it suffices to show that for each $j = 1, \ldots, M$, every $\eta < \eta_0$ and every $t \in \mathbb{R}$ we can find a point $y_j(t) \in \mathcal{A}_\eta(t)$ such that

$$\|x_j - y_j(t)\| < 3\epsilon/4. \tag{7}$$

Since $x_j \in \mathcal{A}_0$ and

$$\mathcal{A}_0 = \overline{\bigcup_{k=1}^n W^{\mathrm{u}}(e_k)},$$

there exists a point $z_j \in W^{\mathrm{u}}(e_{k_j})$ (for some integer k_j with $1 \leq k_j \leq n$) such that $\|x_j - z_j\| \leq \epsilon/4$. Since $z_j \in W^{\mathrm{u}}(e_{k_j})$, there exist $t_j > 0$ and $\zeta_j \in W^{\mathrm{u}}(e_{k_j}) \cap B(e_{k_j}, \rho/2)$ such that $z_j = S_0(t_j)\zeta_j$. Since there are only a finite number of the

x_j there exists a fixed time T such that $t_j \leq T$ for all $j = 1, \ldots, M$. Note that the choices of ζ_j, t_j, and T depend only on the autonomous system, i.e. are independent of t and η.

Since S_0 is continuous and $\mathcal{A}_0$ is compact there exists a $\mu > 0$ such that

$$\sup_{t \in [0,T]} \|S_0(t)z - S_0(t)u\| < \epsilon/4$$

for all $u \in \mathcal{A}_0$ and all z with $\|z - u\| < \mu$.

By assumption the local unstable manifolds near the hyperbolic stationary points $\{e_j\}$ of $\mathcal{A}_0$ perturb continuously. Let $\delta_j > 0$ be the δ occurring in Definition 2.2 applied near e_j, and let $\delta = \min_j \delta_j$. For simplicity of notation we write e_j^η for the time-dependent complete trajectory, and set

$$W_{\mathrm{loc}}^{\mathrm{u}}(e_j^\eta) = W^{\mathrm{u}}(e_j^\eta(\cdot)) \cap B(e_j, \delta)$$

(and similarly for $e_j^0 = e_j$). Then there exists an $\eta_1 > 0$ such that for each j and for every $\eta < \eta_1$ there exists a complete trajectory $e_j^\eta(\cdot)$ such that $W_{\mathrm{loc}}^{\mathrm{u}}(e_j^\eta)$ lies within μ of $W_{\mathrm{loc}}^{\mathrm{u}}(e_j)$.

It follows that for each $t \in \mathbb{R}$ and for each $\eta < \eta_1$ there exists a $\zeta_j^\eta(t) \in W_{\mathrm{loc}}^{\mathrm{u}}(e_k^\eta)$ with

$$\|\zeta_j^\eta(t) - \zeta_j\| < \mu.$$

We also know that $S_\eta \Rightarrow S_0$; since $\mathcal{A}_0$ is bounded so is $N(\mathcal{A}_0, \mu)$, and thus there exists an $\eta_2 > 0$ such that

$$\sup_{u_0 \in N(\mathcal{A}_0, \mu)} \sup_{0 \leq t - s \leq T} \|S_\eta(t, s)u_0 - S_0(t - s)u_0\| < \epsilon/4$$

for all $\eta < \eta_2$. Set $\eta_0 = \min(\eta_1, \eta_2)$.

Our candidate point in $\mathcal{A}_\eta(t)$ close to x_j is $y_j(t) = S_\eta(t, t - t_j)\zeta_j(t - t_j)$. This is contained in $\mathcal{A}_\eta(t)$ since $\zeta_j(t - t_j) \in W^{\mathrm{u}}(e_j^\eta)(t - t_j) \subset \mathcal{A}_\eta(t - t_j)$ and $\mathcal{A}_\eta(\cdot)$ is positively invariant.

Since $t_j \in [0, T]$ it follows for every j that for $\eta < \eta_0$ we have

$$\begin{aligned}
\|y_j(t) - z_j\| &= \|S_\eta(t, t - t_j)\zeta_j(t - t_j) - S_0(t_j)\zeta_j\| \\
&\leq \|S_\eta(t, t - t_j)\zeta_j(t - t_j) - S_0(t_j)\zeta_j(t - t_j)\| \\
&\quad + \|S_0(t_j)\zeta_j(t - t_j) - S_0(t_j)\zeta_j\| \\
&< \epsilon/4 + \epsilon/4 = \epsilon/2,
\end{aligned}$$

and since $\|x_j - z_j\| < \epsilon/4$ we obtain (7). $\square$

Combining the previous two results we obtain continuity of gradient-like attractors under perturbation:

Corollary 2.4 *Suppose that $\mathcal{A}_0$ is gradient-like and that $S_\eta \Rightarrow S_0$. Assume further that there exists a bounded subset B of $\mathscr{B}$ such that*

$$\mathcal{A}_\eta(t) \subseteq B \qquad \text{for all} \qquad t \in \mathbb{R} \quad \text{and all } 0 \leq \eta \leq \eta_0$$

and that the unstable manifolds near the stationary points $\{e_j\}$ are stable under perturbation. Then

$$\sup_{t \in \mathbb{R}} \text{dist}_{\text{H}}(\mathcal{A}_\eta(t), \mathcal{A}_0) \to 0 \qquad \text{as} \qquad \eta \to 0.$$

3 Asymptotic behaviour of individual trajectories and the structure of the non-autonomous attractor

We now show that for sufficiently small perturbations of gradient-like systems every trajectory is asymptotic (as $t \to \infty$) to a complete trajectory that corresponds to an equilibrium point of the unperturbed autonomous system. We also show that in finite-dimensional systems a similar result holds for complete bounded trajectories as $t \to -\infty$. This latter result is perhaps more significant, since it enables us to show that the perturbed attractors still have a 'gradient-like' structure, at least in the finite-dimensional case.

We say that $S(\cdot)$ is *gradient-like* (cf. Hale, 1988) if there exists a continuous Lyapunov function $V : \mathscr{B} \to \mathbb{R}$ such that

  (i) V is bounded below and $V(u) \to \infty$ as $\|u\| \to \infty$,
 (ii) $V(S(t)u_0) \leq V(u_0)$ for all $t \geq 0$, and
(iii) if $V(S(t)u_0)$ is constant for all $t \geq 0$ then u_0 is an equilibrium.

A semigroup $S(\cdot)$ is said to be *asymptotically compact* (cf. Ladyzhenskaya, 1991; and the 'asymptotically smooth' systems of Hale, LaSalle, & Slemrod 1972) if given any bounded subset X of $\mathscr{B}$ and sequences $x_k \in X$ and $t_k \to \infty$ there exists a subsequence
$$S(t_{k_j})x_{k_j}$$
that converges. Any finite-dimensional semigroup with a bounded absorbing set is asymptotically compact; in an infinite-dimensional system this is much weaker than the existence of a compact absorbing set.

3.1 Properties of gradient-like autonomous systems

In the proof of our result on omega (and alpha) limit sets we will need some more detailed properties of the dynamics of gradient-like autonomous systems. In all the results that follow we assume that

- $S(\cdot)$ is asymptotically compact;
- $S(\cdot)$ is gradient-like (as above);
- there are only a finite number of equilibria $\{e_j\}$; and
- all of the equilibria are hyperbolic.

We begin with the following theorem which, given our terminology, is unsurprising. For a proof see Theorem 3.8.5 in Hale (1988).

Theorem 3.1 *The attractor $\mathcal{A}_0$ of a gradient-like system $S(\cdot)$ is gradient-like.*

We quote the following standard result, guaranteeing that forwards and backwards in time all trajectories are asymptotic to equilibria (for a proof see Lemmas 3.1.2 and 3.8.2 in Hale, 1988).

Lemma 3.2 *Given any $u_0 \in \mathscr{B}$ we have*

$$S(t)u_0 \to e_j$$

as $t \to +\infty$ for some j. If in fact $u_0 \in \mathcal{A}_0$ and is not an equilibrium point then we also have

$$S(t)u_0 \to e_k$$

as $t \to -\infty$, for some $k \neq j$.

We now show that all trajectories enter a neighbourhood of one of the equilibria in a uniform time.

Lemma 3.3 *Given any bounded set B and any $\delta > 0$ there exists a time $T_{B,\delta}$ such that if $u_0 \in B$ then for some $0 \leq t \leq T_{B,\delta}$ and some k we have $S(t)u_0 \in B(e_k, \delta)$.*

In the proof we write $N_\delta(E)$ for $\cup_j B(e_j, \delta)$ (the δ-neighbourhood of the equilibria).

Proof. Suppose that the result is not true. Then there must exist a sequence $u_n \in B$ and $t_n \to \infty$ such that $S(t)u_n \notin N_\delta(E)$ for all $t \leq t_n$.

Since $S(\cdot)$ is asymptotically compact, it follows that $S(t_n/2)u_n \to u^*$. However, it cannot be the case that $S(T)u^* \in N_{\delta/2}(E)$ for any $T > 0$: for n large enough $T < t_n/2$ and one can use continuous dependence on initial conditions to ensure that
$$\|S(T)u^* - S(T)[S(t_n/2)u_n]\| < \delta/2.$$
By assumption $\|S(T+t_n/2)u_n - e_j\| > \delta$ for each j, and so $\|S(T)u^* - e_j\| > \delta/2$ for each j, contradicting Lemma 3.2. $\square$

The next result (whose proof follows that of Lemma 3.8.4 in Hale (1988) very closely) shows (essentially) that if a trajectory moves *out of* a neighbourhood of one of the equilibria then it can never return.

Lemma 3.4 *For each equilibrium e_j there exist ρ_j and σ_j with $0 < \rho_j < \sigma_j$ such that if for some $t_0 > 0$*

$$u_0 \in B(e_j, \rho_j) \qquad and \qquad S(t_0)u_0 \notin B(e_j, \sigma_j)$$

then $S(t)u_0 \notin B(e_j, \rho_j)$ for all $t \geq t_0$.

Proof. Choose $\sigma_j > 0$ such that $S(t)u_0 \in B(e_j, \sigma_j)$ for all $t \leq 0$ implies that $u_0 \in W^{\mathrm{u}}_{\mathrm{loc}}(e_j)$. There exist $K, \alpha > 0$ such that

$$\mathrm{dist}(S(t)u_0, W^{\mathrm{u}}_{\mathrm{loc}}(e_j)) \leq K\mathrm{e}^{-\alpha t} \tag{8}$$

while $S(t)u_0 \in B(e_j, \sigma_j)$.

For any δ with $0 < \delta < \sigma_j$ there exists a $t_2(\delta)$ such that

$$S(t)B(e_j, \delta) \subseteq B(e_j, \sigma_j) \qquad \text{for all} \qquad 0 \leq t \leq t_2.$$

Define
$$W_\eta = \{x : \ \delta \leq \|x - e_j\| \leq \sigma_j, \ \mathrm{dist}(x, W^{\mathrm{u}}_{\mathrm{loc}}(e_j)) < \eta\},$$
and choose η such that
$$\sup_{x \in W_\eta} V(x) < V(e_j).$$

Now choose t_1 such that $K\mathrm{e}^{-\alpha t_1} < \eta$, and choose ρ_j small enough that

$$S(t)B(e_j, \rho_j) \subseteq B(e_j, \delta) \qquad \text{for all} \qquad 0 \leq t \leq t_1 \tag{9}$$

and

$$\sup_{x \in W_\eta} V(x) < \inf_{y \in B(e_j, \rho_j)} V(y). \tag{10}$$

Now suppose that $u_0 \in B(e_j, \rho_j)$ but $S(t_0)u_0 \notin B(e_j, \sigma_j)$. Then there must exist a $t_0^* \leq t_0$ and an $\epsilon > 0$ such that

$$\|S(t)u_0 - e_j\| \leq \sigma_j \qquad \text{for all} \qquad 0 \leq t \leq t_0^*$$

and

$$\|S(t)u_0 - e_j\| > \sigma_j \qquad \text{for all} \qquad t_0^* < t < t_0^* + \epsilon.$$

In particular it follows from (9) that $t_0^* > t_1$, and so, using (8), for some t_3 with $t_1 < t_3 < t_0^*$ we have $S(t_3)u_0 \in W_\eta$. It follows from (10) and the fact that V is non-increasing that we must have

$$V(S(t)u_0) < \inf_{y \in B(e_j, \rho_j)} V(y)$$

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for all $t \geq t_3$, and in particular for all $t \geq t_0$. Therefore $S(t)u_0 \notin B(e_j, \rho_j)$ for all $t \geq t_0$. $\square$

The following corollary is closer to the statement in Hale (1988), but contains an additional observation that will be important in the proof of Lemma 3.6.

Corollary 3.5 *If $u_0 \in B(e_j, \rho_j) \setminus W_{\mathrm{loc}}^{\mathrm{s}}(e_j)$ and $u_n \to u_0$ then there is a time t_0 and an n_0 such that for all $t \geq t_0$*

$$S(t)u_0 \notin B(e_j, \rho_j) \qquad and \qquad S(t)u_n \notin B(e_j, \rho_j) \quad for \ all \quad n \geq n_0.$$

Proof. Note that in the proof of Lemma 3.4 one can decrease σ_j if necessary so that $S(t)u_0 \in B(e_j, \sigma_j)$ for all $t \geq 0$ implies that $u_0 \in W_{\mathrm{loc}}^{\mathrm{s}}(e_j)$. It then follows that if $u_0 \in B(e_j, \rho_j) \setminus W_{\mathrm{loc}}^{\mathrm{s}}(e_j)$ then there must exist a t_0 such that $S(t)u_0 \notin B(e_j, \sigma_j)$, from which it is immediate using Lemma 3.4 that $S(t)u_0 \notin B(e_j, \sigma_j)$ for all $t \geq t_0$, while the result for the sequence u_n follows since continuous dependence on initial conditions implies that $S(t)u_n \notin B(e_j, \sigma_j)$ for all n sufficiently large, and one can then apply Lemma 3.4 once more. $\square$

Finally, we show that if a trajectory passes from a small neighbourhood of one equilibrium e_j to a small neighbourhood of another (e_k) then this is in fact sufficient to imply that there is a heteroclinic orbit (or perhaps a chain of heteroclinic orbits) from e_j to e_k, and so in particular $V(e_k) < V(e_j)$. This is a key fact in the proof of the main result.

Lemma 3.6 *There exists a $\gamma > 0$ such that if for $k \neq j$*

$$u_0 \in B(e_j, \gamma) \quad and \quad S(t_0)u_0 \in B(e_k, \gamma)$$

for some $t_0 > 0$, then there exists a chain of heteroclinic orbits between equilibria joining e_j to e_k. In particular $V(e_k) < V(e_j)$.

Proof. If for some $\gamma > 0$ there are no trajectories joining $B(e_j, \gamma)$ to $B(e_k, \gamma)$ then the result claimed in the statement is not violated. So we can assume that there exists a sequence of trajectories $u_n(\cdot)$ and times t_n such that

$$\|u_n(0) - e_j\| \leq \frac{1}{n} \qquad and \qquad \|u_n(t_n) - e_k\| \leq \frac{1}{n}, \tag{11}$$

with $t_n > 0$. We show that there must therefore exist a trajectory that is heteroclinic between e_j and e_k, and hence that $V(e_k) < V(e_j)$.

First note that we can assume in addition to (11) that for some $\eta > 0$

$$\|u_n(t) - e_i\| \geq \eta \qquad \text{for all} \qquad t \in [0, t_n], \ i \neq j, k,$$

since otherwise we could find a finite chain of equilibria $\{e_{j_n}\}$ and trajectories $u_n(\cdot)$ that move between successive $1/n$ neighbourhoods of the e_{j_n}, avoiding

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the η neighbourhoods of all other equilibria. In this case we would apply the following argument to each transition from e_{j_n} to $e_{j_{n+1}}$.

Choose $\beta > 0$ such that $\{B(e_j, 2\beta)\}_{j=1\dots n}$ are disjoint. Noting that one can decrease σ_j in the proof of Lemma 3.4 if required, choose $\sigma_j < \beta$ such that

$$S(t)u_0 \in B(e_j, \sigma_j) \text{ for all } t \leq 0 \quad \Rightarrow \quad u_0 \in W^{\mathrm{u}}_{\mathrm{loc}}(e_j),$$

there are constants $K, \alpha > 0$ such that

$$\mathrm{dist}(S(t)u_0, W^{\mathrm{u}}_{\mathrm{loc}}(e_j)) \leq K\mathrm{e}^{-\alpha t} \tag{12}$$

as long as $S(t)u_0$ remains inside $B(e_j, \sigma_j)$, and

$$S(t)u_0 \in B(e_j, \sigma_j) \text{ for all } t \geq 0 \quad \Rightarrow \quad u_0 \in W^{\mathrm{s}}_{\mathrm{loc}}(e_j)$$

(which ensures that Corollary 3.5 also holds). Choose $\sigma_k < \beta$ similarly, so that Lemma 3.4 and Corollary 3.5 are valid near e_k, and let ρ_k be the corresponding radius of the inner ball.

Now consider t_n such that $\|u_n(t_n) - e_j\| = \sigma_j$ and $\|u_n(t) - e_j\| < \sigma_j$ for all $t \leq t_n$. Then the sequence $\{t_n\}$ cannot be bounded (by T say), for otherwise one could deduce the existence of a time $t \leq T$ such that $\|S(T)e_j - e_j\| = \sigma_j$, which contradicts the fact that e_j is an equilibrium. So there exists a subsequence (which we relabel) such that $t_n \to \infty$, for which $u_n(t) \in B(e_j, \sigma_j)$ for all $0 \leq t \leq t_n$.

Since $S(\cdot)$ is asymptotically compact, there exists a subsequence n_j such that $u_{n_j}(t_{n_j})$ converges to some u^* with $\|u^* - e_j\| = \sigma_j$. Using (12) it follows that $u^* \in W^{\mathrm{u}}_{\mathrm{loc}}(e_j)$.

Now relabel and consider again the sequence u_n that gives rise to $u^* \in W^{\mathrm{u}}_{\mathrm{loc}}(e_j)$ via $u^* = \lim_{n \to \infty} S(t_n)u_n$. Since $W^{\mathrm{u}}_{\mathrm{loc}}(e_j) \cap W^{\mathrm{s}}_{\mathrm{loc}}(e_j) = \{e_j\}$, Lemma 3.4 implies that there is an n_0 and a uniform time t_1 such that $S(t)u_n(t_n) \notin B(e_j, \sigma_j)$ for all $t \geq t_1$, $n \geq n_0$.

It follows from Lemma 3.3 that $\|S(\tau_n)u_n(t_n) - e_k\| = \rho_k/2$ for some $\tau_n \leq T$, where T does not depend on n. Since trajectories converge uniformly on compact time intervals, there is a subsequence such that $\tau_n \to \tau$ with $\tau \leq T$ and $S(\tau_n)u_n(t_n) \to S(\tau)u^*$ with $\|S(\tau)u^* - e_k\| = \rho_k/2$.

Now, suppose that $v^* := S(\tau)u^* \notin W^{\mathrm{s}}_{\mathrm{loc}}(e_k)$. Then it follows from Corollary 3.5 that there exists some time t_0 such that $S(t)v^* \notin U_k$ for all $t \geq t_0$, and $S(t + \tau_n)u_n(t_n) \notin U_k$ for all $n \geq n_0$ and $t \geq t_0$.

But this is a contradiction, since the trajectories passing through $S(\tau_n)u_n(t_n)$ approach e_k arbitrarily closely. If these times of closest approach are bounded

one can easily find a subsequence joining v^* to e_k in a finite time; so these times must be unbounded, but this contradicts the uniformity of t_0 over n.

It follows that there exists a heteroclinic orbit joining e_j to e_k as claimed, and so $V(e_k) < V(e_j)$. $\square$

3.2 Limit sets in the non-autonomous system

We are now in a position to prove the main theorem of this section on the asymptotic behaviour of trajectories forwards in time.

Theorem 3.7 *Suppose that $S_0(\cdot)$ is gradient-like, that $\mathcal{A}_\eta$ is contained in some fixed bounded set B for all $0 \leq \eta < \eta^*$, that $S_\eta \Rightarrow S_0$, and that the manifold structure near the fixed points perturbs continuously. Then for η sufficiently small, for any initial condition $u_0 \in \mathscr{B}$ and any $s \in \mathbb{R}$ we have*

$$\lim_{t \to \infty} \|S_\eta(t,s)u_0 - e_j^\eta(t)\| = 0,$$

where $e_j^\eta(\cdot)$ is the unique complete trajectory in a neighbourhood of e_j.

Proof. The idea of the proof is simple, but the details are a little messy. Essentially there are two steps: first, since non-autonomous trajectories follow autonomous trajectories, every trajectory must end in a neighbourhood of a stationary point (Lemma 3.2). Then, if a trajectory leaves a neighbourhood of a stationary point it must follow an autonomous trajectory that leaves the neighbourhood of the same stationary point and hence moves to a different stationary point (Lemma 3.4). Since the Lyapunov function decreases along orbits of the autonomous system orbits can only move from one such neighbourhood to another a finite number of times (Lemma 3.6).

Noting that since $\mathcal{A}_\eta(t) \subseteq B$ for all t, it follows (enlarging B if necessary) that there is a t_0 such that for $t \geq t_0$ we have

$$S_\eta(t,s)u_0 \in B.$$

Since $S_\eta(t,s)u_0 = S_\eta(t,t_0)[S_\eta(t_0,s)u_0]$, we can assume without loss of generality that $u_0 \in B$.

Let $\delta = \min_{j=1,\dots,n} \delta_j$, where the δ_js are those from Definition 2.2 (the continuous perturbation of the manifold structure). Fix $\sigma < \min(\delta/2, \gamma/2)$, where γ is as in the statement of Lemma 3.6. Note that one can choose $\sigma_j = \sigma$ in Lemma 3.4 (independent of j), giving rise to a corresponding set of ρ_j (which may vary from one equilibrium to another) – set $\rho = \min(\sigma/4, \rho_1, \dots, \rho_j)$.

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We therefore know that if an autonomous trajectory moves from within $B(e_j, \rho)$ to outside $B(e_j, \sigma)$, it can never enter $B(e_j, \rho)$ again, whatever the value of j.

Now using Lemma 3.3 find a T^* such that if $u_0 \in B$ then for some $0 \le t \le T^*$ and some k we have

$$S(t)u_0 \in B(e_j, \rho/4) \qquad \text{for some } j \in \{1, \ldots, n\}, \ 0 \le t \le T^*, \qquad (13)$$

and choose η_0 such that for every $\eta < \eta_0$

$$\|S_0(t)u_0 - S_\eta(t+s, s)u_0\| < \rho/4 \qquad \text{for all} \qquad t \in [0, T^*], \ u_0 \in B. \qquad (14)$$

Now take $u_0 \in B$. Using the definition of σ and (iii) from Definition 2.2, we know that if for some j and t_* we have

$$S_\eta(t, s)u_0 \in B(e_j, 2\sigma) \qquad \text{for all} \qquad t \ge t_* \qquad (15)$$

then

$$\|S_\eta(t, s)u_0 - e_j^\eta(t)\| \to 0 \qquad \text{as} \qquad t \to \infty. \qquad (16)$$

We show that (15) holds for all $u_0 \in B$. Given such a u_0, combining (13) and (14) it follows that for some $0 \le t_0 \le T^*$ we have $S_\eta(t_0 + s, s)u_0 \in B(e_j, \rho/2)$ for some j. Either $S_\eta(t + s, s)u_0 \in B(e_j, 2\sigma)$ for all $t \ge t_0$, in which case we are done (using (16)); or the trajectory leaves $B(e_j, 2\sigma)$.

If the non-autonomous trajectory $S_\eta(\cdot + s, s)u_0$ moves from the interior of $B(e_j, \rho/2)$ to the exterior of $B(e_j, 2\sigma)$ we argue as follows. First, set

$$t_2 = \sup\{t_* > t_0 : \ S_\eta(t+s, s)u_s \in B(e_j, 2\sigma) \text{ for all } t \in [t_0, t_*]\},$$

and then

$$t_1 = \inf\{t_* < t_2 : \ S_\eta(t+s, s)u_s \notin B(e_j, 3\rho/4) \text{ for all } t \in [t_*, t_2]\}.$$

We know that for $t \in [t_1, t_1 + T^*]$ we have

$$\|S_\eta(t+s, s)u_s - u(t)\| < \rho/4,$$

where $u(\cdot)$ is a trajectory of the autonomous system which therefore satisfies

- $u(t) \notin B(e_j, \rho/2)$ for all $t \in [t_1, t_2]$;
- $u(t_1) \in B(e_j, \rho)$;
- $u(t_2) \notin B(e_j, \sigma)$.

Since $u(t_1) \notin B(e_j, \sigma)$, it follows that $u(t_1 + t) \in B(e_k, \rho/4)$ for some e_k for some $0 < t \le T^*$. However, the trajectory $u(t_1 + \cdot)$ moves from within $B(e_j, \rho)$

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to the complement of the σ neighbourhood of e_j, and hence can never reenter $B(e_j, \rho)$. It follows that $k \neq j$, i.e. within a time T^* the autonomous trajectory enters $B(e_k, \rho/4)$ with $k \neq j$. It follows that the non-autonomous trajectory must enter $B(e_k, \rho/2)$.

Now, by our choice of $\rho < \gamma$ it follows from Lemma 3.6 that $V(e_k) < V(e_j)$. This process cannot continue indefinitely since there are only a finite number of equilibrium points, and so eventually (15) holds, and it follows that

$$\|S_\eta(t,s)u_0 - e_j(t)\| \to 0 \qquad \text{as} \qquad t \to \infty.$$

$\square$

4   Gradient semilinear equations in Banach spaces

We now consider gradient semilinear equations on Banach spaces, following Carvalho & Langa (2006) whose recent work provides the continuity of stable and unstable manifolds under perturbation that we require for the application of our results.

Given a Banach space $\mathscr{B}$, let $A : D(A) \subset \mathscr{B} \to \mathscr{B}$ be the generator of a C^0-semigroup of bounded linear operators and f_0 a differentiable function that is Lipschitz continuous in bounded subsets of $\mathscr{B}$.

We take as our underlying autonomous system the equation

$$\dot{y} = Ay + f_0(y) \qquad y(s) = y_0 \in \mathscr{B}, \tag{17}$$

and consider in addition the family of non-autonomous problems

$$\dot{y} = Ay + f_\eta(t, y) \qquad y(s) = y_0 \in \mathscr{B}, \tag{18}$$

where f_η is a differentiable function that is Lipschitz continuous in bounded subsets of $\mathscr{B}$ with Lipschitz constant independent of η and t.

Assume that, for each $\tau \in \mathbb{R}$ and $y_0 \in \mathscr{B}$, unique solutions of (17) and (18) exist for all $t \geq s$.

If the family of non-autonomous terms f_η converge to f_0 in the sense that

$$\lim_{\eta \to 0} \sup_{t \in \mathbb{R}} \sup_{z \in B(0,r)} \|f_\eta(t,z) - f_0(z)\| = 0, \quad \text{for each } r > 0. \tag{19}$$

then, for each $r > 0$ and $T > 0$, it is relatively straightforward to show that

$$\sup\{\|T_\eta(t+\tau, \tau)z - T_0(t)z\|, \tau \in \mathbb{R}, \ t \in [0,T] \text{ and } \|z\| \leq r\} \to 0, \tag{20}$$

as $\eta \to 0$, i.e. that $T_\eta \Rightarrow T_0$ as $\eta \to 0$ in the sense of (1).

Carvalho & Langa (2006) showed recently that if we assume in addition that for any $r > 0$ that the derivatives of f_η converge to those of f,

$$\limsup_{\eta \to 0} \sup_{t \in \mathbb{R}} \sup_{y \in B(0,r)} \{\|f_\eta(t,y) - f_0(y)\| + \|(f_\eta)_y(t,y) - f_0'(y)\|\} = 0, \qquad (21)$$

then the invariant manifold structure is stable under non-autonomous perturbations near hyperbolic fixed points.

The continuity of gradient-like attractors follows (Corollary 2.4), and all trajectories in small non-autonomous perturbations of the underlying equation are forwards asymptotic to one of the hyperbolic trajectories (Theorem 3.7).

4.1 Structure Theorem in finite dimensions

Provided that one can reverse the sense of time, a very similar argument to that used to prove Theorem 3.7 shows convergence of bounded trajectories to one of the the convergence of bounded trajectories to one of the $e_j^\eta(t)$ as $t \to -\infty$, and provide a characterisation of the structure of the attractors in such systems.

However, to do this in general requires the phase space to be finite-dimensional. We therefore state our 'Structure Theorem' for non-autonomous perturbations of gradient ODEs.

Theorem 4.1 ('Structure Theorem for gradient ODEs') *Suppose that $V : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function such that*

  (i) $V(x) \to \infty$ as $|x| \to \infty$;
 (ii) V has a finite number of critical points $\{e_j\}_{j=1}^n$, and at each critical point D^2V is of full rank.

Then the attractor of
$$\dot{x} = -\nabla V(x) \qquad x \in \mathbb{R}^n$$
is gradient-like: $\mathcal{A}_0 = \cup_{j=1}^n W^{\mathrm{u}}(e_j)$.

If $g(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1 in x, C^0 in t, and uniformly bounded on sets of the form $\mathbb{R} \times K$ where $K \subset \mathbb{R}^n$ is compact, then for ϵ sufficiently small every bounded complete trajectory of

$$\dot{x} = -\nabla V(x) + \epsilon g(t,x)$$

is backwards asymptotic to one of the $e_j^\eta(\cdot)$, i.e.

$$\lim_{t\to-\infty} \|u(t) - e_j^\eta(t)\| = 0. \tag{22}$$

In particular it follows that the pullback attractor for the non-autonomous system has the same structure as the underlying autonomous attractor:

$$\mathcal{A}_\eta(t) = \overline{\bigcup_{j=1}^{n} W^{\mathrm{u}}(e_j^\eta)(t)}. \tag{23}$$

Proof. The argument proceeds exactly as in the proof of Theorem 3.7, except that one should consider always the time-reversed flow $S(-t)$. In order to ensure that all autonomous trajectories exist backwards in time, they must be taken within the attractor $\mathcal{A}_0$; but this can be done due to the lower semicontinuity result of Theorem 2.3. From (22) the characterisation (23) is immediate. $\square$

A proof of the structure theorem valid in the infinite-dimensional case this case will be given in Carvalho et al. (2006).

4.2 Asymptotically autonomous systems

As a further (infinite-dimensional) application we now consider the asymptotically autonomous equation

$$\dot{y} = Ay + f(t, y) \qquad y(s) = y_0 \in \mathscr{B}, \tag{24}$$

i.e. when there exists an f_0 such that

$$\lim_{t\to\infty} \sup_{B(0,r)} \|f(t, y) - f_0(y)\| + \|f_y(t, y) - f_0'(y)\| = 0. \tag{25}$$

We write $S(\cdot, \cdot)$ for the process generated by solutions of (24).

In order to apply our previous results, we consider equation (24) with $f(t, y)$ replaced by

$$f_\tau(t, y) = \begin{cases} f(\tau, y) & t \le \tau \\ f(t, y) & t > \tau, \end{cases}$$

i.e.

$$\dot{y} = Ay + f_\tau(t, y) \qquad y(s) = y_0 \in \mathscr{B}. \tag{26}$$

We compare the solutions of this equation, and its corresponding process $S_\tau(\cdot, \cdot)$, to those of

$$\dot{y} = Ay + f_0(y) \qquad y(s) = y_0 \in \mathscr{B} \tag{27}$$

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and its semigroup $T(\cdot)$.

Note that in this case a version of (21) holds, namely

$$\lim_{\tau \to \infty} \sup_{t \in \mathbb{R}} \sup_{y \in B(0,r)} \{\|f_\tau(t,y) - f_0(y)\| + \|(f_\eta)_y(t,y) - f_0'(y)\|\} = 0.$$

This allows us to obtain the following result as a corollary of our previous Theorem 3.7.

Proposition 4.2 *Suppose that (25) holds and that the semigroup $T(\cdot)$ is gradient-like with a finite set of hyperbolic equilibria $\{e_j\}_{j=1}^n$. Then for every $s \in \mathbb{R}$ and $u_0 \in \mathcal{B}$ there exists a $j \in \{1, \ldots, n\}$ such that*

$$\|S(t,s)u_0 - e_j\| \to 0 \qquad \text{as} \qquad t \to \infty,$$

where $S(\cdot, \cdot)$ is the process arising from (18).

Proof. Take $\epsilon > 0$. Since the manifold structure near the equilibria of $T(\cdot)$ perturbs continuously, it follows that there exists a τ_ϵ such that the adjusted processes S_{τ_ϵ} defined above has a set of complete trajectories $e_j^\tau(\cdot)$ that lie within $\epsilon/2$ of the equilibria e_j of $T(\cdot)$.

Applying Theorem 3.7, every trajectory of S_τ (for $\tau \geq \tau_\epsilon$) converges towards one of these complete trajectories. Since trajectories of S agree with those of S_τ for $t \geq \tau$, every trajectory of S converges towards one of the complete trajectories of $e_j^\tau(\cdot)$, and hence

$$\lim_{t \to \infty} \|S(t,s)u_0 - e_j\| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\lim_{t \to \infty} \|S(t,s)u_0 - e_j\| = 0.$$

$\square$

We note that Ball & Peletier (1977) have proved the same result, but their argument is much simpler since they consider only the asymptotically autonomous case and can make strong use of the Lyapunov function for $T(\cdot)$.

5 Conclusion

Generalising results for autonomous systems, we have shown that many of the properties of gradient-like attractors are preserved under small non-autonomous perturbations.

In particular, for the first time we provide a class of examples (perturbations of gradient ODEs) in which the structure of the non-autonomous pullback attractor is non-trivial but nevertheless well understood. The important problem of proving a similar structure theorem for infinite-dimensional examples will be treated in Carvalho et al. (2006).

Acknowledgements

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