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Abstract

Let k be a perfect field of characteristic $p > 0$, $k(t)_{per}$ the perfect closure of $k(t)$ and A a k-algebra. We characterize whether the ring

$$A \otimes_k k(t)_{per} = \bigcup_{m \geq 0} (A \otimes_k k(t^{\frac{1}{p^m}}))$$

is noetherian or not. As a consequence, we prove that the ring $A \otimes_k k(t)_{per}$ is noetherian when A is the ring of formal power series in n indeterminates over k.

Keywords: perfect–power series ring–noetherian ring– perfect extension–complete local ring.

Introduction

Motivated by the generalization of the results in [7] (for the case of a perfect base field k of characteristic $p > 0$) in this paper we study the conservation of noetherianity by the base field extension $k \to k(t)_{per}$, where $k(t)_{per}$ is the

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perfect closure of $k(t)$. Since this extension is not finitely generated, the conservation of noetherianity is not clear *a priori* for k-algebras which are not finitely generated.

Our main result states that $k(t)_{per} \otimes_k A$ is noetherian if and only if A is noetherian and for every prime ideal $\mathfrak{p} \subset A$ the field $\bigcap_{m\geq 0} Qt(A/\mathfrak{p})^{p^m}$ is algebraic over k (see theorem 3.6). In particular, we are able to apply this result to the case where A is the ring of formal power series in n indeterminates over k

We are indebted to J. M. Giral for giving us the proof of proposition 2.5 and for other helpful comments.

1 Preliminaries and notations

All rings and algebras considered in this paper are assumed to be commutative with unit element. If B is a ring, we shall denote by $\dim(B)$ its Krull dimension and by $\Omega(B)$ the set of its maximal ideals. We shall use the letters K, L, k to denote fields and $\mathbb{F}_p$ to denote the finite field of p elements, for p a prime number. If $\mathfrak{p} \in \mathrm{Spec}(B)$, we shall denote by $\mathrm{ht}(\mathfrak{p})$ the height of $\mathfrak{p}$. Remember that a ring B is said to be *equicodimensional* if all its maximal ideals have the same height. Also, B is said to be *biequicodimensional* if all its saturated chains of prime ideals have the same length.

If B is an integral domain, we shall denote by $Qt(B)$ its quotient field.

For any $\mathbb{F}_p$-algebra B, we denote $B^\sharp = \bigcap_{m\geq 0} B^{p^m}$.

We shall first study the contraction-extension process for prime ideals relative to the ring extension $K[t] \subset K[t^{\frac{1}{p}}]$, K being a field of characteristic $p > 0$. Let us recall the following well known result (cf. for example [4], th. 10.8):

Proposition. 1.1 *Let K be a field of characteristic $p > 0$. Let $g(X)$ be a monic polynomial of $K[X]$. Then, the polynomial $f(X) = g(X^p)$ is irreducible in $K[X]$ if and only if $g(X)$ is irreducible in $K[X]$ and not all its coefficients are in K^p.*

From the above result, we deduce the following corollary.

Corollary. 1.2 *Let K be a field of characteristic $p > 0$. Let P be a non zero prime ideal in $K[t^{\frac{1}{p}}]$ and let $F(t) \in K[t]$ be the monic irreducible generator of the contraction $P^c = P \cap K[t]$. Then the following conditions hold:*

1. *If $F(t) = a_0^p + a_1^p t + \cdots + t^d \in K^p[t]$, then $P = (a_0 + a_1 t^{\frac{1}{p}} + \cdots + t^{\frac{d}{p}})$.*
2. *The equality $P = P^{ce}$ holds if and only if $F(t) \notin K^p[t]$.*

Proof:

1. Consider the polynomial $G(\tau) = a_0 + a_1\tau + \cdots + \tau^d \in K[\tau](\tau = t^{\frac{1}{p}})$ and the ring homomorphism $\mu : K[\tau] \to K[t]$ defined by
$$\mu(\sum a_i \tau^i) = \sum a_i^p t^i.$$
From the identity $\mu(G) = F$ we deduce that $G(\tau)$ is irreducible. Since $G(t^{\frac{1}{p}})^p = F(t) \in P$, we deduce that $G(t^{\frac{1}{p}}) \in P$ and then $P = (G(t^{\frac{1}{p}}))$.
2. The equality $P = P^{ce}$ means that $F(t) = F(\tau^p) \in K[\tau]$ generates the ideal P, but that is equivalent to saying that $F(\tau^p)$ is irreducible in $K[\tau]$. To conclude, we apply proposition 1.1.

$\blacksquare$

For each k-algebra A, we define $A(t) := k(t) \otimes_k A$. We also consider the field extension
$$k_{(\infty)} = \bigcup_{m \geq 1} k(t^{\frac{1}{p^m}}).$$
If k is perfect, $k_{(\infty)}$ coincides with the perfect closure of $k(t)$, $k(t)_{per}$.

For the sake of brevity, we will write $t_m = t^{\frac{1}{p^m}}$. We also define
$$A_{(m)} := A(t_m) := A \otimes_k k(t_m) = A(t) \otimes_{k(t)} k(t_m), \quad A_{[m]} := A[t_m]$$
and
$$A_{(\infty)} := A \otimes_k k_{(\infty)} = \bigcup_{m \geq 0} A_{(m)}, \quad A_{[\infty]} := \bigcup_{m \geq 0} A[t_m].$$

Each $A_{(m)}$ (resp. $A_{[m]}$) is a free module over $A(t)$ (resp. over $A[t]$) of rank p^m (because $(t_m)^{p^m} - t = 0$).

For each prime ideal P of $A_{(\infty)}$ we denote $P_{[\infty]} := P \cap A_{[\infty]}$, $P_{[m]} := P \cap A_{[m]} \in \operatorname{Spec}(A_{[m]})$ and $P_{(m)} := P \cap A_{(m)} \in \operatorname{Spec}(A_{(m)})$.

In a similar way, if Q is a prime ideal of $A_{[\infty]}$ we denote $Q_{[m]} := Q \cap A_{[m]} \in \operatorname{Spec}(A_{[m]})$.

We have:

- $P = \bigcup_{m\geq 0} P_{(m)}$, $P_{[\infty]} = \bigcup_{m\geq 0} P_{[m]}$, (resp. $Q = \bigcup_{m\geq 0} Q_{[m]}$).
- $P_{(n)} \cap A_{(m)} = P_{(m)}$ and $P_{[n]} \cap A_{[m]} = P_{[m]}$ for all $n \geq m$ (resp. $Q_{[n]} \cap A_{[m]} = Q_{[m]}$ for all $n \geq m$).

The following properties are straightforward:

1. The k-algebras $A_{[m]}$ (respectively $A_{(m)}$) are isomorphic to each other.
2. If $S_m = k[t_m] - \{0\}$, then $A_{(m)} = S_m^{-1} A_{[m]}$.
3. Since $(S_m)^{p^m} \subset S_0 \subset S_m$, we have $A_{(m)} = S_0^{-1} A_{[m]}$ for $m \geq 0$. Consequently $A_{(\infty)} = S_0^{-1} A_{[\infty]}$.
4. If A is a domain (integrally closed), then $A_{[m]}$ and $A_{(m)}$ are domains (integrally closed) for all $m \geq 0$ or $m = \infty$.
5. If A is a noetherian k-algebra, then $A_{[m]}$ and $A_{(m)}$ are noetherian rings, for every $m \geq 0$.
6. If $A = k[\underline{X}] = k[X_1, \ldots, X_n]$, then $A_{[\infty]}$ is not noetherian (the ideal generated by the t_m, $m \geq 0$, is not finitely generated).
7. If $I \subset A$ is an ideal, then $(A/I)_{(\infty)} = A_{(\infty)}/A_{(\infty)} I$.
8. If $T \subset A$ is a multiplicative subset, then $(T^{-1}A)_{(\infty)} = T^{-1} A_{(\infty)}$.
9. If $A = k[\underline{X}]$, then $A_{(\infty)} = k_{(\infty)}[\underline{X}]$, hence $A_{(\infty)}$ is noetherian. Moreover, $A_{(\infty)}$ is noetherian for every finitely generated k-algebra A.

The main goal of this paper is to characterize whether the ring $A_{(\infty)}$ is noetherian (see th. 3.6 and corollary 3.8).

Proposition. 1.3 *With the above notations, the following properties hold:*

1. *The extensions* $A_{[m-1]} \subset A_{[m]}$ *and* $A_{(m-1)} \subset A_{(m)}$ *are finite free, and therefore integral and faithfully flat.*

2. *The corresponding extensions to their quotient fields are purely inseparable.*

Proof: Straightforward. ■

Corollary. 1.4 *$A_{[\infty]}$ (resp. $A_{(\infty)}$) is integral and faithfully flat over each $A_{[m]}$ (resp. over each $A_{(m)}$).*

From the properties above, we obtain the following lemmas:

Lemma. 1.5 *Let $P' \subseteq P$ be prime ideals of $A_{(\infty)}$ (resp. of $A_{[\infty]}$). The following conditions are equivalent:*

(a) $P' \subsetneq P$

(b) There exists an $m \geq 0$ *such that* $P'_{(m)} \subsetneq P_{(m)}$ *(resp.* $P'_{[m]} \subsetneq P_{[m]}$*).*

(c) For every $m \geq 0$, $P'_{(m)} \subsetneq P_{(m)}$ *(resp.* $P'_{[m]} \subsetneq P_{[m]}$*).*

Lemma. 1.6 *Let P prime ideal of $A_{(\infty)}$ (resp. of $A_{[\infty]}$). The following conditions are equivalent:*

(a) P is maximal.

(b) $P_{(m)}$ *(resp.* $P_{[m]}$*) is maximal for some* $m \geq 0$.

(c) $P_{(m)}$ *(resp.* $P_{[m]}$*) is maximal for every* $m \geq 0$.

Corollary. 1.7 *With the notations above, for every prime ideal P of $A_{(\infty)}$ we have* $\mathrm{ht}(P) = \mathrm{ht}(P_{(m)}) = \mathrm{ht}(P_{[m]})$ *for all* $m \geq 0$*. Moreover,* $\dim(A_{(\infty)}) = \dim(A_{(m)})$.

Proof: Since flat ring extensions satisfy the "going down" property, corollary 1.4 implies that $\mathrm{ht}(P \cap A_{(m)}) \leq \mathrm{ht}(P)$. By corollary 1.4 again, $A_{(\infty)}$ is integral over $A_{(m)}$, then $\mathrm{ht}(P) \leq \mathrm{ht}(P \cap A_{(m)})$.

The equality $\mathrm{ht}(P_{(m)}) = \mathrm{ht}(P_{[m]})$ comes from the fact that $A_{(m)}$ is a localization of $A_{[m]}$.

The last relation is a standard consequence of the "going up" property. ■

Remark. 1.8 *Corollary 1.7 remains true if we replace $A_{(m)} \subset A_{(\infty)}$ by $A_{[m]} \subset A_{[\infty]}$.*

Corollary. 1.9 *With the notations above, for every $Q \in \mathrm{Spec}(A_{(m)})$ there is a unique $\widetilde{Q} \in \mathrm{Spec}(A_{(m+1)})$ such that $\widetilde{Q}^c = Q$. Moreover, the ideal $\widetilde{Q}$ is given by $\widetilde{Q} = \{y \in A_{(m+1)} \mid y^p \in Q\}$.*

Proof: This is an easy consequence of the fact that $(A_{(m+1)})^p \subset A_{(m)}$. ■

Corollary. 1.10 *Let us assume that A is noetherian and for every maximal ideal $\mathfrak{m}$ of A, the residue field $A/\mathfrak{m}$ is algebraic over k. Then for every $m \geq 0$ we have:*

1. $\dim(A_{[\infty]}) = \dim(A_{[m]}) = \dim(A[t]) = n + 1.$
2. $\dim(A_{(\infty)}) = \dim(A_{(m)}) = \dim(A(t)) = n.$

Proof: The first relation comes from remark 1.8 and the noetherianity hypothesis.

The second relation comes from corollary 1.7 and proposition (1.4) of [7]. ■

The following result is a consequence of theorem (1.6) of [7], lemma 1.6 and corollary 1.10.

Corollary. 1.11 *Let A be a noetherian, biequidimensional, universally catenarian k-algebra of Krull dimension n, and that for any maximal ideal $\mathfrak{m}$ of A, the residue field $A/\mathfrak{m}$ is algebraic over k. Then every maximal ideal of $A_{(\infty)}$ has height n.*

2 The biggest perfect subfield of a formal functions field

Throughout this section, k will be a perfect field of characteristic $p > 0$, $A = k[[\underline{X}]]$, $\mathfrak{p} \subset A$ a prime ideal, $R = A/\mathfrak{p}$ and $K = Qt(R)$.

The aim of this section is to prove that the biggest perfect subfield of K, $K^\sharp = \bigcap_{e \geq 0} K^{p^e}$, is an algebraic extension of the field of constants, k. This result is proved in prop. 2.5 and it is one of the ingredients in the proof of corollary 3.8.

Proposition. 2.1 *Under the above hypothesis, it follows that $k = R^\sharp$.*

Proof: Let $\mathfrak{m}$ be the maximal ideal of R. It suffices to prove that $R^\sharp \subseteq k$. If $f \in R^\sharp$, then for every $e > 0$ there exists an $f_e \in R$ such that $f = f_e^{p^e}$.

- Suppose at first that f is not a unit, then f_e is not a unit for any $e > 0$, and $f_e \in \mathfrak{m}$ for every $e > 0$. Thus, $f \in \mathfrak{m}^{p^e}$ for every $e > 0$ and by Krull's intersection theorem,
$$f \in \bigcap_{e \geq 0} \mathfrak{m}^{p^e} = \bigcap_{r \geq 0} \mathfrak{m}^r = (0).$$
- If f is unit, then $f = f_0 + \widetilde{f}$, with $f_0 \in k \subset R^\sharp$ and $\widetilde{f} \in R^\sharp$ and f_0 is unit. By the above case $\widetilde{f} = 0$, hence $f \in k$.

■

Proposition. 2.2 *If $\mathfrak{p} = (0)$, that is $R = k[[\underline{X}]]$, $K = k((\underline{X}))$, then $k = K^\sharp$.*

Proof: It is a consequence of prop. 2.1 and the fact that R is a unique factorization domain. ■

In order to treat the general case, let us look at some general lemmas.

Lemma. 2.3 *(cf. [3] Chap. 5, § 15, ex. 8) If L is a separable algebraic extension of a field K of characteristic $p > 0$, then $L^\sharp$ is an algebraic extension of $K^\sharp$.*

Proof: If $x \in L^\sharp$, then $x = y_e^{p^e}$ with $y_e \in L$ for all $e \geq 0$. Since y_e is separable over K, $K(y_e) = K(y_e^{p^e}) = K(x)$, it follows that $y_e = x^{p^{-e}} \in K(x)$ and then $x \in K^{p^e}(x^{p^e})$. Therefore
$$[K^{p^e}(x) : K^{p^e}] = [K^{p^e}(x^{p^e}) : K^{p^e}] = [K(x) : K].$$
Thus x satisfies the same minimal polynomial over K^{p^e} and over K for all $e \geq 0$, and the coefficients of this minimal polynomial must be in $K^\sharp$. So x is algebraic over $K^\sharp$. ■

Lemma. 2.4 *Every algebraic extension of a perfect field is perfect.*

Proof: This is obvious because this is true for the finite algebraic extensions. ■

Proposition. 2.5 *Let k be a perfect field of characteristic $p > 0$, $A = k[[\underline{X}]] = k[[X_1, \ldots, X_n]]$, $\mathfrak{p} \subset A$ a prime ideal, $R = A/\mathfrak{p}$ and $K = Qt(R)$. Then $K^\sharp$ is an algebraic extension of k.*

Proof: [1] Let $r = \dim(A/\mathfrak{p}) \leq n$. By the normalization lemma for power series rings (cf. [1], 24.5 and 23.7)[2], there is a new system of formal coordinates $Y_1, \ldots, Y_n$ of A, such that

- $\mathfrak{p} \cap k[[Y_1, \ldots, Y_r]] = \{0\}$,
- $k[[Y_1, \ldots, Y_r]] \hookrightarrow \dfrac{A}{\mathfrak{p}} = R$ is a finite extension, and
- $k((Y_1, \ldots, Y_r)) \hookrightarrow K$ is a separable finite extension.

The proposition is then a consequence of proposition 2.2 and lemma 2.3[3]. ■

Remark. 2.6 *Actually, under the hypothesis of proposition 2.5, J.M. Giral and the authors have proved that the following stronger properties hold:*

(1) If R is integrally closed in K, then $K^\sharp = k$.

(2) In the general case, $K^\sharp$ is a <u>finite</u> extension of k.

3 Noetherianity of $A \otimes_k k(t)_{per}$

Throughout this section, k will be a perfect field of characteristic $p > 0$, keeping the notations of section 1.

[1] Due to J. M. Giral.

[2] The proof of the normalization lemma for power series rings in [1] uses generic linear changes of coordinates and needs the field k to be infinite. This proof can be adapted for an arbitrary perfect coefficient field (infinite or not) by using non linear changes of the form $Y_i = X_i + F_i(X_{i+1}^p, \ldots, X_n^p)$, where the F_i are polynomials with coefficients in $\mathbb{F}_p$.

[3] In particular, if k is algebraically closed, we would have $K^\sharp = k$.

Proposition. 3.1 *Let K be a field extension of k and suppose that $K^\sharp$ is algebraic over k. For every prime ideal $\mathcal{P} \in \operatorname{Spec}(K_{[\infty]})$ such that $\mathcal{P} \cap k[t] = 0$ there exists an $m_0 \geq 0$ such that $\mathcal{P}_{[m]}$ is the extended ideal of $\mathcal{P}_{[m_0]}$ for all $m \geq m_0$.*

Proof: The extension $k[t] \subset K^\sharp[t]$ is integral and then $\mathcal{P} \cap K^\sharp[t] = 0$.

We can suppose $\mathcal{P} \neq (0)$. From Remark 1.8, we have $\operatorname{ht}(\mathcal{P}_{[i]}) = \operatorname{ht}(\mathcal{P}) = 1$ for every $i \geq 0$. Let $F_i(t_i) \in K[t_i]$ be the monic irreducible generator of $\mathcal{P}_{[i]}$. From 1.2, for each $i \geq 0$ there are two possibilities:

(1) $F_i \in K^p[t_i]$, then $F_{i+1}(t_{i+1}) = F_i(t_i)^{1/p}$.

(2) $F_i \notin K^p[t_i]$, then $\mathcal{P}_{[i+1]} = (\mathcal{P}_{[i]})^e$ and $F_{i+1}(t_{i+1}) = F_i(t_i) = F_i(t_{i+1}^p)$.

Since $\mathcal{P} \cap K^\sharp[t] = (0)$, $F_0(t_0) \notin (\bigcap_{m\geq 0} K^{p^m})[t_0] = \bigcap_{m\geq 0} K^{p^m}[t_0]$ and there exists an $m_0 \geq 0$ such that $F_0(t_0) \in K^{p^{m_0}}[t_0]$ and $F_0(t_0) \notin K^{p^{m_0+1}}[t_0]$.

From (1) we have $F_i(t_i) = F_0(t_0)^{1/p^i} \in K^{p^{m_0-i}}[t_i]$ for $i = 0, \ldots, m_0 - 1$ and $F_{m_0}(t_{m_0}) \notin K^p[t_{m_0}]$. Hence, applying (2) repeatedly we find $F_{j+m_0}(t_{j+m_0}) = F_{m_0}(t_{m_0}) = F_{m_0}(t_{j+m_0}^{p^j})$ and $\mathcal{P}_{[j+m_0]}$ is the extended ideal of $\mathcal{P}_{[m_0]}$ for all $j \geq 1$. ■

Corollary. 3.2 *Under the same hypothesis of proposition 3.1, $\mathcal{P}$ is the extended ideal of some $\mathcal{P}_{m_0}$.*

Proof: This is a consequence of prop. 3.1 and the equality $\mathcal{P} = \bigcup_{m\geq 0} \mathcal{P}_m$. ■

Let B be a free algebra over a ring A and $S \subset A$ a multiplicative subset. We denote by $I \mapsto I^E, J \mapsto J^C$ (resp. $I \mapsto I^e, J \mapsto J^c$) the extension-contraction process between the rings A or $S^{-1}A$ (resp. A or B) and the rings B or $S^{-1}B$ (resp. $S^{-1}A$ or $S^{-1}B$).

Proposition. 3.3 *With the notations above, let $\mathcal{P}_1$ be a prime ideal in B such that $\mathcal{P}_1 \cap S = \emptyset$. Let $\mathcal{P}_0 = \mathcal{P}_1^C$, $\mathscr{P}_1 = \mathcal{P}_1^e$ and $\mathscr{P}_0 = \mathscr{P}_1^C$. If $\mathscr{P}_1 = \mathscr{P}_0^E$, then $\mathcal{P}_1 = \mathcal{P}_0^E$.*

Proof: Let $\{e_i\}$ be a A–basis of B. Since $\mathcal{P}_1 \cap S = \emptyset$, it is clear that $\mathscr{P}_1^c = \mathcal{P}_1$, $\mathscr{P}_0^c = \mathcal{P}_0$ and $\mathscr{P}_0 = \mathcal{P}_0^e$. If $\mathscr{P}_1 = \mathscr{P}_0^E$, we have

$$\mathcal{P}_1 = \mathcal{P}_1^{ec} = \mathscr{P}_1^c = (\mathscr{P}_0^E)^c = (\mathcal{P}_0^{eE})^c = (\mathcal{P}_0^{Ee})^c = (\mathcal{P}_0^E)^{ec} = \sum_{s\in S}(\mathcal{P}_0^E : s)_B \supset \mathcal{P}_0^E.$$

To prove the other inclusion, take an $s \in S$ and let $f = \sum a_i e_i$ be an element of $(\mathcal{P}_0^E : s)_B$ with $a_i \in A$. Then, $sf = \sum (sa_i)e_i \in \mathcal{P}_0^E$ and from the equality $\mathcal{P}_0^E = \{\sum b_i e_i \mid b_i \in \mathcal{P}_0\}$ we deduce that $sa_i \in \mathcal{P}_0$ and $a_i \in (\mathcal{P}_0^E : s)_A = \mathcal{P}_0$. Therefore $f \in \mathcal{P}_0^E$.

■

Proposition. 3.4 *Let R be an integral k-algebra, $K = Qt(R)$, and suppose that $K^\sharp$ is algebraic over k. Then any prime ideal $\mathcal{P} \in \operatorname{Spec}(R_{[\infty]})$ with $\mathcal{P} \cap k[t] = 0$ and $\mathcal{P} \cap R = 0$ is the extended ideal of some $\mathcal{P}_{[m_0]}$, $m_0 \geq 0$.*

Proof: Let us write $T = R - \{0\}$. We have $K = T^{-1}R$ and $K_{[m]} = T^{-1}R_{[m]}$ for all $m \geq 0$ or $m = \infty$. We define $\mathscr{P} = T^{-1}\mathcal{P}$. We easily deduce that $\mathscr{P}_{[m]} = T^{-1}\mathcal{P}_{[m]}$ for all $m \geq 0$.

From proposition 3.1, there exists an $m_0 \geq 0$ such that $\mathscr{P}_{[m]}$ is the extended ideal of $\mathscr{P}_{[m_0]}$ for every $m \geq m_0$. Then, proposition 3.3 tells us that $\mathcal{P}_{[m]}$ is the extended ideal of $\mathcal{P}_{[m_0]}$ for every $m \geq m_0$, so $\mathcal{P} = \bigcup \mathcal{P}_{[m]}$ is the extended ideal of $\mathcal{P}_{[m_0]}$. ■

Proposition. 3.5 *Let K be a field extension of k and suppose that $K^\sharp$ is not algebraic over k. Then $K_{(\infty)}$ is not noetherian.*

Proof: Let $s \in K^\sharp$ be a transcendental element over k.

For each $m \geq 0$, let $s_m = s^{\frac{1}{p^m}} \in K$ and $\alpha_m = t_m - s_m$. Let P be the ideal in $K_{(\infty)}$ generated by the $\alpha_m, m \geq 0$. We have $\alpha_m = \alpha_{m+1}^p$ and $P_{(m)} = K_{(m)}\alpha_m$ for all $m \geq 0$.

Suppose that P is finitely generated. Then, there exists an $m_0 \geq 0$ such that $P = K_{(\infty)}\alpha_{m_0}$. By faithful flatness, we deduce that $\alpha_{m_0+1} \in K_{(m_0+1)}\alpha_{m_0}$. Let us write $\tau = t_{m_0+1}, \sigma = s_{m_0+1}$. Then, $\alpha_{m_0+1} = \tau - \sigma$ and there exist $\psi(\tau) \in K[\tau] = K_{[m_0+1]}$, $\varphi(\tau) \in k[\tau] \setminus \{0\}$ such that

$$\varphi(\tau)(\tau - \sigma) = \psi(\tau)(\tau - \sigma)^p.$$

Simplifying and making $\tau = \sigma$ we obtain

$$\varphi(\sigma) = \psi(\sigma)(\sigma - \sigma)^{p-1} = 0$$

contradicting the fact that s is transcendental over k.

We conclude that P is not finitely generated and $K_{(\infty)}$ is not noetherian.

■

Theorem. 3.6 *Let k be a perfect field of characteristic $p > 0$ and let A be a k-algebra. The following properties are equivalent:*

(a) The ring A is noetherian and for any $\mathfrak{p} \in \operatorname{Spec}(A)$, the field $Qt(A/\mathfrak{p})^\sharp$ is algebraic over k.

(b) The ring $A_{(\infty)}$ is noetherian.

Proof: Let first prove (a) $\Rightarrow$ (b). By Cohen's theorem (cf. [6], (3.4)), it is enough to prove that any $P \in \operatorname{Spec}(A_{(\infty)}) - \{(0)\}$ is finitely generated.

From corollaries 1.7 and 1.10, we have

$$\operatorname{ht}(P_{[m]}) = \operatorname{ht}(P_{(m)}) = \operatorname{ht}(P_{[\infty]}) = \operatorname{ht}(P) = r \leq n.$$

Consider the prime ideal of A:

$$\mathfrak{p} := A \cap P = A \cap P_{[\infty]} = A \cap P_{[m]} = A \cap P_{(m)}.$$

There are two possibilities (cf. [5], prop. (5.5.3)):

(i) $\operatorname{ht}(\mathfrak{p}) = r = \operatorname{ht}(P_{[m]})$ and $P_{[m]} = \mathfrak{p}[t_m]$, for every $m \geq 0$.

(ii) $\operatorname{ht}(\mathfrak{p}) = r - 1 = \operatorname{ht}(P_{[m]}) - 1$, $\mathfrak{p}[t_m] \subsetneq P_{[m]}$ and $A/\mathfrak{p} \subsetneq A[t_m]/P_{[m]}$ is algebraic generated by t_m mod $P_{[m]}$, for every $m \geq 0$.

In case (i), $P_{[\infty]}$ and P are the extended ideals of $\mathfrak{p}$ and they are finitely generated.

Suppose we are in case (ii). We denote $R = A/\mathfrak{p}$, $K = Qt(R)$.

Then:

$$R_{[m]} = A_{[m]}/\mathfrak{p}[t_m], \quad R_{[\infty]} = A_{[\infty]}/A_{[\infty]}\mathfrak{p} = A_{[\infty]}/\bigcup_{m \geq 0} \mathfrak{p}[t_m].$$

Define $\mathcal{P} := R_{[\infty]} P_{[\infty]} = P_{[\infty]} / \bigcup_{m \geq 0} \mathfrak{p}[t_m] \in \mathrm{Spec}(R_{[\infty]})$. We have $\mathcal{P}_{[m]} = \mathcal{P} \cap R_{[m]} = P_{[m]}/\mathfrak{p}[t_m]$, $\mathcal{P} \cap R = \mathcal{P} \cap k[t] = 0$ and

$$\mathrm{ht}(\mathcal{P}_{[m]}) = \mathrm{ht}\left(P_{[m]}/\mathfrak{p}[t_m]\right) = 1, \quad \mathrm{ht}(\mathcal{P}) = \mathrm{ht}\left(P_{[\infty]} / \bigcup_{m \geq 0} \mathfrak{p}[t_m]\right) = 1.$$

We conclude by applying proposition 3.4: there exists an $m_0 \geq 0$ such that $\mathcal{P}$ is the extended ideal of $\mathcal{P}_{[m_0]}$. Then, $P_{[\infty]}$ is the extended ideal of $P_{[m_0]}$ and $P = A_{(\infty)} P_{[\infty]} = A_{(\infty)} P_{[m_0]}$ is finitely generated.

Let us prove now (b) $\Rightarrow$ (a). Since $A_{(\infty)}$ is faithfully flat over A, we deduce that A is noetherian.

Let $\mathfrak{p} \in \mathrm{Spec}(A)$ and let $R = A/\mathfrak{p}$, $K = Qt(R)$. Noetherianity of $A_{(\infty)}$ implies, first, noetherianity of $R_{(\infty)}$, and second, noetherianity of $K_{(\infty)}$. To conclude we apply proposition 3.5. ■

Corollary. 3.7 *Let k be a perfect field of characteristic $p > 0$ and let A be a noetherian k-algebra. The following properties are equivalent:*

(a) The ring $A_{(\infty)}$ is noetherian.

(b) The ring $(A_{\mathfrak{m}})_{(\infty)}$ is noetherian for any maximal ideal $\mathfrak{m} \in \Omega(A)$.

Proof: For (a) $\Rightarrow$ (b) we use the fact that $(A_{\mathfrak{m}})_{(\infty)} = A_{\mathfrak{m}} \otimes_A A_{(\infty)}$.

For (b) $\Rightarrow$ (a), let $\mathfrak{p} \subset A$ be a prime ideal and let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. From hypothesis (b), the ring $(A_{\mathfrak{m}})_{(\infty)}$ is noetherian. Then, from theorem 3.6 we deduce that the field $Qt(A/\mathfrak{p})^\sharp = Qt\left(A_{\mathfrak{m}}/A_{\mathfrak{m}}\mathfrak{p}\right)^\sharp$ is algebraic over k. From theorem 3.6 again we obtain (a). ■

Corollary. 3.8 *Let k be a perfect field of characteristic $p > 0$, k' an algebraic extension of k and $A = k'[[X_1, \ldots, X_n]]$. Then, the ring $A_{(\infty)} = k(t)_{per} \otimes_k A$ is noetherian.*

Proof: It is a consequence of lemma 2.4, proposition 2.5 and theorem 3.6. ■

Corollary. 3.9 *Let k be a perfect field of characteristic $p > 0$. If $(B, \mathfrak{m})$ is a local noetherian k-algebra such that $B/\mathfrak{m}$ is algebraic over k, then $B_{(\infty)} = k(t)_{per} \otimes_k B$ is noetherian. In particular, the field $Qt(B/\mathfrak{p})^\sharp$ is algebraic over k for every prime ideal $\mathfrak{p} \subset B$.*

Proof: Let $k' = B/\mathfrak{m}$. By Cohen structure theorem (cf. [5], Chap. 0, Th. (19.8.8)), the completion $\widehat{B}$ of B is a quotient of a power-series ring A with coefficients in k'. Since $\widehat{B}_{(\infty)}$ is also a quotient of $A_{(\infty)}$, we deduce from corollary 3.8 that $\hat{B}_{(\infty)}$ is noetherian. Since $\widehat{B}$ is faithfully flat over B, the ring $\widehat{B}_{(\infty)}$ is also faithfully flat over $B_{(\infty)}$. So, B_∞ is noetherian.

The last assertion is a consequence of theorem 3.6. ■

Corollary. 3.10 *Let k be a perfect field of characteristic $p > 0$. For any noetherian k-algebra A such that the residue field $A/\mathfrak{m}$ of every maximal ideal $\mathfrak{m} \in \Omega(A)$ is algebraic over k, the ring $A_{(\infty)}$ is noetherian. Furthermore, if A is regular and equicodimensional then $A_{(\infty)}$ is also regular and equicodimensional of the same dimension as A.*

Proof: The first part is a consequence of corollaries 3.7 and 3.9. For the last part, we use corollary 1.11, the fact that all $A_{(m)}, m \geq 0$ are regular and of the same (global homological = Krull) dimension ([7], th. (1.6)) and [2]. ■

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