Thin sets of integers in Harmonic analysis and p-stable random Fourier series

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Dedicated to the memory of Daniel Rider

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Abstract. *We investigate the behavior of some thin sets of integers defined through random trigonometric polynomial when one replaces Gaussian or Rademacher variables by p-stable ones, with $1 < p < 2$. We show that in one case this behavior is essentially the same as in the Gaussian case, whereas in another case, this behavior is entirely different.*

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1 Introduction

Let G be a compact, abelian group (which will be mostly the circle $\mathbb{T}$), equipped with its normalized Haar measure m, and Γ its (discrete) dual. We will denote by $\mathcal{P}$ the set of finite sums $\sum_{\gamma\in\Gamma} c_\gamma \gamma$, i.e. the vector space generated by Γ, and by $\mathcal{P}_\Lambda$ the set of finite sums $\sum_{\gamma\in\Lambda} c_\gamma \gamma$, where Λ is a subset of Γ. We recall ([22]) that Λ is called a Sidon set if, for some constant C, we have the following *a priori* inequality:

$$\|f\|_{F_1} := \sum_{\gamma} |\hat{f}(\gamma)| \leq C\|f\|_\infty, \quad \forall f \in \mathcal{P}_\Lambda. \tag{1.1}$$

The best constant C in (1.1) is called the Sidon constant of Λ. A long standing problem, solved in the positive by Drury ([4]) at the beginning of the seventies, was whether the union of two Sidon sets is again a Sidon set. A little after Drury's result, Rider ([19]) gave the following necessary and sufficient condition for Sidonicity, from which the result becomes obvious:

$$\|f\|_{F_1} \leq C\,[\![f]\!], \quad \forall f \in \mathcal{P}_\Lambda. \tag{1.2}$$

Here, we have set:

$$[\![f]\!] = \mathbb{E}\left\| \sum \varepsilon_\gamma \hat{f}(\gamma)\gamma \right\|_\infty, \tag{1.3}$$

where $(\varepsilon_\gamma)_\gamma$ is a sequence of i.i.d. Rademacher random variables, defined on some probability space Ω, i.e. independent and taking the values $+1$ and -1 with equal probability $\frac{1}{2}$, and where $\mathbb{E}$ stands for expectation on Ω. This norm was thoroughly studied by Marcus and Pisier ([13]) and is called the $\mathcal{C}^{as}$-norm in the space of almost surely continuous random Fourier series. These two authors proved in particular the non-trivial fact that one could as well use a standard gaussian sequence instead of a Rademacher one, and obtain an equivalent norm (see [16], Théorème 7.1). Pisier ([16]) realized that Sidonicity can also be characterized by the *a priori* inequality:

$$\|f\|_\infty \leq C\,[\![f]\!], \quad \forall f \in \mathcal{P}_\Lambda. \tag{1.4}$$

This is a general fact, the proof of which we recall for the convenience of the reader, and which motivates the forthcoming definition of *stationarity*: let $(Z_\gamma)_{\gamma\in\Gamma}$ be a collection of i.i.d. copies of a complex-valued, centered and integrable random variable Z, and set, for every trigonometric polynomial f,

$$[\![f]\!]_Z = \mathbb{E}\left\| \sum Z_\gamma \hat{f}(\gamma)\gamma \right\|_\infty.$$

We have the following simple proposition.

Proposition 1.1. *Let $\Lambda \subset \Gamma$ be such that*

$$\|f\|_\infty \leq C\,[\![f]\!]_Z\,, \quad \forall f \in \mathcal{P}_\Lambda. \tag{1.5}$$

Then, Λ is a Sidon set.

Proof. Let $(\widetilde{Z}_\gamma)_{\gamma\in\Gamma})$ be an independent family, with each $\widetilde{Z}_\gamma$ a symmetrization of Z_γ. Since the latter variables are centered, we have

$$\mathbb{E}\left\| \sum Z_\gamma \hat{f}(\gamma)\gamma \right\|_\infty \leq 2\,\mathbb{E}\left\| \sum \widetilde{Z}_\gamma \hat{f}(\gamma)\gamma \right\|_\infty,$$

so we may as well assume the Z_γ's symmetric from the beginning. If $\varepsilon_\gamma = \pm 1$, we therefore have

$$\Big|\sum_\gamma \varepsilon_\gamma \hat{f}(\gamma)\Big| \leq \Big\|\sum_\gamma \varepsilon_\gamma \hat{f}(\gamma)\gamma\Big\|_\infty \leq C\,\mathbb{E}\,\Big\|\sum Z_\gamma \hat{f}(\gamma)\gamma\Big\|_\infty.$$

Taking the supremum on all choices of ± 1 gives classically ([10], Chapitre 5, Proposition IV.2):

$$\sum_\gamma |\hat{f}(\gamma)| \leq D\,\mathbb{E}\,\Big\|\sum Z_\gamma \hat{f}(\gamma)\gamma\Big\|_\infty,$$

with $D = \frac{\pi}{2}\,C$. Now, we truncate our variables Z_γ at a level M: $Z_\gamma = Z'_\gamma + Z''_\gamma$, where

$$Z'_\gamma = Z_\gamma \mathbb{1}_{\{|Z_\gamma| \leq M\}}, \quad Z''_\gamma = Z_\gamma \mathbb{1}_{\{|Z_\gamma| > M\}};$$

M being adjusted so as to have $\mathbb{E}\,|Z''_\gamma| \leq \frac{1}{2D}$. We now see that

$$\begin{aligned}
\sum_\gamma |\hat{f}(\gamma)| &\leq D\,\mathbb{E}\,\Big\|\sum_\gamma Z'_\gamma \hat{f}(\gamma)\gamma\Big\|_\infty + D\,\mathbb{E}\,\Big\|\sum_\gamma Z''_\gamma \hat{f}(\gamma)\gamma\Big\|_\infty \\
&\leq D\,\mathbb{E}\,\Big\|\sum_\gamma Z'_\gamma \hat{f}(\gamma)\gamma\Big\|_\infty + D\sum_\gamma |\hat{f}(\gamma)|\,\mathbb{E}\,|Z''_\gamma| \\
&\leq D\,\mathbb{E}\,\Big\|\sum_\gamma Z'_\gamma \hat{f}(\gamma)\gamma\Big\|_\infty + \frac{1}{2}\sum_\gamma |\hat{f}(\gamma)| \\
&\leq 4MD\,\mathbb{E}\,\Big\|\sum_\gamma \varepsilon_\gamma \hat{f}(\gamma)\gamma\Big\|_\infty + \frac{1}{2}\sum_\gamma |\hat{f}(\gamma)|
\end{aligned}$$

(here, $(\varepsilon_\gamma)_\gamma$ is a Rademacher sequence, and we used the usual "contraction principle": see [10], Chapitre 3, Théorème III.3); whence

$$\sum_\gamma |\hat{f}(\gamma)| \leq 8MD\,\mathbb{E}\,\Big\|\sum_\gamma \varepsilon_\gamma \hat{f}(\gamma)\gamma\Big\|_\infty.$$

Now, we are in position to apply Rider's Theorem ([10], Chapitre 5, Théorème IV.18) to conclude that Λ is Sidon. □

Note that Proposition 1.1 has an easy converse (which we state for further reference).

Proposition 1.2. *If Λ is a Sidon set, then $\|\ \|_\infty$ and $[\![\ ]\!]_Z$ are equivalent norms on $\mathcal{P}_\Lambda$.*

Proof. Let $f \in \mathcal{P}_\Lambda$ and $f_Z^\omega = \sum_\gamma Z_\gamma(\omega) \widehat{f}(\gamma)\, \gamma$. On one hand, one has

$$\mathbb{E}_\omega \|f\|_\infty \leq \sum_{\gamma\in\Lambda} \mathbb{E}(|Z_\gamma|)\, |\widehat{f}(\gamma)| = \|Z_1\|_1 \sum_{\gamma\in\Lambda} |\widehat{f}(\gamma)| \leq C\, \|Z\|_1 \|f\|_\infty,$$

and, on the other hand, $\sum_\gamma |Z_\gamma(\omega) \widehat{f}(\gamma)| \leq C\, \|f_Z^\omega\|_\infty$, since Λ is Sidon and $f_Z^\omega \in \mathcal{P}_\Lambda$; hence, by integrating, $\sum_\gamma \mathbb{E}(|Z_\gamma|)\, |\widehat{f}(\gamma)| \leq C\, [\![f]\!]_Z$, and

$$\|f\|_\infty \leq \sum_\gamma |\widehat{f}(\gamma)| \leq \frac{C}{\|Z_1\|_1}\, [\![f]\!]_Z. \qquad \square$$

Pisier ([16]) also studied the subsets Λ of Γ verifying the reverse inequality of (1.5) (see [16], Définition 6.2), namely:

$$[\![f]\!] \leq C\, \|f\|_\infty, \quad \forall f \in \mathcal{P}_\Lambda. \tag{1.6}$$

and he called those sets *stationary*, proving in particular ([16], Proposition 6.2) that the cartesian product of d Sidon sets is always stationary (the first named author [7] proved that, for example, $\{3^{k_1} + \cdots + 3^{k_d}\,;\ 1 \leq k_1 < \cdots < k_d\}$ is also a stationary set). Another well-known notion is that of q-Sidonicity, $1 \leq q < 2$ ([12], [16], Définition 6.1). The subset Λ is called *q-Sidon* if, for some constant C, we have:

$$\|f\|_{F_q} := \Big(\sum_\gamma |\hat{f}(\gamma)|^q\Big)^{\frac{1}{q}} \leq C\, \|f\|_\infty, \quad \forall f \in \mathcal{P}_\Lambda. \tag{1.7}$$

After the work of Rider, the following notion was also introduced ([11] and [21]): the subset Λ is called *q-Rider* if, for some constant C, we have this time:

$$\|f\|_{F_q} \leq C\, [\![f]\!], \qquad \forall f \in \mathcal{P}_\Lambda. \tag{1.8}$$

It is immediate to see that every q-Sidon set is a q-Rider set, and Rider's result can be formulated in saying that the converse holds for $q = 1$. Whether this converse holds for each $q \in (1, 2)$ is an open problem, in spite of several non-trivial partial results ([8]). Let us mention that the cartesian product of d infinite Sidon sets is q-Sidon with $q = \frac{2d}{d+1}$ and not better ([12]).

We could of course study those notions for other probability laws than the (subgaussian) Rademacher laws or gaussian ones. This is precisely the aim of this work, where we will be interested in the complex, symmetric, p-stable random variables Z, $1 < p < 2$, which can be defined through their characteristic function:

$$\mathbb{E}\left(\mathrm{e}^{i\mathrm{Re}\,\overline{z}Z}\right) = \exp(-|z|^p), \quad \forall z \in \mathbb{C}.$$

The case $p = 2$ is the gaussian case already studied. The case $1 < p < 2$ is in some sense more delicate, because in spite of the nice *stability* property:

$$\sum_{n=1}^{N} a_n Z_n \sim \Big(\sum_{n=1}^{N} |a_n|^p\Big)^{\frac{1}{p}} Z_1$$

from which those variables borrow their name, their integrability properties are fairly poor: $Z \in L^s$ for each $s < p$, but $Z \notin L^p$ (in fact, $Z \in L^{p,\infty}$). Yet, this case has also been studied in great detail by Marcus and Pisier in [14], who in particular introduced the following p-stable norm on the space of trigonometric polynomials

$$[\![f]\!]_p = \mathbb{E}\Big\|\sum_{\gamma\in\gamma} Z_\gamma \hat{f}(\gamma)\gamma\Big\|_\infty, \quad \forall f \in \mathcal{P}, \tag{1.9}$$

where $(Z_\gamma)_\gamma$ is a family of independent copies of a complex p-stable, symmetric, random variables. Observe that this has a meaning, since the Z_γ's are integrable. Moreover, due to a general comparison principle of Jain and Marcus ([5]), one has the following inequality, where the implied constants only depend on p_1 and p_2:

$$1 < p_1 < p_2 \leq 2 \quad \Rightarrow \quad [\![f]\!]_{p_2} \leq C(p_1, p_2)\, [\![f]\!]_{p_1},\ \forall f \in \mathcal{P}. \tag{1.10}$$

In other terms, the smaller p, the bigger the corresponding $[\![\ ]\!]_p$-norm. In particular, those new norms are bigger than the previously mentioned Rademacher and Gaussian norms on $\mathcal{P}$. The questions which we examine in this work are the following: what do the notions of stationarity, q-Riderness, become if we replace the gaussian variables by p-stable ones?

After having established, in Section 2, one basic property of the $[\![\ ]\!]_p$-norm, namely a lower p-estimate, we prove in Section 3, that a p-stationary set is in fact Sidon (and, of course, conversely) as soon as $p < 2$, and we study, in Section 4, several equivalent forms of p-stable q-Riderness, and show that this apparently new notion coincides with that of s-Riderness for an appropriate value of the parameter s, depending on p and q. We end with some comments.

2 Basic properties of the p-stable norm

We will need the following two theorems on p-stable norms. They are more or less straightforward consequences of a basic result of Marcus and Pisier.

First, we introduce some notation: F_p will denote the set of functions $f \in L^2 = L^2(G,m)$ such that their Fourier transform is in $\ell_p = \ell_p(\Gamma)$, equipped with the norm

$$\|f\|_{F_p} := \|\hat{f}\|_p \,,$$

which we already encountered in Section 1 (see (1.7)).

We shall denote by $\|.\|_\psi$ the Luxemburg norm in the Orlicz space associated to an Orlicz function ψ. Let $r > 0$, we shall be mainly interested in the Orlicz function φ_r, where

$$\varphi_r(x) = x\big(1 + \log(1+x)\big)^{\frac{1}{r}},$$

and the conjugate Orlicz function ψ_r, where

$$\psi_r(x) = \mathrm{e}^{x^r} - 1.$$

Finally let $A(p, \varphi_{p'})$ (where p' is the conjugate exponent of p) be the space of all functions in $L^2(G)$ which can be written as

$$f = \sum_{n=1}^{\infty} h_n * k_n \,,$$

with:

$$\sum_{n=1}^{\infty} \|h_n\|_{F_p} \|k_n\|_{\varphi_{p'}} < \infty,$$

and equipped with the norm

$$\|f\|_{A(p,\varphi_{p'})} = \inf \Big\{ \sum_{n=1}^{\infty} \|h_n\|_{F_p} \|k_n\|_{\varphi_{p'}} \Big\} \,,$$

where the infimum runs over all possible representations of f.

With those notations, the basic result alluded to above stands as follows, under a simplified form which will be sufficient for us ([14], Theorem 5.1):

Theorem 2.1 (**Marcus-Pisier**). *The norms $[\![f]\!]_p$ and $\|f\|_{A(p,\varphi_{p'})}$ are equivalent on the space $\mathcal{P}$ of trigonometric polynomials on G.*

Two important consequences of that theorem, which are not explicited in [14] in the p-stable case, are the following (see [16], Proposition 7.1 for the case $p = 2$).

Theorem 2.2 (**Contraction principle for the p-stable norm**). *Let $f \in \mathcal{P}$ and $(\xi_\gamma)_\gamma$ be a collection of functions in $L^p(0,1)$, bounded in L^p. Denote the integral over $(0,1)$ by $\mathbb{E}'$. Then, we have for some positive constant a:*

$$\left(\mathbb{E}' \left[\!\!\left[\sum_\gamma \xi_\gamma \hat{f}(\gamma)\gamma \right]\!\!\right]_p^p \right)^{\frac{1}{p}} \leq a \sup_\gamma \|\xi_\gamma\|_{L^p(0,1)} [\![f]\!]_p. \tag{2.1}$$

Proof. Let $f = \sum_{n=1}^\infty h_n * k_n$ be an admissible decomposition of f, and let $\omega' \in (0,1)$, as well as $f_{\omega'} = \sum_\gamma \xi_\gamma(\omega')\hat{f}(\gamma)\gamma$. We can write $f_{\omega'} = \sum_{n=1}^\infty H_n(\omega') * k_n$, with $H_n(\omega') \in F_p$ and $\widehat{H_n(\omega')}(\gamma) = \xi_\gamma(\omega')\hat{h}_n(\gamma)$. Set for convenience $X(\omega') = [\![f_{\omega'}]\!]_p$ and $Y_n(\omega') = \|H_n(\omega')\|_{F_p}$. We see that

$$X(\omega') \leq \sum_{n=1}^\infty [\![H_n(\omega') * k_n]\!]_p \leq a \sum_{n=1}^\infty Y_n(\omega')\|k_n\|_{\varphi_{p'}} ,$$

where a is some constant given by the Marcus-Pisier Theorem above. Now taking L^p-norms in $L^p(0,1)$ and using the triangle inequality, we get:

$$\|X\|_{L^p(0,1)} \leq a \sum_{n=1}^\infty \|Y_n\|_{L^p(0,1)}\|k_n\|_{\varphi_{p'}} \leq aC \sum_{n=1}^\infty \|h_n\|_{F_p}\|k_n\|_{\varphi_{p'}} ,$$

where $C = \sup_\gamma \|\xi_\gamma\|_{L^p(0,1)}$. Taking the infimum over all possible representations of f gives us the result, possibly changing the constant a. □

The second basic consequence is:

Theorem 2.3 (**Lower p-estimate for the $[\![\ ]\!]_p$-norm**). *Let $f, f_1, \ldots, f_N$ be trigonometric polynomials such that:*

$$|\hat{f}(\gamma)| \geq \Big(\sum_{j=1}^{N} |\hat{f}_j(\gamma)|^p \Big)^{\frac{1}{p}}, \qquad \forall \gamma \in \Gamma.$$

Then, the constant a being as in (2.1)*:*

$$[\![f]\!]_p \geq a^{-1} \Big(\sum_{j=1}^{N} [\![f_j]\!]_p^p \Big)^{\frac{1}{p}}. \tag{2.2}$$

Proof. Let $A_1, \ldots, A_N$ be a partition of $(0,1)$ in sets of Lebesgue measure $1/N$ and, for each $\gamma \in \Gamma$, $\xi_\gamma \in L^p(0,1)$ be defined by:

$$\xi_\gamma = N^{\frac{1}{p}} \sum_{j=1}^{N} \frac{\hat{f}_j(\gamma)}{\hat{f}(\gamma)} \, \mathbb{I}_{A_j}.$$

It is clear that $\|\xi_\gamma\|_{L^p(0,1)} \leq 1$ by our assumption, and by definition we have

$$\mathbb{E}' \, \Big[\!\!\Big[\sum_\gamma \xi_\gamma \hat{f}(\gamma) \gamma \Big]\!\!\Big]_p^p = \sum_{j=1}^{N} \Big[\!\!\Big[\sum_\gamma \frac{\hat{f}_j(\gamma)}{\hat{f}(\gamma)} \hat{f}(\gamma) \gamma \Big]\!\!\Big]_p^p = \sum_{j=1}^{N} [\![f_j]\!]_p^p \,,$$

so that an application of (2.1) gives the result. $\square$

We will end this section with the following estimate. This is undoubtedly known, but we did not find any explicit mention; so we are going to give some words of explanation.

Lemma 2.4. *Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n \in \mathbb{N}$, with $\lambda_n \geq 2$. If $f(t) = \sum_{j=1}^{n} \mathrm{e}^{i\lambda_j t}$, $t \in \mathbb{T}$, one has, for some constant $C > 0$:*

$$[\![f]\!]_p \leq C \, n^{1/p} (\log \lambda_n)^{1/p'}.$$

Proof. From (4.6) of [14] (or Remark 1.7, page 186 of [15]), there is a constant $K > 0$ such that

$$[\![g]\!]_p \leq K \sum_{k=2}^{\infty} \frac{1}{k (\log k)^{1/p}} \Big(\sum_{j=k}^{\infty} |\widehat{g}(j)|^p \Big)^{1/p}$$

for every trigonometric polynomial g with spectrum in $\mathbb{N}$. Here one has $\sum_{j=k}^{\infty} |\widehat{f}(j)|^p \leq n$ for $k \leq \lambda_n$ and $\sum_{j=k}^{\infty} |\widehat{f}(j)|^p = 0$ for $k > \lambda_n$; hence

$$[\![f]\!]_p \leq K\, n^{1/p} \sum_{k=2}^{\lambda_n} \frac{1}{k(\log k)^{1/p}} \leq C\, n^{1/p} (\log \lambda_n)^{1/p'}. \qquad \square$$

3 p-stable stationary sets are Sidon for $p < 2$

The aim of this Section is to prove the following:

Theorem 3.1. *Let $\Lambda \subset \Gamma$ be a p-*stationary *set ($1 < p \leq 2$), i.e. a set satisfying the following inequality, for some constant $C = C_p$:*

$$[\![f]\!]_p \leq C\, \|f\|_\infty, \quad \forall f \in \mathcal{P}_\Lambda.$$

Then, if $p < 2$, Λ is a Sidon set.

The difference between the cases $1 < p < 2$ and $p = 2$ (for which non-Sidon stationary sets exist), comes from the fact the p-stable norm for $p < 2$ is bigger than the usual Pisier norm; hence having an upper estimate for it on some space forces the smallness of this space.

In order to prove Theorem 3.1, we will need the following simple lemma :

Lemma 3.2. *If $\Lambda \subset \Gamma$ is a p-stationary set, the norms $[\![\ ]\!]_2$ and $[\![\ ]\!]_p$ are equivalent on the space $\mathcal{P}_\Lambda$.*

Proof of Lemma 3.2. Let $f \in \mathcal{P}_\Lambda$, $(\varepsilon_\gamma)_\gamma$ a Rademacher sequence, and $f^\omega = \sum_\gamma \varepsilon_\gamma(\omega)\hat{f}(\gamma)\gamma$. By symmetry, we have $[\![f]\!]_p = [\![f^\omega]\!]_p \leq C\, \|f^\omega\|_\infty$, where C is the stationarity constant of Λ. Integrating over ω gives: $[\![f]\!]_p \leq C\, \mathbb{E}\, \|f^\omega\|_\infty = C[\![f]\!]_2$, which finishes the proof, since we know from (1.10) that the reverse inequality always holds (we wrote $[\![\ ]\!]_2$ instead of $[\![\ ]\!]$ to make a clear distinction between the subgaussian (i.e. Rademacher) or gaussian case, and the p-stable one). $\square$

Proof of Theorem 3.1. From the previous Lemma 3.2, it will be enough to show that, if Λ is not Sidon, then the norms $[\![\ ]\!]_2$ and $[\![\ ]\!]_p$ are not equivalent on $\mathcal{P}_\Lambda$. It will be convenient to introduce first some notation. Recall that a subset B of Γ is called *quasi-independent* if, for any finite subset $\{\gamma_1, \ldots, \gamma_r\}$ of distinct elements of B, a relation $\sum_{i=1}^r \theta_i \gamma_i = 0$, with $\theta_i = 0, \pm 1$ implies $\theta_i = 0$ for each i. The quasi-independent sets are the prototypes of Sidon sets, in that their Sidon constant S is bounded by an absolute constant ([10], Chapitre 12, Proposition I.1, or [12]; in [6], Kahane found that $S \leq 4.7$). Now, if A is a finite subset of Γ:

1. $|A|$ will denote the (finite) cardinality of A;
2. $q(A)$ will denote the biggest possible cardinality of a quasi-independent subset $B \subset A$;
3. $[\![A]\!]_2$ (respectively $[\![A]\!]_p$) will denote the quantity $\left[\!\left[\sum_{\gamma \in A} \gamma \right]\!\right]_2$ (respectively $\left[\!\left[\sum_{\gamma \in A} \gamma \right]\!\right]_p$);
4. $i_{A,p}$ will be the canonical injection of $(\mathcal{P}_A, \| \, \|_{F_p})$ in $(\mathcal{P}_A, [\![\,]\!]_p)$, and $\|i_{A,p}\|$ its norm.

We will make use of the two following theorems, the first of which is an improvement of Rider's one, since it claims that it is enough to test the assumptions of that theorem on polynomials with coefficients 0 or 1.

Theorem 3.3 (**Pisier** [18]). *The subset Λ is Sidon if and only if there exists a constant $c > 0$ such that*

$$[\![A]\!]_2 \geq c\,|A|, \quad \text{for all finite } A \subset \Lambda. \tag{3.1}$$

(see [10], Chapitre 12, Théorème I.2).

Theorem 3.4. *(***Rodríguez-Piazza** *[20]; [21], Teorema IV.1.3) There exists a numerical constant K such that the following inequality holds :*

$$K^{-1} q(A) \leq \|i_{A,2}\|^2 \leq K q(A). \tag{3.2}$$

In particular, we have :

$$q(A) \geq K^{-1} \frac{[\![A]\!]_2^2}{|A|}\,. \tag{3.3}$$

(see also [10], Chapitre 12, Exercice 12.1, for a proof). To get (3.3) from (3.2), it suffices to observe that $\| \sum_{\gamma \in A} \gamma \|_2^2 = |A|$.

Let us go back to the proof of Theorem 3.1. If Λ is not Sidon, by (3.1), we have:

$$\inf_{A \subset \Lambda,\, |A| < \infty} \frac{[\![A]\!]_2}{|A|} = 0,$$

or equivalently:

$$C_N := \inf_{\substack{A \subset \Lambda \\ 1 \leq |A| \leq N}} \frac{[\![A]\!]_2}{|A|} \underset{N \to \infty}{\longrightarrow} 0.$$

Let now $\delta > 0$ and very small, and N_0 be the smallest integer such that $C_{N_0} < \delta$. Then, there exists a finite $A_0 \subset \Lambda$, with $|A_0| \leq N_0$ (and actually $|A_0| = N_0$ by definition of N_0) such that $\frac{[\![A_0]\!]_2}{|A_0|} < \delta$. We claim that we can find an integer N and disjoint, quasi-independent subsets $B_1, \ldots, B_N$ of A_0 such that (K being as in (3.2)) :

$$K^{-1} \delta \, [\![A_0]\!]_2 \geq |B_j| \geq K^{-1} \frac{\delta}{2} \, [\![A_0]\!]_2, \quad \forall \; 1 \leq j \leq N; \tag{3.4}$$

$$|B_1 \cup \cdots \cup B_N| \geq \frac{|A_0|}{2} \cdot \tag{3.5}$$

Let us first see how (3.4) and (3.5) allow us to finish the proof. They imply together:

$$|A_0| \leq 2 \sum_{j=1}^{N} |B_j| \leq 2K^{-1} N \delta \, [\![A_0]\!]_2 \leq 2K^{-1} N \delta^2 |A_0|,$$

and so:

$$N \geq \frac{K}{2} \, \delta^{-2}. \tag{3.6}$$

Now, the lower p-estimate of Theorem 2.3 as well as the fact that $|B| \leq c_p^{-1} [\![B]\!]_p$ for quasi-independent sets (which are uniformy Sidon as we already mentioned), where c_p is a constant, gives us, a being as in (2.2):

$$\begin{aligned} [\![A_0]\!]_p^p &\geq a^{-p} \sum_{j=1}^{N} [\![B_j]\!]_p^p \geq a^{-p} c_p^p \sum_{j=1}^{N} |B_j|^p \geq a^{-p} \Big(\frac{c_p}{2K} \Big)^p N \delta^p [\![A_0]\!]_2^p \\ &\geq b_p \, \delta^{p-2} [\![A_0]\!]_2^p, \end{aligned}$$

where b_p only depends on p, and where we used (3.4) and (3.6). Now, since $p < 2$ and since δ is arbitrarily small, this inequality proves that $[\![A_0]\!]_p$ is much bigger than $[\![A_0]\!]_2$ and ends the proof.

It remains to show (3.4) and (3.5). We first observe that

$$A' \subsetneqq A_0 \quad \Longrightarrow \quad [\![A']\!]_2 \geq \delta|A'|, \tag{3.7}$$

and then:

$$\frac{[\![A_0]\!]_2}{|A_0|} \geq \frac{\delta}{2}\,. \tag{3.8}$$

Indeed, the inequality (3.7) follows from the fact that N_0 is the smallest integer such that $C_{N_0} < \delta$. For the second inequality (3.8), let $\emptyset \neq A' \subsetneqq A_0$, and set $A'' = A_0 \setminus A'$. The first inequality, applied to A' and A'', as well as the unconditionality of the $[\![\ ]\!]_2$-norm, give:

$$[\![A_0]\!]_2 \geq [\![A']\!]_2 \geq \delta|A'| \qquad \text{and} \qquad [\![A_0]\!]_2 \geq [\![A'']\!]_2 \geq \delta|A''|.$$

Adding those two inequalities gives (3.8).

Now, (3.3) and (3.8) allow us to find a quasi-independent set $B_1 \subset A_0$ such that

$$|B_1| \geq K^{-1}\frac{[\![A_0]\!]_2^2}{|A_0|} \geq K^{-1}\frac{\delta}{2}\,[\![A_0]\!]_2.$$

We first notice that we can assume that $|B_1| \leq K^{-1}\delta[\![A_0]\!]_2$, provided that we reduce B_1. Indeed, this can be done as far as we are sure that $K^{-1}\delta[\![A_0]\!]_2 \geq 1$. This latter fact can be proved in the following way: if we had $1 \geq K^{-1}\delta[\![A_0]\!]_2$ then $K\delta^{-1} \geq [\![A_0]\!]_2 \geq |A_0|^{1/2}$. Since $[\![A_0]\!]_2 \geq c|A_0|^{1/2}(\log|A_0|)^{1/2}$, for some absolute constant $c > 0$, and $\delta \geq \dfrac{[\![A_0]\!]_2}{|A_0|}$, we would have $\delta \geq c|A_0|^{-1/2}(\log|A_0|)^{1/2} \geq cK^{-1}\delta(\log|A_0|)^{1/2}$. Hence $|A_0|$ would be less than $e^{(Kc^{-1})^2}$. We conclude that δ would be greater than an absolute constant $e^{-(Kc^{-1})^2} = \delta_0 > 0$, which is wrong up to a choice of δ small enough (for instance less than $\delta_0/2$) at the beginning of the proof.

If $|B_1| \geq |A_0|/2$, we stop.

Otherwise, we proceed as follows: suppose more generally that we have found disjoint quasi-independent sets $B_1, \dots, B_N \subset A_0$ satisfying (3.4). If they also verify (3.5), we stop. If they do not, we set:

$$A' = A_0 \setminus (B_1 \cup \cdots \cup B_N), \qquad |A'| \geq \frac{|A_0|}{2}\,.$$

As before, we can find $B_{N+1} \subset A'$, quasi-independent, such that:

$$|B_{N+1}| \geq K^{-1}\frac{[\![A']\!]_2^2}{|A'|} \geq K^{-1}\delta^2|A'| \geq K^{-1}\delta^2\frac{|A_0|}{2} \geq \frac{K^{-1}\delta}{2}\,[\![A_0]\!]_2,$$

and we can also assume that $|B_{N+1}| \leq K^{-1}\delta[\![A_0]\!]_2$. Therefore, after a finite number of steps, we will have performed (3.4) and (3.5). And, as we already said, this ends the proof of Theorem 3.1. $\square$

We can extend Theorem 3.4 from the gaussian to the general p-stable framework.

Theorem 3.5. *There exists a numerical constant $K = K_p$ (depending only on p) such that the following inequality holds :*

$$K^{-1}q(A) \leq \|i_{A,p}\|^{p'} \leq Kq(A). \tag{3.9}$$

In particular, we have :

$$q(A) \geq K^{-1}\left(\frac{[\![A]\!]_p}{|A|^{1/p}}\right)^{p'}. \tag{3.10}$$

Remark. The lower bound given by this theorem has to be compared to the one of a different kind (involving Orlicz funcions) given by [9], Theorem 3.2. Actually, both inequalities are useful as this will be the case in the proof of Theorem 5.1.

Proof. The lower bound is easy to obtain: let $B \subset A$ be a quasi-independent set such that $|B| = q(A)$. We have $[\![B]\!]_p \geq c|B|$, for some $c > 0$ (depending only on p), since B is a Sidon set with a universal constant.

Then, testing the norm of $i_{A,p}$ on the function $f = \sum_{\gamma\in B}\gamma$, we have

$$\|i_{A,p}\| \geq \frac{[\![B]\!]_p}{|B|^{1/p}} \geq c|B|^{\frac{1}{p'}} = c(q(A))^{\frac{1}{p'}}$$

which was the claim.

For the upper bound, take any $f \in F_p$. There exists a polynomial P such that $\hat{P} \geq 1/4e$ on A, $\|P\|_1 = 1$ and $\log_2\|P\|_\infty \leq 5eq(A)$ (see [21], Lema 1.2 or [10], p. 513). Then $[\![f]\!]_p \leq 4e[\![f * P]\!]_p$ by the contraction principle.

Thanks to Theorem 2.1, we have

$$[\![f * P]\!]_p \leq C\|f\|_{F_p}.\|P\|_{L^{\varphi_{p'}}},$$

for some $C > 0$ (depending on p only).

Finally,

$$\|P\|_{L^{\varphi_{p'}}} \approx \int_G |P|\big(1+\log(1+|P|)\big)^{1/p'} dx \leq \|P\|_1.\big(1+\log(1+\|P\|_\infty)\big)^{1/p'}$$

Hence, we have some $k > 0$ (depending on p only) such that

$$\|P\|_{L^{\varphi_{p'}}} \leq k\big(q(A)\big)^{1/p'}$$

The conclusion follows: $[\![f]\!]_p \leq K\|f\|_{F_p}.\big(q(A)\big)^{1/p'}$. □

Remark. Let us emphasize that Proposition 1.1 can be proved very quickly using the previous theorem in the particular case of p-stable variables. Indeed, testing the hypothesis of this proposition with $f = \sum_{\gamma \in A} \gamma$, where A is any finite subset of Λ, we have: $[\![f]\!]_p \geq C^{-1}|A|$ so that $q(A) \geq c|A|^{p'(1-1/p))} = c|A|$ for some constant $c > 0$ (depending only on p).

4 p-stable q-Rider sets

Our aim in this Section is to introduce an apparently new notion of thin set, that of *p-stable q-Rider set*, and to compare it with the previously known notion of *q-Rider set*, which might be called *2-stable q-Rider set* to recall that it is defined with Gaussian (equivalently Rademacher) variables. We will always assume that p, q are given in such a way that

$$1 \leq q < p \leq 2,$$

and we will say that $\Lambda \subset \Gamma$ is a *p-stable q-Rider set* if the following a priori inequality holds:

$$\|f\|_{F_q} \leq C\,[\![f]\!]_p, \qquad \forall f \in \mathcal{P}_\Lambda. \tag{4.1}$$

The reader should compare with (1.8) to see what is new here. It should be emphasized that the behaviour of the $[\![\ ]\!]_p$ -norm is very different from that of the $[\![\ ]\!]_2$-one: for example, if $G = \mathbb{T}$, $e_n(t) = \mathrm{e}^{int}$ and $f(t) = \sum_{n=1}^{N} a_n e_n(t)$, Marcus and Pisier ([14], Remark 5.7), extending a result of Salem and Zygmund for $p = p' = 2$ (see [10], Chapitre 13, Proposition III.13), proved that, if $a = (a_1, \ldots, a_N)$ and $N \geq 2$:

$$[\![f]\!]_p \geq C_0\, N^{1/p} (\log N)^{1/p'}\, \frac{1}{N} \sum_{n=1}^{N} |a_n| \tag{4.2}$$

and that the reverse inequality holds for $a_1 = \cdots = a_N = 1$. So, we might expect that the new notion introduced is highly depending on p (and on q, of course!). This is not quite the case, as will be apparent in our results, which will mainly consist in two theorems: a "functional-type" condition, indicating that those sets can be defined by other a priori inequalities, and an *equivalence Theorem* showing that indeed this notion is nothing but s-Riderness for some value of s, depending on p and q. It will be convenient to recall the following definitions and facts:

1. For $r > 0$, ψ_r will denote the Orlicz function $x \mapsto \mathrm{e}^{x^r} - 1$, $x \geq 0$.
   If $A \subset \Gamma$ is finite, we set

   $$\psi_r(A) = \Big\| \sum_{\gamma \in A} \gamma \Big\|_{\psi_r}. \tag{4.3}$$

2. The space F_p and its norm have already been defined in the Introduction: see (1.7).

3. If X, Y are two Banach spaces continuously contained in $L^1(G)$, the set $\mathcal{M}(X, Y)$ of multipliers of X to Y is the set of families $m = (m_\gamma)_{\gamma \in \Gamma}$ of complex numbers such that, whenever $f = \sum_\gamma a_\gamma \gamma \in X$, then

   $$g = \sum_\gamma m_\gamma a_\gamma \gamma \in Y, \qquad \text{with} \quad \|g\|_Y \leq C \|f\|_X.$$

   The best constant C being called the multiplier-norm of m and being denoted by $\|m\|_{\mathcal{M}(X,Y)}$.

4. We denote by $\mathcal{C}^{p-as}$ the completion of $\mathcal{P}$ with respect to the $[\![\ ]\!]_p$-norm; it is the so-called Banach space of almost surely continuous p-stable random Fourier series ([14]). Then, the dual space of $\mathcal{C}^{p-as}$ is

isomorphic to $\mathcal{M}(F_p, L^{\psi_{p'}})$, the set of multipliers from F_p to $L^{\psi_{p'}}$. This result (well-known for $p = 2$: see [13]) follows in a standard way from the delicate Theorem 2.1, as it is already the case for $p = 2$ and we will not detail that formal extension.

5. Once and for all, we set

$$\varepsilon = \frac{p-q}{q(p-1)} = 1 - \frac{p'}{q'}\cdot \tag{4.4}$$

$$\frac{1}{\alpha} = \frac{1}{p} + \frac{1}{q'} \tag{4.5}$$

$$\beta = \frac{\varepsilon}{p'} + \frac{1}{p} - \frac{1}{2} = \frac{1}{q} - \frac{1}{2}\cdot \tag{4.6}$$

We first prove the following simple proposition (the case $q = 2$ being already known [11]), which will actually follow from Theorem 4.3 below, but which will motivate this theorem.

Proposition 4.1 (Mesh condition). *Let $\Lambda \in \mathbb{Z}$ be a p-stable q-Rider set. Then, there exists a constant K such that, for each integer $N \geq 2$, one has :*

$$|\Lambda \cap \{1, \ldots, N\}| \leq K(\log N)^{\frac{(p-1)q}{p-q}} = K(\log N)^{\frac{1}{\varepsilon}}. \tag{4.7}$$

Proof. Set $B = \Lambda \cap \{1, \ldots, N\} = \{\lambda_1 < \cdots < \lambda_n\}$, and $f = \sum_{j=1}^{n} e_{\lambda_j}$. The assumption and Lemma 2.4 give us, for some constant C, that $n^{\frac{1}{q}} \leq C[\![f]\!]_p \leq CC_0 n^{\frac{1}{p}} (\log N)^{\frac{1}{p'}}$. Grouping terms gives $n \leq K(\log N)^{\frac{(p-1)q}{p-q}}$, where we set $K = (CC_0)^{\frac{pq}{p-q}}$, proving our claim. □

We will now prove the main result of this section:

Theorem 4.2. *Let $\Lambda \subset \Gamma$.*
Λ is a p-stable q-Rider set if and only if Λ is an s-Rider set, with $s = \frac{2q'}{2q'-p'}\cdot$

Actually we are going to prove the following more precise theorem:

Theorem 4.3. *Let $\Lambda \subset \Gamma$. Then, the following conditions are equivalent:*

(1) Λ *is a p-stable q-Rider set;*

(2) $\ell_{q'}(\Lambda) \hookrightarrow \mathcal{M}(F_p, L^{\psi_{p'}})$*;*

(3) $\ell_\alpha(\Lambda) \hookrightarrow L^{\psi_{p'}}$*;*

(4) $\psi_{p'}(A) \leq C\,|A|^{1/\alpha}$ *for some constant $C > 0$, and every finite subset A of Λ;*

(5) $q(A) \geq c\,|A|^\varepsilon$*, for some constant $c > 0$, and for all finite subsets $\{0\} \neq A \subset \Lambda$;*

(6) Λ *is an s-Rider set, with $s = \frac{2q'}{2q'-p'} = \frac{2q(p-1)}{p-2q+pq}$.*

Point out the simple relation: $2q' = s'p'$. Moreover, to precise the behavior for the "degenerate" cases: when $q = 1$, we have $s = 1$ as well (remember Proposition 1.1). On the other hand, when $s = 1$, $q = 1$ and any p fits.

Let us make some comments. This result is known for $p = 2$ and has been proved by the fourth-named author ([20], [21], Teorema III.2.3). The symbol $\hookrightarrow$ means that the left-hand space is mapped to the right-hand one by means of the Fourier transform or of its inverse. Recall that $q(A)$ denotes the largest possible cardinality of a quasi-independent subset of A, and that the definition of quasi-independent sets is given at the beginning of the proof of Theorem 3.1.

Remark 1. We know (see [11]) that the *mesh condition* for s-Rider reads as $|\Lambda \cap \{1, \ldots, N\}| \leq K(\log N)^{\frac{s}{2-s}}$. But, $\frac{s}{2-s} = \frac{1}{\varepsilon}$, and we fall again on (4.7) of Proposition 4.1.

Remark 2. The preceding theorem can be read in two ways. First, any p-stable q-Rider set is actually an s-Rider set for the right value of s. But on the other hand, if one fixes some $s \in (1, 2)$ and a set Λ which is an s-Rider set, one can choose either p or q in order to realize Λ as a p-stable q-Rider set. Nevertheless, one has to be careful. Let us precise this:

a. If one fixes $p \in (1, 2]$, then one can choose q such that $q' = s'p'/2$ and Λ is a p-stable q-Rider set. Point out that $q < p \leq 2$.

b. If one fixes $q \in (1, 2)$, then one can choose $p \in (1, 2]$ such that $p' = \frac{2q'}{s'}$ if and only if we have $s \geq q$. In that case, Λ is a p-stable q-Rider set.

Proof of Theorem 4.3.
(1) $\Leftrightarrow$ (2). The Fourier transform maps X_Λ to $\ell_q(\Lambda)$ if and only if its transpose maps $\ell_{q'}(\Lambda)$ to the dual of X_Λ. The result now easily follows from the previous description of the dual of X.
(2) $\Rightarrow$ (3). Let $f = \sum_{\gamma\in\Lambda} c_\gamma \gamma \in \mathcal{P}_\Lambda$. We can write (in short: see (4.5), we have $\ell_\alpha = \ell_p \,.\, \ell_{q'}$):

$$c_\gamma = a_\gamma b_\gamma,$$

where

$$|a_\gamma| = |c_\gamma|^{1-\theta}, \qquad |b_\gamma| = |c_\gamma|^{\theta},$$

with $\theta = \frac{\alpha}{p} = 1 - \frac{\alpha}{q'}$ and

$$\|a\|_{q'} = \|c\|_\alpha^{1-\theta}, \qquad \|b\|_p = \|c\|_\alpha^{\theta}.$$

If we set $P = \sum_\gamma b_\gamma \gamma$, we have from (2) that $\|f\|_{\psi_{p'}} \leq C\,\|a\|_{q'} \|P\|_{F_p}$, where C is some constant. Equivalently: $\|f\|_{\psi_{p'}} \leq C\,\|c\|_\alpha^{1-\theta+\theta} = C\,\|c\|_\alpha$, which was our claim.
(3) $\Rightarrow$ (2). This is obvious since $\frac{1}{\alpha} = \frac{1}{p} + \frac{1}{q'}$.
(3) $\Rightarrow$ (4). Indeed, $\psi_{p'}(A) \leq C\,\|\widehat{1\!\!1_A}\|_\alpha = C\,|A|^{\frac{1}{\alpha}}$.
(4) $\Rightarrow$ (5). We use a result that we proved in ([9], Proposition 3.2), namely that, for any finite set $A \subset \Gamma$, $A \neq \{0\}$, and any $r > 0$, we have:

$$q(A) \geq C_r \Big(\frac{|A|}{\psi_r(A)}\Big)^r \tag{4.8}$$

We use (4.8) with $r = p'$. Our assumption implies that $q(A) \geq C|A|^{(1-\frac{1}{\alpha})p'}$, and

$$\Big(1 - \frac{1}{\alpha}\Big)p' = \Big(1 - \frac{1}{q'} - \frac{1}{p}\Big)p' = \Big(\frac{1}{p'} - \frac{1}{q'}\Big)p' = 1 - \frac{p'}{q'} = \varepsilon.$$

(5) $\Rightarrow$ (6). By [20] or [21], Teorema III.2.3, we know that this condition is equivalent to the fact that Λ is an s-Rider set where $\varepsilon = \frac{2}{s} - 1$. As $\varepsilon = 1 - \frac{p'}{q'}$, we have $s = \frac{2q'}{2q'-p'}$.
(6) $\Rightarrow$ (3). Either using [20], [21], Teorema III.2.3 or applying the equivalence of (1) and (3) when $p = 2$ and s instead of q, we know that $\ell_{\tilde{\alpha}}(\Lambda) \hookrightarrow L^{\psi_2}$ where $\frac{1}{\tilde{\alpha}} = \frac{1}{2} + \frac{1}{s'}$. On the other hand, we obviously have $\ell_1(\Lambda) \hookrightarrow L^\infty$. As

$p > q$, we have $\tilde{\alpha} > \alpha$, so a standard interpolation argument implies that $\ell_\alpha(\Lambda) \hookrightarrow L^{\psi_r}$ with

$$\frac{1}{\alpha} = \frac{\theta}{\tilde{\alpha}} + \frac{1-\theta}{1} \qquad \text{and} \qquad \frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{\infty}$$

We obtain $\theta = \Big(\frac{1}{q'} - \frac{1}{p'}\Big)\Big(\frac{1}{s'} - \frac{1}{2}\Big)^{-1} = \frac{2}{p'}$, since $s'p' = 2q'$. Hence $r = \frac{2}{\theta} = p'$.
□

The following theorem has two aims. First the proof of the preceding theorem is not self-contained: to prove that (5) ⇒ (6), we used the fact that the theorem was already known for $p = 2$. The following theorem provides a proof of this result as well.

On the other hand, this will actually provide a stronger result, which cannot a priori be obtained just using the case $p = 2$ (i.e. assuming the results of [20] or [21]). Nevertheless, the proof proceeds as in [21], using a difficult lemma of Bourgain on quasi-independent sets.

Though there is essentially no change with regard to [21], we will give the details, for the convenience of the reader. The links beetween the values of the parameters are the same as before.

Theorem 4.4. *Let $\Lambda \subset \Gamma$. The following conditions are equivalent:*

(i) $q(A) \geq c\,|A|^\varepsilon$, for some constant $c > 0$, and for all finite subsets $\{0\} \neq A \subset \Lambda$;

(ii) $\mathcal{C}_\Lambda^{p-as} := X_\Lambda \hookrightarrow \ell_{q,1}(\Lambda)$;

(iii) Λ is a p-stable q-Rider set;

(iv) $\mathcal{C}_\Lambda^{p-as} := X_\Lambda \hookrightarrow \ell_{q,\infty}(\Lambda)$.

Recall that $\ell_{q,1} = \ell_{q,1}(\Lambda)$ (resp. $\ell_{q,\infty} = \ell_{q,\infty}(\Lambda)$) is the Lorentz space of families $a = (a_\gamma)_{\gamma\in\Lambda}$ tending to 0 whose decreasing rearrangement $(a_n^*)_n$ satisfies $\displaystyle\sum_{n\geq 1} \frac{a_n^*}{n^{1/q'}} < \infty$ (resp. $\displaystyle\sup_{n\geq 1} n^{1/q} a_n^* < \infty$).

One has $\ell_{q,1}(\Lambda) \hookrightarrow \ell_q(\Lambda) \hookrightarrow \ell_{q,\infty}(\Lambda)$.

Proof. The implications $(ii) \Rightarrow (iii) \Rightarrow (iv)$ are obvious.

$(iv) \Rightarrow (i)$ is easy with help of Theorem 3.5: for any finite subset A of Λ, we have

$$q(A) \geq K^{-1}\bigg(\frac{[\![A]\!]_p}{|A|^{1/p}}\bigg)^{p'} \geq K_1\big(|A|^{1/q-1/p}\big)^{p'}$$

which gives (iv), since $p'\left(\frac{1}{q}-\frac{1}{p}\right)=p'\left(\frac{1}{p'}-\frac{1}{q'}\right)=\varepsilon$.

It remains to prove the difficult part ($(i)\Rightarrow(ii)$) of this theorem. We first need the following lemma.

Lemma 4.5 ([21], Lema III.2.6)**.** *Let Λ be a set satisfying the condition* (5) *of Theorem 4.3. For any finite subset A of Λ such that $A\neq\{0\}$ and $c\,|A|^{\varepsilon}\geq 2$, there exist N pairwise disjoint quasi-independent sets $B_1,\ldots,B_N\subset A$ such that:*

a) $\frac{2}{c}\,|A|^{1-\varepsilon}\geq N\geq\frac{1}{2c}\,|A|^{1-\varepsilon}$*;*

b) $c\,|A|^{\varepsilon}\geq|B_j|\geq\frac{c}{2}\,|A|^{\varepsilon}$ *for every $j=1,\ldots,N$.*

Proof of Lemma 4.5. By our assumption, there is a quasi-independent subset B of A such that $|B|\geq c\,|A|^{\varepsilon}$. Since $\frac{c}{2}\,|A|^{\varepsilon}\geq 1$, we can find a (quasi-independent) subset B_1 of B such that $\frac{c}{2}\,|A|^{\varepsilon}\leq|B_1|\leq c\,|A|^{\varepsilon}$.

Assume now that pairwise disjoint quasi-independent subsets $B_1,\ldots,B_n$ of A has been constructed such that $c\,|A|^{\varepsilon}\geq|B_j|\geq\frac{c}{2}\,|A|^{\varepsilon}$ for $1\leq j\leq n$. There are two possibilities:

(i) If $|A|/2>\left|\bigcup_{j=1}^{n}B_j\right|$, we choose in $A\setminus\bigcup_{j=1}^{n}B_j$ a quasi-independent subset B_{n+1} whose cardinal is

$$|B_{n+1}|\geq c\left|A\setminus\bigcup_{j=1}^{n}B_j\right|^{\varepsilon}\geq c\left(\frac{|A|}{2}\right)^{\varepsilon}\geq\frac{c}{2}\,|A|^{\varepsilon},$$

and which we can also choose such that $|B_{n+1}|\leq c\,|A|^{\varepsilon}$.

(ii) If $|A|/2\leq\left|\bigcup_{j=1}^{n}B_j\right|$, we stop the process at $N=n$. Indeed, we then have:

$$\frac{|A|}{2}\leq\left|\bigcup_{j=1}^{N}B_j\right|\leq N\,c\,|A|^{\varepsilon}$$

and hence $N\geq\frac{1}{2c}\,|A|^{1-\varepsilon}$. On the other hand, since $B_1,\ldots,B_N$ are disjoint, one has

$$|A|\geq\left|\bigcup_{j=1}^{N}B_j\right|=\sum_{j=1}^{N}|B_j|\geq N\,\frac{c}{2}\,|A|^{\varepsilon};$$

hence $N\leq\frac{2}{c}\,|A|^{1-\varepsilon}$.

That ends the proof of Lemma 4.5. $\square$

Now, we have to show that there exists a constant $C>0$ such that

$$\|\widehat{f}\|_{q,1}\leq C\,[\![f]\!]_p$$

for every trigonometric polynomial $f \in \mathcal{P}_\Lambda$.

By homogeneity, we may assume that $\|\hat{f}\|_\infty = 1$, and we consider the level sets

$$A_j = \{\gamma \in \Lambda\,;\; 2^{-j+1} \geq |\widehat{f}(\gamma)| > 2^{-j}\}\,, \qquad j = 1, 2, \ldots$$

Recall now the following result of Bourgain ([1], Lemma 2; see also [2], and [10], Chapitre 12, Lemme I.10).

Lemma 4.6 (**Bourgain**). *There exists a numerical constant $R > 10$ such that for every family $B_1, \ldots, B_L$ of pairwise disjoint finite quasi-independent sets such that*

$$\frac{|B_{l+1}|}{|B_l|} \geq R\,, \qquad \textit{for } l = 1, \ldots, L-1,$$

one can find, for each $l = 1, \ldots, L$, a subset $C_l \subset B_l$ such that:

$$|C_l| \geq \frac{1}{10}\,|B_l|, \qquad \textit{for all } l = 1, \ldots, L,$$

and the union $\bigcup_{l=1}^{L} C_l$ is quasi-independent.

Setting $R_1 = \max\{R^{1/2\beta}, (2R)^\varepsilon\}$, where R is the above constant and β is given by (4.6), we define $j_1 = 1$ and

$$j_{l+1} = \min\{j > j_l\,;\; |A_j| > R_1\,|A_{j_l}|\}, \tag{4.9}$$

whenever this last set is nonempty; we stop and take $L = l$ when it is empty (this eventually happens since f is a trigonometric polynomial). Set

$$N_l = |A_{j_l}|.$$

We have the following upper estimate:

$$\|\widehat{f}\|_{q,1} \leq \sum_{j\geq 1} 2^{-j+1} \|\mathbb{1}_{A_j}\|_{q,1} \leq \sum_{l=1}^{L} \sum_{j_l \leq j < j_{l+1}} 2^{-j+1} q\,|A_j|^{1/q}\,,$$

since

$$\|\mathbb{1}_A\|_{q,1} = \sum_{n=1}^{|A|} \frac{1}{n^{1/q'}} \leq \int_0^{|A|} \frac{dx}{x^{1/q'}} = q\,|A|^{1/q}.$$

Now, we have $|A_j| \leq R_1 |A_{j_l}|$ for $j_l \leq j < j_{l+1}$; hence we get

$$\begin{aligned}\|\widehat{f}\|_{q,1} &\leq q \sum_{l=1}^{L} R_1^{1/q} 2^{-j_l} |A_{j_l}|^{1/q} \sum_{j_l \leq j < j_{l+1}} 2^{-j+1+j_l} \\ &\leq 4q R_1^{1/q} \sum_{l=1}^{L} 2^{-j_l} N_l^{1/q}. \end{aligned} \tag{4.10}$$

We are now going to use Lemma 4.5. For this purpose, let us denote by T the first index l such that $c\, N_l^{2\beta} \geq 2$. If no such an index exists, we will set $T = L + 1$.

We will split the sum (4.10) into two parts.

First, the Cauchy-Schwarz inequality gives

$$\begin{aligned}\sum_{l=1}^{T-1} 2^{-j_l} N_l^{1/q} &\leq \Big(\sum_{l=1}^{T-1} 2^{-2j_l} N_l\Big)^{1/2} \Big(\sum_{l=1}^{T-1} N_l^{2\beta}\Big)^{1/2} \\ &\leq \Big(\sum_{\gamma \in \Lambda} |\widehat{f}(\gamma)|^2\Big)^{1/2} \left[\sum_{l=1}^{T-1} \Big(\frac{N_{T-1}}{R_1^{T-l-1}}\Big)^{2\beta}\right]^{1/2} \\ &= \|f\|_2 N_{T-1}^{\beta} \Big(\sum_{k \geq 0} R_1^{-2k\beta}\Big)^{1/2}.\end{aligned}$$

Since $N_{T-1}^{2\beta} < 2/c$ and $R_1^{2\beta} \geq R > 10 > 4$, we get

$$\sum_{l=1}^{T-1} 2^{-j_l} N_l^{1/q} \leq \|f\|_2 \Big(\frac{2}{c}\Big)^{1/2} \Big(\frac{4}{3}\Big)^{1/2}$$

and

$$\sum_{l=1}^{T-1} 2^{-j_l} N_l^{1/q} \leq \frac{2}{\sqrt{c}} [\![f]\!]_p. \tag{4.11}$$

For $l \geq T$, we apply Lemma 4.5 to the set A_{j_l}: we get M_l pairwise disjoint quasi-independent subsets $B_{l,1}, \ldots, B_{l,M_l} \subset A_{j_l}$ such that

$$\frac{c}{2} N_l^{\varepsilon} \leq |B_{l,m}| \leq c N_l^{\varepsilon}\,, \qquad m = 1, \ldots, M_l, \tag{4.12}$$

with

$$\frac{1}{2c}\,N_l^{1-\varepsilon} \le M_l \le \frac{2}{c}\,N_l^{1-\varepsilon}\,, \qquad l = T, \ldots, L. \tag{4.13}$$

Since $N_{l+1} \ge R_1 N_l$, we get, for $T \le l < L$:

$$\frac{M_L}{M_l} \ge \frac{R_1^{1-\varepsilon}}{4} \ge \frac{1}{4} \tag{4.14}$$

and

$$\frac{|B_{l+1,m}|}{|B_{l,m'}|} \ge \frac{R_1^{\varepsilon}}{2} \ge R. \tag{4.15}$$

Using (4.14), we can find, for each $l = T, \ldots, L-1$, a map

$$\phi_l \colon \{1, \ldots, M_L\} \to \{1, \ldots, M_l\}$$

such that

$$|\phi_l^{-1}(m)| \le 4\,\frac{M_L}{M_l}\,, \qquad m = 1, \ldots, M_l.$$

Applying, for each $m = 1, \ldots, M_L$, Lemma 4.6 to the sequence

$$B_{T,\phi_T(m)}, \ldots, B_{L-1,\phi_{L-1}(m)}, B_{L,m}\,,$$

thanks to (4.15), we get, for each $l = T, \ldots, L$ and each $m = 1, \ldots, M_L$, a quasi-independent set $C_{l,m}$ such that

$$|C_{l,m}| > \frac{1}{10}\,|B_{l,\phi_l(m)}| \ge \frac{c}{20}\,N_l^{\varepsilon}\,, \quad l = T, \ldots, L, \tag{4.16}$$

and such that

$$\bigcup_{l=T}^{L} C_{l,m} \text{ is quasi-independent for } m = 1, \ldots, M_L. \tag{4.17}$$

We now introduce, for $m = 1, \ldots, M_L$, the trigonometric polynomial

$$g_m = \sum_{l=T}^{L} \Big(\frac{M_l}{4M_L}\Big)^{1/p} 2^{-j_l} \sum_{\gamma \in C_{l,m}} \gamma.$$

We claim that, for every $\gamma \in \Gamma$, one has

$$|\widehat{f}(\gamma)|^p \ge \sum_{m=1}^{M_L} |\widehat{g}_m(\gamma)|^p. \tag{4.18}$$

Indeed, it suffices to check that for γ in the spectra of the g_m's, and if γ is in some set B_{l,m_0}, it cannot be in another one, because these sets are pairwise disjoint; hence

$$\begin{aligned}\sum_{m=1}^{M_L} |\widehat{g}_m(\gamma)|^p &= 2^{-pj_l}\frac{M_l}{4M_L}\left|\{m\,;\ \gamma\in C_{l,m}\}\right| \leq 2^{-pj_l}\frac{M_l}{4M_L}\,|\phi_l^{-1}(m_0)| \\ &\leq 2^{-pj_l} \leq |\widehat{f}(\gamma)|^p.\end{aligned}$$

It follows from the lower p-estimate of the norm $[\![\ ]\!]_p$ (Theorem 2.3) that

$$a\,[\![f]\!]_p \geq \Big(\sum_{m=1}^{M_L}[\![g_m]\!]_p\Big)^{1/p}. \tag{4.19}$$

But, by (4.17), the spectrum of g_m is quasi-independent, and every quasi-independent set is a Sidon set, with constant ≤ 5 (see the beginning of the proof of Theorem 3.1), so $\|\widehat{g}_m\|_1 \leq 5\,\|g_m\|_\infty$, and $\|\widehat{g}_m\|_1 \leq 5\,[\![g_m]\!]_p$, by Proposition 1.2. It follows, from (4.13) and (4.16), that:

$$\begin{aligned}5\,[\![g_m]\!]_p &\geq \sum_{l=T}^{L}|C_{l,m}|\Big(\frac{M_l}{4M_L}\Big)^{1/p}2^{-j_l} \geq \frac{1}{4^{1/p}M_L^{1/p}}\sum_{l=T}^{L}\frac{c}{20}\,N_l^{\varepsilon}\Big(\frac{N_l^{1-\varepsilon}}{2c}\Big)^{1/p}2^{-j_l} \\ &= \frac{c^{1/p'}}{20(8M_L)^{1/p}}\sum_{l=T}^{L}N_l^{\varepsilon+\frac{1-\varepsilon}{p}}2^{-j_l}.\end{aligned}$$

Since

$$\varepsilon+\frac{1-\varepsilon}{p} = \Big(1-\frac{1}{p}\Big)\varepsilon+\frac{1}{p} = \frac{\varepsilon}{p'}+\frac{1}{p} = \Big(\frac{1}{q}-\frac{1}{p}\Big)+\frac{1}{p} = \frac{1}{q}\,,$$

we get

$$[\![g_m]\!]_p \geq \frac{c^{1/p'}}{100(8M_L)^{1/p}}\sum_{l=T}^{L}N_l^{1/q}2^{-j_l}. \tag{4.20}$$

Therefore (4.19) gives

$$\begin{aligned}[\![f]\!]_p &\geq \frac{c^{1/p'}}{100a(8M_L)^{1/p}}\left[\sum_{m=1}^{M_L}\Big(\sum_{l=T}^{L}N_l^{1/q}2^{-j_l}\Big)^p\right]^{1/p} \\ &= \frac{c^{1/p'}}{100a.8^{1/p}}\sum_{l=T}^{L}N_l^{1/q}2^{-j_l}. \qquad (4.21)\end{aligned}$$

Putting (4.21) together with (4.11), we get:

$$\sum_{l=1}^{L} N_l^{1/q} 2^{-j_l} \leq \Big(\frac{2}{\sqrt{c}} + \frac{100a.8^{1/p}}{c^{1/p'}}\Big) [\![f]\!]_p.$$

It remains to use (4.10) to obtain:

$$\|\widehat{f}\|_{q,1} \leq 4qR_1^{1/q} \Big(\frac{2}{\sqrt{c}} + \frac{100a.8^{1/p}}{c^{1/p'}}\Big) [\![f]\!]_p,$$

and achieve the proof of Theorem 4.4. $\square$

5 The link with Orlicz spaces

We are going to characterize s-Rider sets in terms of continuous mapping to Orlicz spaces (remember the beginning of Section 2).

Theorem 5.1. *Let $\Lambda \subset \Gamma$, $s \in (1,2)$ and r be greater than both 2 and $\rho = \frac{2-s}{s-1}$. Let $\tilde{p} = \frac{2r}{2r-\rho}\cdot$*

The following conditions are equivalent:

(i) Λ is an s-Rider set;

(ii) $\mathcal{C}_\Lambda^{\tilde{p}-as} \hookrightarrow L^{\psi_r}$;

(iii) For every finite subset A of Λ, we have $\psi_r(A) \leq C[\![A]\!]_{\tilde{p}}$, where C does not depend on A.

Proof. $(i) \Rightarrow (ii)$. We already know that we can realize Λ as a p-stable q-Rider set with $p = r' \leq 2$. Then the value of q is fixed by the relation $q' = rs'/2$. By (3) of Theorem 4.3, we know that $\ell_\alpha(\Lambda) \hookrightarrow L^{\psi_r}$ with $\frac{1}{\alpha} = \frac{1}{p} + \frac{1}{q'} = 1 - \frac{2-s}{sr}$. Let us point out that $\alpha < 2$.

Now we can use Theorem 4.3 again to realize Λ as a $\tilde{p}$-stable α-Rider set but only (see Remark 2 after that theorem) when $s \geq \alpha$. This condition is fulfilled since it is equivalent to the condition $r \geq \rho$.

Moreover the value of $\tilde{p}$ is fixed by the relation $\tilde{p}\,' = \frac{2\alpha'}{s'} = \frac{2r}{2r-\rho}\cdot$ The conclusion follows.

$(ii) \Rightarrow (iii)$. This is obvious.

$(iii) \Rightarrow (i)$. Fix any finite subset A of Λ. We shall use $\varepsilon = \frac{2}{s} - 1\cdot$

If $[\![A]\!]_{\tilde{p}} \leq |A|^{1-\frac{\varepsilon}{r}}$. Then $\psi_r(A) \leq C|A|^{1-\frac{\varepsilon}{r}}$. Hence, by [9], Proposition 3.2,

$$q(A) \geq c_r \left(\frac{|A|}{\psi_r(A)} \right)^r \geq C'_r |A|^{\varepsilon}.$$

If not, then $[\![A]\!]_{\tilde{p}} \geq |A|^{1-\frac{\varepsilon}{r}}$; but

$$q(A) \geq K^{-1} \left(\frac{[\![A]\!]_{\tilde{p}}}{|A|^{1/\tilde{p}}} \right)^{\tilde{p}'}$$

by Theorem 3.5, so we obtain that $q(A) \geq c|A|^{(1-\frac{\varepsilon}{r}-\frac{1}{\tilde{p}})\tilde{p}'}$.

Now, a quick computation gives $\tilde{p}\,' = \frac{2r}{\rho}$ and we conclude that

$$q(A) \geq c|A|^{\varepsilon}.$$

So, in every case, we have $q(A) \geq c|A|^{\varepsilon}$ and this characterizes the fact that Λ is an s-Rider set. □

Remark. The preceding theorem extend Theorem 3.1 of [9]. More precisely, when $s \leq \frac{4}{3}$, we can choose $r = \rho$, so $\tilde{p} = 2$ and we recover the version of [9].

When $s \geq \frac{4}{3}$, we can take $r = 2$ and this gives $\tilde{p} = \frac{4(s-1)}{5s-6}$.

The previous result leads naturally to investigate more specifically thin sets involving both random norms and Orlicz spaces. This was done for instance by the authors in [9], where (among other things) they studied the notion of $\Lambda^{as}(q)$-sets. This latter notion is actually weaker than of the notion of s-Rider, or equivalently (through the previous theorem) the notion of what we could call $\Lambda^{p-as}(\psi_r)$-set. It is not known whether the notion of $\Lambda^{as}(q)$-sets is actually different of the usual notion of $\Lambda(q)$-sets.

In the following, we add several results on $\Lambda^{as}(q)$-sets. Let us first precise the definition.

Definition 5.2. *A subset Λ of Γ is said to be a $\Lambda^{p-as}(q)$-set, $1 < p \leq 2$, $q > 2$, if there is a constant $C > 0$ such that, for every trigonometric polynomial $f \in \mathcal{P}_\Lambda$, one has:*

$$\|f\|_q \leq C\,[\![f]\!]_p.$$

For $p = 2$, this is the notion of $\Lambda^{as}(q)$-set introduced in [9]. Since the p-stable norms $[\![\ ]\!]_p$ dominate the Gaussian norm $[\![\ ]\!]$, which dominates the norm $\|\ \|_2$, every $\Lambda(q)$-set is a $\Lambda^{as}(q)$-set, and every $\Lambda^{as}(q)$-set is a $\Lambda^{p-as}(q)$-set for $1 < p < 2$. However:

Proposition 5.3. *If Λ is a $\Lambda(q)$-set, with $q > 2$, then, for $1 < p < 2$, it is a $\Lambda^{p-as}(r)$-set, with $r = \frac{p'}{2}\, q > q$.*

This is particularly interesting when Λ is a "true" $\Lambda(q)$-set, i.e. a $\Lambda(q)$-set which is not a $\Lambda(s)$-set for any $s > q$ (see [3] and [25]).

It is worth to note that this differs from the case $p = 2$, since we proved in [9], Theorem 4.3, that for any $r > 2$, there exist sets which are not $\Lambda^{as}(r)$-sets, though they are $\Lambda(q)$-sets for every $q < r$.

Proof. Since Λ is a $\Lambda(q)$-set, the inverse Fourier transform is continuous from $\ell_2(\Lambda)$ to L^q_Λ. Since it is trivially continuous from $\ell_1(\Lambda)$ to L^∞_Λ, we get, by interpolation, that is is continuous from $\ell_p(\Lambda)$ to L^r_Λ, with $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}$ and $\frac{1}{r} = \frac{1-\theta}{\infty} + \frac{\theta}{q}$, i.e. $r = \frac{p'}{2}\, q$. Hence, for every $f \in F_p(\Lambda)$ ($a \lesssim b$ meaning $a = O\,(b)$ and $a \approx b$ that $a \lesssim b$ and $b \lesssim a$):

$$\begin{aligned} \|f\|_r \lesssim \Big(\sum_\Lambda |\widehat{f}(\gamma)|^p\Big)^{1/p} &\approx \mathbb{E}\,\Big|\sum_\Lambda Z_\gamma \widehat{f}(\gamma)\,\gamma\Big| \\ &\leq \mathbb{E}\,\Big\|\sum_\Lambda Z_\gamma \widehat{f}(\gamma)\,\gamma\Big\|_\infty = [\![f]\!]_p. \end{aligned}$$

□

Remarks. Since Hausdorff-Young inequality asserts that $\|f\|_{p'} \leq \|\widehat{f}\|_p$ and since, as seen above, $\|\widehat{f}\|_p \lesssim [\![f]\!]_p$, every subset of Γ is a $\Lambda^{p-as}(q)$-set for $q \leq p'$. Hence this notion is only interesting for $q > p'$. Note that in Proposition 5.3, one has $r > p'$.

For $1 \leq q < p \leq 2$, the same inequality $\|f\|_{q'} \leq \|\widehat{f}\|_q = \|f\|_{F_q}$ shows that every p-stable q-Rider set is a $\Lambda^{p-as}(q')$-set.

We will not investigate further this notion here, but only give two results about their thinness.

Proposition 5.4. *For every $\Lambda^{p-as}(q)$-set $\Lambda \subset \mathbb{Z}$, there is $\kappa > 0$ such that, for every $N \geq 1$:*

$$|\Lambda \cap [1, N]| \leq \kappa\, N^{p'/q} \log N.$$

It follows that the set $\mathbb{S}$ of squares is not a $\Lambda^{p-as}(q)$-set of $\mathbb{Z}$ when $q > 2p'$.

Proof. It follows the classical one. Write $\Lambda \cap [1, N] = \{\lambda_1, \ldots, \lambda_n\}$, and consider the trigonometric polynomial $f = \sum_{j=1}^n e_{\lambda_j}$, where $e_{\lambda_j}(t) = \mathrm{e}^{i\lambda_j t}$. By Lemma 2.4, one has $[\![f]\!]_p \leq C\, n^{1/p} (\log \lambda_n)^{1/p'}$.

Now, on the other hand, $n = f * D_N(0)$, where D_N is the N^{th} Dirichlet kernel; hence

$$n \leq \|f\|_q \|D_N\|_{q'} \leq K\,[\![f]\!]_p N^{1/q}.$$

Hence, since $\lambda_n \leq N$, we get $n \leq \kappa\, n^{1/p} (\log N)^{1/p'} N^{1/q}$, and the result follows. $\square$

Corollary 5.5. *Let α be an integer ≥ 2, and $r_\alpha(j)$ is the number of ways to write j as a sum of α elements of Λ. If Λ is an $\Lambda^{p-as}(2\alpha)$-set of $\mathbb{N}$, then*

$$\frac{1}{n} \sum_{j=1}^{n} r_\alpha^2(j) \lesssim n^{\frac{2-p}{p-1}} (\log n)^{2\alpha}.$$

Proof. We follow Rudin's proof of Theorem 4.5 of [23]. Writing $\Lambda = \{n_1, n_2, \ldots\}$, one consider the trigonometric polynomial

$$f(t) = \mathrm{e}^{in_1 t} + \cdots + \mathrm{e}^{in_k t}.$$

One has

$$f^\alpha(t) = r_\alpha(0) + r_\alpha(1)\mathrm{e}^{it} + \cdots,$$

and so $\sum_{j=1}^{n_k} r_\alpha^2(j) \leq \|f\|_{2\alpha}^{2\alpha}$. But:

$$\|f\|_{2\alpha} \lesssim [\![f]\!]_p \lesssim k^{1/p} (\log n_k)^{1/p'},$$

and, by Proposition 5.4, one has $k \lesssim n_k^{p'/2\alpha} \log n_k$; we get hence $[\![f]\!]_p^{2\alpha} \lesssim n_k^{p'/p} (\log n_k)^{2\alpha}$, and the result follows. $\square$

Of course other random variables might be used instead of p-stable ones. In particular, 1-stable ones, for which one has the quasi-norm (see [14], page 296):

$$[\![f]\!]_1 = \sup_{c>0} c\, \mathbb{P}\Big(\Big\| \sum_\gamma Z_\gamma \widehat{f}(\gamma)\, \gamma \Big\|_\infty > c\Big),$$

where $(Z_\gamma)_\gamma$ is an i.i.d. family of 1-stable random variables. A characterization of the continuity of 1-stable random Fourier series is given in [15] and [24].

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