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ATRACTORES PARA EDP PARABÓLICAS NO LINEALES Y NO AUTÓNOMAS EN DOMINIOS NO ACOTADOS

Attractors for nonlinear and non-autonomous parabolic PDEs in unbounded domains

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A mi madre y a mi padre

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Introduction

Our understanding of the fundamental processes of the natural world is based to a large extent on partial differential equations. Examples are the vibrations of solids, the flow of fluids, the diffusion of chemicals, the spread of heat, the structure of molecules, the interactions of photons and electrons, and the radiation of electromagnetic waves. Partial differential equations also play a central role in modern mathematics, especially in geometry and analysis.

The study of partial differential equations (PDE's) started in the 18th century in the work of Euler, d'Alembert, Lagrange and Laplace as a central tool in the description of mechanics of continua and more generally, as the principal mode of analytical study of models in the physical science.

The analysis of physical models has remained to the present day as one of the fundamental concerns of the development of PDE's. Beginning in the middle of the 19th century, particularly with the work of Riemann, PDE's also became an essential tool in other branches of mathematics.

This duality of viewpoints has been central to the study of PDE's through the 19th and 20th century. On the one side, one always has the relationship to models in physics, engineering and other applied disciplines. On the other side, there are the potential applications, which have often turned out to be quite revolutionary, of PDE's as an instrument in the development of other branches of mathematics. This dual perspective was clearly stated for the first time by H. Poincaré in his prophetic paper *Sur les équations aux dérivées partielles de la physique mathématique* in 1890.

The major example of second-order parabolic PDE's, the heat equation,

$$\frac{\partial u}{\partial t} = \Delta u,$$

was introduced by Fourier in his celebrated memoir *Théorie analytique de la chaleur* in the first decade of the 19th century.

Besides this classical example, a profusion of equations, associated with major physical phenomena, appeared in the period between 1750 and 1900: the wave equation, the Euler equation, the Laplace equation, the Poisson equation, and the Navier-Stokes equations.

When a phenomenon from Physics, Chemistry, Biology, Economics can be described by a system of partial differential equations where the existence of solution can be assured, one of the most interesting problems is to know what is the asymptotic behaviour of the system when time grows to infinity. The study of the asymptotic behaviour of the system is giving us relevant information about the future of the phenomenon described in the model. In this context, the concept of *global attractor* has become a very useful tool to describe the long-time behaviour of many important dynamical systems: deterministic systems in both the autonomous and non-autonomous cases and the stochastic flows generated by stochastic differential equations. An important step in all these theories is the development of a general abstract framework in which to express the underlying dynamics of the problem (semiflows, processes and cocycles respectively).

The study of the global attractor in autonomous systems has been developed extensively over the past thirty years and has become now a classical theory with books like Babin and Vishik [16], Hale [38], Ladyzhenskaya [52], Robinson [68] or Temam [81]. In the autonomous case, the global attractor is an invariant compact set which attracts all the trajectories of the system, uniformly on bounded sets for the initial data. This set has, in general, a very complicated geometry which reflects the complexity of the long-time behaviour of the systems.

However, given the underlying models that arise in various branches of the sciences, it is very natural to extend the theory to treat non-autonomous systems. In the finite-dimensional framework, i.e., for non-autonomous ordinary differential equations in $\mathbb{R}^N$, the long-time behaviour of non-autonomous dynamical systems has been studied by means of the theory of skew-product flows (see the pioneering works by Miller [61] and Sell [72]). However, most of the progress in the infinite-dimensional context, i.e., for non-autonomous partial differential equations, has been made during the last two decades.

The first attempts to extend the notion of global attractor to the non-autonomous case led to the concept of the so-called uniform attractor (see [26]). It is remarkable that the conditions ensuring the existence of the uniform attractor parallel those for autonomous systems. To this end, non-autonomous systems are lifted in [83] to autonomous ones by expanding the phase space. Then, the existence of uniform attractors relies on some compactness property of the solution operator associated to the systems. However, one disadvantage of this uniform attractor is that it need not be invariant unlike the global attractor for autonomous systems.

At the same time, the theory of pullback (or cocycle) attractors has been developed for both the non-autonomous and random dynamical systems (see

Crauel *et al.* [28], Kloeden and Schmalfuß [51], Langa and Schmalfuß [54] and Schmalfuß [71]), and has shown to be very useful in the understanding of the dynamics of non-autonomous dynamical systems.

In this case, the concept of pullback (or cocycle or non-autonomous) attractor provides a time-dependent family of compact sets which attracts families of sets in a certain universe. This concept is an appropriate extension of the autonomous concept of attractor.

There exists the pullback attractor of fixed bounded sets as the most usual option. See, for instance [28] for a brief description on deterministic pullback attractors for non-autonomous systems. On the other hand, several authors use the concept of attraction in a *tempered universe*, i.e. a universe $\mathcal{D}$ not only composed by fixed sets, but also moving in time, which usually appears in applications and is defined in terms of a tempered condition. We will denote $\mathcal{A}_{\mathcal{D}_F^X}$ the attractor in the first case and $\mathcal{A}_{\mathcal{D}}$ the attractor in the second one.

This work is divided in five chapters. The first is a developed of the theory of pullback attractors and the third is a generalization of this theory, where we show the main results of the theory of set-valued non-autonomous dynamical systems.

Chapter 1 is split into two sections. In Section 1.1 we recall some concepts from the framework of process and give some relevant results that will be necessary on the next section. The most relevant existence result for minimal pullback attractors, Theorem 1.11, is given in Section 1.2. In this section we also relate the pullback attractor for a tempered universe, with the usual attractor of fixed bounded sets.

Our aim in Chapter 3 is to give a generalization of the results from Chapter 1 to the set-valued framework. In Section 3.1 we give the definitions and concepts related to multi-valued non-autonomous dynamical systems. In Section 3.2 we show Theorem 3.10, the most important existence result for minimal pullback attractors, and, as in Chapter 1, we relate the two notions of attractors.

The other three chapters are devoted to three problems related to a non-autonomous reaction-diffusion equation.

In Chapter 2, we will give an example of non-autonomous reaction-diffusion equation with zero Dirichlet boundary condition, in a bounded domain containing a non-autonomous forcing term taking values in the space L^2, and with a continuous nonlinearity which ensures uniqueness of solution. We will apply results of Chapter 1 to prove Theorem 2.14, where we show the existence of minimal pullback attractors for this problem.

In Chapter 4, we will give a more general problem, a non-autonomous reaction-diffusion equation with zero Dirichlet boundary condition, in an unbounded domain containing a non-autonomous forcing term taking values in the space H^{-1}, and with a continuous nonlinearity which does not ensure uniqueness of solution. Using results of Chapter 3, in Theorem 4.14 we prove the existence of minimal pullback attractors for this problem. We ensure in Theorem 4.25 and Remark 4.26 that the pullback attractors are connected.

Finally, in Chapter 5, we will study a non-autonomous reaction-diffusion equation with dynamical boundary conditions, in a bounded domain and with a continuous nonlinearity which ensures uniqueness of solution. We will ensure, in Theorem 5.14, the existence of minimal pullback attractors for this problem applying results of Chapter 1.

In these last three chapters, we ensure the existence of minimal pullback attractors in the frameworks of universes of fixed bounded sets and tempered universes. We also establish the relation between both attractors.

A *process* U on the metric space X is a map $U : (t, \tau, x) \in \mathbb{R}^2_d \times X \mapsto U(t, \tau)x \in X$ such that

(a) $U(\tau, \tau)x = x$,

(b) $U(t, \tau)x = U(t, r)U(r, \tau)x \quad$ for all $\tau \leq r \leq t$ and $x \in X$,

where $\mathbb{R}^2_d := \{(t, \tau) \in \mathbb{R}^2 : t \geq \tau\}$.

If we have the following general non-autonomous system with uniqueness of solution

$$\begin{cases} \dfrac{\partial u}{\partial t} - A(t)u = f(u) + h(t), \\ u(\tau) = u_\tau, \end{cases} \tag{1}$$

we can write the unique solution of the system as $u(t; \tau, u_\tau) = U(t, \tau)u_\tau$.

A family of compact sets $\mathcal{A}_\mathcal{D} = \{\mathcal{A}_\mathcal{D}(t) : t \in \mathbb{R}\}$ is called the *pullback* $\mathcal{D}-$*attractor* for the process U, if it possesses the following properties:

(a) $\mathcal{A}_\mathcal{D}$ attracts all subsets of $\mathcal{D}$ in the pullback sense , i.e.

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(U(t, \tau)D(\tau), \mathcal{A}_\mathcal{D}(t)) = 0 \quad \text{for all } \widehat{D} \in \mathcal{D}, \quad t \in \mathbb{R},$$

where dist_X denotes the Hausdorff semi-distance in X.

(b) $\mathcal{A}_\mathcal{D}$ is invariant, i.e. $U(t, \tau)\mathcal{A}_\mathcal{D}(\tau) = \mathcal{A}_\mathcal{D}(t) \quad$ for all $\tau \leq t$.

A result for the existence of minimal pullback $\mathcal{D}-$attractor of this work is Theorem 1.11 in Chapter 1, where, under several assumptions, we have guaranteed the existence of the minimal pullback $\mathcal{D}-$attractor, $\mathcal{A}_\mathcal{D}$, which is defined by

$$\mathcal{A}_\mathcal{D}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}^X \quad t \in \mathbb{R},$$

where $\overline{\{\cdots\}}^X$ is the closure in X and $\Lambda(\widehat{D},t) := \bigcap_{s\leq t} \overline{\bigcup_{\tau\leq s} U(t,\tau)D(\tau)}^X$ for all $t \in \mathbb{R}$. In addition, if $\widehat{D}_0 = \{D_0(t) : \ t \in \mathbb{R}\}$ is a family of nonempty sets of X, Theorem 1.11 provides a structure of the pullback attractor, ensuring that if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0,t) \subset \overline{D_0(t)}^X$, for all $t \in \mathbb{R}$.

In Corollary 1.13 and Remark 1.14 we relate this concept with the usual attractor of fixed bounded sets, based on the paper of Marín-Rubio and Real [58].

However, these results may not be applied to a wide class of initial-boundary problems, in which the solution may not be unique. Typical and important examples of set-valued problems are those arising in control theory and viability theory, or from partial differential equations without uniqueness or without known uniqueness (as the three-dimensional Navier-Stokes equations) or differential inclusions.

When one is interested in studying the asymptotic behaviour of solutions to set-valued problems, we can still expect the attractor to be useful. The development of a theory of set-valued dynamical systems is a necessary first step in the study of attractors for such problems.

In recent years, set-valued semiflows on general Banach spaces have been considered by various authors, often in the context of partial differential equations or inclusions, as for instance Elmounir and Simondon [29], Kapustyan *et al.* [45], Kapustyan and Valero [46] or Melnik and Valero [60].

Many interesting systems are in fact non-autonomous, although most concepts have been developed only in the more convenient setting of autonomous systems. It is of practical importance as well as intellectual interest to see how such concepts generalize to non-autonomous systems.

The theory of set-valued non-autonomous dynamical systems is now well established as has been extensively developed over the last one and a half decades. We can find results about this theory in the work of Caraballo and Kloeden [20], among others. Most results in Chapter 3 are slight modifications and generalizations of results of this paper. We establish a sufficient condition for the existence of pullback attractors for set-valued non-autonomous systems.

As in the single-valued case, there exist pullback attractors of fixed bounded sets and in the framework of a universe $\mathcal{D}$ where the families of time-dependent sets are given by a tempered condition on their growth in time.

A multi-valued map, on the metric space X, $U : \mathbb{R}^2_d \times X \mapsto \mathcal{P}(X)$ is called a *multi-valued non-autonomous dynamical system* (*MNDS*) on X if

(a) $U(\tau,\tau,x) = \{x\}$ for all $\tau \in \mathbb{R}$, $x \in X$,

(b) $U(t,\tau,x) \subset U(t,s,U(s,\tau,x))$ for all $\tau \leq s \leq t$, $x \in X$,

where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X.

If we have the following general non-autonomous system without uniqueness of solution

$$\begin{cases} \dfrac{\partial u}{\partial t} - A(t)u = f(u) + h(t), \\ u(\tau) = u_\tau, \end{cases} \tag{2}$$

we can denote by $S(\tau, u_\tau)$ the set of all solutions of this system defined for all $t \geq \tau$.

We define a multi-valued non-autonomous dynamical system by

$$U(t,\tau,u_\tau) = \{u(t) : \ u \in S(\tau,u_\tau)\}, \quad \tau \leq t. \tag{3}$$

.

A family of compact sets $\mathcal{A}_\mathcal{D} = \{\mathcal{A}_\mathcal{D}(t) : t \in \mathbb{R}\}$ is called the *global pullback $\mathcal{D}-$attractor* for the MNDS U, if has the following properties:

(a) $\mathcal{A}_\mathcal{D}$ attracts all subsets of $\mathcal{D}$ in the pullback sense , i.e.

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau,D(\tau)), \mathcal{A}_\mathcal{D}(t)) = 0 \quad \text{for all } \widehat{D} \in \mathcal{D}, \quad t \in \mathbb{R}.$$

(b) $\mathcal{A}_\mathcal{D}$ is negatively invariant, i.e.

$$\mathcal{A}_\mathcal{D}(t) \subset U(t,\tau,\mathcal{A}_\mathcal{D}(\tau)), \text{ for any } (t,\tau) \in \mathbb{R}^2_d.$$

In Chapter 3, we developed the theory of set-valued non-autonomous dynamical systems, which is a generalization of the theory given in Chapter 1. In particular, Theorem 3.10 shows the existence of minimal global pullback $\mathcal{D}$-attractor for the MNDS, $\mathcal{A}_\mathcal{D}$, providing the same structure than in the univalued case. Moreover, under several assumptions we can ensure a new result about the connectedness of the minimal global pullback $\mathcal{D}$-attractor.

As in the univalued case, in Corollary 3.11 and Remark 3.12 we relate this concept with the usual attractor of fixed bounded sets, based on the paper of Marín-Rubio and Real [58].

In this thesis we present three problems related to a non-autonomous reaction-diffusion equation. The first two ones are considered with zero Dirichlet boundary conditions, one with uniqueness of solution in a bounded domain and with the non-autonomous term belonging to the space L^2, and the other one without uniqueness of solution in an unbounded domain with the non-autonomous term belonging to the space H^{-1}. The third one is considered with dynamical boundary conditions and with uniqueness of solution in a bounded domain.

We prove the existence of minimal pullback attractors for these equations. We use the theory of pullback attractors (developed in Chapter 1) for the equations with uniqueness of solution and the theory of set-valued non-autonomous dynamical systems (developed in Chapter 3) for the equation without uniqueness of solution.

The first problem which has been studied in the present work is the following,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(u) + h(t) \ \text{ in } \ \Omega \times (\tau, +\infty)\,, \\ u = 0 \ \text{ on } \ \partial\Omega \times (\tau, +\infty)\,, \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $f \in C^1(\mathbb{R})$ and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$. We assume that there exist positive constants α_1, α_2, k, l, and $p > 2$ such that

$$-k - \alpha_1 |s|^p \leq f(s)s \leq k - \alpha_2 |s|^p\,, \ \ \forall s \in \mathbb{R}, \tag{5}$$

$$f'(s) \leq l, \quad \forall s \in \mathbb{R}. \tag{6}$$

Under these assumptions, in Section 2.2 we prove that (4) has a unique solution. It is also worth mentioning that our problem has received much attention over the last years, as we will recall now.

In [22] it is proved the existence of pullback attractor in the space $L^2(\Omega)$ when the domain is bounded and h is unbounded but with polynomial growth, i.e

$$\|h(t)\|_{L^2(\Omega)} \leq k_1 |t|^\alpha + k_2,$$

where k_1, k_2 and α are nonnegative constants.

When Ω is bounded and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and is translation bounded, i.e.

$$\sup_{m \in \mathbb{R}} \int_m^{m+1} \|h(s)\|_{L^2(\Omega)}\, ds < \infty, \tag{7}$$

the existence of a pullback attractor in the space $H^1_0(\Omega)$ is proved in [74], while in [55] the translation bounded condition (7) is weakened to

$$\|h(s)\|^2_{L^2(\Omega)} \leq M e^{\alpha |s|},$$

where $0 \leq \alpha \leq \lambda_1$, and λ_1 denotes the first eigenvalue of the negative Laplacian.

In [84], the existence of pullback attractor in $H^1_0(\Omega)$ is shown for a bounded domain and for a $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ such that

$$\int_{-\infty}^t e^{\sigma s} \left(\|h(s)\|^2_{L^2(\Omega)} + \|h'(s)\|^2_{L^2(\Omega)} \right) ds < +\infty,$$

for all $t \in \mathbb{R}$ and certain $\sigma \geq 0$. For a bounded domain Ω, and a translation bounded function $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, the existence of a uniform attractor in $L^p(\Omega)$ is proved in [75].

Finally, the reader can find similar results for several variants of our model in the references [2], [3], [67], [77], among others.

We prove in Section 2.4 a sufficient condition ensuring the existence of minimal pullback attractors in $L^2(\Omega)$ for (4) in the framework of universes of fixed bounded sets and given by a tempered growth condition. We discuss the relation between these families. In Sections 2.5 and 2.6, we also prove some regularity results and some exponential growth results for pullback attractors of our model when time goes to $-\infty$. These results can be found in Anguiano *et al.* [4, 5].

Now, we consider a more general problem, namely

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(x,u) + h(t) \quad \text{in} \quad \Omega \times (\tau, +\infty), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (\tau, +\infty), \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{8}$$

where $\Omega \subset \mathbb{R}^N$ is a nonempty open set, not necessarily bounded, and suppose that Ω satisfies the Poincaré inequality, $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and satisfies that there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$, and $p \geq 2$ and positive functions $C_1(x)$, $C_2(x) \in L^1(\Omega)$ such that

$$|f(x,s)|^{\frac{p}{p-1}} \leq \alpha_1 |s|^p + C_1(x) \quad \forall s \in \mathbb{R},\, x \in \Omega, \tag{9}$$

$$f(x,s)s \leq -\alpha_2 |s|^p + C_2(x) \quad \forall s \in \mathbb{R},\, x \in \Omega. \tag{10}$$

In Section 4.2 we prove the existence of solution, but these assumptions do not ensure uniqueness of solution of (8). We mention below some papers which are significant contributions in any of the cases considered when uniqueness of solutions cannot be ensured or does not hold.

The study of reaction-diffusion equations without uniqueness of solutions in a bounded domain in the autonomous case (i.e. h does not depend on the time t), or in the non-autonomous case but with strong uniformity properties on the time dependent terms, can be found in [7], [29], [41], [42], [44], [47], [48], [70], [78], [82], where the classical theory of global attractor is adapted to handle this set-valued case.

In the autonomous case, when the domain Ω is unbounded, but we have non-uniqueness of solutions, several studies on the problem can be found in [63] and [64], among others.

Due to the non-autonomous character of the problem, we have to use an appropriate framework. Being possible to choose amongst several theories (skew-product flows, uniform attractors, trajectory attractors, pullback attractors) we will use the theory of set-valued non-autonomous dynamical systems (developed in Chapter 3) since this allows for more generality in the non-autonomous terms (see [3], [20], [21], [23], [24], [58], [59], for some results concerning pullback attractors and several reasons justifying the interest of using this theory).

We show in Section 4.3 a sufficient condition ensuring the existence of minimal pullback attractors in $L^2(\Omega)$ for (8), in the frameworks of universes of fixed bounded sets and families of sets depending on time. We also discuss the relation between these attractors. These results can be found in Anguiano *et al.* [6].

When we consider a partial differential equation with non-uniqueness of the Cauchy problem it is interesting to consider the *Kneser property*, that is, the connectedness and compactness of the set of values attained by the solutions at any moment of time. In particular, this problem has been studied for reaction-diffusion equations by several authors so far.

In this direction some results are known for scalar reaction-diffusion equations in bounded domains in the case where the nonlinearity has at most linear growth [43]. Such results were extended later in [47, 48] for systems of reaction-diffusion equations with nonlinearities having more than linear growth (see also [45]), with applications to the complex Ginzburg-Landau equation and the Lotka-Volterra system with diffusion, among others. Also, the Kneser property for degenerate reaction-diffusion equations was considered in [10, 11].

When the domain is unbounded the problem has additional technical difficulties. A first result in this direction was given in [49], in which it is studied a scalar reaction-diffusion on unbounded domains in which the nonlinear term is equal to $|u|^{\frac{1}{2}}$. In [64] the results of [48] were extended to unbounded domains. However, due to technical difficulties it was necessary to assume an additional condition concerning the derivatives of the nonlinear terms. We improve the method of the proof given in [64] in order to avoid such condition. This improvement has not been considered in the literature yet, as far as we know.

We observe that the function $C_1(x)$ in (9) belongs just to $L^1(\Omega)$, but for the Kneser property we need the stronger condition $C_1(x) \in L^1(\Omega) \cap L^\infty(\Omega)$.

In Section 4.4 we prove the Kneser property for our model (8) and using this property of solutions in Section 4.5 we also check the connectedness of the associated global pullback attractor.

In the sixth section of Chapter 4 we consider a system of reaction-diffusion equations with $\Omega = \mathbb{R}^N$, which was studied before in [64]. Using a similar technique as for problem (8) we improve the results of that paper, proving the Kneser property and the connectedness of the global attractor without using an extra condition on the derivative of the nonlinear term of the equation. Instead, as in problem (8), we have to assume that the functions appearing in the growth condition of the nonlinear term belong to $L^\infty(\Omega)$.

In the last section of Chapter 4 we apply these results to a generalized logistic equation. All these results can be found in Anguiano *et al.* [9].

The next step is to consider a non-autonomous reaction-diffusion equation with other boundary conditions. In this sense, partial differential equations with dynamical boundary conditions arise for example in hydrodynamics and the heat transfer theory. For instance, they allow to model heat flow inside the considered domain subject to nonlinear heating or cooling at the boundary, or heat transfer in a solid in contact with a moving fluid, in thermoelasticity, heat transfer in two mediums, etc.

This type of problems has been studied by many authors (e.g., cf. [1, 13, 30, 31, 36, 40, 56, 66] and the references therein).

Several approaches have been used for these problems, like the theory of semigroups, with Bessel potential and Besov spaces, and of course the variational setting as well. Some questions addressed concerning these models are the local and global existence of solutions or blow-up phenomena.

Another question is the study of these problems under the introduction of singular perturbations. For instance, in [66] the behaviour of solutions of a singularly perturbed model (damped wave equation) when the introduced parameter goes to zero and the relation with the limit problem is analyzed.

A different sort of question, with a great variety of results, is again the long-time behaviour of the (global) solutions. For an autonomous model, the existence of a global attractor is, for instance, studied in [33], although the nonlinearity is the same in the domain and in the boundary (see also [86]).

For a non-autonomous reaction-diffusion equation and using the approach of skew-product formulation, the existence of a uniform attractor is established.

But to our knowledge, there does not seem to be in the literature any study of the existence of pullback attractors for non-autonomous dynamical systems associated to this kind of problems (up to the stochastic framework, e.g., cf. [27]).

In this sense, the last non-autonomous reaction-diffusion equation which has been studied in the present work is the following,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta\, u + \kappa u + f(u) = h(t) & \text{in } \ \Omega \times (\tau, \infty), \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial \vec{n}} + g(u) = \rho(t) & \text{on } \ \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u_\tau(x), & \text{for } \ x \in \Omega, \\ u(x, \tau) = \psi_\tau(x), & \text{for } \ x \in \partial\Omega, \end{cases} \tag{11}$$

where $\vec{n}$ is the outer normal to $\partial\Omega$, $\tau \in \mathbb{R}$ is an initial time, and

$$\kappa > 0, \quad u_\tau \in L^2(\Omega), \quad \psi_\tau \in L^2(\partial\Omega), \tag{12}$$

$$h \in L^2_{loc}\left(\mathbb{R}; L^2(\Omega)\right), \quad \rho \in L^2_{loc}\left(\mathbb{R}; L^2(\partial\Omega)\right), \tag{13}$$

are given.

We also assume that the functions f and $g \in C(\mathbb{R})$ are given, and satisfy that there exist constants $p \geq 2$, $q \geq 2$, $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta > 0$, and $l > 0$, such that

$$\alpha_1 |s|^p - \beta \leq f(s)s \leq \alpha_2 |s|^p + \beta, \quad \text{for all } s \in \mathbb{R}, \tag{14}$$

$$\alpha_1 |s|^q - \beta \leq g(s)s \leq \alpha_2 |s|^q + \beta, \quad \text{for all } s \in \mathbb{R}, \tag{15}$$

$$(f(s) - f(r))(s - r) \geq -l(s - r)^2, \quad \text{for all } s, r \in \mathbb{R}, \tag{16}$$

and

$$(g(s) - g(r))(s - r) \geq -l(s - r)^2, \quad \text{for all } s, r \in \mathbb{R}. \tag{17}$$

In Section 5.2 we prove that (11) has a unique solution.

As we mentioned before, we only have references in the literature of this approach in the stochastic context, with the help of random dynamical systems. In that sense, a particularly interesting situation is treated in [27]. There, the authors obtain the existence of a random attractor for a general class of stochastic parabolic equations with dynamical boundary conditions, under the restrictive assumptions $p = q$ and $|f(s) - g(s)| \leq c(1 + |s|)$.

In Section 5.4 we obtain, without these assumptions, the existence of minimal pullback attractors for (11), in the frameworks of universes of fixed bounded sets and that given by a tempered growth condition, using a continuous dependence result which is proved using an energy method in Section 5.3. Moreover, we discussed the relation between these two notions of attractors. These results can be found in Anguiano *et al.* [8].

Spanish Summary

Este trabajo está dividido en cinco capítulos. En los Capítulos 1 y 3, se trata la parte teórica de los sistemas dinámicos no autónomos dentro del marco de los procesos y de los sistemas dinámicos no autónomos multivaluados. Por otro lado, en los Capítulos 2, 4 y 5, se aplica esta teoría a tres problemas para una ecuación de reacción-difusión no autónoma.

En el Capítulo 1 se ofrece una visión de la teoría sobre atractores pullback. Este capítulo está dividido en dos secciones. En la Sección 1.1 recordamos algunas definiciones y conceptos dentro del marco de los procesos y desarrollamos algunos resultados previos necesarios para demostrar el teorema principal de existencia de atractores pullback. En la Sección 1.2 demostraremos el resultado más relevante de este capítulo sobre la existencia de atractores pullback minimales, el Teorema 1.11, y relacionamos el atractor pullback de fijos acotados con el atractor pullback para una clase, $\mathcal{D}$, de familias parametrizadas en tiempo.

Nuestro objetivo en el Capítulo 3 es exponer una generalización de los resultados del Capítulo 1 dentro del marco de los sistemas multivaluados. Para ello, es necesario redefinir los conceptos dentro del marco de los sistemas dinámicos no autónomos multivaluados (ver Sección 3.1). En la Sección 3.2 se muestra un resultado de existencia de atractores pullback minimales, el Teorema 3.10, donde el nuevo resultado es el apartado sobre el carácter conexo del atractor pullback. Se relacionan, igual que en el Capítulo 1, el atractor pullback de fijos acotados con el atractor pullback para una clase de familias parametrizadas en tiempo.

En el Capítulo 2 se considera la siguiente ecuación

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(u) + h(t) \quad \text{in} \quad \Omega \times (\tau, +\infty)\,, \\ u = 0 \quad \text{on} \quad \partial\Omega \times (\tau, +\infty)\,, \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{18}$$

donde $\Omega \subset \mathbb{R}^N$ es un conjunto abierto acotado, $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $f \in C^1(\mathbb{R})$ y $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$. Asumimos ciertas hipótesis sobre el término no lineal f de tal manera que (18) tenga una única solución, lo cual se prueba en la Sección 2.2. Gracias a la unicidad de solución, podemos definir un proceso para nuestro problema y a partir de ahí, en el Teorema 2.14 demostramos la existencia de atractores pullback minimales para el proceso asociado a (18). En las Secciones 2.5 y 2.6 probamos algunos resultados de regularidad y de crecimiento exponencial para los atractores pullback asociados a nuestro modelo. Estos resultados pueden ser encontrados en Anguiano *et al.* [4, 5].

En el Capítulo 4 consideramos un problema mucho más general,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(x,u) + h(t) \quad \text{in} \quad \Omega \times (\tau, +\infty), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (\tau, +\infty), \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{19}$$

donde $\Omega \subset \mathbb{R}^N$ es un conjunto abierto no vacío, no necesariamente acotado, y suponemos que en Ω se satisface la desigualdad de Poincaré, $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ y $f : \Omega \times \mathbb{R} \to \mathbb{R}$ es una función Carathéodory satisfaciendo ciertas hipótesis que no aseguran unicidad de solución del problema (19). En la Sección 4.2 probamos la existencia de solución para este problema. Definiendo el sistema dinámico no autónomo multivaluado como el conjunto de todas las soluciones de (19), se prueba en el Teorema 4.14 la existencia de atractores pullback minimales para el sistema dinámico no autónomo multivaluado asociado a (19). Estos resultados se encuentran publicados en Anguiano *et al.* [6].

Al no tener unicidad en el problema de Cauchy, es interesante considerar la *Propiedad Kneser* para este problema, es decir, la compacidad y conexión del conjunto de valores alcanzados por las soluciones en cualquier instante de tiempo. En este sentido, asumiendo una condición adicional en las hipótesis del término no lineal f, en la Sección 4.4 probamos la propiedad Kneser para nuestro problema (19) mejorando el método usado en el artículo [64]. Usando esta propiedad somos capaces de demostrar, en el Teorema 4.25 y la Nota 4.26, la conexión de los atractores pullback asociados al problema (19). Un caso que no ha sido considerado todavía en la literatura matemática, hasta donde nosotros conocemos.

En la Sección 4.6, usando una técnica similar a la usada para el problema (19), mejoramos los resultados del artículo [64]. Finalmente, en la última sección del Capítulo 4, aplicamos estos resultados a una ecuación logística generalizada. Todos estos resultados relacionados con la Propiedad Kneser pueden encontrarse en Anguiano *et al.* [9].

El siguiente objetivo que nos planteamos es considerar una ecuación de reacción-difusión no autónoma con condiciones dinámicas en la frontera. Desde nuestro conocimiento, no parece haber en la literatura ningún estudio sobre la existencia de atractores pullback para este tipo de problema. En ese sentido, el problema que consideramos en el Capítulo 5 es el siguiente.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta\, u + \kappa u + f(u) = h(t) & \text{in } \ \Omega \times (\tau, \infty), \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial \vec{n}} + g(u) = \rho(t) & \text{on } \ \partial\Omega \times (\tau, \infty), \\ u(x,\tau) = u_\tau(x), & \text{for } \ x \in \Omega, \\ u(x,\tau) = \psi_\tau(x), & \text{for } \ x \in \partial\Omega, \end{cases} \tag{20}$$

donde $\vec{n}$ es el vector unitario exterior normal a $\partial\Omega$, $\tau \in \mathbb{R}$ es un tiempo inicial, y

$$\kappa > 0, \quad u_\tau \in L^2(\Omega), \quad \psi_\tau \in L^2(\partial\Omega), \tag{21}$$

$$h \in L^2_{loc}\left(\mathbb{R}; L^2(\Omega)\right), \quad \rho \in L^2_{loc}\left(\mathbb{R}; L^2(\partial\Omega)\right), \tag{22}$$

son dados. Asumiendo ciertas hipótesis sobre f y g, probamos en la Sección 5.2 la existencia y unicidad de solución del problema (20). Usando un resultado de dependencia continua, que se prueba en la Sección 5.3, concluimos en el Teorema 5.14 la existencia de atractores pullback minimales para (20). Para ello definimos previamente el proceso como la única solución del problema y usamos la teoría pullback desarrollada en el Capítulo 1. Estos resultados pueden encontrarse en Anguiano *et al.* [8].

Cabe destacar que tanto en el Capítulo 2, como en los Capítulos 4 y 5, demostramos la existencia de atractores pullback minimales en el marco de los universos de conjuntos acotados fijos y de las familias parametrizadas en tiempo. La relación entre estas familias también se discute en dichos capítulos.

Chapter 1

Theory of Pullback Attractors

The understanding of the asymptotic behaviour of dynamical systems is one of the most important problems of modern mathematical physics. One way to treat this problem for systems having some dissipativity properties is to analyse the existence and structure of its global attractor.

On some occasions, some phenomena are modelled by nonlinear evolutionary equations which do not take into account all the relevant information of the real systems. Instead some neglected quantities can be modelled as an external force which in general becomes time-dependent. For this reason, non-autonomous systems are of great importance and interest.

In particular, most of the progress in the infinite-dimensional context, i.e., for non-autonomous partial differential equations has been made during the last two decades.

The theory of nonlinear process of operators in Banach spaces is an important mathematical tool for studying the qualitative behavior of infinite-dimensional dynamical systems.

In this chapter we are going to give a global view of the theory of pullback attractors. We will introduce the concept of pullback attractor for a dynamical system and we will establish a general result ensuring the existence of pullback attractor in a variety of systems. There are many works on this subject (see, e.g., Caraballo *et al.* [23, 24] and Anguiano *et al.* [3]). Most results in this chapter are slight modifications and generalizations of the ones in Marín-Rubio and Real [58] and are due to García-Luengo *et al.* [37].

1.1 Basic concepts

Let $X = (X, d_X)$ be a metric space, and let us denote $\mathbb{R}^2_d := \{(t,\tau) \in \mathbb{R}^2 : t \geq \tau\}$.

A **process** (also called a two-parameter semigroup) U on the metric space X is a map $U : (t,\tau,x) \in \mathbb{R}^2_d \times X \mapsto U(t,\tau)x \in X$ such that $U(\tau,\tau)x = x$, and

$$U(t,\tau)x = U(t,r)U(r,\tau)x \quad \text{for all } \tau \leq r \leq t \text{ and } x \in X. \tag{1.1}$$

Definition 1.1 *Let U be a process on X.*

a) U is said to be ***continuous*** *if for any pair $\tau \leq t$, the mapping $U(t,\tau) : X \to X$ is continuous.*

b) U is said to be ***closed*** *if for any $\tau \leq t$, and any sequence $\{x_n\} \subset X$, such that $x_n \to x \in X$ and $U(t,\tau)x_n \to y \in X$, then $U(t,\tau)x = y$.*

c) U is said to be ***strong-weak continuous*** *if for any pair $t \geq \tau \in \mathbb{R}$, the map $U(t,\tau)$ is continuous from X with the strong topology into X with the weak topology.*

Remark 1.2 *It is clear that every continuous process is closed. More generally, every strong-weak continuous process is a closed process.*

Let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ (observe that we do not require any additional condition on these sets as compactness or boundedness).

Definition 1.3 *We say that a process U on X is* ***pullback $\widehat{D}_0$-asymptotically compact*** *if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.*

Denote the omega-limit set of $\widehat{D}_0$ by

$$\Lambda(\widehat{D}_0, t) := \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t,\tau)D_0(\tau)}^X \quad \text{for all } t \in \mathbb{R}, \tag{1.2}$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

We denote by $\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2)$ the Hausdorff semi-distance in X between two sets $\mathcal{O}_1$ and $\mathcal{O}_2$, defined as

$$\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2) = \sup_{x \in \mathcal{O}_1} \inf_{y \in \mathcal{O}_2} d_X(x,y) \quad \text{for } \mathcal{O}_1,\, \mathcal{O}_2 \subset X.$$

Proposition 1.4 *If the process U on X is pullback $\widehat{D}_0$-asymptotically compact, then, for any $t \in \mathbb{R}$, the set $\Lambda(\widehat{D}_0, t)$ given by (1.2) is a nonempty compact subset of X, and*

$$\lim_{\tau \to -\infty} dist_X(U(t,\tau)D_0(\tau), \Lambda(\widehat{D}_0, t)) = 0. \tag{1.3}$$

Moreover, the family $\{\Lambda(\widehat{D}_0, t) : t \in \mathbb{R}\}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that

$$\lim_{\tau \to -\infty} dist_X(U(t,\tau)D_0(\tau), C(t)) = 0,$$

then $\Lambda(\widehat{D}_0, t) \subset C(t)$.

Proof Fix $t \in \mathbb{R}$, and consider two sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ such that $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n. Since the process U is pullback $\widehat{D}_0$-asymptotically compact, then there exist two subsequences $\{\tau_{n'}\}$ and $\{x_{n'}\}$, and a point $y \in X$, such that $U(t, \tau_{n'})x_{n'}$ converges to y in X. Then, $y \in \Lambda(\widehat{D}_0, t)$ and therefore $\Lambda(\widehat{D}_0, t)$ is a nonempty subset of X.

On the other hand, due to its own definition, it is evident that the set $\Lambda(\widehat{D}_0, t)$ is closed. Then, to prove that it is compact, it is sufficient to prove that it is relatively compact in X. To this end, let us consider a sequence $\{y_n\} \subset \Lambda(\widehat{D}_0, t)$ and prove that it is possible to extract a convergent subsequence in X.

From the characterization of $\Lambda(\widehat{D}_0, t)$, for every integer n there exist $\tau_n \le t - n$ and $x_n \in D_0(\tau_n)$, such that $d_X(y_n, U(t, \tau_n)x_n) \le 1/n$. Since the process U is pullback $\widehat{D}_0$-asymptotically compact, the sequence $\{U(t, \tau_n)x_n\}$ possesses a convergent subsequence in X. Therefore, it is evident that the corresponding subsequence of $\{y_n\}$ converges in X to the same point.

Now, we will prove (1.3). We suppose that there exists $t \in \mathbb{R}$ such that (1.3) it is not true. Then, there exist $\varepsilon > 0$, $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$, with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, such that

$$dist_X(U(t, \tau_n)x_n, \Lambda(\widehat{D}_0, t)) \ge \varepsilon \quad \forall n \ge 1. \tag{1.4}$$

But, since the process U is pullback $\widehat{D}_0$-asymptotically compact, we can extract a subsequence of $\{U(t, \tau_n)x_n\}$ which converges to a point $x \in X$, and, therefore, belongs to $\Lambda(\widehat{D}_0, t)$. This is a contradiction at light of (1.4).

Finally, suppose that there exists a family $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ of closed sets such that

$$\lim_{\tau \to -\infty} dist_X(U(t,\tau)D_0(\tau), C(t)) = 0. \tag{1.5}$$

We consider $x \in \Lambda(\widehat{D}_0, t)$, then there exist $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, and $x_n \in D_0(\tau_n)$ for all n, such that $U(t, \tau_n)x_n \to x$. By (3.3) we have $x \in \overline{C(t)} = C(t)$, and then $\Lambda(\widehat{D}_0, t) \subset C(t)$.

□

Proposition 1.5 *If the process U on X is pullback $\widehat{D}_0$-asymptotically compact and closed, then the family of sets $\{\Lambda(\widehat{D}_0, t) : t \in \mathbb{R}\}$ is invariant for U, that is*

$$\Lambda(\widehat{D}_0, t) = U(t, \tau)\Lambda(\widehat{D}_0, \tau), \quad \forall \tau \leq t.$$

Proof Let us consider $\tau < t$ and $y \in \Lambda(\widehat{D}_0, \tau)$. Then, there exist $\{\tau_n\} \subset (-\infty, \tau]$ and $\{x_n\} \subset X$, with $\lim_n \tau_n = -\infty$ and $x_n \in D_0(\tau_n)$ for all n, such that $U(\tau, \tau_n)x_n \to y$.

On the other hand, since the process U is pullback $\widehat{D}_0$-asymptotically compact, we can extract a subsequence of $\{U(t, \tau_n)x_n\}$ that converges to a point $z \in X$, which belongs to $\Lambda(\widehat{D}_0, t)$. As $U(t, \tau_n) = U(t, \tau)U(\tau, \tau_n)$ for all n and U is closed, then $z = U(t, \tau)y$. Thus, we have proved that $U(t, \tau)\Lambda(\widehat{D}_0, \tau) \subset \Lambda(\widehat{D}_0, t)$.

On the other hand, we consider $z \in \Lambda(\widehat{D}_0, t)$ and $\{\tau_n\} \subset (-\infty, \tau]$ with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, such that $U(t, \tau_n)x_n \to z$. Since $\{U(\tau, \tau_n)x_n\}$ is relatively compact, there exists a subsequence such that $U(\tau, \tau_{n'})x_{n'} \to y \in \Lambda(\widehat{D}_0, \tau)$. As U is closed, and $U(t, \tau_n) = U(t, \tau)U(\tau, \tau_n)$ for all n, we have that $z = U(t, \tau)y$. Then, we have proved that $\Lambda(\widehat{D}_0, t) \subset U(t, \tau)\Lambda(\widehat{D}_0, \tau)$.

□

Let $\mathcal{D}$ denote a given nonempty class of families parameterized in time

$$\widehat{D} = \{D(t) \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(X).$$

The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 1.6 *It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is **pullback** $\mathcal{D}-$**absorbing** for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists $\tau_0(t, \widehat{D}) \leq t$ such that*

$$U(t, \tau)D(\tau) \subset D_0(t) \quad \textit{for all } \tau \leq \tau_0(t, \widehat{D}).$$

Observe that, in the previous definition, $\widehat{D}_0$ does not belong necessarily to the universe $\mathcal{D}$.

Proposition 1.7 *If $\widehat{D}_0$ is pullback $\mathcal{D}-$absorbing for the process U on X, then*

$$\Lambda(\widehat{D}, t) \subset \Lambda(\widehat{D}_0, t) \quad \text{for all } \widehat{D} \in \mathcal{D},\ t \in \mathbb{R}.$$

Moreover, if $\widehat{D}_0 \in \mathcal{D}$, then

$$\Lambda(\widehat{D}_0, t) \ \subset \overline{D_0(t)}^X \quad \text{for all } t \in \mathbb{R}.$$

Proof Fix $\widehat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$. For any $y \in \Lambda(\widehat{D}, t)$, there exist two sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$, with $\lim_n \tau_n = -\infty$ and $x_n \in D(\tau_n)$ for all n, such that $U(t, \tau_n)x_n \to y$.

Since $\widehat{D}_0$ is pullback $\mathcal{D}-$absorbing for the process U, for each integer $k \geq 1$, there exist $\tau_{n_k} \in \{\tau_n\}$ with $\tau_{n_k} \leq t - k$, and $y_{n_k} = U(t-k, \tau_{n_k})x_{n_k} \in D_0(t-k)$. As $U(t, t-k)y_{n_k} = U(t, \tau_{n_k})x_{n_k} \to y$, then $y \in \Lambda(\widehat{D}_0, t)$.

Finally, consider $t \in \mathbb{R}$, and suppose that $\widehat{D}_0 \in \mathcal{D}$. Then, we observe that for any $y \in \Lambda(\widehat{D}_0, t)$, there exist $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$ and $\{x_n\} \subset X$ with $x_n \in D_0(\tau_n)$ for all n, such that $U(t, \tau_n)x_n \to y$. Since $\widehat{D}_0$ is pullback $\mathcal{D}-$absorbing for the process U, then from certain $n \in \mathbb{N}$, $U(t, \tau_n)x_n \in D_0(t)$. Consequently, $y \in \overline{D_0(t)}^X$.

□

Definition 1.8 *A process U on X is said to be **pullback $\mathcal{D}-$asymptotically compact** if it is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$, i.e. if for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}$, and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ for all n, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X.*

Proposition 1.9 *If $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}-$absorbing for the process U on X and the process U is pullback $\widehat{D}_0$-asymptotically compact. Then, the process U is also pullback $\mathcal{D}-$asymptotically compact.*

Proof Fix $\widehat{D} \in \mathcal{D}$, $t \in \mathbb{R}$ and $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$, with $\lim_n \tau_n = -\infty$ and $x_n \in D(\tau_n)$ for all n. We have to prove that the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X.

Taking into account that $\widehat{D}_0$ is pullback $\mathcal{D}-$absorbing for the process U on X, we can deduce that for any integer number $k \geq 1$, there exists $\tau_{n_k} \in \{\tau_n\}$ such that $\tau_{n_k} \leq t - k$, and

$$y_{n_k} = U(t-k, \tau_{n_k})x_{n_k} \in D_0(t-k).$$

Since U is pullback $\widehat{D}_0$-asymptotically compact, we can extract a subsequence $\left\{U(t, t-k')y_{n_{k'}}\right\}$ of $\{U(t, t-k)y_{n_k}\}$ which converges in X.

But as

$$U(t,t-k')y_{n_{k'}} = U(t,t-k')U(t-k',\tau_{n_{k'}})x_{n_{k'}} = U(t,\tau_{n_{k'}})x_{n_{k'}},$$

we can extract a subsequence $\{U(t,\tau_{n_{k'}})x_{n_{k'}}\}$ of $\{U(t,\tau_n)x_n\}$ that converges in X.

□

1.2 Existence of pullback attractors

In order to prove the main theorem which ensures the existence of minimal pullback attractors, we first need the following result, which is an immediate consequence of Propositions 1.4 and 1.5,

Proposition 1.10 *If the process U on X is pullback $\mathcal{D}$-asymptotically compact and closed, then, for each $\widehat{D} \in \mathcal{D}$ and for any $t \in \mathbb{R}$, the set $\Lambda(\widehat{D},t)$ is a nonempty compact subset of X, invariant for the process, and attracts $\widehat{D}$ in the pullback sense, that is,*

$$\lim_{\tau\to-\infty} dist_X(U(t,\tau)D(\tau),\Lambda(\widehat{D},t)) = 0. \tag{1.6}$$

Moreover, for each $\widehat{D} \in \mathcal{D}$, the family $\{\Lambda(\widehat{D},t) : t \in \mathbb{R}\}$ is minimal amongst all the families of closed sets that satisfy (1.6).

Using the previous results and definitions, we obtain the following relevant result,

Theorem 1.11 *Consider a closed process $U : \mathbb{R}^2_d \times X \to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ which is pullback $\mathcal{D}-$absorbing for U, and assume also that U is pullback $\widehat{D}_0-$asymptotically compact.*

Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined by

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda(\widehat{D},t)}^X \quad t \in \mathbb{R},$$

possesses the following properties:

(a) For any $t \in \mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X, and

$$\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda(\widehat{D}_0, t).$$

(b) $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}-$attracting, i.e.

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0 \quad \text{for all } \widehat{D} \in \mathcal{D}, \quad t \in \mathbb{R}.$$

(c) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e. $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $\tau \leq t$.

(d) If $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$, for all $t \in \mathbb{R}$.

The family $\mathcal{A}_{\mathcal{D}}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$,

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), C(t)) = 0,$$

then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Proof Since $\widehat{D}_0$ is pullback $\mathcal{D}-$absorbing for U, by Proposition 1.7 we have that $\Lambda(\widehat{D}, t) \subset \Lambda(\widehat{D}_0, t)$ for all $\widehat{D} \in \mathcal{D}$, and $t \in \mathbb{R}$. If in addition $\widehat{D}_0 \in \mathcal{D}$, then $\Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.

On the other hand, as U is pullback $\widehat{D}_0-$asymptotically compact, by Proposition 1.4, the set $\Lambda(\widehat{D}_0, t)$ is a nonempty compact set, for all $t \in \mathbb{R}$.

Thanks to Proposition 1.9, U is also pullback $\mathcal{D}-$asymptotically compact, then by the Proposition 1.4, we have that the set $\Lambda(\widehat{D}, t)$ is a nonempty compact set, for all $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}$. Consequently, (a) and (d) are proved.

On the other hand, (b) is a consequence of Proposition 1.4 and the fact that, for any $\widehat{D} \in \mathcal{D}$, we have

$$\operatorname{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) \leq \operatorname{dist}_X(U(t,\tau)D(\tau), \Lambda(\widehat{D}, t)).$$

On the other hand, on account of Proposition 1.5, we obtain

$$\Lambda(\widehat{D}, t) = U(t,\tau)\Lambda(\widehat{D}, \tau) \quad \text{for all } \tau \leq t, \text{ and any } \widehat{D} \in \mathcal{D}. \tag{1.7}$$

If $y \in \mathcal{A}_{\mathcal{D}}(t)$, there exist two sequences $\{\widehat{D}_n\} \subset \mathcal{D}$ and $\{y_n\} \subset X$, such that $y_n \in \Lambda(\widehat{D}_n, t)$ and $y_n \to y$. By (1.7), $y_n = U(t,\tau)x_n$, with $x_n \in \Lambda(\widehat{D}_n, \tau) \subset \mathcal{A}_{\mathcal{D}}(\tau)$. Since $\mathcal{A}_{\mathcal{D}}(\tau)$ is a compact set then there exists a sequence $\{x_{n'}\} \subset \{x_n\}$ such that $x_{n'} \to x \in \mathcal{A}_{\mathcal{D}}(\tau)$. As U is closed, $y = U(t,\tau)x$ and we can then deduce that $\mathcal{A}_{\mathcal{D}}(t) \subset U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau)$. The opposite inclusion can be proved analogously.

Finally, we can prove that the family $\mathcal{A}_{\mathcal{D}}$ is minimal taking into account Proposition 1.10 and the definition of $\mathcal{A}_{\mathcal{D}}$.

□

Remark 1.12 *Under the assumptions of Theorem 1.11, the family* $\mathcal{A}_{\mathcal{D}}$ *is called the* ***minimal pullback*** $\mathcal{D}-$***attractor*** *for the process* U.

If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, *then it is the unique family of closed subsets in* $\mathcal{D}$ *that satisfies (b)–(c).*

If we have that $\widehat{D}_0 \in \mathcal{D}$, *the set* $D_0(t)$ *is closed for all* $t \in \mathbb{R}$, *and the family* $\mathcal{D}$ *is inclusion-closed (i.e. if* $\widehat{D} \in \mathcal{D}$, *and* $\widehat{D'} = \{D'(t) \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ *with* $D'(t) \subset D(t)$ *for all* t, *then* $\widehat{D'} \in \mathcal{D}$*), then it follows that* $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$.

We will denote $\mathcal{D}_F^X$ the universe of fixed nonempty bounded subsets of X, i.e. the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X. In the particular case of the universe $\mathcal{D}_F^X$, the corresponding minimal pullback $\mathcal{D}_F^X-$attractor for the process U is the pullback attractor defined by Crauel, Debussche, and Flandoli, [28, Th.1.1, p. 311], and will be denoted $\mathcal{A}_{\mathcal{D}_F^X}$.

Now, it is easy to conclude the following result.

Corollary 1.13 *Under the assumptions of Theorem 1.11, if the universe* $\mathcal{D}$ *contains the universe* $\mathcal{D}_F^X$, *then both attractors,* $\mathcal{A}_{\mathcal{D}_F^X}$ *and* $\mathcal{A}_{\mathcal{D}}$, *exist, and the following relation holds:*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) \subset \mathcal{A}_{\mathcal{D}}(t) \quad \text{for all } t \in \mathbb{R}.$$

Remark 1.14 *It can be proved (see [58]) that, under the assumptions of the preceding corollary, if, moreover, for some* $T \in \mathbb{R}$ *the set* $\cup_{t \le T} D_0(t)$ *is a bounded subset of* X, *then*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) = \mathcal{A}_{\mathcal{D}}(t) \quad \text{for all } t \le T.$$

Chapter 2
Non-autonomous reaction-diffusion equation with uniqueness of solution

Several aspects of reaction-diffusion equations are being analyzed over the last years, particularly, their asymptotic behaviour. In this chapter we are dealing with a reaction-diffusion equation in a bounded domain containing a non-autonomous forcing term taking values in the space L^2, and with a continuous nonlinearity term which ensures uniqueness of solutions. We will show the existence of pullback attractors, with respect to two different universes, in the phase space L^2 for our problem. It is also worth mentioning that our problem has received much attention over the last years (see for instance [22], [55], [74], [75] and [84]).

We also prove some regularity results for the pullback attractor. First, we establish a general result about H^2-boundedness of invariant sets for the associated evolution process. Then, as a consequence, we deduce that the pullback attractor of our non-autonomous reaction-diffusion equation is bounded, not only on $L^2 \cap H_0^1$, but in H^2.

Finally, we will prove some exponential growth results for the pullback attractor of our model when time goes to $-\infty$. First we establish a general result about $L^p \cap H_0^1$ exponential growth condition. Then, under additional assumptions, we deduce an exponential growth condition in H^2 for the pullback attractor.

Most results in this chapter can be found in [4], [5], [68] and [81].

2.1 Setting of the problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let us consider the following problem for a non-autonomous reaction-diffusion equation with zero Dirichlet boundary condition in Ω,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(u) + h(t) \quad \text{in} \quad \Omega \times (\tau, +\infty)\,, \\ u = 0 \quad \text{on} \quad \partial\Omega \times (\tau, +\infty)\,, \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{2.1}$$

where $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $f \in C^1(\mathbb{R})$ and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$. We assume that there exist positive constants α_1, α_2, k, l, and $p > 2$ such that

$$-k - \alpha_1 |s|^p \le f(s)s \le k - \alpha_2 |s|^p\,, \quad \forall s \in \mathbb{R}, \tag{2.2}$$

$$f'(s) \le l, \quad \forall s \in \mathbb{R}. \tag{2.3}$$

Taking into account (2.2), we can deduce that there exists $c > 0$ such that

$$|f(s)| \le c(|s|^{p-1} + 1) \quad \forall s \in \mathbb{R}. \tag{2.4}$$

Let us denote

$$\mathcal{F}(s) := \int_0^s f(r)dr.$$

Then, there exist positive constants $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$ and $\widetilde{k}$ such that

$$-\widetilde{k} - \widetilde{\alpha}_1 |s|^p \le \mathcal{F}(s) \le \widetilde{k} - \widetilde{\alpha}_2 |s|^p\,, \qquad \forall s \in \mathbb{R}. \tag{2.5}$$

An example of function satisfying the precedent conditions is any polynomial of odd degree,

$$f(s) = \sum_{j=0}^{2k+1} c_j s^j,$$

with $c_{2k+1} < 0$. The typical example is $f(s) = -s^3 + \lambda s$.

We will denote by $(\cdot,\cdot)$ the scalar product in $L^2(\Omega)$, by $|\cdot|$ the norm in $L^2(\Omega)$, by $\|\cdot\|_{H^2(\Omega)}$ the norm in $H^2(\Omega)$, and by $\|\cdot\|_{L^p(\Omega)}$ the norm in $L^p(\Omega)$.

We will use $\langle\cdot,\cdot\rangle$ to denote the duality product between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$, and we will use $(\cdot,\cdot)$ to denote the duality product between $L^{p'}(\Omega)$ and $L^p(\Omega)$, where $p' = \frac{p}{p-1}$ is the conjugate exponent of p.

On $H_0^1(\Omega)$ we consider the scalar product $((\cdot,\cdot))$ given by

$$((u,v)) = \int_\Omega \nabla u \cdot \nabla v dx, \ \ \forall u,v \in H_0^1(\Omega),$$

and we will denote by $\|\cdot\| = |\nabla \cdot|$ the norm in $H_0^1(\Omega)$.

The aim of this chapter is to show the existence of a pullback attractor in the phase space $L^2(\Omega)$ for the problem (2.1) using the Theorem 1.11 of Section 1.2.

To do this we have to guarantee the existence and uniqueness of solution of the problem (2.1), which will be analyzed in the next Section.

2.2 Existence and Uniqueness of Solution

We state in this section a result on the existence and uniqueness of solution of problem (2.1). First, we give the definition of weak solution of the problem (2.1).

Definition 2.1 *A weak solution of (2.1) is any function u such that*

(a) $u \in L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$, *for all* $T > \tau$,

(b) $\frac{d}{dt}(u(t), v) + ((u(t), v)) = (f(u(t)), v) + (h(t), v)$ *in* $\mathcal{D}'(\tau, \infty)$, *for all* $v \in H_0^1(\Omega) \cap L^p(\Omega)$,

(c) $u(\tau) = u_\tau$.

Remark 2.2 *In view of Theorem 3 in Chapter 5 subsection 5.9.2 in [32], it is not difficult to prove that if u satisfies (a) and (b), then*

$$u \in C\left([\tau, T]; L^2(\Omega)\right),$$

and the function $t \mapsto |u(t)|^2$ is absolutely continuous on every interval $[\tau, T]$ and

$$\frac{d}{dt}|u(t)|^2 = 2\left\langle \frac{du}{dt}, u \right\rangle \text{ for a.a. } t \in (\tau, T).$$

Hence, it satisfies the energy equality

$$|u(t)|^2 + 2\int_\tau^t |\nabla u(s)|^2 \, ds = |u_\tau|^2 + 2\int_\tau^t (f(u(s)) + h(s), u(s)), ds \quad \forall t \geq \tau.$$

Our goal now is to prove the following result.

Theorem 2.3 *Assume that $f \in C^1(\mathbb{R})$ satisfies (2.2) and (2.3). Then for any initial condition $u_\tau \in L^2(\Omega)$ and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, there exists a unique weak solution $u(\cdot) = u(\cdot; \tau, u_\tau)$ of (2.1), i.e., a unique function $u \in L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap C([\tau, T]; L^2(\Omega))$ for all $T > \tau$, such that*

$$u(t) - \int_\tau^t \Delta u(s)\, ds = u_\tau + \int_\tau^t (f(u(s)) + h(s))\, ds \quad \forall\, t \geq \tau,$$

where the equality must be understood in the sense of the dual of $H_0^1(\Omega) \cap L^p(\Omega)$.

Finally, if $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$, then the weak solution u also satisfies

$$u \in L^\infty\left(\tau, T; H_0^1(\Omega)\right) \cap L^\infty(\tau, T; L^p(\Omega)), \quad u' \in L^2\left(\tau, T; L^2(\Omega)\right). \qquad (2.6)$$

Proof Thanks to the Remark 2.2, every weak solution of (2.1), if there exists any, belongs to $C([\tau, T]; L^2(\Omega))$.

Now, we will prove the uniqueness of solution and then the existence.

Uniqueness

Suppose that u and $\overline{u}$ are two weak solutions of (2.1) corresponding to the same initial condition u_τ, and the same non-autonomous term h. If we denote $w = u - \overline{u}$, thanks to the energy equality, we obtain

$$|w(t)|^2 + 2\int_\tau^t |\nabla w(s)|^2\, ds = 2\int_\tau^t (f(u(s)) - f(\overline{u}(s)), u(s) - \overline{u}(s)) ds \quad \forall\, t \geq \tau,$$

and taking into account (2.3), applying the Mean Value Theorem to f, in particular we obtain

$$|w(t)|^2 \leq 2l \int_\tau^t |w(s)|^2 ds \quad \forall\, t \geq \tau,$$

and thanks to the Gronwall Lemma we have $w(t) = 0$ for all $t \geq \tau$.

Existence of solution when u_τ belongs to $H_0^1(\Omega) \cap L^p(\Omega)$

Let us suppose that $u_\tau \in H_0^1(\Omega) \cap L^p(\Omega)$. To prove the existence of a weak solution u of (2.1), we use the Galerkin method, and we pass to the limit by compactness.

For each integer $n \geq 1$, we denote by $u_n(t) = u_n(t; \tau, u_\tau)$ the Galerkin approximation of the solution $u(t; \tau, u_\tau)$ of (2.1), which is given by

$$u_n(t) = \sum_{j=1}^n \gamma_{nj}(t) w_j, \qquad (2.7)$$

and satisfies

$$\begin{cases} \dfrac{d}{dt}(u_n(t), w_j) = \langle \Delta u_n(t), w_j\rangle + (f(u_n(t)), w_j) + (h(t), w_j), \ (\tau, +\infty), \\ u_n(\tau) = u_{\tau_n} \qquad j = 1, .., n, \end{cases} \tag{2.8}$$

where $\{w_j : j \geq 1\} \subset H_0^1(\Omega) \cap L^p(\Omega)$ is a Hilbert basis of $L^2(\Omega)$ such that *span* $\{w_j\}_{j\geq 1}$ is dense in $H_0^1(\Omega) \cap L^p(\Omega)$, and where the sequence u_{τ_n} converges to u_τ in $H_0^1(\Omega)$ and in $L^p(\Omega)$, with $u_{\tau_n} \in span\{w_j : 1 \leq j \leq n\}$.

Multiplying by γ_{nj} in (2.8), and summing from $j = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|^2 + |\nabla u_n(t)|^2 = (f(u_n(t)), u_n(t)) + (h(t), u_n(t)). \tag{2.9}$$

Using (2.2),

$$\begin{aligned}(f(u_n(t)), u_n(t)) &\leq \int_\Omega (k - \alpha_2 |u_n(x,t)|^p)\, dx \\ &= k|\Omega| - \alpha_2 \|u_n(t)\|^p_{L^p(\Omega)},\end{aligned}$$

where $|\Omega|$ denotes the measure of Ω.

On the other hand,

$$\begin{aligned}(h(t), u_n(t)) &\leq \frac{1}{2\lambda_1}|h(t)|^2 + \frac{\lambda_1}{2}|u_n(t)|^2 \\ &\leq \frac{1}{2\lambda_1}|h(t)|^2 + \frac{1}{2}|\nabla u_n(t)|^2,\end{aligned}$$

where λ_1 denotes the first eigenvalue of the negative Laplacian with zero Dirichlet boundary condition in Ω.

Thus, from (2.9) we deduce

$$\frac{d}{dt}|u_n(t)|^2 + |\nabla u_n(t)|^2 + 2\alpha_2\|u_n(t)\|^p_{L^p(\Omega)} \leq \frac{1}{\lambda_1}|h(t)|^2 + 2k|\Omega|,$$

and integrating between τ and t

$$\begin{aligned}|u_n(t)|^2 &+ \int_\tau^t |\nabla u_n(s)|^2\, ds + 2\alpha_2 \int_\tau^t \|u_n(s)\|^p_{L^p(\Omega)}\, ds \\ &\leq |u_n(\tau)|^2 + \frac{1}{\lambda_1}\int_\tau^t |h(s)|^2\, ds + 2k|\Omega|(t-\tau), \ \ \forall n \geq 1.\end{aligned} \tag{2.10}$$

Then, we deduce that

$$\{u_n\} \text{ is bounded in } L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap C([\tau, T]; L^2(\Omega)), \tag{2.11}$$

for all $T > \tau$.

Taking into account (2.4), it follows that

$$f(u_n) \text{ is bounded in } L^{p'}(\tau, T; L^{p'}(\Omega)),$$

for all $T > \tau$. Then, there exists a subsequence $\{u_\mu\} \subset \{u_n\}$ such that

$$u_\mu \overset{*}{\rightharpoonup} u \text{ weak-star in } L^\infty(\tau, T; L^2(\Omega)),$$

$$u_\mu \rightharpoonup u \text{ weakly in } L^p(\tau, T; L^p(\Omega)), \tag{2.12}$$

$$u_\mu \rightharpoonup u \text{ weakly in } L^2(\tau, T; H_0^1(\Omega)), \tag{2.13}$$

$$f(u_\mu) \rightharpoonup \chi \text{ weakly in } L^{p'}(\tau, T; L^{p'}(\Omega)), \tag{2.14}$$

for all $T > \tau$. Now (2.13) implies that

$$\Delta u_\mu \rightharpoonup \Delta u \text{ weakly in } L^2(\tau, T; H^{-1}(\Omega)).$$

On the other hand, to prove that $\chi(t) = f(u(t))$, we will use Lemma 1.3 from Chapter 1 in [56].

Multiplying (2.8) by γ'_{nj}, and summing from $j = 1$ to n,

$$\begin{aligned}|u_n'(t)|^2 + \frac{1}{2}\frac{d}{dt}|\nabla u_n(t)|^2 &= (f(u_n(t)), u_n'(t)) + (h(t), u_n'(t)) \\ &\leq \frac{1}{2}|h(t)|^2 + \frac{1}{2}|u_n'(t)|^2 + \frac{d}{dt}\int_\Omega \mathcal{F}(u_n(x,t))\,dx.\end{aligned}$$

Integrating now between τ and t, we obtain

$$\begin{aligned}\int_\tau^t |u_n'(s)|^2\,ds + |\nabla u_n(t)|^2 &\leq |\nabla u_n(\tau)|^2 + \int_\tau^t |h(s)|^2\,ds \\ &\quad + 2\int_\Omega \mathcal{F}(u_n(x,t))\,dx - 2\int_\Omega \mathcal{F}(u_n(x,\tau))\,dx,\end{aligned}$$

which, jointly with (2.5), yields that

$$\begin{aligned}&\int_\tau^t |u_n'(s)|^2\,ds + |\nabla u_n(t)|^2 + 2\widetilde{\alpha}_2\|u_n(t)\|_{L^p(\Omega)}^p \\ &\quad \leq |\nabla u_n(\tau)|^2 + \int_\tau^t |h(s)|^2\,ds + 4\widetilde{k}\,|\Omega| + 2\widetilde{\alpha}_1\,\|u_n(\tau)\|_{L^p(\Omega)}^p,\end{aligned} \tag{2.15}$$

for all $\tau \leq t$.

As the sequence $u_n(\tau)$ is bounded in $H_0^1(\Omega)$ and in $L^p(\Omega)$, then, it follows that

$$\{u_n\} \text{ is bounded in } L^\infty(\tau, T; H_0^1(\Omega)) \cap L^\infty(\tau, T; L^p(\Omega)), \tag{2.16}$$

and

$$\{u_n'\} \text{ is bounded in } L^2(\tau,T;L^2(\Omega)), \text{ for all } T>\tau. \tag{2.17}$$

Then, there exists a subsequence $\{u_\mu\}\subset\{u_n\}$ such that

$$u_\mu \overset{*}{\rightharpoonup} u \text{ weak-star in } L^\infty(\tau,T;H_0^1(\Omega)),$$

$$u_\mu \overset{*}{\rightharpoonup} u \text{ weak-star in } L^\infty(\tau,T;L^p(\Omega)),$$

$$u_\mu' \rightharpoonup u' \text{ weakly in } L^2(\tau,T;L^2(\Omega)), \text{ for all } T>\tau.$$

In particular, we obtain (2.6).

On the other hand, observing that

$$\begin{aligned}|u_n(t_2)-u_n(t_1)|^2 &= \left|\int_{t_1}^{t_2} u_n'(s)ds\right|^2 \\ &\le \|u_n'\|^2_{L^2(\tau,T;L^2(\Omega))}\,|t_2-t_1| \quad \forall t_1,\, t_2\in[\tau,T],\end{aligned}$$

from (2.16), (2.17) and the compactness of the injection of $H_0^1(\Omega)$ into $L^2(\Omega)$, by the Ascoli's Theorem, we can suppose that $\{u_\mu\}$ converges strongly in $\mathcal{C}([\tau,T];L^2(\Omega))$ for all $T>\tau$, and *a.e.* in $\Omega\times(\tau,+\infty)$ to u.

Then, as f is continuous,

$$f(u_\mu)\longrightarrow f(u) \text{ } a.e. \text{ in } \Omega\times(\tau,+\infty),$$

and, as $\{f(u_\mu)\}$ is bounded in $L^{p'}(\Omega\times(\tau,T))$, by Lemma 1.3, Chapter 1 in [56], we obtain

$$f(u_\mu)\rightharpoonup f(u) \text{ weakly in } L^{p'}\left(\tau,T;L^{p'}(\Omega)\right) \quad \forall T>\tau.$$

From (2.14) and by the uniqueness of the weak limit, we have

$$\chi=f(u) \quad a.e. \text{ in } \Omega\times(\tau,T) \quad \forall T>\tau.$$

Thanks to the equation satisfied by u_μ' and the fact that $span\,\{w_j\}_{j\ge1}$ is dense in $H_0^1(\Omega)\cap L^p(\Omega)$, we can pick an element in the equivalence class of u satisfying

$$(u(t),w)=(u_\tau,w)+\int_\tau^t\langle\Delta u(s),w\rangle ds+\int_\tau^t(f(x,u(s))+h(s),w)ds, \tag{2.18}$$

for all $t\ge\tau$, for any $w\in H_0^1(\Omega)\cap L^p(\Omega)$.

Existence of solution when u_τ belongs to $L^2(\Omega)$

Now, we suppose that $u_\tau \in L^2(\Omega)$. To prove the existence of weak solution u of (2.1), taking into account that $H_0^1(\Omega) \cap L^p(\Omega)$ is dense in $L^2(\Omega)$, we consider a sequence $\{u_{\tau_m}\} \subset H_0^1(\Omega) \cap L^p(\Omega)$ such that $u_{\tau_m} \to u_\tau$ strongly in $L^2(\Omega)$ when m tends to ∞.

For each initial condition u_{τ_m}, we consider the corresponding weak solution of (2.1), which we denote by u_m. Then, we have a sequence

$$\{u_m\} \subset C\left([\tau, T]; L^2(\Omega)\right) \cap L^2\left(\tau, T; H_0^1(\Omega)\right) \cap L^p(\tau, T; L^p(\Omega)),$$

such that

$$(u_m(t), v) + \int_\tau^t ((u_m(s), v))\, ds = (u_{\tau_m}, v) + \int_\tau^t (f(u_m(s)), v)\, ds + \int_\tau^t (h(s), v)\, ds,$$

for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$, for all $t \in [\tau, T]$ and for all $m \geq 1$.

If we apply the energy equality to u_m, we have

$$\begin{aligned} |u_m(t)|^2 + 2\int_\tau^t |\nabla u_m(s)|^2\, ds = |u_{\tau_m}|^2 &+ 2\int_\tau^t (f(u_m(s)), u_m(s))\, ds \\ &+ 2\int_\tau^t (h(s), u_m(s))\, ds, \end{aligned}$$

for all $t \in [\tau, T]$. From this equality, and taking into account (2.2), it is not difficult to prove that

$$\{u_m\} \text{ is bounded in } L^\infty\left(\tau, T; L^2(\Omega)\right) \cap L^2\left(\tau, T; H_0^1(\Omega)\right) \cap L^p(\tau, T; L^p(\Omega)). \quad (2.19)$$

On the other hand, if we argue as in the proof of the uniqueness of solution, we obtain

$$\begin{aligned} &|u_m(t) - u_n(t)|^2 + 2\int_\tau^t |\nabla u_m(s) - \nabla u_n(s)|^2\, ds \\ &\leq |u_{\tau_m} - u_{\tau_n}|^2 + 2l\int_\tau^t |u_m(s) - u_n(s)|^2\, ds, \end{aligned}$$

for all $t \in [\tau, T]$, and for all $m,\, n \geq 1$. Using the Gronwall Lemma, we have that

$$\{u_m\} \text{ is a Cauchy sequence in } L^2\left(\tau, T; H_0^1(\Omega)\right) \cap C\left([\tau, T]; L^2(\Omega)\right). \quad (2.20)$$

Thanks to (2.19) and (2.20) we can extract a subsequence $\{u_\mu\} \subset \{u_m\}$ such that $u_\mu \to u$ strongly in $L^2(\tau, T; H_0^1(\Omega)) \cap C([\tau, T]; L^2(\Omega))$ when $\mu \to +\infty$, with $u \in L^p(\tau, T; L^p(\Omega))$, and $u_\mu \to u$ *a.e.*, in $\Omega \times (\tau, +\infty)$. And, reasoning as before, we also obtain that

$$f(u_\mu) \rightharpoonup f(u) \text{ weakly in } L^{p'}\left(\tau, T; L^{p'}(\Omega)\right) \quad \forall T > \tau.$$

Then, we can pass to the limit in the equation which satisfies u_μ, and we obtain that u is a weak solution of (2.1).

□

2.3 A regularity result

In this section we prove a regularity result for the solution of (2.1). If we denote

$$D(-\Delta) = \left\{v \in H_0^1(\Omega) : \Delta v \in L^2(\Omega)\right\},$$

with the scalar product

$$(v, w)_{D(-\Delta)} = (\Delta v, \Delta w) \quad \forall v, w \in D(-\Delta),$$

then $D(-\Delta)$ is a Hilbert space, and $D(-\Delta)$ is included in $H_0^1(\Omega)$ with continuous and dense injection.

The following result shows a regularity result for the solution of our reaction-diffusion model.

Theorem 2.4 *Assume that $f \in C^1(\mathbb{R})$ satisfies (2.2) and (2.3), and either $N \leq 2p/(p-2)$ or $\Omega \subset \mathbb{R}^N$ is a bounded C^k domain, with $k \geq 2$ such that $k \geq N(p-2)/2p$. Then for any initial condition $u_\tau \in H_0^1(\Omega) \cap L^p(\Omega)$ and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, the weak solution of (2.1), whose existence and uniqueness is guaranteed by Theorem 2.3, satisfies in addition*

$$u \in C\left([\tau, T]; H_0^1(\Omega)\right) \cap L^2(\tau, T; D(-\Delta)), \text{ for all } T > \tau. \tag{2.21}$$

Proof If we add $f(0)$ in both members of the equation in (2.1), we can suppose that $f(0) = 0$.

Let $u_\tau \in H_0^1(\Omega) \cap L^p(\Omega)$, and we consider an orthonormal basis of Hilbert $\{w_j : j \geq 1\}$ of $L^2(\Omega)$ that is formed by the eigenvectors associated with eigenvalues $\{\lambda_j : j \geq 1\}$ of the operator $-\Delta$ with zero Dirichlet boundary condition in Ω. Then, it is not difficult to conclude that $w_j \in D(-\Delta)$, and if we denote

$$u_{\tau_n} = \sum_{j=1}^{n} (u_\tau, w_j)\, w_j,$$

the sequence $\{u_{\tau_n}\}$ converges to u_τ in $H_0^1(\Omega)$, and then, by the Sobolev injection Theorem, if $N \leq 2p/(p-2)$, the sequence $\{u_{\tau_n}\}$ converges to u_τ in $L^p(\Omega)$.

On the other hand, if $\Omega \subset \mathbb{R}^N$ is a bounded C^k domain, with $k \geq 2$ such that $k \geq N(p-2)/2p$, then we can prove (see [68]) that *span* $\{w_j : j \geq 1\}$ is also dense in $L^p(\Omega)$.

For each integer $n \geq 1$, we consider the sequence $\{u_n\}$ defined by (2.7) and (2.8).

Using the proof of Theorem 2.3 and taking into account the uniqueness of solution, we have that the sequence $\{u_n\}$ converges weakly-star in $L^\infty(\tau, T; H_0^1(\Omega)) \cap L^\infty(\tau, T; L^p(\Omega))$ to the solution u of (2.1), and we also obtain that $u' \in L^2(\tau, T; L^2(\Omega))$.

Now, we will see that the sequence $\{u_n\}$ is bounded in $L^2(\tau, T; D(-\Delta))$, and in which case we will have that $u \in L^2(\tau, T; D(-\Delta))$.

As $u \in L^\infty(\tau, T; H_0^1(\Omega)) \cap L^2(\tau, T; D(-\Delta))$ and $u' \in L^2(\tau, T; L^2(\Omega))$, by Theorem 2.1 in [76], we can deduce that $u \in C([\tau, T]; H_0^1(\Omega))$.

To prove that the sequence $\{u_n\}$ is bounded in $L^2(\tau, T; D(-\Delta))$, multiplying in (2.8) by $\lambda_j \gamma_{nj}$, where λ_j is the eigenvalue associated to the eigenfunction w_j, and summing once more from $j = 1$ to n, we obtain

$$(u_n'(t), \Delta u_n(t)) = |\Delta u_n(t)|^2 + (f(u_n(t)), \Delta u_n(t)) + (h(t), \Delta u_n(t)). \quad (2.22)$$

Taking into account that $f(0) = 0$, and then $f(u_n(t)) \in H_0^1(\Omega)$ for any $t \in (\tau, T)$, we have, using (2.3), that

$$\begin{aligned} -(f(u_n(t)), \Delta u_n(t)) &= -\int_\Omega f(u_n(x,t)) \Delta u_n(x,t) dx \\ &= \sum_{i=1}^N \int_\Omega \partial_i(f(u_n(x,t))) \partial_i(u_n(x,t)) dx \\ &= \int_\Omega f'(u_n(x,t)) |\nabla u_n(x,t)|^2 dx \\ &\leq l |\nabla u_n(t)|^2, \end{aligned}$$

and, using the last inequality in (2.22), we obtain

$$\frac{d}{dt}\left(|\nabla u_n(t)|^2\right) + |\Delta u_n(t)|^2 \leq 2l |\nabla u_n(t)|^2 + |h(t)|^2, \quad a.e., \text{in } (\tau, T).$$

Finally, integrating the last inequality between τ and T, and taking into account that $\{u_n\}$ is bounded in $L^\infty(\tau, T; H_0^1(\Omega))$, we obtain that $\{u_n\}$ is also bounded in $L^2(\tau, T; D(-\Delta))$.

□

Remark 2.5 *We note that if $\Omega \subset \mathbb{R}^N$ is a bounded C^2 domain, then we have that $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, and moreover the norm induced by $(\cdot,\cdot)_{D(-\Delta)}$ in $D(-\Delta)$ and the norm of $H^2(\Omega)$ are equivalent. Thus, in this case, to write $u \in L^2(\tau, T; D(-\Delta))$ is equivalent to write $u \in L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega))$.*

As a consequence of Theorems 2.3 and 2.4, we can now establish the following result.

Theorem 2.6 *Under the assumptions in Theorem 2.4, for any initial condition $u_\tau \in L^2(\Omega)$ and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, the weak solution u of (2.1) satisfies*

$$u \in C\left((\tau, T]; H_0^1(\Omega)\right) \cap L^2(\tau+\varepsilon, T; D(-\Delta)) \cap L^\infty(\tau+\varepsilon, T; L^p(\Omega)),$$
$$u' \in L^2\left(\tau+\varepsilon, T; L^2(\Omega)\right),$$

for all $T > \tau + \varepsilon > \tau$.

The following Lemma is due to W.A. Strauss [76], and its proof can be seen in [79].

Lemma 2.7 *Let X, Y be Banach spaces such that the inclusion $X \subset Y$ is continuous. Assume that $u \in L^\infty(a, b; X)$ is weakly continuous in $[a, b]$ with values in Y, then u is also weakly continuous in $[a, b]$ with values in X.*

Taking into account this Lemma and Theorem 2.6, we obtain the following result.

Corollary 2.8 *Under the assumptions in Theorem 2.6, for any initial condition $u_\tau \in L^2(\Omega)$ and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, the weak solution u of (2.1) is weakly continuous in $(\tau, +\infty)$ with values in $L^p(\Omega)$. In particular, $u(t) \in L^p(\Omega)$ for all $t \in (\tau, +\infty)$.*

2.4 Existence of Pullback Attractors

In this section, we will show the existence of pullback attractors in $L^2(\Omega)$ of problem (2.1).

First, thanks to Theorem 2.3, we can define a process $\{U(t,\tau), \tau \le t\}$ in $L^2(\Omega)$, as

$$U(t,\tau)u_\tau = u(t; \tau, u_\tau) \quad \forall u_\tau \in L^2(\Omega), \forall \tau \le t, \tag{2.23}$$

where $u(t; \tau, u_\tau)$ is the unique weak solution of (2.1).

From the uniqueness of solution to problem (2.1), it follows that (2.23) defines a process in $L^2(\Omega)$.

Remark 2.9 *It can be proved that the process defined by (2.23) is continuous in* $L^2(\Omega)$.

To do this, we denote

$$w(t;\tau,u_\tau - v_\tau) := u(t;\tau,u_\tau) - v(t;\tau,v_\tau) = U(t,\tau)u_\tau - U(t,\tau)v_\tau,$$

where u is the weak solution of (2.1) for the initial condition u_τ and for the non-autonomous term h and v is the weak solution of (2.1) for the initial condition v_τ and for the same non-autonomous term h.

Then $w(t;\tau,u_\tau - v_\tau)$ is weak solution of the following problem

$$\begin{cases} \dfrac{\partial w}{\partial t} - \triangle w = f(u) - f(v) \;\text{ in }\; \Omega \times (\tau,+\infty), \\ w = 0 \;\text{ on }\; \partial\Omega \times (\tau,+\infty), \\ w(x,\tau) = u_\tau(x) - v_\tau(x), \quad x \in \Omega. \end{cases}$$

Using the energy equality, we have

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + |\nabla w(t)|^2 = \int_\Omega (f(u(t)) - f(v(t)))w(t)dx,$$

and by (2.3), in particular, we obtain

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 \le l\,|w(t)|^2.$$

Integrating between τ and t, and by the Gronwall Lemma, we can deduce

$$|w(t)|^2 \le |w(\tau)|^2 e^{2l(t-\tau)} \quad \forall t \ge \tau.$$

Thus,

$$|U(t,\tau)u_\tau - U(t,\tau)v_\tau|^2 \le |u_\tau - v_\tau|^2 e^{2l(t-\tau)} \quad \forall t \ge \tau,$$

and, thus we have shown that the process defined by (2.23) is continuous in $L^2(\Omega)$.

Now, we define the universe in $\mathcal{P}(L^2(\Omega))$. Let us denote λ_1 the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition in Ω. We denote by $\mathcal{D}_{\lambda_1}$ the class of all families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ such that

$$\lim_{\tau\to-\infty}\left(e^{\lambda_1\tau}\sup_{v\in D(\tau)}|v|^2\right) = 0.$$

According to the notation introduced in the previous chapter, $\mathcal{D}_F^H$ will denote the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of $L^2(\Omega)$.

Remark 2.10 *We note that $\mathcal{D}_F^H \subset \mathcal{D}_{\lambda_1}$ and both universes are inclusion-closed.*

Now, our aim is to prove the existence of minimal pullback $\mathcal{D}_{\lambda_1}$-attractor and minimal pullback $\mathcal{D}_F^H$-attractor for the process U defined by (2.23).

To do this, we will use Theorem 1.11 and Corollary 1.13 of Chapter 1.

First, we need the following results.

Proposition 2.11 *Assume that $f \in C^1(\mathbb{R})$ satisfies (2.2) and (2.3). Then for any initial condition $u_\tau \in L^2(\Omega)$ and any $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, the solution u of (2.1) satisfies*

$$|u(t;\tau,u_\tau)|^2 \leq \frac{2k|\Omega|}{\lambda_1} + \frac{1}{\lambda_1} e^{-\lambda_1 t} \int_{-\infty}^{t} e^{\lambda_1 s} |h(s)|^2 \, ds + e^{\lambda_1(\tau - t)} |u_\tau|^2, \quad (2.24)$$

for all $t \geq \tau$.

Proof From the energy equality, we deduce that

$$\frac{d}{dt}|u(t)|^2 + 2|\nabla u(t)|^2 = 2(f(u(t)),u(t)) + 2(h(t),u(t)). \quad (2.25)$$

Multiplying by $e^{\lambda_1 t}$ and taking into account that

$$e^{\lambda_1 t}\frac{d}{dt}|u(t)|^2 = \frac{d}{dt}\left(e^{\lambda_1 t}|u(t)|^2\right) - \lambda_1 e^{\lambda_1 t}|u(t)|^2,$$

we obtain

$$\begin{aligned}\frac{d}{dt}\left(e^{\lambda_1 t}|u(t)|^2\right) + 2e^{\lambda_1 t}|\nabla u(t)|^2 &= \lambda_1 e^{\lambda_1 t}|u(t)|^2 + 2e^{\lambda_1 t}(f(u(t)),u(t)) \\ &\quad + 2e^{\lambda_1 t}(h(t),u(t)).\end{aligned}$$

Using the Poincaré Inequality, we have

$$\begin{aligned}\frac{d}{dt}\left(e^{\lambda_1 t}|u(t)|^2\right) + 2e^{\lambda_1 t}\lambda_1|u(t)|^2 &\leq \lambda_1 e^{\lambda_1 t}|u(t)|^2 \\ &\quad + 2e^{\lambda_1 t}(f(u(t)),u(t)) + 2e^{\lambda_1 t}(h(t),u(t)).\end{aligned} \quad (2.26)$$

Using (2.2)

$$\begin{aligned}(f(u(t)),u(t)) &\leq \int_\Omega (k - \alpha_2|u(t)|^p)\,dx \\ &= k|\Omega| - \alpha_2\int_\Omega |u(t)|^p\,dx \\ &\leq k|\Omega|.\end{aligned} \quad (2.27)$$

On the other hand,

$$\begin{aligned}(h(t),u(t)) &\le |h(t)|\,|u(t)| \\ &\le \frac{\lambda_1}{2}|u(t)|^2+\frac{1}{2\lambda_1}|h(t)|^2 .\end{aligned} \tag{2.28}$$

Taking into account (2.27) and (2.28) in (2.26), we can deduce

$$\frac{d}{dt}\left(e^{\lambda_1 t}|u(t)|^2\right)\le 2e^{\lambda_1 t}k|\Omega|+\frac{1}{\lambda_1}e^{\lambda_1 t}|h(t)|^2,$$

and integrating between τ and t

$$\begin{aligned}e^{\lambda_1 t}|u(t)|^2 &\le \frac{2k|\Omega|}{\lambda_1}\left(e^{\lambda_1 t}-e^{\lambda_1\tau}\right)+\frac{1}{\lambda_1}\int_\tau^t e^{\lambda_1 s}|h(s)|^2\,ds+e^{\lambda_1\tau}|u_\tau|^2 \\ &\le \frac{2k|\Omega|}{\lambda_1}e^{\lambda_1 t}+\frac{1}{\lambda_1}\int_{-\infty}^t e^{\lambda_1 s}|h(s)|^2\,ds+e^{\lambda_1\tau}|u_\tau|^2 .\end{aligned}$$

Therefore, we obtain (2.24).

□

Now, we prove that there exists a family pullback $\mathcal{D}_{\lambda_1}$-absorbing for the process U defined by (2.23).

Proposition 2.12 *Assume that $f\in C^1(\mathbb{R})$ satisfies (2.2) and (2.3). Let $h\in L^2_{loc}(\mathbb{R};L^2(\Omega))$, such that*

$$\int_{-\infty}^t e^{\lambda_1 s}|h(s)|^2\,ds<+\infty \quad \forall t\in\mathbb{R}. \tag{2.29}$$

Then, the family $\widehat{D}_0=\{D_0(t):t\in\mathbb{R}\}$ defined by $D_0(t)=\overline{B}_{L^2(\Omega)}(0,R_{\lambda_1}(t))$, where $R_{\lambda_1}(t)$ is the nonnegative number given for each $t\in\mathbb{R}$ by

$$R^2_{\lambda_1}(t)=1+\frac{2k|\Omega|}{\lambda_1}+\frac{1}{\lambda_1}e^{-\lambda_1 t}\int_{-\infty}^t e^{\lambda_1 s}|h(s)|^2\,ds, \tag{2.30}$$

form a family $\widehat{D}_0\in\mathcal{D}_{\lambda_1}$ which is pullback $\mathcal{D}_{\lambda_1}$-absorbing for the process U defined by (2.23) (therefore pullback $\mathcal{D}^H_F$-absorbing).

Moreover for every $t\in\mathbb{R}$, there exist a positive constant $I_V(t)$ such that for every $\widehat{D}\in\mathcal{D}_{\lambda_1}$ there exists $\tau_{\widehat{D}}(t)<t$ such that

$$\int_t^{t+1}\|u(s;\tau,u_\tau)\|^2\,ds\le I_V(t)\quad \forall\tau\le\tau_{\widehat{D}}(t),$$

for all $u_\tau\in D(\tau)$.

Proof As a consequence of (2.29) and (2.30), it is evident that $\widehat{D}_0 \in \mathcal{D}_{\lambda_1}$.

Let $\widehat{D} \in \mathcal{D}_{\lambda_1}$ be fixed. Thanks to Proposition 2.11, we deduce that for every $\tau \leq t$ and any $u_\tau \in D(\tau)$,

$$\begin{aligned} |U(t,\tau)u_\tau|^2 &\leq \frac{2k\,|\Omega|}{\lambda_1} + \frac{1}{\lambda_1} e^{-\lambda_1 t} \int_{-\infty}^{t} e^{\lambda_1 s} |h(s)|^2 \, ds + e^{\lambda_1(\tau - t)} |u_\tau|^2 \\ &\leq \frac{2k\,|\Omega|}{\lambda_1} + \frac{1}{\lambda_1} e^{-\lambda_1 t} \int_{-\infty}^{t} e^{\lambda_1 s} |h(s)|^2 \, ds + e^{-\lambda_1 t} e^{\lambda_1 \tau} \sup_{v \in D(\tau)} |v|^2 . \end{aligned}$$

We can see that there exists $\tau_{\widehat{D}}(t) \leq t$ such that $e^{\lambda_1 \tau} \sup_{v \in D(\tau)} |v|^2 \leq e^{\lambda_1 t}$ for all $\tau \leq \tau_{\widehat{D}}(t)$.

Thus, we obtain

$$|U(t,\tau)u_\tau|^2 \leq \frac{2k\,|\Omega|}{\lambda_1} + \frac{1}{\lambda_1} e^{-\lambda_1 t} \int_{-\infty}^{t} e^{\lambda_1 s} |h(s)|^2 \, ds + 1,$$

for all $\tau \leq \tau_{\widehat{D}}(t)$ and for all $u_\tau \in D(\tau)$.

Consequently the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_{L^2(\Omega)}(0, R_{\lambda_1}(t))$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing for the process U defined by (2.23).

Finally, to prove the last part of our proposition, by (2.25), (2.27) and (2.28), we can deduce

$$\begin{aligned} \frac{d}{dt} |u(t)|^2 + 2\,|\nabla u(t)|^2 &= 2(f(u(t)), u(t)) + 2(h(t), u(t)) \\ &\leq 2k\,|\Omega| + \lambda_1 |u(t)|^2 + \frac{1}{\lambda_1} |h(t)|^2 . \end{aligned}$$

Using the Poincaré Inequality

$$\frac{d}{dt} |u(t)|^2 + 2\,|\nabla u(t)|^2 \leq 2k\,|\Omega| + |\nabla u(t)|^2 + \frac{1}{\lambda_1} |h(t)|^2 .$$

Therefore

$$\frac{d}{dt} |u(t)|^2 + |\nabla u(t)|^2 \leq 2k\,|\Omega| + \frac{1}{\lambda_1} |h(t)|^2 .$$

Integrating between t and $t+1$, we have

$$\begin{aligned} &|u(t+1;\tau,u_\tau)|^2 - |u(t;\tau,u_\tau)|^2 + \int_t^{t+1} |\nabla u(s;\tau,u_\tau)|^2 \, ds \\ &\leq 2k\,|\Omega| + \frac{1}{\lambda_1} \int_t^{t+1} |h(s)|^2 \, ds. \end{aligned}$$

Taking into account that $|u(t+1;\tau,u_\tau)|^2 \geq 0$, $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and using the first part of this proposition, we obtain

$$\begin{aligned}\int_t^{t+1} \|u(s;\tau,u_\tau)\|^2\, ds &\leq 2k\,|\Omega| + |u(t;\tau,u_\tau)|^2 + \frac{1}{\lambda_1}\int_t^{t+1} |h(s)|^2\, ds \\ &\leq 2k\,|\Omega| + R^2_{\lambda_1}(t) + \frac{1}{\lambda_1}\int_t^{t+1} |h(s)|^2\, ds \\ &=: I_V(t),\end{aligned}$$

for all $\tau \leq \tau_{\widehat{D}}(t)$.

□

Proposition 2.13 *Under the assumptions in Theorem 2.4, assume that h satisfies (2.29). Then, the process U defined by (2.23) is pullback $\mathcal{D}_{\lambda_1}$−asymptotically compact (therefore, pullback $\widehat{D}_0$−asymptotically compact).*

Proof Fix $\widehat{D} \in \mathcal{D}_{\lambda_1}$, $t \in \mathbb{R}$, $\{\tau_n\} \subset (-\infty, t]$ and $\{u_{\tau_n}\} \subset L^2(\Omega)$, with $\lim_n \tau_n = -\infty$ and $u_{\tau_n} \in D(\tau_n)$ for all n.

We have to prove that the sequence $\{U(t,\tau_n)u_{\tau_n}\}$ is relatively compact in $L^2(\Omega)$. Now, we proceed formally. The rigorous proof should be made using the Galerkin approximations used in Theorem 2.4.

We can assume without loss of generality that $f(0) = 0$ and $\tau_n < t-1$.

We denote

$$u_n(t) = U(t,\tau_n)u_{\tau_n}. \tag{2.31}$$

Thanks to Remark 2.5, Theorem 2.6, and the fact that $\tau_n < t-1$, we deduce

$$u_n \in C\left([t-1,t]\,; H^1_0(\Omega)\right) \cap L^2\left(t-1,t; H^2(\Omega)\right).$$

and

$$u_n' \in L^2\left(t-1,t; L^2(\Omega)\right).$$

From the equation,

$$\frac{du_n(t)}{dt} - \Delta u_n(t) = f(u_n(t)) + h(t),$$

taking scalar products in $L^2(\Omega)$, we obtain

$$\left(\frac{du_n(t)}{dt}, -\Delta u_n(t)\right) + |\Delta u_n(t)|^2 = (f(u_n(t)), -\Delta u_n(t)) + (h(t), -\Delta u_n(t)). \tag{2.32}$$

Observe that we can deduce

$$\int_\Omega -\Delta u_n(x,t)\frac{du_n(x,t)}{dt}dx = \sum_{i=1}^n \int_\Omega \frac{\partial u_n(x,t)}{\partial x_i}\frac{\partial^2 u_n(x,t)}{\partial t \partial x_i}dx \tag{2.33}$$
$$= \int_\Omega \frac{d}{dt}\frac{1}{2}\left(\nabla u_n(x,t)\right)^2 dx$$
$$= \frac{1}{2}\frac{d}{dt}\left|\nabla u_n(t)\right|^2.$$

By (2.3), we have

$$(f(u_n(t)), -\Delta u_n(t)) = -\int_\Omega \Delta u_n(x,t) f(u_n(x,t))dx \tag{2.34}$$
$$= \int_\Omega \sum_{j=1}^n f'(u_n(x,t))\left|\frac{\partial u_n(x,t)}{\partial x_j}\right|^2 dx$$
$$\leq l\left|\nabla u_n(t)\right|^2,$$

and

$$(h(t), -\Delta u_n(t)) = \int_\Omega -\Delta u_n(x,t)h(t)dx \tag{2.35}$$
$$\leq \left|\Delta u_n(t)\right|\left|h(t)\right|$$
$$\leq \frac{1}{2}\left|\Delta u_n(t)\right|^2 + \frac{1}{2}\left|h(t)\right|^2.$$

Therefore, taking into account (2.33)-(2.35) in (2.32) we obtain

$$\frac{1}{2}\frac{d}{dt}\left|\nabla u_n(t)\right|^2 + \left|\Delta u_n(t)\right|^2 \leq l\left|\nabla u_n(t)\right|^2 + \frac{1}{2}\left|\Delta u_n(t)\right|^2 + \frac{1}{2}\left|h(t)\right|^2.$$

In particular, we can deduce that

$$\frac{d}{dt}\left|\nabla u_n(t)\right|^2 \leq 2l\left|\nabla u_n(t)\right|^2 + \left|h(t)\right|^2.$$

Integrating between z and t,

$$\left|\nabla u_n(t)\right|^2 \leq 2l\int_z^t \left|\nabla u_n(s)\right|^2 ds + \int_z^t \left|h(s)\right|^2 ds + \left|\nabla u_n(z)\right|^2,$$

for all $t-1 \leq z \leq t$.

Now, integrating with respect to z between $t-1$ and t,

$$\left|\nabla u_n(t)\right|^2 \leq 2l\int_{t-1}^t \left|\nabla u_n(s)\right|^2 ds + \int_{t-1}^t \left|h(s)\right|^2 ds + \int_{t-1}^t \left|\nabla u_n(s)\right|^2 ds,$$

i.e.,

$$|\nabla u_n(t)|^2 \leq (2l+1)\int_{t-1}^{t} |\nabla u_n(s)|^2\, ds + \int_{t-1}^{t} |h(s)|^2\, ds.$$

As τ_n tends to $-\infty$, using (2.31) and thanks to Proposition 2.12, we have that there exists $\tau_{\widehat{D}}(t)$ such that for all $\tau_n \leq \tau_{\widehat{D}}(t)$ we have

$$\|U(t,\tau_n)u_{\tau_n}\|^2 \leq (2l+1)I_V(t) + \int_{t-1}^{t} |h(s)|^2\, ds =: \rho_V^2(t),$$

i.e.,

$$U(t,\tau_n)u_{\tau_n} \in \overline{B}_{H_0^1(\Omega)}\left(0, \rho_V(t)\right),$$

for all $\tau_n \leq \tau_{\widehat{D}}(t)$.

As $H_0^1(\Omega)$ is imbedded in $L^2(\Omega)$ with compact injection, then $\overline{B}_{H_0^1(\Omega)}(0,\rho_V(t))$ is a compact subset in $L^2(\Omega)$. Therefore we can extract a subsequence of $\{U(t,\tau_n)u_{\tau_n}\}$ that converges strongly in $L^2(\Omega)$. Thus, it is proved that the process U defined by (2.23) is pullback $\mathcal{D}_{\lambda_1}-$asymptotically compact.

□

Now, as a direct consequence of the preceding results, Theorem 1.11 and Corollary 1.13 of Chapter 1, we have the existence of the minimal pullback $\mathcal{D}_{\lambda_1}-$attractor and the minimal pullback $\mathcal{D}_F^H-$attractor for the process U defined by (2.23).

Theorem 2.14 *Under the assumptions in Theorem 2.4, assume that h satisfies (2.29). Then, the process U defined by (2.23) has a unique pullback $\mathcal{D}_{\lambda_1}-$attractor $\mathcal{A}_{\mathcal{D}_{\lambda_1}}$ belonging to $\mathcal{D}_{\lambda_1}$, which is given by*

$$\mathcal{A}_{\mathcal{D}_{\lambda_1}}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}_{\lambda_1}} \Lambda\left(\widehat{D},t\right)}^{L^2(\Omega)} = \Lambda\left(\widehat{D}_0,t\right),$$

where $\widehat{D}_0$ was defined in Proposition 2.12. Moreover, there exists the minimal pullback $\mathcal{D}_F^H-$attractor, $\mathcal{A}_{\mathcal{D}_F^H}$, and we have the following relation

$$\mathcal{A}_{\mathcal{D}_F^H}(t) \subset \mathcal{A}_{\mathcal{D}_{\lambda_1}}(t) \subset \overline{B}_{L^2(\Omega)}\left(0, R_{\lambda_1}(t)\right) \quad \text{for all } t\in\mathbb{R}.$$

Remark 2.15 *We note that if we also assume that*

$$\sup_{t\leq 0}\left(\int_{t-1}^{t} |h(s)|^2\, ds\right) < \infty,$$

then we have that $\cup_{t\leq T} D_0(t) = \cup_{t\leq T}\overline{B}_{L^2(\Omega)}(0, R_{\lambda_1}(t))$ *is a bounded subset of* $L^2(\Omega)$.

Therefore, taking into account Remark 1.14 in Chapter 1, we can deduce that

$$\mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_{\lambda_1}}(t) \quad \text{for all } t \leq T.$$

As a final comment, we have the following remark.

Remark 2.16 *Let* $\mu \in (0, 2\lambda_1)$ *and denote by* $\mathcal{D}_\mu$ *the class of all families* $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ *such that*

$$\lim_{\tau\to-\infty}\left(e^{\mu\tau}\sup_{v\in D(\tau)}|v|^2\right) = 0.$$

Under the assumptions in Theorem 2.4, assume that

$$\int_{-\infty}^{t} e^{\mu s}|h(s)|^2\, ds < +\infty \quad \forall t \in \mathbb{R}. \tag{2.36}$$

Then, with slight changes in the preceding reasoning, we can also deduce that the process U *defined by (2.23) possesses a minimal pullback* $\mathcal{D}_\mu-$*attractor,* $\mathcal{A}_{\mathcal{D}_\mu}$*, which is given by*

$$\mathcal{A}_{\mathcal{D}_\mu}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}_\mu}\Lambda\left(\widehat{D}, t\right)}^{L^2(\Omega)} = \Lambda\left(\widehat{D}_{0,\mu}, t\right),$$

where, in this case, the family $\widehat{D}_{0,\mu}$ *is defined by* $D_{0,\mu}(t) = \overline{B}_{L^2(\Omega)}(0, R_\mu(t))$, *where* $R_\mu(t)$ *is the nonnegative number given for each* $t \in \mathbb{R}$ *by*

$$R_\mu^2(t) = 1 + \frac{2k|\Omega|}{\mu} + \frac{1}{\mu}e^{-\mu t}\int_{-\infty}^{t} e^{\mu s}|h(s)|^2\, ds. \tag{2.37}$$

Moreover, we have the following relation

$$\mathcal{A}_{\mathcal{D}_F^H}(t) \subset \mathcal{A}_{\mathcal{D}_\mu}(t) \subset \overline{B}_{L^2(\Omega)}(0, R_\mu(t)) \quad \text{for all } t \in \mathbb{R}.$$

Remark 2.16 is important because we can observe that if $\mu \in (\lambda_1, 2\lambda_1)$, there are more functions that satisfy assumption (2.36) than assumption (2.29).

Now, in the next section we prove some results which, in particular, imply that, under suitable assumptions, for any $\mu \in (0, 2\lambda_1)$, the minimal pullback $\mathcal{D}_\mu-$attractor, $\mathcal{A}_{\mathcal{D}_\mu}$, for the process U defined by (2.23) satisfies that $\mathcal{A}_{\mathcal{D}_\mu}(t)$ is a bounded subset of $H^2(\Omega) \cap H_0^1(\Omega) \cap L^p(\Omega)$, for every $t \in \mathbb{R}$.

2.5 H^2-Boundedness of Pullback Attractors

In this section we prove that, under suitable assumptions, every family of bounded subsets of $L^2(\Omega)$ which is invariant for the process U, is in fact bounded in $H^2(\Omega)$.

Fist, we recall a lemma which we will use for the proof of our results.

Lemma 2.17 *Let X, Y be Banach spaces such that X is reflexive, and the inclusion $X \subset Y$ is continuous. Assume that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(t_0, T; X)$ such that $u_n \rightharpoonup u$ weakly in $L^p(t_0, T; X)$ for some $p \in [1, +\infty)$ and $u \in C([t_0, T]; Y)$.*

Then, for every $t \in [t_0, T]$, $u(t)$ belongs to X and satisfies

$$\|u(t)\|_X \leq \sup_{n\geq 1} \|u_n\|_{L^\infty(t_0,T;X)}.$$

Proof We denote

$$C := \sup_{n\geq 1} \|u_n\|_{L^\infty(t_0,T;X)}.$$

As $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(t_0, T; X)$, there exist a subsequence (u_μ) and $v \in L^\infty(t_0, T; X)$ such that $u_\mu \overset{*}{\rightharpoonup} v$ weak-star in $L^\infty(t_0, T; X)$, i.e.,

$$\int_{t_0}^T \langle w^*(t), u_\mu(t)\rangle\, dt \longrightarrow \int_{t_0}^T \langle w^*(t), v(t)\rangle\, dt \quad \forall\ w^* \in L^1(t_0, T; X'),$$

where by $\langle\cdot,\cdot\rangle$ we denote the duality product between X' and X.
In particular, we have this convergence for all $w^* \in L^{p'}(t_0, T; X')$. Then, $u_\mu \rightharpoonup v$ weakly in $L^p(t_0, T; X)$, and as we also have $u_n \rightharpoonup u$ weakly in $L^p(t_0, T; X)$, then $v = u$.
Then, $u_\mu \overset{*}{\rightharpoonup} u$ weak-star in $L^\infty(t_0, T; X)$ and by the $*-$weak lower semicontinuity of the norm, we obtain

$$\|u\|_{L^\infty(t_0,T;X)} \leq \liminf_{\mu\to\infty} \|u_\mu\|_{L^\infty(t_0,T;X)} \leq C. \tag{2.38}$$

Now fix $t \in [t_0, T]$. By (2.38) there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $[t_0, T]$ such that $t_n \to t$ and $u(t_n) \in X$ with $\|u(t_n)\|_X \leq C$ for all $n \in \mathbb{N}$.

As X is reflexive, there exist a subsequence (t_μ) and $x \in X$ such that $u(t_\mu) \rightharpoonup x$ weakly in X. The inclusion $X \subset Y$ is assumed to be continuous and so,

$$u(t_\mu) \rightharpoonup x \text{ weakly in } Y. \tag{2.39}$$

As $u \in C([t_0, T]; Y)$, we have in addition that $u(t_n) \to u(t)$ strongly in Y. This implies $u(t) = x \in X$.
Finally, the weak lower semi-continuity of the norm implies, for every $t \in [t_0, T]$,

$$\|u(t)\|_X \;=\; \|x\|_X \;\leq\; \liminf_{\mu \to \infty} \|u(t_\mu)\|_X \;\leq\; C.$$

□

For each integer $n \geq 1$, we denote by $u_n(t) = u_n(t; \tau, u_\tau)$ the Galerkin approximation of the solution $u(t; \tau, u_\tau)$ of (2.1), which is given by

$$u_n(t) = \sum_{j=1}^{n} \gamma_{nj}(t) w_j, \tag{2.40}$$

and is the solution of

$$\begin{cases} \dfrac{d}{dt}(u_n(t), w_j) = \langle \Delta u_n(t), w_j \rangle + (f(u_n(t)), w_j) + (h(t), w_j), \\ (u_n(\tau), w_j) = (u_\tau, w_j) \qquad j = 1, .., n, \end{cases} \tag{2.41}$$

where $\{w_j : \ j \geq 1\}$ is the Hilbert basis of $L^2(\Omega)$ formed by the eigenfunctions associated to $-\Delta$ in $H_0^1(\Omega)$.

We first prove the following result.

Proposition 2.18 *Assume that $f \in C^1(\mathbb{R})$ satisfies* (2.2) *and* (2.3). *Suppose moreover that $\Omega \subset \mathbb{R}^N$ is a bounded C^k domain, with $k \geq \max(2, N(p-2)/2p)$, and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$. Then, for any bounded set $B \subset L^2(\Omega)$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$ and any $t > \tau + \varepsilon$, the set $\{u_n(r; \tau, u_\tau) : \ r \in [\tau + \varepsilon, t], \ u_\tau \in B, \ n \geq 1\}$ is a bounded subset of $H_0^1(\Omega) \cap L^p(\Omega)$.*

Proof Observe that by the regularity of Ω, all the eigenfunctions w_j associated to $-\Delta$ in $H_0^1(\Omega)$ belong to $H^2(\Omega) \cap H_0^1(\Omega) \cap L^p(\Omega)$.

Let us fix a bounded set $B \subset L^2(\Omega)$, $\tau \in \mathbb{R}$, $\varepsilon > 0$, $t > \tau + \varepsilon$, and $u_\tau \in B$.

Multiplying by γ_{nj} in (2.41), and summing from $j = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dr}|u_n(r)|^2 + |\nabla u_n(r)|^2 = (f(u_n(r)), u_n(r)) + (h(r), u_n(r)). \tag{2.42}$$

Using (2.2),

$$\begin{aligned}(f(u_n(r)), u_n(r)) &\leq \int_\Omega (k - \alpha_2 |u_n(x,r)|^p)\, dx \\ &= k\,|\Omega| - \alpha_2 \|u_n(r)\|^p_{L^p(\Omega)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(h(r), u_n(r)) &\leq \frac{1}{2\lambda_1} |h(r)|^2 + \frac{\lambda_1}{2} |u_n(r)|^2 \\ &\leq \frac{1}{2\lambda_1} |h(r)|^2 + \frac{1}{2} |\nabla u_n(r)|^2 .\end{aligned}$$

Thus, from (2.42) we deduce

$$\frac{d}{dr} |u_n(r)|^2 + |\nabla u_n(r)|^2 + 2\alpha_2 \|u_n(r)\|^p_{L^p(\Omega)} \leq \frac{1}{\lambda_1} |h(r)|^2 + 2k\,|\Omega|\,,$$

and integrating between τ and r

$$\begin{aligned}&|u_n(r)|^2 + \int_\tau^r |\nabla u_n(s)|^2\, ds + 2\alpha_2 \int_\tau^r \|u_n(s)\|^p_{L^p(\Omega)}\, ds \\ &\quad\leq |u_\tau|^2 + \frac{1}{\lambda_1}\int_\tau^t |h(s)|^2\, ds + 2k\,|\Omega|\,(t-\tau),\ \ \forall r \in [\tau, t],\ \ \forall n \geq 1.\end{aligned} \tag{2.43}$$

Now, multiplying by γ'_{nj} in (2.41), and summing from $j = 1$ to n,

$$\begin{aligned}|u'_n(r)|^2 + \frac{1}{2}\frac{d}{dr} |\nabla u_n(r)|^2 &= (f(u_n(r)), u'_n(r)) + (h(r), u'_n(r)) \\ &\leq \frac{1}{2} |h(r)|^2 + \frac{1}{2} |u'_n(r)|^2 + \frac{d}{dr} \int_\Omega \mathcal{F}\left(u_n(x,r)\right) dx.\end{aligned}$$

Integrating now between $s \in [\tau, r]$ and $r \leq t$, we obtain

$$\begin{aligned}\int_s^r |u'_n(\theta)|^2\, d\theta + |\nabla u_n(r)|^2 &\leq |\nabla u_n(s)|^2 + \int_\tau^t |h(\theta)|^2\, d\theta \\ &\quad + 2\int_\Omega \mathcal{F}\left(u_n(x,r)\right) dx - 2\int_\Omega \mathcal{F}\left(u_n(x,s)\right) dx,\end{aligned}$$

which, jointly with (2.5), yields that

$$\begin{aligned}&\int_s^r |u'_n(\theta)|^2\, d\theta + |\nabla u_n(r)|^2 + 2\widetilde{\alpha}_2 \|u_n(r)\|^p_{L^p(\Omega)} \\ &\quad\leq |\nabla u_n(s)|^2 + \int_\tau^t |h(\theta)|^2\, d\theta + 4\widetilde{k}\,|\Omega| + 2\widetilde{\alpha}_1 \left\|u_n(s)\right\|^p_{L^p(\Omega)}\,,\end{aligned} \tag{2.44}$$

for all $s \in [\tau, r]$, and any $r \in [\tau, t]$.

Integrating in this last inequality with respect to s from τ to r, we, in particular, obtain

$$(r-\tau)\left(|\nabla u_n(r)|^2 + 2\widetilde{\alpha}_2\|u_n(r)\|^p_{L^p(\Omega)}\right) \leq \int_\tau^t |\nabla u_n(s)|^2\, ds + (t-\tau)\int_\tau^t |h(s)|^2\, ds$$
$$+ 4\widetilde{k}\,|\Omega|\,(t-\tau) + 2\widetilde{\alpha}_1\int_\tau^t \|u_n(s)\|^p_{L^p(\Omega)}\, ds,$$

for all $r \in [\tau, t]$, and for any $n \geq 1$. From this inequality and (2.43), our result holds.

□

Corollary 2.19 *Under the assumptions in Proposition 2.18, for any bounded set $B \subset L^2(\Omega)$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$, and any $t > \tau + \varepsilon$, the set $\bigcup_{r\in[\tau+\varepsilon,t]} U(r,\tau)B$ is a bounded subset of $H_0^1(\Omega) \cap L^p(\Omega)$.*

Proof This is a straightforward consequence of Lemma 2.17, Proposition 2.18, and the fact that $u_n(\cdot;\tau,u_\tau)$ converges weakly to $u(\cdot;\tau,u_\tau)$ in the space $L^2(\tau,t;H_0^1(\Omega)) \cap L^p(\tau,t;L^p(\Omega))$ (see (2.12) and (2.13)).

□

Proposition 2.20 *In addition to the assumptions in Proposition 2.18, assume that $h \in W^{1,2}_{loc}(\mathbb{R}; L^2(\Omega))$. Then, for any bounded set $B \subset L^2(\Omega)$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$, and any $t > \tau+\varepsilon$, the set $\{u_n(r;\tau,u_\tau) : r \in [\tau+\varepsilon, t],\ u_\tau \in B,\ n \geq 1\}$ is a bounded subset of $H^2(\Omega)$.*

Proof Let us fix a bounded set $B \subset L^2(\Omega)$, $\tau \in \mathbb{R}$, $\varepsilon > 0$, $t > \tau + \varepsilon$, and $u_\tau \in B$.

As we are assuming that $h \in W^{1,2}_{loc}(\mathbb{R}; L^2(\Omega))$, we can differentiate with respect to time in (2.41), and then, multiplying by γ'_{nj}, and summing from $j = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dr}|u'_n(r)|^2 + |\nabla u'_n(r)|^2 = (f'(u_n(r))u'_n(r), u'_n(r)) + (h'(r), u'_n(r))$$
$$\leq l\,|u'_n(r)|^2 + \frac{1}{2}|u'_n(r)|^2 + \frac{1}{2}|h'(r)|^2.$$

In particular, integrating in the last inequality,

$$|u'_n(r)|^2 \leq |u'_n(s)|^2 + (2l+1)\int_{\tau+\varepsilon/2}^t |u'_n(\theta)|^2\, d\theta + \int_{\tau+\varepsilon/2}^t |h'(\theta)|^2\, d\theta,$$

for all $\tau + \varepsilon/2 \leq s \leq r \leq t$. Now, integrating with respect to s between $\tau + \varepsilon/2$ and r,

$$(r - \tau - \varepsilon/2) |u_n'(r)|^2 \leq [(2l + 1)(t - \tau - \varepsilon/2) + 1] \int_{\tau+\varepsilon/2}^{t} |u_n'(\theta)|^2 \, d\theta + (r - \tau - \varepsilon/2) \int_{\tau+\varepsilon/2}^{t} |h'(\theta)|^2 \, d\theta,$$

for all $\tau + \varepsilon/2 \leq r \leq t$, and, in particular,

$$|u_n'(r)|^2 \leq 2\varepsilon^{-1}[(2l + 1)(t - \tau - \varepsilon/2) + 1] \int_{\tau+\varepsilon/2}^{t} |u_n'(\theta)|^2 \, d\theta + \int_{\tau+\varepsilon/2}^{t} |h'(\theta)|^2 \, d\theta, \tag{2.45}$$

for all $r \in [\tau + \varepsilon, t]$.

On the other hand, multiplying in (2.41) by $\lambda_j \gamma_{nj}$, where λ_j is the eigenvalue associated to the eigenfunction w_j, and summing once more from $j = 1$ to n, we obtain

$$(u_n'(r), \Delta u_n(r)) = |\Delta u_n(r)|^2 + (f(u_n(r)), \Delta u_n(r)) + (h(r), \Delta u_n(r)) . \tag{2.46}$$

But, it follows from (2.3) that

$$\begin{aligned}
-(f(u_n(r)), \Delta u_n(r)) &= -\int_\Omega (f(u_n(x, r)) - f(0)) \, \Delta u_n(x, r) dx \\
&\quad - f(0) \int_\Omega \Delta u_n(x, r) dx \\
&\leq l \, |\nabla u_n(r)|^2 + \frac{1}{4} |\Delta u_n(r)|^2 + (f(0))^2 |\Omega| \\
&= l \, (u_n(r), -\Delta u_n(r)) + \frac{1}{4} |\Delta u_n(r)|^2 + (f(0))^2 |\Omega| \\
&\leq l^2 |u_n(r)|^2 + \frac{1}{2} |\Delta u_n(r)|^2 + (f(0))^2 |\Omega| ,
\end{aligned}$$

and thus, from (2.46) we obtain

$$|\Delta u_n(r)|^2 \leq 8 |u_n'(r)|^2 + 8 |h(r)|^2 + 4l^2 |u_n(r)|^2 + 4 (f(0))^2 |\Omega| , \tag{2.47}$$

for all $r \geq \tau$.

Finally, observe that by (2.44)

$$\int_{\tau+\varepsilon/2}^{t} |u_n'(\theta)|^2 \, d\theta \le |\nabla u_n(\tau+\varepsilon/2)|^2 + \int_{\tau}^{t} |h(\theta)|^2 \, d\theta + 4\widetilde{k}\,|\Omega| \qquad (2.48)$$
$$+ 2\widetilde{\alpha}_1 \, \|u_n(\tau+\varepsilon/2)\|_{L^p(\Omega)}^p .$$

Taking into account that, in particular, $h \in C([\tau,t];L^2(\Omega))$, the result is a direct consequence of Proposition 2.18 and estimates (2.45), (2.47) and (2.48).

□

Corollary 2.21 *Under the assumptions of Proposition 2.20, for any bounded set $B \subset L^2(\Omega)$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$, and any $t > \tau+\varepsilon$, the set $\bigcup_{r\in[\tau+\varepsilon,t]} U(r,\tau)B$ is a bounded subset of $H^2(\Omega)$.*

Proof This follows from Lemma 2.17, propositions 2.18 and 2.20, and the facts that $u_n(\cdot;\tau,u_\tau)$ converges weakly to $u(\cdot;\tau,u_\tau)$ in $L^2(\tau,t;H_0^1(\Omega))$ (see (2.13)) and $u(\cdot;\tau,u_\tau) \in C([\tau+\varepsilon,t];H_0^1(\Omega))$ (see Theorem 2.6).

□

As a direct consequence of the above results, we can now establish our main results of this subsection.

Theorem 2.22 *Under the assumptions in Proposition 2.20, if $\mathcal{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ is a family of bounded subsets of $L^2(\Omega)$, such that $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for any $\tau \le t$, then for any $T_1 < T_2$, the set $\bigcup_{t\in[T_1,T_2]} \mathcal{A}(t)$ is a bounded subset of $H^2(\Omega) \cap H_0^1(\Omega) \cap L^p(\Omega)$.*

In particular, we have the following result.

Corollary 2.23 *Under the assumptions in Proposition 2.20, if $\mathcal{A}_{\mathcal{D}_{\lambda_1}} = \{\mathcal{A}_{\mathcal{D}_{\lambda_1}}(t) : t \in \mathbb{R}\}$ is a minimal pullback $\mathcal{D}_{\lambda_1}-$attractor and $\mathcal{A}_{\mathcal{D}_F^H} = \{\mathcal{A}_{\mathcal{D}_F^H}(t) : t \in \mathbb{R}\}$ is a minimal pullback $\mathcal{D}_F^H-$attractor for the process defined by (2.23), then for any $T_1 < T_2$, the set $\bigcup_{t\in[T_1,T_2]} \mathcal{A}_{\mathcal{D}_{\lambda_1}}(t)$ is a bounded subset of $H^2(\Omega) \cap H_0^1(\Omega) \cap L^p(\Omega)$, and therefore the set $\bigcup_{t\in[T_1,T_2]} \mathcal{A}_{\mathcal{D}_F^H}(t)$ is also a bounded subset of $H^2(\Omega) \cap H_0^1(\Omega) \cap L^p(\Omega)$.*

Moreover, more generally, for any $\mu \in (0, 2\lambda_1)$, if $\mathcal{A}_{\mathcal{D}_\mu} = \{\mathcal{A}_{\mathcal{D}_\mu}(t) : t \in \mathbb{R}\}$ is a minimal pullback $\mathcal{D}_\mu -$attractor for the process defined by (2.23), then for any $T_1 < T_2$, the set $\bigcup_{t\in[T_1,T_2]} \mathcal{A}_{\mathcal{D}_\mu}(t)$ is a bounded subset of $H^2(\Omega) \cap H_0^1(\Omega) \cap L^p(\Omega)$.

2.6 An exponential growth condition in H^2 for the Pullback Attractor

The aim of this section is to continue with the analysis of our model in the sense of proving that the family $\mathcal{A}_{\mathcal{D}_{\lambda_1}}$ satisfies also an exponential growth condition on the space $L^p(\Omega) \cap H_0^1(\Omega)$, and finally in $H^2(\Omega)$ provided some additional assumptions are fulfilled.

We have already shown in Section 2.4 that, under the condition (2.29), there exists the minimal pullback $\mathcal{D}_{\lambda_1}-$attractor for the process U defined by (2.23), and satisfies

$$\lim_{\tau\to-\infty} \left(e^{\lambda_1 \tau} \sup_{v\in\mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau)} |v|^2 \right) = 0. \tag{2.49}$$

Now, we prove the following result.

Theorem 2.24 *Assume that $f \in C^1(\mathbb{R})$ satisfies (2.2) and (2.3). Suppose moreover that $\Omega \subset \mathbb{R}^N$ is a bounded C^κ domain, with $\kappa \geq \max(2, N(p-2)/2p)$, $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, and condition (2.29) holds. Then, $\mathcal{A}_{\mathcal{D}_{\lambda_1}}$ satisfies*

$$\lim_{\tau\to-\infty} \left\{ e^{\lambda_1 \tau} \left(\sup_{v\in\mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau)} \|v\|^2 + \sup_{v\in\mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau)} \|v\|^p_{L^p(\Omega)} \right) \right\} = 0. \tag{2.50}$$

Proof From the inequality (2.43), for any $t \geq \tau$ we have

$$\begin{aligned} |u_n(r)|^2 + \int_\tau^r |\nabla u_n(s)|^2 \, ds + \int_\tau^r \|u_n(s)\|^p_{L^p(\Omega)} \, ds \\ \leq C_1 \left(|u_\tau|^2 + \int_\tau^t |h(s)|^2 \, ds + (t-\tau) \right), \end{aligned} \tag{2.51}$$

for all $r \in [\tau, t]$, and all $n \geq 1$, where $C_1 := \dfrac{\max\{1, \lambda_1^{-1}, 2k|\Omega|\}}{\min\{1, 2\alpha_2\}}$.

Also, integrating inequality (2.44) with respect to s from τ to r, we obtain

$$(r-\tau)\left(|\nabla u_n(r)|^2+\|u_n(r)\|^p_{L^p(\Omega)}\right) \tag{2.52}$$
$$\le C_2\left(\int_\tau^r |\nabla u_n(s)|^2\,ds+\int_\tau^r \|u_n(s)\|^p_{L^p(\Omega)}\,ds\right)$$
$$+\frac{(t-\tau)}{\min\{1,2\widetilde{\alpha}_2\}}\int_\tau^t |h(s)|^2\,ds$$
$$+\frac{4\widetilde{k}}{\min\{1,2\widetilde{\alpha}_2\}}|\Omega|\,(t-\tau),$$

for any $t\ge\tau$, all $r\in[\tau,t]$, and all $n\ge 1$, where $C_2:=\dfrac{\max\{1,2\widetilde{\alpha}_1\}}{\min\{1,2\widetilde{\alpha}_2\}}$.

From (2.51) and (2.52) we now obtain that

$$(r-\tau)\left(|\nabla u_n(r)|^2+\|u_n(r)\|^p_{L^p(\Omega)}\right)\le C_1C_2\left(|u_\tau|^2+\int_\tau^t |h(s)|^2\,ds+(t-\tau)\right)$$
$$+\frac{(t-\tau)}{\min\{1,2\widetilde{\alpha}_2\}}\int_\tau^t |h(s)|^2\,ds$$
$$+\frac{4\widetilde{k}}{\min\{1,2\widetilde{\alpha}_2\}}|\Omega|\,(t-\tau), \tag{2.53}$$

for any $t\ge\tau$, all $r\in[\tau,t]$, and all $n\ge 1$.
In particular, from (2.53) we deduce

$$|\nabla u_n(r)|^2+\|u_n(r)\|^p_{L^p(\Omega)}\le C_3\left(|u_\tau|^2+\int_\tau^{\tau+2}|h(s)|^2\,ds+1\right), \tag{2.54}$$

for all $r\in[\tau+1,\tau+2]$, and any $n\ge 1$, where

$$C_3:=\max\left\{C_1C_2+\frac{2}{\min\{1,2\widetilde{\alpha}_2\}},\,2C_1C_2+\frac{8\widetilde{k}}{\min\{1,2\widetilde{\alpha}_2\}}|\Omega|\right\}.$$

By (2.12) and (2.13), we have that $u_n(\cdot)=u_n(\cdot;\tau,u_\tau)$ converges weakly to $u(\cdot)=u(\cdot;\tau,u_\tau)$ in $L^2(\tau,t;H_0^1(\Omega))\cap L^p(\tau,t;L^p(\Omega))$, for all $t>\tau$.

Thus, from (2.54) and Lemma 2.17, we in particular obtain

$$|\nabla u(\tau+1)|^2+\|u(\tau+1)\|^p_{L^p(\Omega)}\le C_3\left(|u_\tau|^2+\int_\tau^{\tau+2}|h(s)|^2\,ds+1\right).$$

Multiplying this inequality by $e^{\lambda_1(\tau+1)}$ and using (2.23), we have

$$e^{\lambda_1(\tau+1)}\left(\|U(\tau+1,\tau)u_\tau\|^2+\|U(\tau+1,\tau)u_\tau\|^p_{L^p(\Omega)}\right) \tag{2.55}$$
$$\le C_3e^{\lambda_1}\left(e^{\lambda_1\tau}|u_\tau|^2+\int_\tau^{\tau+2}e^{\lambda_1 s}|h(s)|^2\,ds+e^{\lambda_1\tau}\right),$$

for all $\tau \in \mathbb{R}$, and all $u_\tau \in L^2(\Omega)$.

As $\mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau+1) = U(\tau+1,\tau)\mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau)$, it follows from (2.55) that

$$\begin{aligned} & e^{\lambda_1(\tau+1)}\left(\|v\|^2 + \|v\|^p_{L^p(\Omega)}\right) \\ & \leq C_3 e^{\lambda_1}\left(e^{\lambda_1\tau}\sup_{w\in\mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau)}|w|^2 + \int_\tau^{\tau+2} e^{\lambda_1 s}|h(s)|^2\,ds + e^{\lambda_1\tau}\right), \end{aligned}$$

for all $v \in \mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau+1)$, and any $\tau \in \mathbb{R}$.

Finally, this inequality implies

$$\begin{aligned} & e^{\lambda_1\tau}\left(\|v\|^2 + \|v\|^p_{L^p(\Omega)}\right) \\ & \leq C_3 e^{\lambda_1}\left(e^{\lambda_1(\tau-1)}\sup_{w\in\mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau-1)}|w|^2 + \int_{\tau-1}^{\tau+1} e^{\lambda_1 s}|h(s)|^2\,ds + e^{\lambda_1(\tau-1)}\right), \end{aligned} \tag{2.56}$$

for all $v \in \mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau)$, and any $\tau \in \mathbb{R}$. Taking into account (2.29) and (2.49), from (2.56) we obtain (2.50).

□

Theorem 2.25 *In addition to the assumptions in Theorem 2.24, assume moreover that* $h \in W^{1,2}_{loc}(\mathbb{R}; L^2(\Omega))$, *and satisfies*

$$\lim_{\tau\to-\infty} e^{\lambda_1\tau}\int_\tau^{\tau+1}|h'(s)|^2\,ds = 0 \tag{2.57}$$

and

$$\lim_{\tau\to-\infty} e^{\lambda_1\tau}|h(\tau)|^2 = 0. \tag{2.58}$$

Then, $\mathcal{A}_{\mathcal{D}_{\lambda_1}}$ *satisfies that*

$$\lim_{\tau\to-\infty}\left(e^{\lambda_1\tau}\sup_{v\in\mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau)}\|v\|^2_{H^2(\Omega)}\right) = 0. \tag{2.59}$$

Proof From inequality (2.45), taking $t = \tau+3$ and $\varepsilon = 2$, we have

$$\begin{aligned} |u_n'(r)|^2 \leq (4l+3)\int_{\tau+1}^{\tau+3}|u_n'(s)|^2\,ds \\ + \int_{\tau+1}^{\tau+3}|h'(s)|^2\,ds, \end{aligned} \tag{2.60}$$

for all $r \in [\tau+2, \tau+3]$, and any $n \geq 1$.

Analogously, and if we take $s = \tau+1$ and $r = t = \tau+3$ in inequality (2.44), we have

$$\int_{\tau+1}^{\tau+3} |u_n'(s)|^2\, ds + |\nabla u_n(\tau+3)|^2 + 2\widetilde{\alpha}_2 \|u_n(\tau+3)\|_{L^p(\Omega)}^p \tag{2.61}$$
$$\leq |\nabla u_n(\tau+1)|^2 + \int_{\tau}^{\tau+3} |h(s)|^2\, ds + 4\widetilde{k}\, |\Omega| + 2\widetilde{\alpha}_1 \, \|u_n(\tau+1)\|_{L^p(\Omega)}^p ,$$

for all $n \geq 1$.

From (2.60) and (2.61), we obtain

$$\begin{aligned} |u_n'(r)|^2 \leq{}& (4l+3) \left(|\nabla u_n(\tau+1)|^2 + 2\widetilde{\alpha}_1 \, \|u_n(\tau+1)\|_{L^p(\Omega)}^p \right) \\ &+ (4l+3) \left(\int_{\tau}^{\tau+3} |h(s)|^2\, ds + 4\widetilde{k}\, |\Omega| \right) \\ &+ \int_{\tau+1}^{\tau+3} |h'(s)|^2\, ds, \end{aligned}$$

for all $r \in [\tau+2, \tau+3]$, and any $n \geq 1$.

Owing to this inequality and (2.54), there exists a constant $\widetilde{C}_1 > 0$ such that

$$|u_n'(r)|^2 \leq \widetilde{C}_1 \left(|u_\tau|^2 + \int_{\tau}^{\tau+3} \left(|h(s)|^2 + |h'(s)|^2 \right) ds + 1 \right), \tag{2.62}$$

for all $r \in [\tau+2, \tau+3]$, and any $n \geq 1$.

From inequality (2.47), and thanks to (2.62), we have

$$\begin{aligned} |\Delta u_n(r)|^2 \leq{}& 8\widetilde{C}_1 \left(|u_\tau|^2 + \int_{\tau}^{\tau+3} \left(|h(s)|^2 + |h'(s)|^2 \right) ds + 1 \right) + 8\, |h(r)|^2 \\ &+ 4l^2\, |u_n(r)|^2 + 4\, (f(0))^2\, |\Omega| , \end{aligned}$$

for all $r \in [\tau+2, \tau+3]$, and any $n \geq 1$, and therefore, by (2.51) we obtain that there exists a constant $\widetilde{C}_2 > 0$ such that

$$|\Delta u_n(r)|^2 \tag{2.63}$$
$$\leq \widetilde{C}_2 \left(|u_\tau|^2 + \int_{\tau}^{\tau+3} \left(|h(s)|^2 + |h'(s)|^2 \right) ds + 1 + \sup_{r \in [\tau+2, \tau+3]} |h(r)|^2 \right),$$

for all $r \in [\tau+2, \tau+3]$, and any $n \geq 1$.

Using (2.13), in particular, we have that $u_n(\cdot) = u_n(\cdot;\tau,u_\tau)$ converges weakly to $u(\cdot) = u(\cdot;\tau,u_\tau)$ in $L^2(\tau+2,\tau+3;H_0^1(\Omega))$ and thanks to Theorem 2.6 we have that $u(\cdot;\tau,u_\tau) \in C([\tau+2,\tau+3];H_0^1(\Omega))$. Then, by Lemma 2.17, inequality (2.63) and the equivalence of the norms $|\Delta v|$ and $\|v\|_{H^2(\Omega)}$, we have that there exists a constant $\widetilde{C}_3 > 0$ such that

$$\|u(r;\tau,u_\tau)\|^2_{H^2(\Omega)} \le \widetilde{C}_3\left(|u_\tau|^2 + \int_\tau^{\tau+3}\left(|h(s)|^2 + |h'(s)|^2\right)ds + 1 + \sup_{r\in[\tau+2,\tau+3]}|h(r)|^2\right), \tag{2.64}$$

for all $r \in [\tau+2,\tau+3]$, any $\tau \in \mathbb{R}$, and $u_\tau \in L^2(\Omega)$.

Now, observe that by the Cauchy inequality,

$$|h(r)| \le |h(\tau+2)| + \left(\int_{\tau+2}^{\tau+3}|h'(s)|^2\,ds\right)^{1/2},$$

for all $r \in [\tau+2,\tau+3]$. Thus, from (2.64), and using (2.23), we deduce that there exists a constant $\widetilde{C}_4 > 0$ such that

$$\|U(\tau+2,\tau)u_\tau\|^2_{H^2(\Omega)} \le \widetilde{C}_4\left(|u_\tau|^2 + \int_\tau^{\tau+3}\left(|h(s)|^2 + |h'(s)|^2\right)ds + |h(\tau+2)|^2 + 1\right),$$

for all $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$.

From this inequality, and the fact that $\mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau) = U(\tau,\tau-2)\mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau-2)$, we obtain

$$\|v\|^2_{H^2(\Omega)} \le \widetilde{C}_4\left(\sup_{w\in\mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau-2)}|w|^2 + \int_{\tau-2}^{\tau+1}\left(|h(s)|^2 + |h'(s)|^2\right)ds + |h(\tau)|^2 + 1\right), \tag{2.65}$$

for all $v \in \mathcal{A}_{\mathcal{D}_{\lambda_1}}(\tau)$, and any $\tau \in \mathbb{R}$.

Now, thanks to (2.29), (2.49), (5.43) and (2.58), we obtain (2.59) from (2.65).

□

Remark 2.26 *The same kind of results can be obtained for $\mathcal{A}_{\mathcal{D}_\mu}$ with $\mu \in (0, 2\lambda_1)$.*

Remark 2.27 *In Theorems 2.24 and 2.25, the pullback attraction property is not needed. In fact, both theorems are also valid for any family $\{\mathcal{A}(\tau) : \tau \in \mathbb{R}\}$ of nonempty subsets of $L^2(\Omega)$ satisfying (2.49) and the semi-invariance property*

$$\mathcal{A}(\tau+n) \subset U(\tau+n,\tau)\mathcal{A}(\tau),$$

for all $\tau \in \mathbb{R}$ and any integer $n \ge 1$.

Chapter 3

Theory of set-valued non-autonomous dynamical systems

The theory of pullback attractors is an important mathematical tool for studying the qualitative behavior of infinite-dimensional dynamical systems. By using this theory during the last few years, many results concerning attractors for evolution differential equations have been obtained. However, these results can not be applied to a wide class of initial-boundary problems, in which the solution may be not unique. Good examples of such systems are differential inclusions, variational inequalities, control infinite-dimensional systems and also some partial-differential equations as the three-dimensional Navier-Stokes equations or the non-autonomous reaction-diffusion equations without uniqueness of solution.

For the qualitative analysis of the above-mentioned systems from the point of view of the theory of dynamical systems it is necessary to develop the corresponding theory for set-valued non-autonomous dynamical systems.

Our aim in this chapter is to give a theory of the pullback attractors in the framework of set-valued problems. This theory will be a generalization of the theory given in Chapter 1.

First we recall some basic definitions for set-valued non-autonomous dynamical systems and establish a sufficient condition for the existence of a pullback attractor for these systems. The results in this chapter can be found in Anguiano *et al.* [9], Caraballo and Kloeden [20] and Marín-Rubio and Real [59] (see Melnik and Valero [60] for the autonomous case). The new result give in this chapter is about the connectivity of the pullback attractor. For a more general random context the reader is referred to [19].

3.1 Basic concepts

Let $X = (X, d_X)$ be a metric space, let $\mathcal{P}(X)$ denote the family of all nonempty subsets of X, let us denote $\mathbb{R}^2_d := \{(t,s) \in \mathbb{R}^2 : t \geq s\}$.

Definition 3.1 *A multi-valued map $U : \mathbb{R}^2_d \times X \mapsto \mathcal{P}(X)$ is called a* ***multi-valued non-autonomous dynamical system (MNDS)*** *on X (also named a multi-valued process on X) if*

$$U(\tau,\tau,x) = \{x\} \text{ for all } \tau \in \mathbb{R},\, x \in X,$$

$$U(t,\tau,x) \subset U(t,s,U(s,\tau,x)) \text{ for all } \tau \leq s \leq t,\, x \in X,$$

where $U(t,\tau,V) := \bigcup_{x_0 \in V} U(t,\tau,x_0)$ *for any non-empty set $V \subset X$.*

An ***MNDS*** *is said to be* ***strict*** *if*

$$U(t,\tau,x) = U(t,s,U(s,\tau,x)) \text{ for all } \tau \leq s \leq t,\, x \in X.$$

Definition 3.2 *An MNDS U on X is said to be* ***upper-semicontinuous*** *if for all $t \geq \tau$ the mapping $U(t,\tau,\cdot)$ is upper-semicontinuous from X into $\mathcal{P}(X)$, i.e., for any $x_0 \in X$ and for every neighborhood $\mathcal{N}$ in X of the set $U(t,\tau,x_0)$, there exists $\delta > 0$ such that $U(t,\tau,y) \subset \mathcal{N}$ whenever $d_X(x_0,y) < \delta$.*

Let $\mathcal{D}$ be a class of sets parameterized in time, $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

As we saw in Chapter 1, we will say that the class $\mathcal{D}$ is inclusion-closed, if $\widehat{D} \in \mathcal{D}$ and $\emptyset \neq D'(t) \subset D(t)$ for all $t \in \mathbb{R}$, imply that $\widehat{D'} = \{D'(t) : t \in \mathbb{R}\}$ belongs to $\mathcal{D}$.

Definition 3.3 *We say that a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is* ***pullback $\mathcal{D}$-absorbing*** *for the MNDS U if for every $\widehat{D} \in \mathcal{D}$ and every $t \in \mathbb{R}$, there exists $\tau(t,\widehat{D}) \leq t$ such that*

$$U(t,\tau,D(\tau)) \subset D_0(t) \text{ for all } \tau \leq \tau(t,\widehat{D}).$$

Definition 3.4 *The MNDS U is* ***pullback asymptotically compact*** *with respect to a family $\widehat{B} = \{B(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ (or* ***pullback $\widehat{B}$-asymptotically compact****) if for all $t \in \mathbb{R}$ and every sequence $\tau_n \leq t$ tending to $-\infty$, any sequence $y_n \in U(t,\tau_n,B(\tau_n))$ is relatively compact in X.*

Proposition 3.5 *If $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}-$absorbing and the MNDS U is pullback asymptotically compact with respect to $\widehat{D}_0$, then U is also pullback asymptotically compact with respect to $\widehat{D}$, for any $\widehat{D} \in \mathcal{D}$.*

Proof Fix $\widehat{D} \in \mathcal{D}$, $t \in \mathbb{R}$ and $\{\tau_n\} \subset (-\infty, t]$ and $\{y_n\} \subset X$, with $\lim_n \tau_n = -\infty$ and $y_n \in U(t, \tau_n, D(\tau_n))$ for all n. We have to prove that the sequence $\{y_n\}$ is relatively compact in X. Taking into account that $\widehat{D}_0$ is pullback $\mathcal{D}-$absorbing for U, for each integer $k \geq 1$, there exist $\tau_{n_k} \in \{\tau_n\}$ with $\tau_{n_k} \leq t - k$, and $U(t-k, \tau_{n_k}, D(\tau_{n_k})) \subset D_0(t-k)$. Then,

$$\begin{aligned} y_{n_k} &\in U(t, \tau_{n_k}, D(\tau_{n_k})) \\ &\subset U(t, t-k, U(t-k, \tau_{n_k}, D(\tau_{n_k}))) \\ &\subset U(t, t-k, D_0(t-k)), \end{aligned}$$

and therefore, since U is pullback $\widehat{D}_0$-asymptotically compact, we can extract from $\{y_{n_k}\}$ a subsequence $\{y_{n_{k'}}\}$ that converges in X.

□

As in Chapter 1, we denote by $\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2)$ the Hausdorff semi-distance in X between two sets $\mathcal{O}_1$ and $\mathcal{O}_2$, defined as

$$\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2) = \sup_{x \in \mathcal{O}_1} \inf_{y \in \mathcal{O}_2} d_X(x, y) \quad \text{for } \mathcal{O}_1,\ \mathcal{O}_2 \subset X.$$

The following definition provides the main objective of this chapter.

Definition 3.6 *A family $\mathcal{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is said to be a **global pullback $\mathcal{D}$-attractor** for the MNDS U if it satisfies*

1) $\mathcal{A}(t)$ is compact for any $t \in \mathbb{R}$,

2) $\mathcal{A}$ is pullback $\mathcal{D}$-attracting, i.e.

$$\lim_{\tau \to -\infty} dist_X(U(t, \tau, D(\tau)), \mathcal{A}(t)) = 0 \quad \forall t \in \mathbb{R},$$

for all $\widehat{D} \in \mathcal{D}$,

3) $\mathcal{A}$ is negatively invariant, i.e.,

$$\mathcal{A}(t) \subset U(t, \tau, \mathcal{A}(\tau)), \text{ for any } (t, \tau) \in \mathbb{R}^2_d.$$

*$\mathcal{A}$ is said to be a **strict global pullback $\mathcal{D}$-attractor** if the invariance property in the third item is strict, i.e.,*

$$\mathcal{A}(t) = U(t, \tau, \mathcal{A}(\tau)), \text{ for } (t, \tau) \in \mathbb{R}^2_d.$$

The main tool to prove the existence of an attractor is the concept of pullback-omega-limit set.

Definition 3.7 *For any family* $\widehat{B} = \{B(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$*, we define the* ***pullback-omega-limit set*** *as the* t*–dependent set* $\Lambda(\widehat{B}, t)$ *given by*

$$\Lambda(\widehat{B}, t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau, B(\tau))}^{X}.$$

This set is closed, but it may be empty. It can be proved that $y \in \Lambda(\widehat{B}, t)$ if and only if there exist $\tau_n \to -\infty$ and $y_n \in U(t, \tau_n, B(\tau_n))$ such that

$$\lim_{n \to +\infty} y_n = y.$$

Proposition 3.8 *If* $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ *is pullback* $\mathcal{D}-$*absorbing for the MNDS* U*, then*

$$\Lambda(\widehat{D}, t) \subset \Lambda(\widehat{D}_0, t) \quad \textit{for all } \widehat{D} \in \mathcal{D},\ t \in \mathbb{R}.$$

Moreover, if $\widehat{D}_0 \in \mathcal{D}$*, then*

$$\Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^{X} \quad \textit{for all } t \in \mathbb{R}.$$

Proof Fix $\widehat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$. For any $y \in \Lambda(\widehat{D}, t)$, there exist two sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{y_n\} \subset X$, with $\lim_n \tau_n = -\infty$ and $y_n \in U(t, \tau_n, D(\tau_n))$ for all n, such that $y_n \to y$.

Since $\widehat{D}_0$ is pullback $\mathcal{D}-$absorbing for U, for each integer $k \geq 1$, there exist $\tau_{n_k} \in \{\tau_n\}$ with $\tau_{n_k} \leq t - k$, and $U(t-k, \tau_{n_k}, D(\tau_{n_k})) \subset D_0(t-k)$. Then,

$$\begin{aligned} y_{n_k} &\in U(t, \tau_{n_k}, D(\tau_{n_k})) \\ &\subset U(t, t-k, U(t-k, \tau_{n_k}, D(\tau_{n_k})) \\ &\subset U(t, t-k, D_0(t-k)), \end{aligned}$$

with $y_{n_k} \to y$, and therefore, $y \in \Lambda(\widehat{D}_0, t)$.

Finally, we consider $t \in \mathbb{R}$ and we suppose that $\widehat{D}_0 \in \mathcal{D}$. We observe that for any $y \in \Lambda(\widehat{D}_0, t)$, there exist $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$ and $\{y_n\} \subset X$ with $y_n \in U(t, \tau_n, D_0(\tau_n))$ for all n, such that $y_n \to y$. Since $\widehat{D}_0$ is pullback $\mathcal{D}-$absorbing for the MNDS U, then from certain $n \in \mathbb{N}$, $y_n \in D_0(t)$. Thus, $y \in \overline{D_0(t)}^{X}$.

□

3.2 Existence of Pullback attractors for MNDS

In this section, we will establish a sufficient condition ensuring the existence of pullback attractors with respect to a general universe $\mathcal{D}$ (as in [23]). When this universe consists of bounded sets, the results have already been proved in [21].

We have the following lemma, which is a generalization of Theorem 6 and Lemma 8 in Caraballo *et al.* [21].

Lemma 3.9 *Let $\widehat{B} = \{B(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ be a family of sets such that the MNDS U is pullback asymptotically compact with respect to $\widehat{B}$.*

Then, for all $t \in \mathbb{R}$, the pullback–omega–limit set $\Lambda(\widehat{B}, t)$ is non–empty, compact, and

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(U(t, \tau, B(\tau)), \Lambda(\widehat{B}, t)) = 0. \tag{3.1}$$

The family $\{\Lambda(\widehat{B}, t) : t \in \mathbb{R}\}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that

$$\lim_{\tau \to -\infty} dist_X(U(t, \tau, B(\tau)), C(t)) = 0,$$

then $\Lambda(\widehat{B}, t) \subset C(t)$.

Moreover. if we also assume that the MNDS U is upper–semicontinuous with closed values for all $(t, \tau) \in \mathbb{R}^2_d$, then

$$\Lambda(\widehat{B}, t) \subset U(t, \tau, \Lambda(\widehat{B}, \tau)), \text{ for all } (t, \tau) \in \mathbb{R}^2_d. \tag{3.2}$$

Proof Consider a sequence $y_n \in U(t, \tau_n, B(\tau_n))$ with $\tau_n \to -\infty$.

As U is pullback asymptotically compact with respect to $\widehat{B}$, there exists a convergent subsequence and its limit y belongs to $\Lambda(\widehat{B}, t)$, so that $\Lambda(\widehat{B}, t)$ is non–empty.

We now prove that $\Lambda(\widehat{B}, t)$ is compact. For any sequence $\{y_n\} \subset \Lambda(\widehat{B}, t)$ there exist $\tau_n \to -\infty$ and $z_n \in U(t, \tau_n, B(\tau_n))$, such that

$$d_X(y_n, z_n) < \frac{1}{n}.$$

Using again the pullback asymptotic compactness of U the existence of a converging subsequence $z_{n_k} \to z \in \Lambda(\widehat{B}, t)$ follows. Then, $y_{n_k} \to z$, so that $\Lambda(\widehat{B}, t)$ is compact.

We prove (3.1) by contradiction. If (3.1) does not hold, then there exist $\varepsilon > 0$ and $y_n \in U(t, \tau_n, B(\tau_n))$ with $\tau_n \to -\infty$, such that

$$\mathrm{dist}_X(y_n, \Lambda(\widehat{B}, t)) > \varepsilon.$$

As U is pullback asymptotically compact with respect to $\widehat{B}$, it follows that there exists a subsequence (relabelled again the same) $y_n \to y \in \Lambda(\widehat{B}, t)$, which is not possible.

On the other hand, if we suppose that there exists a family $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ of closed sets such that

$$\lim_{\tau \to -\infty} dist_X(U(t, \tau), B(\tau)), C(t)) = 0. \tag{3.3}$$

We consider $y \in \Lambda(\widehat{B}, t)$, then there exist $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, and $y_n \in U(t, \tau_n, B(\tau_n))$ for all n, such that $y_n \to y$. By (3.3) we have $y \in \overline{C(t)}^X = C(t)$, and therefore $\Lambda(\widehat{B}, t) \subset C(t)$.

Finally, let us now prove that if moreover U is upper–semicontinuous with closed values for all $(t, \tau) \in \mathbb{R}^2_d$, then (3.2) holds. Fix $(t, \tau) \in \mathbb{R}^2_d$. Then, if $y \in \Lambda(\widehat{B}, t)$, there exist sequences $y_n \in U(t, \tau_n + \tau, x_n)$, $x_n \in B(\tau_n + \tau)$ with $\tau_n \to -\infty$, such that $y_n \to y$.

The process property implies

$$U(t, \tau_n + \tau, x_n) \subset U(t, \tau, U(\tau, \tau_n + \tau, x_n)),$$

and then $y_n \in U(t, \tau, z_n)$, where $z_n \in U(\tau, \tau_n + \tau, x_n)$.

As before, up to a subsequence, $z_n \to z \in \Lambda(\widehat{B}, \tau)$. Since $x \mapsto U(t, \tau, x)$ is upper–semicontinuous with closed values, so that its graph is closed, and then we have

$$y \in U(t, \tau, z) \subset U(t, \tau, \Lambda(\widehat{B}, \tau)).$$

□

We can now present a sufficient condition ensuring the existence of pullback attractor. This is our main result in this chapter.

Theorem 3.10 *Assume that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}-$absorbing for a MNDS U, which is also pullback $\widehat{D}_0-$asymptotically compact. Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ given by*

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}^X \quad t \in \mathbb{R}, \tag{3.4}$$

satisfies the following properties:

1) For each $t \in \mathbb{R}$ the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X, and

$$\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda(\widehat{D}_0, t).$$

2) $\mathcal{A}_{\mathcal{D}}$ *is pullback* $\mathcal{D}$*-attracting, and in fact is the minimal family of closed sets that attracts pullback to all elements of* $\mathcal{D}$*.*

3) If $\widehat{D}_0 \in \mathcal{D}$*, then* $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$*, for all* $t \in \mathbb{R}$*.*

4) If U *is upper semi-continuous and with closed values,* $\mathcal{A}_{\mathcal{D}}$ *is a global pullback* $\mathcal{D}$*-attractor for* U*.*

5) If U *is upper semi-continuous, with closed and connected values, and for each* $t \in \mathbb{R}$ $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$*, where* $\widehat{C} \in \mathcal{D}$ *and* $C(t)$ *is a connected subset of* X*, then* $\mathcal{A}_{\mathcal{D}}$ *is connected, i.e.* $\mathcal{A}_{\mathcal{D}}(t)$ *is connected for any* $t \in \mathbb{R}$*.*

6) If $\widehat{D}_0 \in \mathcal{D}$*, each* $D_0(t)$ *is closed and the universe* $\mathcal{D}$ *is inclusion-closed, then* $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$*. If moreover* U *is upper semi-continuous and with closed values,* $\mathcal{A}_{\mathcal{D}}$ *is the unique global pullback* $\mathcal{D}$*-attractor belonging to* $\mathcal{D}$*. In this case, if moreover* U *is strict, then* $\mathcal{A}_{\mathcal{D}}$ *is a strict global pullback* $\mathcal{D}$*-attractor for* U*.*

Proof Assertions 1)–4) are immediate consequences of Propositions 3.5 and 3.8, and Lemma 3.9.

For the proof of 5), suppose that $\mathcal{A}_{\mathcal{D}}$ is not connected. Then there exist $t \in \mathbb{R}$ and two open sets $\mathcal{O}_1$, $\mathcal{O}_2$ of X satisfying $A(t) \cap \mathcal{O}_i \neq \emptyset$ for $i = 1, 2$, $A(t) \subset \mathcal{O}_1 \cup \mathcal{O}_2$ and $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$.

It is well known (see [17], [34] or also [47, Theorem 24]) that an upper semicontinuous map with connected values maps any connected set into a connected one. Since the set $C(\tau)$ is connected, then $U(t, \tau, C(\tau))$ is connected for all $\tau \leq t$.

As $\mathcal{A}_{\mathcal{D}}$ is negatively invariant, we have

$$\mathcal{A}_{\mathcal{D}}(t) \subset U(t, \tau, \mathcal{A}_{\mathcal{D}}(\tau)) \subset U(t, \tau, C(\tau)).$$

Hence, $U(t, \tau, C(\tau)) \cap \mathcal{O}_i \neq \emptyset$ for $i = 1, 2$, and by the connectedness of $U(t, \tau, C(\tau))$ we obtain that $\mathcal{O}_1 \cup \mathcal{O}_2$ does not contain $U(t, \tau, C(\tau))$. Thus for any $\tau \leq t$ there exists $\xi_\tau \in U(t, \tau, C(\tau))$ such that $\xi_\tau \notin \mathcal{O}_1 \cup \mathcal{O}_2$.

Now, as $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting and $\widehat{C} \in \mathcal{D}$, for each $n \geq 1$ there exist $\tau_n \leq t$, with $\tau_n \to -\infty$, and $y_n \in \mathcal{A}_{\mathcal{D}}(t)$ such that

$$d_X(\xi_{\tau_n}, y_n) \leq \frac{1}{n}. \tag{3.5}$$

As $\mathcal{A}_{\mathcal{D}}(t)$ is compact, we can extract a converging subsequence

$$y_m \to y \in \mathcal{A}_{\mathcal{D}}(t).$$

By (3.5), we obtain

$$\xi_{\tau_m} \to y \in \mathcal{A}_{\mathcal{D}}(t) \subset \mathcal{O}_1 \cup \mathcal{O}_2.$$

But taking into account that $\mathcal{O}_1 \cup \mathcal{O}_2$ is an open set then there exists m_0 for which $\xi_{\tau_m} \in \mathcal{O}_1 \cup \mathcal{O}_2$, for all $m > m_0$, which is a contradiction.

For the proof of 6), if $\widehat{D}_0 \in \mathcal{D}$, each $D_0(t)$ is closed and the universe $\mathcal{D}$ is inclusion-closed, then the fact that $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is an immediate consequence of 3). If moreover U is upper semi-continuous and with closed values, and we have another pullback D–attractor $\mathcal{A}'_{\mathcal{D}} \in \mathcal{D}$, then as

$$\mathcal{A}'_{\mathcal{D}}(t) \subset U(t, \tau, \mathcal{A}'_{\mathcal{D}}(\tau))$$

and

$$\lim_{\tau \to -\infty} \operatorname{dist}_X \left(U(t, \tau, \mathcal{A}'_{\mathcal{D}}(\tau)), \mathcal{A}_{\mathcal{D}}(t)\right) = 0,$$

we have that $\mathcal{A}'_{\mathcal{D}}(t) \subset \mathcal{A}_{\mathcal{D}}(t)$. interchanging $\mathcal{A}_{\mathcal{D}}$ and $\mathcal{A}'_{\mathcal{D}}$ it follows that $\mathcal{A}_{\mathcal{D}} = \mathcal{A}'_{\mathcal{D}}$.

Finally, assume that moreover U is a strict MNDS. Then, for all $t \geq r$,

$$\begin{aligned} U\left(t, r, \mathcal{A}_{\mathcal{D}}\left(r\right)\right) &\subset U\left(t, r, U\left(r, r+\tau, \mathcal{A}_{\mathcal{D}}(r+\tau)\right)\right) \\ &= U\left(t, r+\tau, \mathcal{A}_{\mathcal{D}}\left(r+\tau\right)\right), \quad \text{for all } \tau \leq 0. \end{aligned}$$

As $\mathcal{A}_{\mathcal{D}}$ pullback attracts itself, it follows that

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t, r+\tau, \mathcal{A}_{\mathcal{D}}(r+\tau)), \mathcal{A}_{\mathcal{D}}(t)) = 0,$$

and, consequently, given $\varepsilon > 0$, there exists $T(\varepsilon, t, r) < 0$ such that, for $\tau \leq T(\varepsilon, t, r)$

$$\operatorname{dist}_X(U(t, r+\tau, \mathcal{A}_{\mathcal{D}}(r+\tau)), \mathcal{A}_{\mathcal{D}}(t)) < \varepsilon,$$

and as $U\left(t, r, \mathcal{A}_{\mathcal{D}}\left(r\right)\right) \subset U\left(t, r+\tau, \mathcal{A}_{\mathcal{D}}\left(r+\tau\right)\right)$, we have

$$\operatorname{dist}_X(U(t, r, \mathcal{A}_{\mathcal{D}}(r)), \mathcal{A}_{\mathcal{D}}(t)) < \varepsilon, \text{ for all } \varepsilon > 0,$$

so $U(t, r, \mathcal{A}_{\mathcal{D}}(r)) \subset \mathcal{A}_{\mathcal{D}}(t)$, as required.

□

Again as in Chapter 1, if we denote $\mathcal{D}_F^X$ the universe of fixed nonempty bounded subsets of X, i.e. the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X it is easy to conclude the following result.

Corollary 3.11 *Assume that* $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ *is pullback* $\mathcal{D}-$*absorbing for a MNDS* U, *which is also pullback* $\widehat{D}_0-$*asymptotically compact, upper semi-continuous and with closed values. Then, if the universe* $\mathcal{D}$ *contains the universe* $\mathcal{D}_F^X$, *both attractors,* $\mathcal{A}_{\mathcal{D}_F^X}$ *and* $\mathcal{A}_{\mathcal{D}}$, *exist, and the following relation holds:*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) \subset \mathcal{A}_{\mathcal{D}}(t) \quad \textit{for all } t \in \mathbb{R}.$$

Remark 3.12 *As in the univalued case, under the assumptions of the preceding corollary, if, moreover, for some* $T \in \mathbb{R}$ *the set* $\cup_{t \le T} D_0(t)$ *is a bounded subset of* X, *then*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) = \mathcal{A}_{\mathcal{D}}(t) \quad \textit{for all } t \le T.$$

Chapter 4

Non-autonomous reaction-diffusion equation without uniqueness of solution

The asymptotic behaviour of equations without uniqueness of the Cauchy problem has been studied by several authors in the last years. There are many important reasons which justify the interest of the researches in such type of equations. On the one hand, they contain important models coming from Mathematical Physics. On the other hand, they allow to weaken the conditions imposed in the nonlinear functions involved in the equations, which are in many cases very restrictive.

In this sense, now, our aim is to consider a much more general problem than the problem considered in Chapter 2: a reaction-diffusion equation in an unbounded domain, with a continuous nonlinearity and a non-autonomous forcing term with values in the space H^{-1} which does not have uniqueness of solutions.

We will use the theory of multi-valued non-autonomous (pullback) dynamical systems, which has been developed in Chapter 3, to prove the existence of a pullback attractor for our problem. Then, we will prove the Kneser property for our problem and using this property of solutions we will check the connectedness of the associated global pullback attractor.

The results of this chapter can be found in [6] and [9].

4.1 Setting of the problem

Let $\Omega \subset \mathbb{R}^N$ be a nonempty open set, not necessarily bounded, and suppose that Ω satisfies the Poincaré inequality, i.e., there exists a constant $\lambda_1 > 0$ such that

$$\int_\Omega |u(x)|^2\, dx \le \lambda_1^{-1} \int_\Omega |\nabla u(x)|^2\, dx \quad \forall u \in H_0^1(\Omega). \tag{4.1}$$

Let us consider the following problem for a non-autonomous reaction-diffusion equation with zero Dirichlet boundary condition in Ω,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(x,u) + h(t) \ \text{ in } \ \Omega \times (\tau, +\infty), \\ u = 0 \ \text{ on } \ \partial\Omega \times (\tau, +\infty), \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{4.2}$$

where $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function, that is, $f(\cdot, u)$ is a measurable function for any $u \in \mathbb{R}$ and $f(x, \cdot) \in C(\mathbb{R})$ for almost every $x \in \Omega$, and satisfies that there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$, and $p \ge 2$ and positive functions $C_1(x)$, $C_2(x) \in L^1(\Omega)$ such that

$$|f(x,s)|^{\frac{p}{p-1}} \le \alpha_1 |s|^p + C_1(x) \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega, \tag{4.3}$$

$$f(x,s)s \le -\alpha_2 |s|^p + C_2(x) \ \ \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega. \tag{4.4}$$

We observe that thanks to (4.3), we can deduce that

$$|f(x,s)| \le \alpha_1^{\frac{p-1}{p}} |s|^{p-1} + C_1(x)^{\frac{p-1}{p}} \qquad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega. \tag{4.5}$$

By $|\cdot|$, $\|\cdot\| = |\nabla \cdot|$, $\|\cdot\|_*$ and $\|\cdot\|_{L^p(\Omega)}$ we denote the norms in the spaces $L^2(\Omega)$, $H_0^1(\Omega)$, $H^{-1}(\Omega)$ and $L^p(\Omega)$, respectively.

We will use $(\cdot,\cdot)$ to denote the scalar product in $L^2(\Omega)$ or $[L^2(\Omega)]^N$, and $\langle \cdot,\cdot \rangle$ to denote either the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ or between $L^{p'}(\Omega)$ and $L^p(\Omega)$, where $p' = \frac{p}{p-1}$ is the conjugate exponent of p.

The aim of this chapter is to show the existence of a pullback attractor, which is connected, in the phase space $L^2(\Omega)$ for the problem (4.2) using Theorem 3.10 of Section 3.2.

To do this we need a theorem on existence of solutions of problem (4.2), which we will see in the next section.

4.2 Existence of Solution

We state in this section a result on the existence of solutions of problem (4.2). First, we give the definition of weak solution of problem (4.2).

Definition 4.1 *A weak solution of (4.2) is a function* $u \in L^p(\tau, T; L^p(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega))$ *for all* $T > \tau$, *and such that*

$$(u(t), w) + \int_\tau^t (\nabla u(s), \nabla w)\, ds = (u_\tau, w) + \int_\tau^t \langle f(x, u(s)) + h(s), w\rangle\, ds \quad \forall\, t \geq \tau, \tag{4.6}$$

for all $w \in L^p(\Omega) \cap H_0^1(\Omega)$.

Remark 4.2 *Definitions 4.1 and 2.1 are equivalent.*

It is well known [26, p.285] that under the above assumptions on u_τ, f and h, if u is a weak solution of (4.2), then $u \in C([\tau, +\infty); L^2(\Omega))$, the function $t \mapsto \|u(t)\|^2$ is absolutely continuous on every interval $[\tau, T]$ and

$$\frac{d}{dt}|u(t)|^2 = 2\left\langle \frac{du}{dt}, u\right\rangle \text{ for a.a. } t \in (\tau, T).$$

Hence, it satisfies the energy equality

$$|u(t)|^2 + 2\int_\tau^t |\nabla u(s)|^2\, ds = |u_\tau|^2 + 2\int_\tau^t \langle f(x, u(s)) + h(s), u(s)\rangle\, ds \quad \forall\, t \geq \tau.$$

From now on, for all $m \geq 1$, we denote

$$\Omega_m = \Omega \cap \left\{x \in \mathbb{R}^N : |x|_{\mathbb{R}^N} < m\right\},$$

where $|\cdot|_{\mathbb{R}^N}$ denotes the Euclidean norm in $\mathbb{R}^N$.

Our goal now is to prove the following result.

Theorem 4.3 *Assume that* Ω *satisfies (4.1),* $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ *and* f *is Carathéodory and satisfies (4.3) and (4.4). Then, for all* $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, *there exists at least a weak solution* u *of (4.2).*

Proof For each integer $n \geq 1$, we denote by

$$u_n(t) = \sum_{j=1}^n \gamma_{nj}(t) w_j,$$

a solution of

$$\begin{cases} \frac{d}{dt}(u_n(t), w_j) = -(\nabla u_n(t), \nabla w_j) + \langle f(x, u_n(t)), w_j\rangle + \langle h(t), w_j\rangle & t > \tau, \\ (u_n(\tau), w_j) = (u_\tau, w_j) \quad j = 1, .., n, \end{cases} \tag{4.7}$$

where $\{w_j : j \geq 1\} \subset H_0^1(\Omega) \cap L^p(\Omega)$ is a Hilbert basis of $L^2(\Omega)$ such that *span* $\{w_j\}_{j\geq 1}$ is dense in $H_0^1(\Omega) \cap L^p(\Omega)$.

Multiplying by $\gamma_{nj}(t)$ in (4.7), and summing from $j = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|^2 + |\nabla u_n(t)|^2 = \langle f(u_n(t)), u_n(t)\rangle + \langle h(t), u_n(t)\rangle . \tag{4.8}$$

Using (4.4),

$$\begin{aligned} \langle f(u_n(t)), u_n(t)\rangle &\leq \int_\Omega (C_2(x) - \alpha_2 |u_n(x,t)|^p)\, dx \\ &= \|C_2\|_{L^1(\Omega)} - \alpha_2 \|u_n(t)\|^p_{L^p(\Omega)}. \end{aligned}$$

On the other hand,

$$\langle h(t), u_n(t)\rangle \leq \frac{1}{2}\|h(t)\|_*^2 + \frac{1}{2}|\nabla u_n(t)|^2 .$$

Thus, from (4.8) we deduce

$$\frac{d}{dt}|u_n(t)|^2 + |\nabla u_n(t)|^2 + 2\alpha_2 \|u_n(t)\|^p_{L^p(\Omega)} \leq \|h(t)\|_*^2 + 2\|C_2\|_{L^1(\Omega)},$$

and integrating between τ and t

$$\begin{aligned} &|u_n(t)|^2 + \int_\tau^t |\nabla u_n(s)|^2\, ds + 2\alpha_2 \int_\tau^t \|u_n(s)\|^p_{L^p(\Omega)}\, ds \\ &\qquad \leq |u_\tau|^2 + \int_\tau^t \|h(s)\|_*^2\, ds + 2\|C_2\|_{L^1(\Omega)}(t-\tau), \ \ \forall n \geq 1. \end{aligned}$$

It is a standard matter to deduce that

$$\{u_n\} \text{ is bounded in } L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap C([\tau, T]; L^2(\Omega)), \tag{4.9}$$

for all $T > \tau$.

Now, taking into account (4.3) and (4.9), it is also a standard matter to deduce that

$$f(x, u_n) \text{ is bounded in } L^{p'}(\tau, T; L^{p'}(\Omega)), \tag{4.10}$$

for all $T > \tau$ (we note that the above estimates allow to extend every local solution of (4.7) to a global one).

Then, there exists a subsequence $\{u_\mu\} \subset \{u_n\}$ such that

$$u_\mu \overset{*}{\rightharpoonup} u \text{ weak-star in } L^\infty(\tau, T; L^2(\Omega)),$$

$$u_\mu \rightharpoonup u \text{ weakly in } L^p(\tau, T; L^p(\Omega)),$$

$$u_\mu \rightharpoonup u \text{ weakly in } L^2(\tau, T; H_0^1(\Omega)), \tag{4.11}$$

$$f(x, u_\mu) \rightharpoonup \chi \text{ weakly in } L^{p'}(\tau, T; L^{p'}(\Omega)), \tag{4.12}$$

for all $T > \tau$.

Now (4.11) implies that

$$\Delta u_\mu \rightharpoonup \Delta u \text{ weakly in } L^2(\tau, T; H^{-1}(\Omega)).$$

On the other hand, to prove that $\chi(t) = f(x, u(t))$, we argue similarly to [69].

Let $w = w_j$ with $j = 1, ..., \mu$. Integrating the equality

$$\begin{aligned} \frac{d}{ds}(u_\mu(s), w) &= -(\nabla u_\mu(s), \nabla w) \\ &\quad + \langle f(u_\mu(s)), w\rangle + \langle h(s), w\rangle, \end{aligned}$$

between t and $t+a$, with $a \in (0, T-\tau)$, $t \in (\tau, T-a)$, and using the Hölder inequality, we obtain

$$\begin{aligned} &(u_\mu(t+a) - u_\mu(t), w) \\ &\quad \leq \int_t^{t+a} |\nabla u_\mu(s)| \, |\nabla w| \, ds \\ &\qquad + \int_t^{t+a} \|f(u_\mu(s)\|_{L^{p'}(\Omega)} \|w\|_{L^p(\Omega)} \, ds \\ &\qquad + \int_t^{t+a} \|h(s)\|_{H^{-1}(\Omega)} |\nabla w| \, ds \\ &\quad \leq |\nabla w| \, a^{1/2} \|u_\mu\|_{L^2\left(\tau,T;H_0^1(\Omega)\right)} \\ &\qquad + \|w\|_{L^p(\Omega)} a^{1/p} \|f(u_\mu)\|_{L^{p'}(\tau,T;L^{p'}(\Omega))} \\ &\qquad + |\nabla w| \, a^{1/2} \|h\|_{L^2(\tau,T;H^{-1}(\Omega))}, \end{aligned}$$

and thanks to (4.9) and (4.10), we deduce that there exists a constant $C^{(1)}$ (depending on T) such that

$$\begin{aligned} &(u_\mu(t+a) - u_\mu(t), w) \\ &\quad \leq C^{(1)}(a^{1/2} + a^{1/p})(|\nabla w| + \|w\|_{L^p(\Omega)}). \end{aligned}$$

As $u_\mu(t+a) - u_\mu(t)$ is a linear combination of the elements $w_1, \ldots, w_\mu$, we can take $w = u_\mu(t+a) - u_\mu(t)$ in the last inequality and obtain

$$\begin{aligned}
&|u_\mu(t+a) - u_\mu(t)|^2 \\
&\leq C^{(1)}(a^{1/2} + a^{1/p})\,|\nabla u_\mu(t+a) - \nabla u_\mu(t)| \\
&\quad + C^{(1)}(a^{1/2} + a^{1/p})\,\|u_\mu(t+a) - u_\mu(t)\|_{L^p(\Omega)},
\end{aligned}$$

for a.e. $t \in (\tau, T-a)$.
Integrating between τ and $T-a$,

$$\begin{aligned}
&\int_\tau^{T-a} |u_\mu(t+a) - u_\mu(t)|^2\,dt \\
&\leq C^{(1)}(a^{1/2} + a^{1/p}) \int_\tau^{T-a} |\nabla u_\mu(t+a)|\,dt \\
&+ C^{(1)}(a^{1/2} + a^{1/p}) \int_\tau^{T-a} |\nabla u_\mu(t)|\,dt \\
&+ C^{(1)}(a^{1/2} + a^{1/p}) \int_\tau^{T-a} \|u_\mu(t+a)\|_{L^p(\Omega)}\,dt \\
&+ C^{(1)}(a^{1/2} + a^{1/p}) \int_\tau^{T-a} \|u_\mu(t)\|_{L^p(\Omega)}\,dt.
\end{aligned}$$

It follows

$$\begin{aligned}
&\int_\tau^{T-a} |u_\mu(t+a) - u_\mu(t)|^2\,dt \\
&\leq 2C^{(1)}(a^{1/2} + a^{1/p}) \int_\tau^{T} |\nabla u_\mu(s)|\,ds \\
&+ 2C^{(1)}(a^{1/2} + a^{1/p}) \int_\tau^{T} \|u_\mu(s)\|_{L^p(\Omega)}\,ds,
\end{aligned}$$

and using the Hölder inequality, we obtain

$$\begin{aligned}
&\int_\tau^{T-a} |u_\mu(t+a) - u_\mu(t)|^2\,dt \\
&\leq 2C^{(1)}(a^{1/2} + a^{1/p})(T-\tau)^{1/2}\,\|u_\mu\|_{L^2\left(\tau,T;H_0^1(\Omega)\right)} \\
&+ 2C^{(1)}(a^{1/2} + a^{1/p})(T-\tau)^{1/p'}\,\|u_\mu\|_{L^p(\tau,T;L^p(\Omega))}.
\end{aligned}$$

Thanks to (4.9) we deduce that there exists a constant $\widetilde{C}_T$ such that

$$\int_\tau^{T-a} |u_\mu(t+a) - u_\mu(t)|^2\,dt \leq \widetilde{C}_T(a^{1/2} + a^{1/p}),$$

for all μ, and all $a \in (0, T - \tau)$, and thus

$$\lim_{a\to 0} \sup_{\mu} \int_{\tau}^{T-a} |u_\mu(t+a) - u_\mu(t)|^2 \, dt = 0, \tag{4.13}$$

for all $T > \tau$.

Now, let $\phi \in C^1([0, +\infty))$ be a function such that

$$0 \le \phi(s) \le 1,$$

$$\phi(s) = 1 \quad \forall s \in [0, 1],$$

$$\phi(s) = 0 \quad \forall s \ge 2.$$

For each μ and $m \ge 1$, we define

$$v_{\mu,m}(x,t) = \phi\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) u_\mu(x,t) \qquad \forall x \in \Omega_{2m}, \forall \mu, \ \ \forall m \ge 1. \tag{4.14}$$

We note that $v_{\mu,m}(x,t)$ verifies

$$|v_{\mu,m}(x,t)| \le |u_\mu(x,t)\ |,$$

$$\partial_i v_{\mu,m}(x,t) = \phi\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) \partial_i u_\mu(x,t) + \frac{2x_i}{m^2}\phi'\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) u_\mu(x,t),$$

and using that $\phi'\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) = 0$ if $|x|_{\mathbb{R}^N} > \sqrt{2}m$, and $\phi'\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) \le C_{\phi'}$ for all x, we obtain

$$|\partial_i v_{\mu,m}(x,t)| \le |\partial_i u_\mu(x,t)| + \frac{4}{m} C_{\phi'} \, |u_\mu(x,t)| \, .$$

Using (4.9), we obtain that, for all $m \ge 1$, the sequence $\{v_{\mu,m}\}_{\mu \ge 1}$ is bounded in $L^\infty(\tau, T; L^2(\Omega_{2m})) \cap L^p(\tau, T; L^p(\Omega_{2m})) \cap L^2(\tau, T; H^1(\Omega_{2m}))$, for all $T > \tau$.

As $u_\mu \in H^1_0(\Omega)$, then there exists $\varphi_n \in \mathcal{D}(\Omega)$ such that $\varphi_n \to u_\mu$ strongly in $H^1(\Omega)$. Then, we have

$$\phi\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) \varphi_n \to \phi\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) u_\mu = v_{\mu,m} \text{ strongly in } H^1(\Omega_{2m}) \, .$$

If we denote $\psi_n = \phi\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) \varphi_n$, we have that $\psi_n \in C^1(\Omega_{2m})$, and *supp* $\psi_n \subset$ *supp* $\varphi_n \subset \Omega$. Then, we can deduce that *supp* ψ_n is a compact subset.

Moreover, we note that if $\psi_n(x) \neq 0$, then $|x|_{\mathbb{R}^N} < \sqrt{2}m < 2m$. Then, we can deduce

$$supp\ \psi_n \subset \Omega \cap \left\{x \in \mathbb{R}^N : |x|_{\mathbb{R}^N} < 2m\right\} = \Omega_{2m}.$$

We have obtained that $\psi_n \in C_c^1(\Omega_{2m})$ and $\psi_n \to v_{\mu,m}$ strongly in $H^1(\Omega_{2m})$, i.e., we have that $v_{\mu,m} \in H_0^1(\Omega_{2m})$.

Therefore, we obtain that, for all $m \geq 1$, the sequence $\{v_{\mu,m}\}_{\mu\geq 1}$ is bounded in $L^\infty(\tau, T; L^2(\Omega_{2m})) \cap L^p(\tau, T; L^p(\Omega_{2m})) \cap L^2(\tau, T; H_0^1(\Omega_{2m}))$, for all $T > \tau$.

In particular, it follows that

$$\lim_{a\to 0} \sup_\mu \left(\int_\tau^{\tau+a} |v_{\mu,m}(x,t)|^2_{L^2(\Omega_{2m})}\, dt + \int_{T-a}^T |v_{\mu,m}(x,t)|^2_{L^2(\Omega_{2m})}\, dt \right) = 0.$$

On the other hand, from (4.13) we deduce that for $m \geq 1$,

$$\lim_{a\to 0} \sup_\mu \left(\int_\tau^{T-a} |v_{\mu,m}(x,t+a) - v_{\mu,m}(x,t)|^2_{L^2(\Omega_{2m})}\, dt \right) = 0.$$

Moreover, as Ω_{2m} is a bounded set, then $H_0^1(\Omega_{2m})$ is included in $L^2(\Omega_{2m})$ with compact injection.

Then, by the compactness Theorem 13.3 and Remark 13.1 of [80] with $X = L^2(\Omega_{2m})$, $Y = H_0^1(\Omega_{2m})$, $r = 2$ and $\mathcal{G} = \{v_{\mu,m}\}_{\mu\geq 1}$, we obtain that

$$\{v_{\mu,m}\}_{\mu\geq 1} \text{ is relatively compact in } L^2\left(\tau, T; L^2(\Omega_{2m})\right),$$

and thus, taking into account that $v_{\mu,m}(x,t) = u_\mu(x,t)$ for all $x \in \Omega_m$, we deduce that, in particular, for all $m \geq 1$

$$\left\{u_{\mu|\Omega_m}\right\}_{\mu\geq 1} \text{ is pre-compact in } L^2\left(\tau, T; L^2(\Omega_m)\right). \tag{4.15}$$

It is not difficult to conclude from (4.11) and (4.15), via a diagonal procedure, the existence of a subsequence $\left\{u_\mu^\mu\right\}_{\mu\geq 1} \subset \{u_\mu\}_{\mu\geq 1}$ such that

$$u_\mu^\mu \to u \ \ a.e. \text{ in } \Omega_m \times (\tau, +\infty) \quad \text{as } \mu \longrightarrow \infty, \quad \forall m \geq 1.$$

Then, as f is continuous,

$$f(x, u_\mu^\mu) \longrightarrow f(x,u)\ a.e. \text{ in } \Omega_m \times (\tau, +\infty),$$

and as $\left\{f(x, u_\mu^\mu)\right\}$ is bounded in $L^{p'}(\Omega_m \times (\tau, T))$, by Lemma 1.3, Chapter 1 in [56], we obtain

$$f(x, u_\mu^\mu) \rightharpoonup f(x,u) \text{ weakly in } L^{p'}\left(\tau, T; L^{p'}(\Omega_m)\right) \quad \forall T > \tau.$$

From (4.12)

$$f(x,u_\mu) \rightharpoonup \chi_{|\Omega_m\times(\tau,T)} \text{ weakly in } L^{p'}\left(\tau,T;L^{p'}(\Omega_m)\right).$$

By the uniqueness of the weak limit, we have

$$\chi = f(x,u) \;\; a.e. \text{ in } \Omega_m\times(\tau,T) \quad \forall T>\tau, \forall m\geq 1,$$

and thus, taking into account that $\bigcup_{m=1}^{\infty}\Omega_m=\Omega$, we obtain

$$\chi = f(x,u) \; a.e. \text{ in } \Omega\times(\tau,+\infty). \tag{4.16}$$

Then, (4.12) and (4.16) yield that

$$f(x,u_\mu) \rightharpoonup f(x,u(t)) \text{ weakly in } L^{p'}(\tau,T;L^{p'}(\Omega)) \quad \forall T>\tau. \tag{4.17}$$

On the other hand, let $\varphi\in H^1(\tau,T)$ such that $\varphi(T)=0$. For each $w\in span\{w_j\}_{j\geq 1}$, we consider the following function

$$(u_\mu(t),w)\,\varphi(t).$$

Thanks to the equation satisfied by u'_μ, we have (see [18] Lemma pg. 131),

$$\frac{d}{dt}\left[(u_\mu(t),w)\,\varphi(t)\right] = \langle \Delta u_\mu(t)+f(x,u_\mu(t))+h(t),w\rangle\,\varphi(t)+(u_\mu(t),w)\,\varphi'(t),$$

integrating between τ and T, and taking into account that $\varphi(T)=0$, we obtain

$$\begin{aligned}-(u_\mu(\tau),w)\,\varphi(\tau) &= \int_\tau^T \langle \Delta u_\mu(s)+f(x,u_\mu(s))+h(s),w\rangle\,\varphi(s)ds\\ &\quad+\int_\tau^T (u_\mu(s),w)\,\varphi'(s)ds.\end{aligned}$$

Taking weak convergence, if μ tends to infinity, we can deduce

$$\begin{aligned}-(u_\tau,w)\,\varphi(\tau) &= \int_\tau^T \langle \Delta u(s)+f(x,u(s))+h(s),w\rangle\,\varphi(s)ds\\ &\quad+\int_\tau^T (u(s),w)\,\varphi'(s)ds,\end{aligned} \tag{4.18}$$

for all $w\in span\{w_j\}_{j\geq 1}$. As $span\{w_j\}_{j\geq 1}$ is dense in $H_0^1(\Omega)\cap L^p(\Omega)$, we have (4.18) for all $w\in H_0^1(\Omega)\cap L^p(\Omega)$.

On the other hand, fix $t \in (\tau, T)$, and for each integer $k \geq 1$ such that $\tau < t - 1/2k < t + 1/2k < T$, we consider the following function

$$\varphi_k(s) = \begin{cases} 1, & \text{if } \tau \leq s < t - 1/2k, \\ k(t-s) + 1/2, & \text{if } t - 1/2k \leq s \leq t + 1/2k, \\ 0, & \text{if } t + 1/2k < s \leq T, \end{cases}$$

which belongs to $H^1(\tau, T)$ and is zero in $s = T$. Taking into account that

$$\varphi_k'(s) = \begin{cases} -k, & \text{if } t - 1/2k < s < t + 1/2k, \\ 0, & \text{otherwise,} \end{cases}$$

and using (4.18) for φ_k, we obtain

$$\begin{aligned} -(u_\tau, w) = & \int_\tau^T \langle \Delta u(s) + f(x, u(s)) + h(s), w \rangle \varphi_k(s) ds \\ & - k \int_{t-1/2k}^{t+1/2k} (u(s), w)\, ds, \end{aligned} \tag{4.19}$$

for all $k \geq 1$, for all $w \in H_0^1(\Omega) \cap L^p(\Omega)$, and with $t \in (\tau, T)$ arbitrary. Using Lebesgue's Theorem, we have

$$-k \int_{t-1/2k}^{t+1/2k} (u(s), w)\, ds \to -(u(t), w) \quad a.e. \;\; t \in (\tau, T),$$

as $k \to \infty$.

Then, passing to the limit in (4.19), we obtain

$$(u(t), w) = (u_\tau, w) + \int_\tau^t \langle \Delta u(s) + f(x, u(s)) + h(s), w \rangle\, ds \quad a.e. \;\; t \in (\tau, T).$$

It is a standard matter to prove that we can pick an element in the equivalence class of u satisfying

$$(u(t), w) = (u_\tau, w) + \int_\tau^t \langle \Delta u(s) + f(x, u(s)) + h(s), w \rangle ds, \tag{4.20}$$

for all $t \geq \tau$, for any $w \in H_0^1(\Omega) \cap L^p(\Omega)$.

□

Remark 4.4 *Observe that the conditions on the function f do not allow to obtain the uniqueness of the Cauchy problem (see [48] for a counterexample in the autonomous case).*

4.3 Existence of Pullback Attractors

In this section we prove our main result in this chapter. First, we need a priori estimates and a continuity result which are established in the next subsections.

4.3.1 A priori estimates

For each $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\Omega)$, let us denote $S(\tau, u_\tau)$ the set of all weak solutions of (4.2) defined for all $t \geq \tau$.

We define a multi-valued map $U : \mathbb{R}^2_d \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$ by

$$U(t,\tau,u_\tau) = \{u(t) : \ u \in S(\tau, u_\tau)\}, \quad \tau \leq t, \quad u_\tau \in L^2(\Omega). \tag{4.21}$$

Lemma 4.5 *Under the assumptions of Theorem 4.3, the multi-valued mapping U defined by (4.21) is a strict MNDS on $L^2(\Omega)$.*

Proof It is easy to check that U is well defined.

Moreover, U satisfies the first part in Definition 3.1 in Chapter 3.

Let us now prove that $U(t,\tau,u_\tau) \subset U(t,s,U(s,\tau,u_\tau))$ also holds for all $\tau \leq s \leq t$, $u_\tau \in L^2(\Omega)$.

Consider $\phi \in U(t,\tau,u_\tau)$. Then from the definition of U, there exists a solution $u \in S(\tau,u_\tau)$ such that $u(t) = \phi$.

If $\tau \leq s$, then $u(s) \in U(s,\tau,u_\tau)$, and as

$$U(t,s,u(s)) = \{z(t) : z \in S(s,u(s))\},$$

obviously,

$$u(t) = \phi \in U(t,s,u(s)) \subset U(t,s,U(s,\tau,u_\tau)).$$

Thus,

$$U(t,\tau,u_\tau) \subset U(t,s,U(s,\tau,u_\tau)) \ \ \forall \tau \leq s \leq t.$$

To prove that the MNDS is strict, let us consider $\phi \in U(t,s,U(s,\tau,u_\tau))$. Then there exists a solution u to (4.2) such that $u(t) = \phi$, and $u(s) = y(s)$, where y is another solution to (4.2) with initial value $y(\tau) = u_\tau$.

We now define

$$z(r) = \begin{cases} y(r) & \text{if } \tau \leq r \leq s, \\ u(r) & \text{if } s \leq r \leq t. \end{cases}$$

It is clear that $z(\cdot)$ is solution to (4.2) (see [63]), and it is also holds that $z(\tau) = y(\tau) = u_\tau$, and $z(t) = u(t) = \phi$, i.e., $\phi \in U(t,\tau,u_\tau)$.

Which means that

$$U(t,s,U(s,\tau,u_\tau)) \subset U(t,\tau,u_\tau) \ \forall \tau \leq s \leq t.$$

□

Let $\mathcal{R}_{\lambda_1}$ be the set of all functions $r : \mathbb{R} \to (0,+\infty)$ such that

$$\lim_{t\to -\infty} e^{\lambda_1 t} r^2(t) = 0,$$

and denote by $\mathcal{D}_{\lambda_1}$ the class of all families

$$\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega)),$$

such that

$$D(t) \subset \overline{B}(0, r_{\widehat{D}}(t)) \text{ for some } r_{\widehat{D}} \in \mathcal{R}_{\lambda_1},$$

where $\overline{B}(0, r_{\widehat{D}}(t))$ denotes the closed ball in $L^2(\Omega)$ centered at zero with radius $r_{\widehat{D}}(t)$.

According to the notation introduced in Chapter 3, $\mathcal{D}_F^H$ will denote the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of $L^2(\Omega)$.

Remark 4.6 *We note that $\mathcal{D}_F^H \subset \mathcal{D}_{\lambda_1}$ and both universes are inclusion-closed.*

Lemma 4.7 *Suppose that Ω satisfies (4.1) and suppose that f is Carathéodory and satisfies (4.3) and (4.4). Let $h = \sum_{i=1}^N \frac{\partial h_i}{\partial x_i}$, with $h_i \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ for all $1 \leq i \leq N$, such that*

$$\sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 \, ds < +\infty \quad \forall t \in \mathbb{R}. \tag{4.22}$$

Then, the balls $B_{\lambda_1}(t) = \overline{B}_{L^2(\Omega)}(0, R_{\lambda_1}(t))$, where $R_{\lambda_1}(t)$ is the nonnegative number given for each $t \in \mathbb{R}$ by

$$R_{\lambda_1}^2(t) = 2e^{-\lambda_1 t} \sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 \, ds + 2\lambda_1^{-1} \|C_2\|_{L^1(\Omega)} + 1, \tag{4.23}$$

form a family $\widehat{B}_{\lambda_1} \in \mathcal{D}_{\lambda_1}$ which is pullback $\mathcal{D}_{\lambda_1}$-absorbing for the MNDS U defined by (4.21).

Proof As a consequence of (4.23), it is evident that $\widehat{B}_{\lambda_1} \in \mathcal{D}_{\lambda_1}$.

On the other hand, taking into account the energy equality, (4.1) and (4.4), if $u \in S(\tau, u_\tau)$ we obtain

$$\frac{d}{dt}\left(e^{\lambda_1 t}\left|u(t)\right|^2\right) + \frac{\lambda_1}{2}e^{\lambda_1 t}\left|u(t)\right|^2 \leq 2e^{\lambda_1 t}\sum_{i=1}^{N}\left|h_i(t)\right|^2 + 2e^{\lambda_1 t}\left\|C_2\right\|_{L^1(\Omega)}, \quad (4.24)$$

for $t \geq \tau$.

In particular, integrating between τ and t, we have

$$e^{\lambda_1 t}\left|u(t)\right|^2 \leq e^{\lambda_1 \tau}\left|u_\tau\right|^2 + 2\sum_{i=1}^{N}\int_{-\infty}^{t} e^{\lambda_1 s}\left|h_i(s)\right|^2 ds + 2\lambda_1^{-1}\left\|C_2\right\|_{L^1(\Omega)} e^{\lambda_1 t},$$

for all $t \geq \tau$.

From this inequality, we deduce that if $\widehat{D} \in \mathcal{D}_{\lambda_1}$ and $y \in U(t, \tau, D(\tau))$, then

$$\left|y\right|^2 \leq e^{\lambda_1(\tau - t)} r_{\widehat{D}}^2(\tau) + 2e^{-\lambda_1 t}\sum_{i=1}^{N}\int_{-\infty}^{t} e^{\lambda_1 s}\left|h_i(s)\right|^2 ds + 2\lambda_1^{-1}\left\|C_2\right\|_{L^1(\Omega)}.$$

Consequently the family $\widehat{B}_{\lambda_1}$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing for U.

□

Lemma 4.8 *Under the assumptions in Lemma 4.7, for any real numbers* $t_1 \leq t_2$ *and any* $\varepsilon > 0$, *there exist* $T = T(t_1, t_2, \varepsilon, \widehat{B}_{\lambda_1}) \leq t_1$ *and* $M = M(t_1, t_2, \varepsilon, \widehat{B}_{\lambda_1}) \geq 1$ *verifying*

$$\int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} u^2(x,t)\, dx \leq \varepsilon, \quad \forall \tau \leq T, \quad t \in [t_1, t_2], \quad m \geq M,$$

for any weak solution $u \in S(\tau, u_\tau)$ *where* $u_\tau \in B_{\lambda_1}(\tau)$.

Proof Let $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$ and $u \in S(\tau, u_\tau)$ be fixed. We take a smooth function $\theta \in C^1([0, +\infty))$ verifying

$$0 \leq \theta(s) \leq 1,$$

$$\theta(s) = 0 \quad \forall s \in [0, 1],$$

$$\theta(s) = 1 \quad \forall s \geq 2.$$

Under the above assumptions on u_τ, f and h, if u is a weak solution of (4.2), the function $\|\theta u(t)\|^2 = \int_{\mathbb{R}^N} \theta^2 \left(\frac{|x|^2}{m^2}\right) |u(x,t)|^2 dx$ is absolutely continuous and $\frac{d}{dt}\|\theta u\|^2 = 2\left\langle \frac{du}{dt}, \theta^2 u \right\rangle$ for a.a. t (see [63, Lemma 3]).

On the other hand (see for example [18, propositions IX.4 and IX.5]) observe that $\theta\left(\frac{|\cdot|^2_{\mathbb{R}^N}}{m^2}\right) u(\cdot,t) \in H^1_0(\Omega)$, a.e. in (τ,∞), with

$$\partial_i\left(\theta\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u(x,t)\right) = \theta\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)\partial_i u(x,t) + \frac{2x_i}{m^2}\theta'\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u(x,t), \quad (4.25)$$

and the same is true replacing θ by θ^2.

Hence, we obtain for every $t \geq \tau$,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u^2(x,t)dx + \int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)|\nabla u(x,t)|^2 dx \quad (4.26)$$

$$+ \frac{4}{m^2}\int_\Omega \theta'\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)\theta\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u(x,t)x\cdot\nabla u(x,t)dx$$

$$= \int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)f(x,u(x,t))u(x,t)dx \quad (4.27)$$

$$- \sum_{i=1}^N \int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_i(x,t)\partial_i u(x,t)dx$$

$$- \sum_{i=1}^N \int_\Omega \frac{4x_i}{m^2}\theta'\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)\theta\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u(x,t)h_i(x,t)dx$$

$$= I_1 + I_2 + I_3.$$

From (4.4), we obtain

$$I_1 \leq \int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)C_2(x)dx. \quad (4.28)$$

Using the Cauchy-Schwarz inequality, we obtain

$$I_2 \leq \frac{1}{4}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)|\nabla u(x,t)|^2 dx \quad (4.29)$$

$$+ \sum_{i=1}^N \int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_i^2(x,t)dx.$$

Using that $\theta'\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) = 0$ if $|x|_{\mathbb{R}^N} > \sqrt{2}m$, $\theta'\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) \leq C_{\theta'}$ for all x, and the Cauchy-Schwarz inequality, we obtain

$$|I_3| \leq \frac{16}{m^2} C^2_{\theta'} N \int_\Omega |u(x,t)|^2 \, dx + \sum_{i=1}^N \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) h_i^2(x,t) dx, \tag{4.30}$$

where we have used that $|x|_{\mathbb{R}^N} \leq \sqrt{2}m < 2m$.

Moreover, we have

$$\begin{aligned} &\left| \frac{4}{m^2} \int_\Omega \theta' \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) \theta \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) u(x,t) x \cdot \nabla u(x,t) dx \right| \qquad (4.31) \\ &\qquad \leq \frac{4}{m} C_{\theta'} \int_\Omega |u(x,t)|^2 \, dx + \frac{4}{m} C_{\theta'} \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) |\nabla u(x,t)|^2 \, dx. \end{aligned}$$

From (4.26)-(4.31) we deduce

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) u^2(x,t) dx + \left(\frac{3}{4} - \frac{4}{m} C_{\theta'}\right) \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) |\nabla u(x,t)|^2 \, dx \\ &\quad \leq \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) C_2(x) dx + \frac{4}{m} C_{\theta'} \int_\Omega u^2(x,t) dx \\ &\qquad + \frac{16}{m^2} C^2_{\theta'} N \int_\Omega u^2(x,t) dx + 2 \sum_{i=1}^N \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) h_i^2(x,t) dx. \end{aligned} \tag{4.32}$$

On the other hand, by (4.25) we have

$$\begin{aligned} \left| \nabla \left(\theta \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) u(x,t) \right) \right|^2 &= \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) |\nabla u(x,t)|^2 \\ &\quad + \frac{4 |x|^2_{\mathbb{R}^N}}{m^4} \left(\theta' \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) \right)^2 u^2(x,t) \\ &\quad + \frac{4}{m^2} \theta' \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) u(x,t) \theta \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) x \cdot \nabla u(x,t), \end{aligned}$$

and therefore

$$\begin{aligned}
&\int_\Omega \left| \nabla \left(\theta \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) u(x,t) \right) \right|^2 dx \\
&\quad \leq \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) |\nabla u(x,t)|^2 \, dx + \frac{8}{m^2} C^2_{\theta'} \int_\Omega u^2(x,t) dx \\
&\qquad + \frac{4}{m} C_{\theta'} \left(\int_\Omega u^2(x,t) dx + \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) |\nabla u(x,t)|^2 \, dx \right) \\
&\quad = \left(1 + \frac{4}{m} C_{\theta'} \right) \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) |\nabla u(x,t)|^2 \, dx \\
&\qquad + \left(\frac{8}{m^2} C^2_{\theta'} + \frac{4}{m} C_{\theta'} \right) \int_\Omega u^2(x,t) dx.
\end{aligned}$$

From this inequality and (4.1) we obtain

$$\begin{aligned}
\int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) |\nabla u(x,t)|^2 \, dx \geq \left(\frac{m}{m + 4C_{\theta'}} \right) \lambda_1 \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) u^2(x,t) dx \\
- \left(\frac{m}{m + 4C_{\theta'}} \right) \left(\frac{8}{m^2} C^2_{\theta'} + \frac{4}{m} C_{\theta'} \right) \int_\Omega u^2(x,t) dx.
\end{aligned} \tag{4.33}$$

Assume that $\frac{3}{4} - \frac{4}{m} C_{\theta'} > 0$ (and this is true for m large enough).
Then, from (4.32) and (4.33), we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) u^2(x,t) dx \\
&+ \left(\frac{3}{4} - \frac{4}{m} C_{\theta'} \right) \left(\frac{m}{m + 4C_{\theta'}} \right) \lambda_1 \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) u^2(x,t) \, dx \\
&\leq \left(\frac{4C_{\theta'}}{m} + \frac{16 C^2_{\theta'} N}{m^2} + \left(\frac{m}{m + 4C_{\theta'}} \right) \left(\frac{8C^2_{\theta'}}{m^2} + \frac{4C_{\theta'}}{m} \right) \left(\frac{3}{4} - \frac{4C_{\theta'}}{m} \right) \right) \int_\Omega u^2(x,t) \, dx \\
&\quad + 2 \sum_{i=1}^N \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) h_i^2(x,t) dx + \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) C_2(x) dx.
\end{aligned} \tag{4.34}$$

Evidently, there exists m_0 such that for all $m \geq m_0$ we have

$$\left(\frac{3}{4} - \frac{4}{m} C_{\theta'} \right) \left(\frac{m}{m + 4C_{\theta'}} \right) > \frac{1}{2}.$$

Then from (4.34), if we denote $\widehat{C} = 14C_{\theta'} + 44C_{\theta'}^2 N$, and multiplying by $e^{\lambda_1 t}$, we obtain

$$\frac{d}{dt}\left(e^{\lambda_1 t}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u^2(x,t)dx\right) \le \frac{\widehat{C}}{m}e^{\lambda_1 t}\int_\Omega u^2(x,t)dx + 4\sum_{i=1}^N e^{\lambda_1 t}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_i^2(x,t)dx + 2e^{\lambda_1 t}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)C_2(x)dx.$$

Integrating now between τ and t, and using the properties of θ, we have

$$\begin{aligned}\int_{\Omega\cap\{|x|_{\mathbb{R}^N}\ge 2m\}} u^2(x,t)dx \le{}& e^{-\lambda_1 t}e^{\lambda_1\tau}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)u_\tau^2(x)\,dx \\ &+ \frac{\widehat{C}}{m}e^{-\lambda_1 t}\int_\tau^t e^{\lambda_1 s}|u(s)|^2\,ds \\ &+ 4\sum_{i=1}^N e^{-\lambda_1 t}\int_{-\infty}^t e^{\lambda_1 s}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_i^2(x,s)dxds \\ &+ 2\lambda_1^{-1}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)C_2(x)dx,\end{aligned} \tag{4.35}$$

for all $m \ge m_0$, $t \ge \tau$.

On the other hand, from (4.24), integrating between τ and t, we have

$$\begin{aligned}\frac{\lambda_1}{2}\int_\tau^t e^{\lambda_1 s}|u(s)|^2\,ds \le{}& e^{\lambda_1\tau}|u_\tau|^2 + 2\sum_{i=1}^N\int_\tau^t e^{\lambda_1 s}|h_i(s)|^2\,ds \\ &+ 2\lambda_1^{-1}\|C_2\|_{L^1(\Omega)}e^{\lambda_1 t}.\end{aligned} \tag{4.36}$$

Thus, if we take $u_\tau \in B_{\lambda_1}(\tau)$, we obtain

$$\begin{aligned}\int_\tau^t e^{\lambda_1 s}|u(s)|^2 ds \le{}& 2\lambda_1^{-1}e^{\lambda_1\tau}R^2_{\lambda_1}(\tau) \\ &+ 4\lambda_1^{-1}\sum_{i=1}^N\int_{-\infty}^t e^{\lambda_1 s}|h_i(s)|^2\,ds \\ &+ 4\lambda_1^{-2}\|C_2\|_{L^1(\Omega)}e^{\lambda_1 t}.\end{aligned} \tag{4.37}$$

Let us fix $t_1 \le t_2 \in \mathbb{R}$.

Observing that

$$\lim_{\tau\to-\infty} e^{\lambda_1\tau} R^2_{\lambda_1}(\tau) = 0,$$

from (4.22) and (4.37), we deduce that there exists a constant $C(t_1, t_2)$ such that

$$e^{-\lambda_1 t}\int_\tau^t e^{\lambda_1 s}|u(s)|^2\, ds \le C(t_1,t_2) \quad \forall\, t\in[t_1,t_2],\ \tau\le t_1,$$

and therefore, by (4.35),

$$\begin{aligned}\int_{\Omega\cap\{|x|_{\mathbb{R}^N}\ge 2m\}} u^2(x,t)dx \le{}& e^{-\lambda_1 t}e^{\lambda_1\tau}R^2_{\lambda_1}(\tau) + \frac{\widehat{C}}{m}C(t_1,t_2)\\ &+4\sum_{i=1}^N e^{-\lambda_1 t}\int_{-\infty}^t e^{\lambda_1 s}\int_\Omega \theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_i^2(x,s)dxds\\ &+2\lambda_1^{-1}\int_\Omega\theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)C_2(x)dx,\end{aligned} \tag{4.38}$$

for all $m \ge m_0$ and $t \in [t_1, t_2]$, for every $u \in S(\tau, u_\tau)$, where $\tau \le t_1$ and $u_\tau \in B_{\lambda_1}(\tau)$.

On the other hand, from (4.22) and Lebesgue's Dominated Convergence Theorem, for every $t \in [t_1, t_2]$ we obtain

$$\begin{aligned}&\int_{-\infty}^t e^{\lambda_1 s}\int_\Omega\theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)h_i^2(x,s)dxds\\ &\quad\le\int_{-\infty}^{t_2}\int_\Omega \chi_{\{|x|_{\mathbb{R}^N}\ge m\}}e^{\lambda_1 s}h_i^2(x,s)dxds \longrightarrow 0 \text{ as } m\to\infty,\end{aligned} \tag{4.39}$$

for all $i = 1, .., N$, where χ is the indicator function.

Analogously,

$$\int_\Omega\theta^2\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right)C_2(x)dx \le \int_\Omega \chi_{\{|x|_{\mathbb{R}^N}\ge m\}}C_2(x)dx \longrightarrow 0 \text{ as } m\to\infty. \tag{4.40}$$

From (4.38), (4.39) and (4.40) we deduce our lemma.

□

Remark 4.9 *It is clear from the proof that Lemma 4.8 above also holds for any* $\widehat{D}\in\mathcal{D}_{\lambda_1}$ *instead of* $\widehat{B}_{\lambda_1}$.

Lemma 4.10 *Under the assumptions in Lemma 4.7, let K be a relatively compact set in $L^2(\Omega)$. Then, for all $\tau \leq T$ and $\varepsilon > 0$ there exists $M = M(\tau, T, \varepsilon, K)$ such that*

$$\int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} u^2(x,t)\, dx \leq \varepsilon, \ \forall t \in [\tau, T], \quad \forall m \geq M,$$

for any $u \in S(\tau, u_\tau)$, where $u_\tau \in K$ is arbitrary.

Proof We note that, as shown in Lemma 4.8, we have

$$\begin{aligned} \int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} u^2(x,t)dx \leq\ & e^{-\lambda_1 t} e^{\lambda_1 \tau} \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) u_\tau^2(x)\, dx \\ & + \frac{\widehat{C}}{m} e^{-\lambda_1 t} \int_\tau^t e^{\lambda_1 s} |u(s)|^2 \, ds \\ & + 4 \sum_{i=1}^N e^{-\lambda_1 t} \int_{-\infty}^t e^{\lambda_1 s} \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) h_i^2(x,s) dx ds \\ & + 2\lambda_1^{-1} \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) C_2(x) dx, \end{aligned} \tag{4.41}$$

for all $m \geq m_0$, and any $u \in S(\tau, u_\tau)$, where $\tau \leq t$ and $u_\tau \in L^2(\Omega)$ are arbitrary, and where m_0 and $\widehat{C}$ are defined in Lemma 4.8. On the other hand, as K is a bounded subset of $L^2(\Omega)$, from (4.36) we deduce that for some constant $c > 0$,

$$\begin{aligned} \int_\tau^t e^{\lambda_1 s} |u(s)|^2 ds \leq\ & 2\lambda_1^{-1} e^{\lambda_1 \tau} c^2 + 4\lambda_1^{-1} \sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 \, ds \\ & + 4\lambda_1^{-2} \|C_2\|_{L^1(\Omega)} e^{\lambda_1 t}, \end{aligned}$$

and thus there exists a constant $C(\tau, T)$ such that

$$e^{-\lambda_1 t} \int_\tau^t e^{\lambda_1 s} \int_\Omega u^2(x,s) dx ds \leq C(\tau, T), \quad \forall t \in [\tau, T], \tag{4.42}$$

for any $u \in S(\tau, u_\tau)$, where $u_\tau \in K$ is arbitrary.

Finally, as K is a relatively compact subset of $L^2(\Omega)$, then for all $\varepsilon > 0$ there exists m_ε such that

$$\int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) u_\tau^2(x) dx < \varepsilon \quad \forall u_\tau \in K, \quad \forall m \geq m_\varepsilon. \tag{4.43}$$

In the contrary case, there would exist an $\varepsilon > 0$ and a sequence $\{u_n\} \subset K$ such that

$$\int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{n^2} \right) u_n^2(x)dx \geq \varepsilon, \, \forall n \geq 1.$$

But then, there would exist a convergent subsequence $\{u_\mu\} \subset \{u_n\}$, with $u_\mu \to u$ in $L^2(\Omega)$ as $\mu \to \infty$. And thus we would have

$$\begin{aligned} \varepsilon &\leq \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{\mu^2} \right) u_\mu^2(x)dx \\ &\leq 2 \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{\mu^2} \right) (u_\mu(x) - u(x))^2 dx + 2 \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{\mu^2} \right) u^2(x)dx \\ &\leq 2 \int_\Omega (u_\mu(x) - u(x))^2 dx + 2 \int_\Omega \theta^2 \left(\frac{|x|^2_{\mathbb{R}^N}}{\mu^2} \right) u^2(x)dx, \end{aligned}$$

and therefore, making $\mu \to \infty$, we would have $\varepsilon \leq 0$, which is a contradiction. From (4.41)-(4.43), and taking into account (4.39) and (4.40), we deduce the assertion of the lemma.

□

4.3.2 A continuity result

Further, we obtain a continuity result leading to the upper semicontinuity of the MNDS U.

Proposition 4.11 *Under the assumptions in Lemma 4.7, let $\tau \in \mathbb{R}$ and $\{u_\tau^n\} \subset L^2(\Omega)$ be a sequence converging weakly in $L^2(\Omega)$ to an element $u_\tau \in L^2(\Omega)$. For each $n \geq 1$ let us fix $u_n \in S(\tau, u_\tau^n)$. Then there exists a subsequence $\{u_\mu\} \subset \{u_n\}$ satisfying that there exists $u \in S(\tau, u_\tau)$ such that*

$$u_\mu(t) \rightharpoonup u(t) \text{ weakly in } L^2(\Omega) \;\; \forall t \geq \tau, \tag{4.44}$$

$$u_\mu \rightharpoonup u \text{ weakly in } L^2(\tau, T; H_0^1(\Omega)) \; \forall T > \tau, \tag{4.45}$$

$$u_\mu \rightharpoonup u \text{ weakly in } L^p(\tau, T; L^p(\Omega)) \; \forall T > \tau, \tag{4.46}$$

$$f(x, u_\mu) \rightharpoonup f(x, u) \text{ weakly in } L^{p'}(\tau, T; L^{p'}(\Omega)) \; \forall T > \tau, \tag{4.47}$$

$$u_{\mu|\Omega_m} \to u_{|\Omega_m} \;\; \text{strongly in } \; L^2(\tau, T; L^2(\Omega_m)) \;\; \forall T > \tau, \quad \forall m \geq 1. \tag{4.48}$$

Finally, if the sequence $\{u_\tau^n\}$ converges strongly in $L^2(\Omega)$ to u_τ, then

$$u_\mu \to u \text{ strongly in } L^2\left(\tau, T; L^2(\Omega)\right) \;\; \forall T > \tau, \tag{4.49}$$

and

$$u_\mu(t) \to u(t) \text{ strongly in } L^2(\Omega) \quad \forall t \geq \tau. \tag{4.50}$$

Proof Taking into account the energy equality

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|^2 + |\nabla u_n(t)|^2 = \langle f(x,u_n(t)), u_n(t)\rangle + \langle h(t), u_n(t)\rangle,$$

if we argue similarly to the proof of Theorem 4.3, we obtain that there exists a subsequence $\{u_\mu\} \subset \{u_n\}$ such that

$$u_\mu \rightharpoonup u \text{ weakly in } L^2(\tau,T;H_0^1(\Omega)), \tag{4.51}$$

$$u_\mu \rightharpoonup u \text{ weakly in } L^p(\tau,T;L^p(\Omega)),$$
$$f(x,u_\mu) \rightharpoonup f(x,u) \text{ weakly in } L^{p'}(\tau,T;L^{p'}(\Omega)),$$

for all $T > \tau$, and $u \in S(\tau, u_\tau)$.

On the other hand, in particular, for a fixed $T > \tau$, the sequence $\{u_\mu(T)\}$ is bounded in $L^2(\Omega)$, then there exists a subsequence $\{u_{\mu'}\} \subset \{u_\mu\}$ such that

$$u_{\mu'}(T) \rightharpoonup \xi \text{ weakly in } L^2(\Omega). \tag{4.52}$$

Let $w \in H_0^1(\Omega) \cap L^p(\Omega)$. From the equation satisfied by $u_{\mu'}$, we obtain

$$(u_{\mu'}(T), w) = \left(u_\tau^{\mu'}, w\right) + \int_\tau^T \langle \Delta u_{\mu'}(s) + f(x, u_{\mu'}(s)) + h(s), w\rangle\, ds,$$

and thus, making $\mu' \to \infty$,

$$(\xi, w) = (u_\tau, w) + \int_\tau^T \langle \Delta u(s) + f(x,u(s)) + h(s), w\rangle\, ds.$$

Consequently, as $u \in S(\tau, u_\tau)$, we obtain

$$(\xi, w) = (u(T), w) \quad \forall\, w \in H_0^1(\Omega) \cap L^p(\Omega),$$

and therefore, by density, it follows

$$\xi = u(T). \tag{4.53}$$

Then, from (4.52), (4.53), we can deduce that the whole sequence $\{u_\mu(T)\}$ satisfies

$$u_\mu(T) \rightharpoonup u(T) \text{ weakly in } L^2(\Omega).$$

As $T > \tau$ has been taken arbitrarily, we see that (4.44) holds.

On the other hand, reasoning as in the proof of (4.15) in Theorem 4.3, we can deduce that for all $m \geq 1$,

$$\left\{u_{\mu|\Omega_m}\right\}_{\mu\geq 1} \text{ is pre-compact in } L^2\left(\tau, T; L^2\left(\Omega_m\right)\right) \quad \forall T > \tau. \tag{4.54}$$

From (4.45) and (4.54), we deduce (4.48).

Assume now that the sequence $\{u_\tau^n\}$ converges strongly in $L^2(\Omega)$ to u_τ, and let us fix $T > \tau$.

Then, by Lemma 4.10, we have that for all $\varepsilon > 0$ there exists $M_\varepsilon \geq 1$ such that

$$\begin{aligned}\int_\tau^T \int_{\Omega\cap\{|x|_{\mathbb{R}^N}\geq 2m\}} (u_n - u)^2\, dxds &\leq 2\int_\tau^T \int_{\Omega\cap\{|x|_{\mathbb{R}^N}\geq 2m\}} u_n^2 dxds \\ &\quad + 2\int_\tau^T \int_{\Omega\cap\{|x|_{\mathbb{R}^N}\geq 2m\}} u^2 dxds \\ &\leq 4\varepsilon(T-\tau),\, \forall\, n\geq 1,\ \forall\, m \geq M_\varepsilon.\end{aligned} \tag{4.55}$$

Moreover, by (4.48),

$$\int_\tau^T \int_{\Omega_{2m}} (u_\mu - u)^2\, dxds \to 0, \text{ as } \mu\to\infty, \text{ for all } m\geq 1. \tag{4.56}$$

From (4.55) and (4.56) we obtain (4.49).

From (4.49) we deduce that from every subsequence of $\{u_\mu\}$ we can extract a subsequence that we will denote by $\{u_\nu\}$, such that

$$|u_\nu(t)| \to |u(t)| \quad a.e. \text{ in } (\tau, T). \tag{4.57}$$

Let us define

$$J_\nu(t) = \frac{1}{2}|u_\nu(t)|^2 - \int_\tau^t \langle h(s), u_\nu(s)\rangle\, ds - \int_\tau^t\int_\Omega C_2(x) dxds,$$

and

$$J(t) = \frac{1}{2}|u(t)|^2 - \int_\tau^t \langle h(s), u(s)\rangle\, ds - \int_\tau^t\int_\Omega C_2(x) dxds,$$

for all $t \geq \tau$.

It is clear that J_ν and J are continuous functions. Also, from (4.45) and (4.57) we see that

$$J_\nu(t) \to J(t)\ a.e.\ t\in(\tau,T) \quad \text{as } \nu\to\infty. \tag{4.58}$$

On the other hand, taking into account the energy equality and (4.4), we obtain

$$\frac{1}{2}\frac{d}{dt}\left|u_\nu(t)\right|^2 \le \int_\Omega C_2(x)dx + \langle h(t), u_\nu(t)\rangle\,, \quad t > \tau. \tag{4.59}$$

Thus, for every ν, the function J_ν is a non-increasing function of t.

We are now in position to show that

$$J_\nu(t) \to J(t) \quad \text{strongly for all } t \in (\tau, T)\,. \tag{4.60}$$

Let $t \in (\tau, T)$ and $\varepsilon > 0$ be fixed. From (4.58) and the continuity of J, we can take $t' > t$ and $t'' < t$ such that

$$J_\nu(t') \to J(t') \quad \text{strongly as } \nu \to \infty, \tag{4.61}$$

$$J_\nu(t'') \to J(t'') \quad \text{strongly as } \nu \to \infty, \tag{4.62}$$

$$|J(t'') - J(t)| \le \varepsilon, \tag{4.63}$$

and

$$|J(t) - J(t')| \le \varepsilon. \tag{4.64}$$

As J_ν is a non-increasing function of t, we obtain

$$J_\nu(t') - J_\nu(t) \le 0, \tag{4.65}$$

and

$$J_\nu(t'') - J_\nu(t) \ge 0, \tag{4.66}$$

for every ν. Using (4.63) and (4.66) we have

$$\begin{aligned} J_\nu(t) - J(t) &= J_\nu(t) - J_\nu(t'') + J_\nu(t'') - J(t'') \\ &\quad + J(t'') - J(t) \\ &\le |J_\nu(t'') - J(t'')| + \varepsilon. \end{aligned} \tag{4.67}$$

Analogously, using (4.64) and (4.65) we obtain

$$\begin{aligned} J(t) - J_\nu(t) &= J(t) - J(t') + J(t') - J_\nu(t') \\ &\quad + J_\nu(t') - J_\nu(t) \\ &\le |J(t') - J_\nu(t')| + \varepsilon. \end{aligned} \tag{4.68}$$

From (4.61), (4.62), (4.67) and (4.68), we have

$$\limsup_{\nu\to\infty} |J(t) - J_\nu(t)| \le \varepsilon, \tag{4.69}$$

and therefore, as $\varepsilon > 0$ is arbitrary, (4.60) follows from (4.69). Thanks to (4.69), and taking into account (4.45), we deduce that

$$|u_\nu(t)| \to |u(t)| \text{ strongly } \ \forall t \in (\tau, T)\,,$$

and then, by (4.44), we obtain

$$u_\nu(t) \to u(t) \text{ strongly in } L^2(\Omega) \quad \forall t \in (\tau, T)\,.$$

Then from a standard contradiction argument combined with the fact that $T > \tau$ has been taken arbitrarily, we deduce that (4.50) holds.

□

4.3.3 Existence of the global pullback attractor

Now, we are ready to obtain the main result of this chapter, that is, the existence of the global pullback attractor.

Lemma 4.12 *Under the assumptions in Lemma 4.7, the MNDS U defined by (4.21) is upper semicontinuous and has closed values.*

Proof If U is not upper semicontinuous, then there exist $\tau \le t$, a point $u_\tau \in L^2(\Omega)$, a neighborhood $\mathcal{N}$ of $U(t,\tau,u_\tau)$ and a sequence $y_n \in U(t,\tau,u_\tau^n)$ with $u_\tau^n \to u_\tau$ in $L^2(\Omega)$, such that $y_n \notin \mathcal{N}$ for all n.

Proposition 4.11 implies that there exists a subsequence $\{y_\mu\} \subset \{y_n\}$ and $y \in U(t,\tau,u_\tau)$ such that $y_\mu \to y$ in $L^2(\Omega)$, which is a contradiction.

The fact that U has closed values follows immediately from Proposition 4.11.

□

Lemma 4.13 *Under the assumptions in Lemma 4.7, the MNDS U defined by (4.21) is pullback asymptotically compact with respect to the family $\widehat{B}_{\lambda_1}$ defined in that lemma.*

Proof Let us fix a sequence $\tau_n \to -\infty$, a sequence $u_{\tau_n} \in B_{\lambda_1}(\tau_n)$ and $t \in \mathbb{R}$. We have to prove that from any sequence $y_n \in U(t,\tau_n,u_{\tau_n})$ we can extract a subsequence that converges in $L^2(\Omega)$.

As $y_n \in U(t,\tau_n,u_{\tau_n})$, there exists $u_n \in S(\tau_n,u_{\tau_n})$ such that $u_n(t) = y_n$.

As the family $\widehat{B}_{\lambda_1}$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing and $\tau_n \to -\infty$, there exists $n_0(t) \ge 1$ such that $\tau_n \le t-1$ and

$$u_n(t-1) \in U(t-1,\tau_n,u_{\tau_n}) \subset U(t-1,\tau_n,B_{\lambda_1}(\tau_n)) \subset B_{\lambda_1}(t-1), \quad (4.70)$$

for all $n \geq n_0(t)$.

From (4.70), we deduce that there exists a subsequence $\{u_\mu\} \subset \{u_n\}$ and $\zeta_0 \in B_{\lambda_1}(t-1)$, such that

$$u_\mu(t-1) \rightharpoonup \zeta_0 \quad \text{weakly in } L^2(\Omega). \tag{4.71}$$

As $u_\mu \in S(t-1, u_\mu(t-1))$, by Proposition 4.11 we have that there exists a subsequence $\{u_{n'}\} \subset \{u_\mu\}$, such that there exists $u \in S(t-1, \zeta_0)$ satisfying in particular

$$y_{n'} = u_{n'}(t) \rightharpoonup u(t) \quad \text{weakly in } L^2(\Omega), \tag{4.72}$$

and

$$u_{n'|\Omega_{2m}} \to u_{|\Omega_{2m}} \quad \text{strongly in } L^2\left(t-1, t, L^2(\Omega_{2m})\right), \quad \forall m \geq 1. \tag{4.73}$$

By Lemma 4.8, for any $\varepsilon > 0$ there exists $T = T(t-1, t, \varepsilon, \widehat{B}_{\lambda_1}) \leq t-1$, and $M = M(t-1, t, \varepsilon, \widehat{B}_{\lambda_1}) \geq 1$, such that

$$\begin{aligned} &\int_{t-1}^{t} \int_{|x|_{\mathbb{R}^N} \geq 2m} \left(u_{n'}(x,s) - u(x,s)\right)^2 dxds \\ &\leq 2\int_{t-1}^{t} \int_{|x|_{\mathbb{R}^N} \geq 2m} u_{n'}^2(x,s)dxds \\ &\qquad + 2\int_{t-1}^{t} \int_{|x|_{\mathbb{R}^N} \geq 2m} u^2(x,s)dxds \leq 4\varepsilon, \end{aligned} \tag{4.74}$$

for all $m \geq M$ and any n' such that $\tau_{n'} \leq T$.

From (4.73) and (4.74) we have

$$u_{n'} \to u \text{ strongly in } L^2\left(t-1, t; L^2(\Omega)\right).$$

Now, if we argue similarly to Proposition 4.11 we obtain

$$y_{n'} = u_{n'}(t) \to u(t) \text{ strongly in } L^2(\Omega).$$

□

Now, as a direct consequence of the preceding results, Theorem 3.10 and Corollary 3.11 in Chapter 3, we have the existence of the minimal pullback $\mathcal{D}_{\lambda_1}-$attractor and the minimal pullback $\mathcal{D}_F^H-$attractor for the MNDS U defined by (4.21).

Theorem 4.14 *Under the assumptions in Lemma 4.7, the MNDS U defined by (4.21) has a unique pullback $\mathcal{D}_{\lambda_1}$-attractor $\mathcal{A}_{\mathcal{D}_{\lambda_1}}$ belonging to $\mathcal{D}_{\lambda_1}$, which is strictly invariant and is given by*

$$\mathcal{A}_{\mathcal{D}_{\lambda_1}}(t) = \Lambda\left(\widehat{B}_{\lambda_1}, t\right) = \bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s} U(t,\tau,B_{\lambda_1}(\tau))}, \tag{4.75}$$

where $\widehat{B}_{\lambda_1}$ was defined in Lemma 4.7, and the closure is taken in $L^2(\Omega)$. Moreover, there exists the minimal pullback $\mathcal{D}_F^H-$attractor, $\mathcal{A}_{\mathcal{D}_F^H}$, and we have the following relation

$$\mathcal{A}_{\mathcal{D}_F^H}(t) \subset \mathcal{A}_{\mathcal{D}_{\lambda_1}}(t) \subset \overline{B}_{L^2(\Omega)}\left(0, R_{\lambda_1}(t)\right) \quad \text{for all } t\in\mathbb{R}.$$

Remark 4.15 *We note that if we also assume that*

$$\sup_{t\leq 0} e^{-\lambda_1 t}\sum_{i=1}^{N}\int_{-\infty}^{t} e^{\lambda_1 s}|h_i(s)|^2\,ds < \infty,$$

then we have that $\cup_{t\leq T}\overline{B}_{L^2(\Omega)}\left(0, R_{\lambda_1}(t)\right)$ is a bounded subset of $L^2\left(\Omega\right)$.

Therefore, taking into account Remark 3.12 in Chapter 3, we can deduce that

$$\mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_{\lambda_1}}(t) \quad \text{for all } t\leq T.$$

4.4 The Kneser Property for reaction-diffusion equations

When we consider a partial differential equation with non-uniqueness of the Cauchy problem it is interesting to consider the Kneser property, that is, the connectedness and compactness of the set of values attained by the solutions at any moment of time.

We observe that when we set the problem in Section 4.1 the function $C_1\left(x\right)$, which appears in (4.3), belongs just to $L^1\left(\Omega\right)$, but for the Kneser property we need the stronger condition $C_1(x)\in L^1\left(\Omega\right)\cap L^\infty\left(\Omega\right)$.

From now on, we assume the same assumptions as in Section 4.1, but we also assume that $C_1(x)\in L^1\left(\Omega\right)\cap L^\infty\left(\Omega\right)$.

In this section we shall prove that the set of values attained by the solutions of equation (4.2) at any moment of time is compact and connected.

Remark 4.16 *We note that the compactness of $U(t,\tau,u_\tau)$ in $L^2(\Omega)$ is a consequence of Proposition 4.11 in Section 4.3, as in that Proposition it is shown that $U(t,\tau,u_\tau)$ is precompact and closed for any u_τ.*

Our aim is to prove the connectedness of the set $U(t,\tau,u_\tau) \subset L^2(\Omega)$, defined in (4.21), for any $t \geq \tau$, and for this aim we need some preliminary lemmas.

We shall obtain now that $U(t,\tau,u_\tau)$ is conected in $L^2(\Omega)$ and for this aim we need some preliminary lemmas.

We take a sequence $0 < \epsilon_k < 1$ converging to 0 as $k \to \infty$ and define a sequence of smooth functions $\psi_k : \mathbb{R}^+ \longrightarrow [0,1]$ satisfying

$$\psi_k(s) := \begin{cases} 1, \text{ if } 0 \leq s \leq \sqrt{\epsilon_k}, \\ 0 \leq \psi_k \leq 1, \text{ if } \sqrt{\epsilon_k} \leq s \leq 2\sqrt{\epsilon_k}, \\ 0, \text{ if } 2\sqrt{\epsilon_k} \leq s \leq 1/\epsilon_k, \\ 0 \leq \psi_k \leq 1, \text{ if } 1/\epsilon_k \leq s \leq 1/\epsilon_k + 1, \\ 1, \text{ if } s \geq 1/\epsilon_k + 1. \end{cases}$$

Let $\rho_{\epsilon_k} : \mathbb{R} \longrightarrow \mathbb{R}^+$ be a mollifier, that is, $\rho_{\epsilon_k} \in \mathbb{C}_0^\infty(\mathbb{R};\mathbb{R})$, supp$\rho_{\epsilon_k} \subset B_{\epsilon_k}$, $\int_{\mathbb{R}} \rho_{\epsilon_k}(s)\,ds = 1$ and $\rho_{\epsilon_k}(s) \geq 0$ for all $s \in \mathbb{R}$, where $B_{\epsilon_k} = \{u \in \mathbb{R} : |u| \leq \epsilon_k\}$.

We define the following approximating function

$$f^k(x,u) := \psi_k(|u|)\left(C_0^1 |u|^{p-2} u + f(x,0)\right) + (1 - \psi_k(|u|)) \int_{\mathbb{R}} \rho_{\epsilon_k}(s) f(x, u-s)\,ds,$$

where $k \geq 1$, $p \geq 2$, and C_0^1 is a negative constant.

Remark 4.17 *It is easy to check that for a.a. $x \in \Omega$,*

$$\sup_{|u|\leq A} \left| f^k(x,u) - f(x,u) \right| \longrightarrow 0, \textit{ as } k \to \infty, \textit{ for any } A > 0.$$

Lemma 4.18 *Assume (4.3) with $C_1(x) \in L^1(\Omega) \cap L^\infty(\Omega)$ and (4.4). Then the function f^k satisfies that there exist constants $\widehat{\alpha}_1$, $\widehat{\alpha}_2 > 0$, and positive functions $\widehat{C}_1(x) \in L^1(\Omega) \cap L^\infty(\Omega)$ and $\widehat{C}_2(x) \in L^1(\Omega)$, not depending on k, such that*

$$\left| f^k(x,u) \right|^{\frac{p}{p-1}} \leq \widehat{\alpha}_1 |u|^p + \widehat{C}_1(x) \quad \forall u \in \mathbb{R},\ x \in \Omega,$$

$$f^k(x,u)u \leq -\widehat{\alpha}_2 |u|^p + \widehat{C}_2(x) \quad \forall u \in \mathbb{R},\ x \in \Omega,$$

for k great enough.

Proof Indeed, for the first property we have the following cases.

1) If $0 \leq |u| \leq \sqrt{\epsilon_k}$ or $|u| \geq 1/\epsilon_k + 1$, then we have

$$\left|f^k(x,u)\right| \leq \left|C_0^1\right| |u|^{p-1} + |f(x,0)|,$$

so (4.3) yields

$$\begin{aligned}\left|f^k(x,u)\right|^{\frac{p}{p-1}} &\leq 2^{\frac{1}{p-1}} \left|C_0^1\right|^{\frac{p}{p-1}} |u|^p + 2^{\frac{1}{p-1}} |f(x,0)|^{\frac{p}{p-1}} \\ &\leq 2^{\frac{1}{p-1}} \left|C_0^1\right|^{\frac{p}{p-1}} |u|^p + 2^{\frac{1}{p-1}} C_1(x).\end{aligned}$$

2) If $\sqrt{\epsilon_k} \leq |u| \leq 2\sqrt{\epsilon_k}$ or $1/\epsilon_k \leq |u| \leq 1/\epsilon_k + 1$, then using (4.5) we have

$$\begin{aligned}\left|f^k(x,u)\right| &\leq \left|C_0^1\right| |u|^{p-1} + |f(x,0)| \\ &+ \int_{\mathbb{R}} \rho_{\epsilon_k}(s) \left(\alpha_1^{\frac{p-1}{p}} |u-s|^{p-1} + C_1(x)^{\frac{p-1}{p}}\right) ds \\ &\leq \left|C_0^1\right| |u|^{p-1} + 2C_1(x)^{\frac{p-1}{p}} \\ &+ \alpha_1^{\frac{p-1}{p}} 2^{p-2} \int_{\mathbb{R}} \rho_{\epsilon_k}(s) \left(|u|^{p-1} + |s|^{p-1}\right) ds \\ &\leq 2C_1(x)^{\frac{p-1}{p}} + \left(\left|C_0^1\right| + \alpha_1^{\frac{p-1}{p}} 2^{p-1}\right) |u|^{p-1},\end{aligned}$$

and then

$$\left|f^k(x,u)\right|^{\frac{p}{p-1}} \leq 2^{\frac{p+1}{p-1}} C_1(x) + 2^{\frac{1}{p-1}} \left(\left|C_0^1\right| + \alpha_1^{\frac{p-1}{p}} 2^{p-1}\right)^{\frac{p}{p-1}} |u|^p.$$

3) If $2\sqrt{\epsilon_k} \leq |u| \leq 1/\epsilon_k$, then arguing as in the previous case we have

$$\left|f^k(x,u)\right| \leq C_1(x)^{\frac{p-1}{p}} + \left(\alpha_1^{\frac{p-1}{p}} 2^{p-1}\right) |u|^{p-1},$$

and then

$$\left|f^k(x,u)\right|^{\frac{p}{p-1}} \leq 2^{\frac{1}{p-1}} C_1(x) + 2^{\frac{1}{p-1}} 2^p \alpha_1 |u|^p.$$

Finally, we obtain the first property,

$$\left|f^k(x,u)\right|^{\frac{p}{p-1}} \leq \widehat{C}_1(x) + \widehat{\alpha}_1 |u|^p,$$

where

$$\widehat{C}_1(x) := 2^{\frac{p+1}{p-1}} C_1(x) \in L^1(\Omega) \cap L^\infty(\Omega),$$

and

$$\widehat{\alpha}_1 := 2^{\frac{1}{p-1}} \left(\left| C_0^1 \right| + \alpha_1^{\frac{p-1}{p}} 2^{p-1} \right)^{\frac{p}{p-1}} > 0.$$

On the other hand, note that

$$f^k(x,u)u = \psi_k(|u|)\, C_0^1 |u|^p + \psi_k(|u|)\, f(x,0)u + (1-\psi_k(|u|)) \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, f(x,u-s)\, u ds. \tag{4.76}$$

Hence, for the second property we have the following cases.

1) If $2\sqrt{\epsilon_k} \leq |u| \leq 1/\epsilon_k$, then using (4.4) and the Young inequality $ab \leq \frac{a^p}{\varepsilon^{p-1}p} + \frac{\varepsilon b^{p'}}{p'}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, f(x,u-s)\, u ds &= \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, f(x,u-s)\,(u-s)\, ds \\ &+ \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, f(x,u-s)\, s ds \\ &\leq \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\,(-\alpha_2 |u-s|^p + C_2(x))\, ds \\ &+ \frac{\alpha_2}{p'\alpha_1} \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, |f(x,u-s)|^{p'} ds \\ &+ \frac{1}{p}\left(\frac{\alpha_1}{\alpha_2}\right)^{p-1} \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, |s|^p ds,\end{aligned}$$

where $p' = \frac{p}{p-1}$ is the conjugate exponent of p.

Taking into account (4.3), we have

$$\begin{aligned}\int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, f(x,u-s)\, u ds &\leq \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\,(-\alpha_2 |u-s|^p + C_2(x))\, ds \\ &+ \frac{\alpha_2}{p'\alpha_1} \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\,(\alpha_1 |u-s|^p + C_1(x))\, ds \\ &+ \frac{1}{p}\left(\frac{\alpha_1}{\alpha_2}\right)^{p-1} \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, |s|^p ds \\ &\leq \left(-\alpha_2 + \frac{\alpha_2}{p'}\right) \int_{\mathbb{R}} \rho_{\epsilon_k}(s) \left(\frac{|u|^p}{2^{p-1}} - |s|^p\right) ds \\ &+ \frac{1}{p}\left(\frac{\alpha_1}{\alpha_2}\right)^{p-1} \int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, |s|^p ds + \overline{C}_2(x),\end{aligned} \tag{4.77}$$

where

$$\overline{C}_2(x) := C_2(x) + \frac{\alpha_2}{p'\alpha_1}C_1(x), \tag{4.78}$$

and where in the last inequality we have used

$$|u|^p = |u-s+s|^p \le 2^{p-1}\left(|u-s|^p + |s|^p\right).$$

We observe that $|u| \ge \sqrt{\epsilon_k}$, so that for k large enough,

$$|s|^p \le \epsilon_k^p \le \frac{1}{2^p}\epsilon_k^{\frac{p}{2}} \le \frac{1}{2^p}|u|^p.$$

Then from (4.77) we obtain

$$\int_{\mathbb{R}} \rho_{\epsilon_k}(s) f(x, u-s)\, u ds \le \left(-\alpha_2 + \frac{\alpha_2}{p'}\right)\frac{1}{2^p}|u|^p \tag{4.79}$$

$$+ \frac{1}{p}\left(\frac{\alpha_1}{\alpha_2}\right)^{p-1}\int_{\mathbb{R}} \rho_{\epsilon_k}(s)\, |s|^p ds + \overline{C}_2(x).$$

Since for k large enough, we have

$$\begin{aligned}|s|^p \le \epsilon_k^p &\le -p\left(\frac{\alpha_2}{\alpha_1}\right)^{p-1}\left(-\alpha_2 + \frac{\alpha_2}{p'}\right)\frac{1}{2^{p+1}}\epsilon_k^{\frac{p}{2}} \\ &\le -p\left(\frac{\alpha_2}{\alpha_1}\right)^{p-1}\left(-\alpha_2 + \frac{\alpha_2}{p'}\right)\frac{1}{2^{p+1}}|u|^p,\end{aligned}$$

we obtain

$$\int_{\mathbb{R}} \rho_{\epsilon_k}(s) f(x, u-s)\, u ds \le \beta |u|^p + \overline{C}_2(x), \tag{4.80}$$

where

$$\beta = \left(-\alpha_2 + \frac{\alpha_2}{p'}\right)\frac{1}{2^{p+1}} < 0, \tag{4.81}$$

and $\overline{C}_2(x) \in L^1(\Omega)$ is a positive function.

2) If $0 \le |u| \le \sqrt{\epsilon_k}$ or $|u| \ge 1/\epsilon_k + 1$, then using the Young inequality $ab \le \frac{\varepsilon a^p}{p} + \frac{b^{p'}}{\varepsilon^{\frac{1}{p-1}}p'}$ and (4.3) we obtain

$$\begin{aligned}f^k(x,u)u &= C_0^1 |u|^p + f(x,0)u \\ &\le C_0^1\left(1 - \frac{1}{p}\right)|u|^p + \frac{1}{p'\left(-C_0^1\right)^{\frac{1}{p-1}}}C_1(x) \\ &\le \frac{C_0^1}{2}|u|^p + \frac{1}{p'\left(-C_0^1\right)^{\frac{1}{p-1}}}C_1(x),\end{aligned}$$

where $\frac{C_0^1}{2}$ is a negative constant and $\frac{1}{p'\left(-C_0^1\right)^{\frac{1}{p-1}}}C_1(x) \in L^1(\Omega)$ is a positive function.

3) If $\sqrt{\epsilon_k} \leq |u| \leq 2\sqrt{\epsilon_k}$ or $1/\epsilon_k \leq |u| \leq 1/\epsilon_k + 1$, we argue as in the first case to obtain (4.80). From (4.76), using the Young inequality and (4.3), we have

$$
\begin{aligned}
f^k(x,u)u &\leq \psi_k(|u|)\, C_0^1 |u|^p - \psi_k(|u|)\, \frac{C_0^1}{p} |u|^p \\
&+ \psi_k(|u|)\, \frac{1}{p'\left(-C_0^1\right)^{\frac{1}{p-1}}} C_1(x) \\
&+ (1-\psi_k(|u|))\left(\beta |u|^p + \overline{C}_2(x)\right) \\
&\leq \psi_k(|u|)\, \frac{C_0^1}{2} |u|^p + (1-\psi_k(|u|))\, \beta |u|^p \\
&+ \left(\overline{C}_2(x) + \frac{1}{p'\left(-C_0^1\right)^{\frac{1}{p-1}}} C_1(x)\right) \\
&\leq \widetilde{\beta}\, |u|^p + \widehat{C}_2(x),
\end{aligned}
$$

where

$$\widetilde{\beta} := \max\left\{\beta, \frac{C_0^1}{2}\right\} < 0, \tag{4.82}$$

and $\widehat{C}_2$ is a positive function and is given by

$$\widehat{C}_2(x) := \overline{C}_2(x) + \frac{1}{p'\left(-C_0^1\right)^{\frac{1}{p-1}}} C_1(x) \in L^1(\Omega), \tag{4.83}$$

where $\overline{C}_2(x)$ is given by (4.78).

Finally, we have the second property,

$$f^k(x,u)u \leq \widetilde{\beta}\, |u|^p + \widehat{C}_2(x),$$

where $\widetilde{\beta}$ is given in (4.82) and $\widehat{C}_2(x) \in L^1(\Omega)$ is given by (4.83).

□

Lemma 4.19 *Assume (4.3) with $C_1(x) \in L^1(\Omega) \cap L^\infty(\Omega)$ and (4.4). Then, there exist D_{ϵ_k} such that*

$$f_u^k(x,u) \leq D_{\epsilon_k},\ \forall u \in \mathbb{R},\ \textit{for a.a. } x \in \Omega, \tag{4.84}$$

where f_u^k is the derivative with respect to u.

Proof We note that $\psi_{ku}(|u|)$ is uniformly bounded on $\mathbb{R}$. We have

$$\begin{aligned} f_u^k(x,u) &= C_0^1\psi_k(|u|)(p-1)|u|^{p-2} + C_0^1|u|^{p-2}u\psi_{ku}(|u|) \\ &+ f(x,0)\psi_{ku}(|u|) + (1-\psi_k(|u|))\int_{\mathbb{R}}\rho'_{\epsilon_k}(u-s)f(x,s)\,ds \\ &- \psi_{ku}(|u|)\int_{\mathbb{R}}\rho_{\epsilon_k}(s)f(x,u-s)\,ds. \end{aligned} \tag{4.85}$$

We consider each term in (4.85).

- As C_0^1 is a negative constant, for the first term, we have

$$C_0^1\psi_k(|u|)(p-1)|u|^{p-2} \leq 0.$$

- For the second term, we get

$$\left|C_0^1|u|^{p-2}u\psi_{ku}(|u|)\right| \leq \left|C_0^1\right|\left(\frac{1}{\epsilon_k}+1\right)^{p-1}C_{\psi_k}.$$

- For the third term using (4.5), we obtain

$$|f(x,0)\psi_{ku}(|u|)| \leq |f(x,0)|\,C_{\psi_k} \leq C_1(x)^{\frac{p-1}{p}}C_{\psi_k} \leq \|C_1\|_\infty^{\frac{p-1}{p}}C_{\psi_k}.$$

- For the fourth term, we have to consider several cases.
  If $0 \leq |u| \leq \sqrt{\epsilon_k}$ or $|u| \geq 1/\epsilon_k + 1$, we obtain

$$(1-\psi_k(|u|))\int_{\mathbb{R}}\rho'_{\epsilon_k}(u-s)f(x,s)\,ds = 0.$$

  If $\sqrt{\epsilon_k} < |u| < 1/\epsilon_k + 1$, then using (4.5) we have

$$\begin{aligned} &\left|(1-\psi_k(|u|))\int_{\mathbb{R}}\rho'_{\epsilon_k}(u-s)f(x,s)\,ds\right| \\ &\leq \int_{B_{\epsilon_k}}\left|\rho'_{\epsilon_k}(s)\right|\left(\alpha_1^{\frac{p-1}{p}}|u-s|^{p-1} + C_1(x)^{\frac{p-1}{p}}\right)ds \\ &\leq \|C_1\|_\infty^{\frac{p-1}{p}}\int_{\mathbb{R}}\left|\rho'_{\epsilon_k}(s)\right|ds + 2^{p-1}\alpha_1^{\frac{p-1}{p}}(1/\epsilon_k+1)^{p-1}\int_{\mathbb{R}}\left|\rho'_{\epsilon_k}(s)\right|ds \\ &\leq D_{\epsilon_k}^1, \end{aligned}$$

  as $\rho_{\epsilon_k} \in \mathbb{C}_0^\infty(\mathbb{R};\mathbb{R})$.

- For the last term, if $\sqrt{\epsilon_k} < |u| < 1/\epsilon_k + 1$, using (4.5) we have

$$\begin{aligned}
&\left| -\psi_{ku}\left(|u|\right) \int_{\mathbb{R}} \rho_{\epsilon_k}\left(s\right) f\left(x, u-s\right) ds \right| \\
&\leq \left|\psi_{ku}\left(|u|\right)\right| \int_{\mathbb{R}} \rho_{\epsilon_k}\left(s\right) \alpha_1^{\frac{p-1}{p}} \left|u-s\right|^{p-1} ds \\
&+ \left|\psi_{ku}\left(|u|\right)\right| \int_{\mathbb{R}} \rho_{\epsilon_k}\left(s\right) C_1(x)^{\frac{p-1}{p}} ds \\
&\leq \left|\psi_{ku}\left(|u|\right)\right| \left\|C_1\right\|_\infty^{\frac{p-1}{p}} \\
&+ \left|\psi_{ku}\left(|u|\right)\right| 2^{p-2} \alpha_1^{\frac{p-1}{p}} \int_{\mathbb{R}} \rho_{\epsilon_k}\left(s\right) \left(|u|^{p-1} + |s|^{p-1}\right) ds \\
&\leq C_{\psi_k} \left\|C_1\right\|_\infty^{\frac{p-1}{p}} + C_{\psi_k} 2^{p-1} \alpha_1^{\frac{p-1}{p}} \left(1/\epsilon_k + 1\right)^{p-1} =: D_{\epsilon_k}^2.
\end{aligned}$$

  In other case $\psi_{ku}\left(|u|\right) = 0$. Then (4.84) holds.

$\square$

Let $T > \tau$ be arbitrary. In order to prove the Kneser property let us consider the following auxiliary problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f^k(x,u) + h(t) \quad \text{in} \quad \Omega \times (\gamma, +\infty), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (\gamma, +\infty), \\ u(x,\gamma) = u^\gamma(x), \quad x \in \Omega, \end{cases} \tag{4.86}$$

where $\gamma \in [\tau, T]$.

In view of Lemma 4.18 for all $k \geq 1$ the function f^k satisfies (4.3) with $C_1(x) \in L^1(\Omega) \cap L^\infty(\Omega)$ and (4.4), so that by the Theorem 4.3 for any $u^\gamma \in L^2(\Omega)$ problem (4.86) has at least one weak solution $u_\gamma^k(\cdot)$ defined on $[\gamma, T]$.

Using Lemma 4.19 it is standard to check that for the difference $w(t)$ of two solutions of (4.86) with the same initial condition u^γ, we have

$$\frac{1}{2}\frac{d}{dt}\left|w(t)\right|^2 + \left|\nabla w(t)\right|^2 \leq D_{\epsilon_k}\left|w(t)\right|^2.$$

Hence, from Gronwall's Lemma we obtain

$$|w(t)| \leq |w(\gamma)|\, e^{D_{\epsilon_k}(t-\gamma)}, \tag{4.87}$$

so that the solution is unique.

We need some preliminary estimates.

Lemma 4.20 *Suppose that* Ω *satisfies (4.1) and suppose that* f *is Carathéodory function and satisfies (4.3) with* $C_1(x) \in L^1(\Omega) \cap L^\infty(\Omega)$ *and (4.4). Let* $h = \sum_{i=1}^N \frac{\partial h_i}{\partial x_i}$, *with* $h_i \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ *for all* $1 \le i \le N$, *such that*

$$\sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 \, ds < +\infty \quad \forall t \in \mathbb{R}. \tag{4.88}$$

Then there exists $R = R(B,T)$ *(not depending neither on* γ *nor* k*), where* B *is a bounded set of* $L^2(\Omega)$, *such that*

$$\left| u^k_\gamma(t) \right| \le R, \quad \forall t \in [\gamma, T], \tag{4.89}$$

and

$$\left\| u^k_\gamma(\cdot) \right\|_{L^p(\gamma,T;L^p(\Omega))} \le R, \tag{4.90}$$

for any $u^\gamma \in B$, *where* $u^k_\gamma(\cdot)$ *is the unique solution to (4.86) with* $u^k_\gamma(\gamma) = u^\gamma$.

Proof We note that using Lemma 4.18 one can easily obtain that the functions u^k_γ satisfy the estimate

$$\frac{1}{2}\frac{d}{dt}\left| u^k_\gamma(t) \right|^2 + \frac{1}{2}\left| \nabla u^k_\gamma(t) \right|^2 + \widehat{\alpha}_2 \left\| u^k_\gamma(t) \right\|^p_{L^p(\Omega)} \le \frac{1}{2}\|h(t)\|^2_* + \left\| \widehat{C}_2 \right\|_{L^1(\Omega)}.$$

Integrating between γ to t, we have

$$\begin{aligned} &\left| u^k_\gamma(t) \right|^2 + \int_\gamma^t \left| \nabla u^k_\gamma(s) \right|^2 ds + 2\widehat{\alpha}_2 \int_\gamma^t \left\| u^k_\gamma(s) \right\|^p_{L^p(\Omega)} ds \\ &\le |u^\gamma|^2 + \int_\gamma^T \|h(s)\|^2_* \, ds + 2\left\| \widehat{C}_2 \right\|_{L^1(\Omega)} (T-\gamma). \end{aligned} \tag{4.91}$$

Hence, (4.89) and (4.90) follow.

□

Lemma 4.21 *Under the assumptions in Lemma 4.20, let* K *be a relatively compact set in* $L^2(\Omega)$. *Then, for all* $\tau \le T$ *and* $\varepsilon > 0$ *there exists* $M = M(\gamma, T, \varepsilon, K)$ *such that*

$$\int_{\Omega \cap \{|x|_{\mathbb{R}^N} \ge 2m\}} u^k_\gamma(x,t)^2 dx \le \varepsilon, \ \forall t \in [\gamma, T], \quad \forall \gamma \in [\tau, T], \ \ \forall m \ge M,$$

for any $u^\gamma \in K$, *where* $u^k_\gamma(\cdot)$ *is the unique solution to (4.86) with* $u^k_\gamma(\gamma) = u^\gamma$.

Proof Thanks to Lemma 4.18 we have that f^k satisfies (4.3) and (4.4). If we argue as in Lemma 4.10, we obtain a similar estimate for u_γ^k.

□

Theorem 4.22 *Under the assumptions in Lemma 4.20, the set $U(t, \tau, u_\tau)$ is connected in $L^2(\Omega)$ for any $t \in [\tau, T]$.*

Proof The case $t = \tau$ is obvious. Suppose that for some $t^* \in (\tau, T]$ the set $U(t^*, \tau, u_\tau)$ is not connected. Then there exist two compact sets A_1, $A_2 \subset L^2(\Omega)$ such that $A_1 \cup A_2 = U(t^*, \tau, u_\tau)$, $A_1 \cap A_2 = \emptyset$.

Let $u_1(\cdot)$, $u_2(\cdot) \in S(\tau, u_\tau)$ be such that $u_1(t^*) \in E_1$, $u_2(t^*) \in E_2$, where E_1, E_2 are disjoint open neighborhoods of A_1, A_2, respectively.

Let $u_i^k(t, \gamma)$, $i = 1, 2$, be equal to $u_i(t)$, if $t \in [\tau, \gamma]$, and let $u_i^k(t, \gamma)$ be the unique solution of problem (4.86), if $t \in [\gamma, T]$.

We shall prove now that the maps $u_i^k(t, \gamma)$ are continuous on γ for each fixed $k \geq 1$ and $t \in [\tau, T]$. We shall omit the index i for simplicity of notation.

Lemma 4.23 *The maps $\gamma \mapsto u^k(t, \gamma)$ are continuous for each fixed $k \geq 1$ and $t \in [\tau, T]$.*

Proof Let $\gamma \to \gamma_0$. Consider first the case where $\gamma > \gamma_0$, i.e., $\gamma \searrow \gamma_0$. If $t \leq \gamma_0 < \gamma$, then $u^k(t, \gamma) = u(t) = u^k(t, \gamma_0)$. We note also that $u^k(t, \gamma) = u(t)$, for all $t \in [\tau, \gamma]$. Now if $t > \gamma_0$, then we can assume that $t > \gamma$, so that $u^k(t, \gamma)$ is the solution of (4.86) on $[\gamma, T]$ such that $u^k(\gamma, \gamma) = u(\gamma)$, and $u^k(t, \gamma_0)$ is the solution of (4.86) on $[\gamma_0, T]$ such that $u^k(\gamma_0, \gamma_0) = u(\gamma_0)$. Further, $u(\gamma) \to u(\gamma_0)$, $u^k(\gamma, \gamma_0) \to u(\gamma_0)$, as $\gamma \to \gamma_0$, by continuity. Using (4.87) for $w(t) = u^k(t, \gamma) - u^k(t, \gamma_0)$ we have

$$\begin{aligned}
\left|u^k(t, \gamma) - u^k(t, \gamma_0)\right| &\leq \left|u^k(\gamma, \gamma) - u^k(\gamma, \gamma_0)\right| e^{D_{\epsilon_k}(t-\gamma)}, \\
&= \left|u(\gamma) - u^k(\gamma, \gamma_0)\right| e^{D_{\epsilon_k}(t-\gamma)} \\
&\leq |u(\gamma) - u(\gamma_0)| \, e^{D_k(t-\gamma)} \\
&\quad + \left|u(\gamma_0) - u^k(\gamma, \gamma_0)\right| e^{D_k(t-\gamma)},
\end{aligned}$$

then, we can deduce that

$$\left|u^k(t, \gamma) - u^k(t, \gamma_0)\right| \longrightarrow 0,$$

as $\gamma \to \gamma_0$.

Let now $\gamma < \gamma_0$, i.e., $\gamma \nearrow \gamma_0$. If $t < \gamma_0$, then we can assume that $t < \gamma$, so that $u^k(t, \gamma) = u(t) = u^k(t, \gamma_0)$. We note also that $u^k(t, \gamma_0) = u(t)$, for all $t \in [\tau, \gamma_0]$. If $t \geq \gamma_0 > \gamma$, then $u^k(t, \gamma)$ is the solution of (4.86) on $[\gamma, T]$,

$u^k(\gamma,\gamma) = u(\gamma)$, and $u^k(t,\gamma_0)$ is the solution of (4.86) on $[\gamma_0, T]$ such that $u^k(\gamma_0,\gamma_0) = u(\gamma_0)$. Hence,

$$\begin{aligned}\left|u^k(t,\gamma) - u^k(t,\gamma_0)\right| &\leq \left|u^k(\gamma_0,\gamma) - u^k(\gamma_0,\gamma_0)\right| e^{D_{\epsilon_k}(t-\gamma_0)} \\ &= \left|u^k(\gamma_0,\gamma) - u(\gamma_0)\right| e^{D_{\epsilon_k}(t-\gamma_0)}.\end{aligned}$$

To finish the proof of continuity, we have to check that

$$\left|u^k(\gamma_0,\gamma) - u(\gamma_0)\right| \to 0,$$

as $\gamma \nearrow \gamma_0$. For the difference $v^k(t,\gamma) = u^k(t,\gamma) - u(t)$ we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left|v^k(t,\gamma)\right|^2 + \left|\nabla v^k(t,\gamma)\right|^2 + \int_\Omega \left(f(x,u)u^k + f^k(x,u^k)u\right) dx \\ &\leq \int_\Omega \left(f^k(x,u^k)u^k + f(x,u)u\right) dx,\end{aligned} \tag{4.92}$$

for a.a. $t \in (\gamma, T)$. Using conditions (4.4) and integrating (4.92) over (γ,γ_0), we obtain

$$\begin{aligned}\left|u^k(\gamma_0,\gamma) - u(\gamma_0)\right|^2 &\leq |u(\gamma) - u(\gamma)|^2 + K(\gamma_0 - \gamma) \\ &\quad + K\left\|f(\cdot,u)\right\|_{L^{p'}\left(\gamma,\gamma_0;L^{p'}(\Omega)\right)} \left\|u^k\right\|_{L^p(\gamma,\gamma_0;L^p(\Omega))} \\ &\quad + K\left\|f^k(\cdot,u^k)\right\|_{L^{p'}\left(\gamma,\gamma_0;L^{p'}(\Omega)\right)} \left\|u\right\|_{L^p(\gamma,\gamma_0;L^p(\Omega))},\end{aligned}$$

where K is a positive constant.

It follows from (4.90) and (4.3) that the norms $\left\|u^k\right\|_{L^p(\gamma,\gamma_0;L^p(\Omega))}$, and $\left\|f^k(\cdot,u^k)\right\|_{L^{p'}\left(\gamma,\gamma_0;L^{p'}(\Omega)\right)}$ are bounded by a constant that does not depend on γ. On the other hand, $u \in L^p\left(\gamma,\gamma_0;L^p(\Omega)\right) \cap L^2\left(\gamma,\gamma_0;L^2(\Omega)\right)$ and $f(\cdot,u) \in L^{p'}\left(\gamma,\gamma_0;L^{p'}(\Omega)\right)$ (again by (4.3)), so that $\left\|f(\cdot,u)\right\|_{L^{p'}\left(\gamma,\gamma_0;L^{p'}(\Omega)\right)} \leq \varepsilon$, $\|u\|_{L^p(\gamma,\gamma_0;L^p(\Omega))} \leq \varepsilon$, as soon as $|\gamma - \gamma_0| < \delta(\varepsilon)$. Therefore,

$$\left|u^k(\gamma_0,\gamma) - u(\gamma_0)\right| \to 0,$$

as $\gamma \nearrow \gamma_0$.

□

Now we put

$$\gamma(\lambda) = \begin{cases} \tau - (T-\tau)\,\lambda, \text{ if } \lambda \in [-1,0], \\ \tau + (T-\tau)\,\lambda, \text{ if } \lambda \in [0,1], \end{cases}$$

and define the function

$$\varphi^k(\lambda)(t) = \begin{cases} u_1^k(t,\gamma(\lambda)) \text{ if } \lambda \in [-1,0], \\ u_2^k(t,\gamma(\lambda)) \text{ if } \lambda \in [0,1]. \end{cases}$$

We have

$$\varphi^k(-1)(t) = u_1^k(t,T) = u_1(t),$$

and

$$\varphi^k(1)(t) = u_2^k(t,T) = u_2(t).$$

The map $\lambda \mapsto \varphi^k(\lambda)(t) \in L^2(\Omega)$ is continuous for any fixed $k \geq 1$, $t \in [\tau, T]$. Note that $u_1^k(t,\tau) = u_2^k(t,\tau)$ and $\varphi^k(-1)(t^*) \in E_1$, $\varphi^k(1)(t^*) \in E_2$, so that there exists $\lambda_k \in [-1,1]$ such that $\varphi^k(\lambda_k)(t^*) \notin E_1 \cup E_2$.

Denote

$$u^k(t) = \varphi^k(\lambda_k)(t).$$

Note that for each $k \geq 1$ either $u^k(t) = u_1^k(t,\gamma(\lambda_k))$ or $u^k(t) = u_2^k(t,\gamma(\lambda_k))$.

For some subsequence it is equal to one of them, say $u_1^k(t,\gamma(\lambda_k))$.

Now we shall consider the function $u_1^k(t,\gamma(\lambda_k))$, $t \in [\tau, T]$. We have

$$u^k(t) = \begin{cases} u_1(t), \text{ if } t \in [\tau, \gamma(\lambda_k)], \\ u_1^k(t,\gamma(\lambda_k)), \text{ if } t \in [\gamma(\lambda_k), T], \end{cases}$$

where $\gamma(\lambda_k) \longrightarrow \gamma_0 \in [\tau, T]$. We define the functions

$$\widetilde{f}_k(t,x,v) = \begin{cases} f(x,v), \text{ if } t \in [\tau, \gamma(\lambda_k)], \\ f^k(x,v), \text{ if } t \in (\gamma(\lambda_k), T]. \end{cases}$$

By continuity $u_1(\gamma(\lambda_k)) \to u_1(\gamma_0)$ strongly, as $k \to \infty$.

Further, by (4.91),

$$\left\{u^k(\cdot)\right\} \text{ is bounded in } L^\infty\left(\tau,T;L^2(\Omega)\right) \cap L^2\left(\tau,T;H_0^1(\Omega)\right) \cap L^p\left(\tau,T;L^p(\Omega)\right). \tag{4.93}$$

It follows also that $\frac{du^k}{dt}$ is bounded in $L^{p'}\left(\tau,T;L^{p'}(\Omega)\right) + L^2\left(\tau,T;H^{-1}(\Omega)\right)$. Then for some function $u = u(x,t)$ we have

$$u^k \rightharpoonup u \text{ weakly in } L^2\left(\tau,T;H_0^1(\Omega)\right), \tag{4.94}$$

$$u^k \rightharpoonup u \text{ weakly in } L^p\left(\tau,T;L^p(\Omega)\right),$$

$$\frac{du^k}{dt} \rightharpoonup \frac{du}{dt} \text{ weakly in } L^{p'}\left(\tau,T;L^{p'}(\Omega)\right) + L^2\left(\tau,T;H^{-1}(\Omega)\right),$$

$$u^k \overset{*}{\rightharpoonup} u \text{ weak-star in } L^\infty\left(\tau,T;L^2(\Omega)\right).$$

Arguing in a similar way as in the proof of Theorem 4.3 we first deduce

$$\lim_{a\to 0}\sup_k \int_\tau^{T-a} \left|u^k(t+a)-u^k(t)\right|^2 dt = 0, \tag{4.95}$$

for all $T > \tau$.

Let $\phi \in C^1([0,+\infty))$ be a function such that

$$0 \leq \phi(s) \leq 1,$$

$$\phi(s) = 1 \quad \forall s \in [0,1],$$

$$\phi(s) = 0 \quad \forall s \geq 2.$$

For each k and $m \geq 1$, we define

$$u^{k,m}(x,t) = \phi\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\right) u^k(x,t) \qquad \forall x \in \Omega_{2m},\ \forall k,\ \ \forall m \geq 1. \tag{4.96}$$

We obtain from (4.93) that, for all $m \geq 1$, the sequence $\left\{u^{k,m}\right\}_{k\geq 1}$ is bounded in $L^\infty(\tau,T;L^2(\Omega_{2m}))\cap L^p(\tau,T;L^p(\Omega_{2m}))\cap L^2(\tau,T;H_0^1(\Omega_{2m}))$, for all $T > \tau$.

In particular, it follows that

$$\lim_{a\to 0}\sup_k \left(\int_\tau^{\tau+a} \left|u^{k,m}(t)\right|^2_{L^2(\Omega_{2m})} dt + \int_{T-a}^{T} \left|u^{k,m}(t)\right|^2_{L^2(\Omega_{2m})} dt\right) = 0.$$

On the other hand, from (4.95) we deduce that for $m \geq 1$,

$$\lim_{a\to 0}\sup_k \left(\int_\tau^{T-a} \left|u^{k,m}(t+a)-u^{k,m}(t)\right|^2_{L^2(\Omega_{2m})} dt\right) = 0.$$

Moreover, as Ω_{2m} is a bounded set, $H_0^1(\Omega_{2m})$ is included in $L^2(\Omega_{2m})$ with compact injection.

Then, by the Compactness Theorem 13.3 and Remark 13.1 of [80] with $X = L^2(\Omega_{2m})$, $Y = H_0^1(\Omega_{2m})$, $p = 2$ and $\mathcal{G} = \left\{u^{k,m}\right\}_{k\geq 1}$, we obtain that

$$\left\{u^{k,m}\right\}_{k\geq 1} \text{ is relatively compact in } L^2\left(\tau,T;L^2(\Omega_{2m})\right),$$

and thus, taking into account that $u^{k,m}(x,t) = u^k(x,t)$ for all $x \in \Omega_m$, we deduce that, in particular, for all $m \geq 1$

$$\left\{u^k_{|\Omega_m}\right\}_{k\geq 1} \text{ is precompact in } L^2\left(\tau,T;L^2(\Omega_m)\right). \tag{4.97}$$

It is not difficult to conclude from (4.94), (4.97), via a diagonal procedure, the existence of a subsequence of $\left\{u^k\right\}_{k\geq 1}$ (which we denote as the sequence) such that

$$u^k_{|\Omega_m}(x,t) \to u_{|\Omega_m}(x,t) \text{ for a.a. } (x,t) \in \Omega_m \times (\tau, T)\,, \tag{4.98}$$

for all $m \geq 1$.

It is easy to obtain (see [62, Chapter 3]) that

$$\frac{du^k_{|\Omega_m}}{dt} \text{ is bounded in } L^{p'}\left(\tau, T; L^{p'}(\Omega_m)\right) + L^2\left(\tau, T; H^{-1}(\Omega_m)\right),$$

which is continuously embedded in the space $L^q\left(\tau, T; H^{-s}(\Omega_m)\right)$ for $s = \max\left\{1, N\left(\frac{1}{p'} - \frac{1}{2}\right)\right\}$.

From (4.98),

$$\widetilde{f}_k(t, x, u^k_{|\Omega_m}(x,t)) \longrightarrow f(x, u_{|\Omega_m}(x,t)) \text{ for a.a. } (x,t) \in \Omega_m \times (\tau, T)\,,$$

and then the boundedness of $\widetilde{f}_k(t, x, u^k_{|\Omega_m})$ in $L^{p'}\left(\tau, T; L^{p'}(\Omega_m)\right)$ implies that $\widetilde{f}_k(t, x, u^k_{|\Omega_m})$ converges to $f(x, u_{|\Omega_m})$ weakly in $L^{p'}\left(\tau, T; L^{p'}(\Omega_m)\right)$ for any $m \geq 1$ (see Lemma 1.3, Chapter 1 in [56]).

Also, we note that (4.97), Lemma 4.10 and Lemma 4.21 imply, up to a subsequence, that

$$u^k \longrightarrow u \text{ strongly in } L^2\left(\tau, T; L^2(\Omega)\right), \tag{4.99}$$

$$u^k(t) \longrightarrow u(t) \text{ in } L^2(\Omega) \text{ for a.a.} \, t \in (\tau, T)\,.$$

Moreover,

$$u^k_{|\Omega_m}(t) \rightharpoonup u_{|\Omega_m}(t) \text{ weakly in } L^2(\Omega)\,,$$

for all $t \in [\tau, T]$ and $m \geq 1$.

Indeed, as $\frac{du^k_{|\Omega_m}}{dt}$ is a bounded sequence of the space $L^q\left(\tau, T; H^{-s}(\Omega_m)\right)$, we have that $u^k_{|\Omega_m}(t) : [\tau, T] \longrightarrow H^{-s}(\Omega_m)$ is an equicontinuous family of functions.

By (4.89) for each fixed $r \in [\tau, T]$ the sequence $u^k_{|\Omega_m}(r)$ is bounded in $L^2(\Omega_m)$, so that the compact embedding $L^2(\Omega_m) \subset H^{-s}(\Omega_m)$, implies that it is precompact in $H^{-s}(\Omega_m)$.

Applying the Ascoli-Arzelà Theorem we deduce that $\left\{u^k_{|\Omega_m}(t)\right\}$ is a precompact sequence in $C\left([\tau, T], H^{-s}(\Omega_m)\right)$.

Hence, since

$$u^k_{|\Omega_m} \rightharpoonup u_{|\Omega_m} \text{ weakly in } L^2\left(\tau, T; H^{-s}(\Omega_m)\right),$$

we have

$$u^k_{|\Omega_m} \longrightarrow u_{|\Omega_m} \text{ strongly in } C\left([\tau, T], H^{-s}(\Omega_m)\right).$$

The boundedness of $u^k_{|\Omega_m}(r)$ in $L^2(\Omega_m)$ implies then by a standard argument that

$$u^k_{|\Omega_m}(r) \rightharpoonup u_{|\Omega_m}(r) \text{ weakly in } L^2(\Omega_m),$$

for all r.

Then it follows easily that

$$u^k(t) \rightharpoonup u(t) \text{ weakly in } L^2(\Omega),$$

for any $t \in [\tau, T]$.

Also, we deduce $u(\tau) = u_\tau$. As $u^k_{|\Omega_m}$ is a weak solution with f replaced by $\widetilde{f}_k$ and Ω by Ω_m, passing to the limit we obtain that u is a weak solution.

Finally, we shall prove the following:

Lemma 4.24 *Then it follows easily using Lemma 4.10 and 4.21 that*

$$u^k(t^*) \to u(t^*) \text{ strongly in } L^2(\Omega).$$

Proof We note that using (4.4) one can easily obtain that any solution v of (4.2) satisfies the estimate

$$\frac{1}{2}\frac{d}{dt}|v(t)|^2 + \frac{1}{2}|\nabla v(t)|^2 + \alpha_2 \|v(t)\|^p_{L^p(\Omega)} \leq \frac{1}{2}\|h(t)\|^2_* + \|C_2\|_{L^1(\Omega)}.$$

By integration and using (4.91) we have

$$\left|u^k(t)\right|^2 \leq \left|u^k(s)\right|^2 + \int_s^t \|h(r)\|^2_* \, dr + 2M(t-s), \tag{4.100}$$

$$|u(t)|^2 \leq |u(s)|^2 + \int_s^t \|h(r)\|^2_* \, dr + 2M(t-s), \tag{4.101}$$

for all $t \geq s$, t, $s \in [\tau, T]$, where the constant $M > 0$ does not depend on k.

From (4.100) and (4.101) the functions

$$J_k(t) = \left|u^k(t)\right|^2 - 2Mt - \int_\tau^t \|h(s)\|^2_* \, ds,$$

and

$$J(t) = |u(t)|^2 - 2Mt - \int_\tau^t \|h(s)\|^2_* \, ds,$$

are continuous and non-increasing on $[\tau, T]$.

We state that

$$\limsup J_k(t^*) \leq J(t^*).$$

We know from (4.98) that $J_k(t) \longrightarrow J(t)$, for a.a. $t \in (\tau, T)$.

Let t_m be a sequence such that $\tau < t_m < t^*$ and

$$t_m \to t^* \text{ as } m \to \infty,$$

and

$$J_k(t_m) \longrightarrow J(t_m) \text{ as } k \to \infty,$$

for any fixed m.

Hence, using the continuity of J and the monotonicity of J_k, J, we have that for any $\varepsilon > 0$ there exist $m(\varepsilon)$ and $K(\varepsilon, m)$ such that

$$\begin{aligned} J_k(t^*) - J(t^*) &= J_k(t^*) - J_k(t_m) + J_k(t_m) - J(t_m) + J(t_m) - J(t^*) \\ &\leq |J_k(t_m) - J(t_m)| + |J(t_m) - J(t^*)| \leq 2\varepsilon, \end{aligned}$$

if $k \geq K$.

Hence,

$$\begin{aligned} \limsup J_k(t^*) &= \limsup \left|u^k(t^*)\right|^2 - 2Mt^* - \int_\tau^{t^*} \|h(s)\|_*^2 \, ds \\ &\leq |u(t^*)|^2 - 2Mt^* - \int_\tau^{t^*} \|h(s)\|_*^2 \, ds. \end{aligned}$$

Therefore,

$$\limsup \left|u^k(t^*)\right| \leq |u(t^*)|.$$

Since

$$u^k(t^*) \rightharpoonup u(t^*) \text{ weakly in } L^2(\Omega),$$

we have

$$\liminf \left|u^k(t^*)\right| \geq |u(t^*)|.$$

Thus,

$$u^k(t^*) \to u(t^*) \text{ strongly in } L^2(\Omega).$$

□

From this we immediately obtain that $u(t^*) \notin E_1 \cup E_2$, which is a contradiction and we conclude the proof of the theorem.

□

4.5 Connectedness of pullback attractors

In Section 4.3 concerning equation (4.2) we proved the existence of a global compact pullback attractor. It is interesting to prove that the global attractor is connected. Using the Kneser property we are able to obtain such result.

To do this, we will use Theorem 3.10 of Chapter 3.

Theorem 4.25 *Under the assumptions in Lemma 4.20, the MNDS U defined by (4.21) has a unique pullback $\mathcal{D}_{\lambda_1}$-attractor $\mathcal{A}_{\mathcal{D}_{\lambda_1}}$ belonging to $\mathcal{D}_{\lambda_1}$, which is strictly invariant and connected.*

Proof In Proposition 4.11 and Lemma 4.12 it is proved that the MNDS U defined by (4.21) has closed values and is upper semicontinuous.

It follows also by Lemma 4.7 that the family $\widehat{B}_{\lambda_1} \in \mathcal{D}_{\lambda_1}$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing, where the family $\widehat{B}_{\lambda_1}$ is defined by $B_{\lambda_1}(t) = \overline{B}_{L^2(\Omega)}(0, R_{\lambda_1}(t))$, and where $R_{\lambda_1}(t)$ is the nonnegative number given for each $t \in \mathbb{R}$ by

$$R_{\lambda_1}^2(t) = 2e^{-\lambda_1 t}\sum_{i=1}^{N}\int_{-\infty}^{t} e^{\lambda_1 s}\left|h_i(s)\right|^2 ds + 2\lambda_1^{-1}\left\|C_2\right\|_{L^1(\Omega)} + 1.$$

In Lemma 4.13 it is proved that the MNDS U defined by (4.21) is pullback asymptotically compact with respect to the family $\widehat{B}_{\lambda_1}$.

Also, in Theorem 4.14 it is shown the existence of a unique pullback $\mathcal{D}_{\lambda_1}$-attractor $\mathcal{A}_{\mathcal{D}_{\lambda_1}}$ for U which is strictly invariant and belongs to $\mathcal{D}_{\lambda_1}$.

Finally, we shall study the connectedness of the pullback $\mathcal{D}_{\lambda_1}$-attractor $\mathcal{A}_{\mathcal{D}_{\lambda_1}}$.

By Theorem 4.22, $U(t, \tau, u_\tau)$ has connected values in $L^2(\Omega)$. On the other hand, as $\widehat{B}_{\lambda_1}$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing, taking into account that $\widehat{B}_{\lambda_1} \in \mathcal{D}_{\lambda_1}$, thanks to the third statement of Theorem 3.10, we have

$$\mathcal{A}_{\mathcal{D}_{\lambda_1}}(t) \subset B_{\lambda_1}(t) = \overline{B}_{L^2(\Omega)}(0, R_{\lambda_1}(t)),$$

where $\widehat{B}_{\lambda_1} \in \mathcal{D}_{\lambda_1}$ is connected.

Hence, all conditions of the fifth statement of Theorem 3.10 are satisfied. Then, we have that $\mathcal{A}_{\mathcal{D}_{\lambda_1}}$ is connected.

□

Remark 4.26 *Under the assumptions in Lemma 4.20, for the MNDS U defined by (4.21) there exists the minimal pullback $\mathcal{D}_F^H-$attractor, $\mathcal{A}_{\mathcal{D}_F^H}$, which is connected.*

4.6 The Kneser property for a system of reaction-diffusion equations

We shall extend now the results of the Section 4.4 to the following system of reaction-diffusion equations

$$u_t = a\Delta u - f(x,u),\ x \in \mathbb{R}^N, t > 0, \tag{4.102}$$

$$u(0) = u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d, \tag{4.103}$$

where u is an unknown vector function, that is, $u(x,t) = (u^1, .., u^d)$, $x \in \mathbb{R}^N$, $t > 0$, $f(x,u) = (f^1, ..., f^d)$, where $d > 0$ is an integer, and $u_t = \dfrac{\partial u}{\partial t}$.

We assume the next conditions:

$(H1)$ The real $d \times d$ matrix a has a positive symmetric part $\frac{1}{2}(a + a^*) \geq AI$, where $A > 0$.

$(H2)$ $f = f_0 + f_1$, $f_0(x,u) = (f_0^1, .., f_0^d)$, $f_1(x,u) = (f_1^1, .., f_1^d)$ and f_i are Carathéodory functions, that is, they are continuous on u and measurable on x.

$(H3)$ There exist positive functions $C_0(x)$, $C_1(x) \in L^1(\mathbb{R}^N)$ and constants $\alpha, \beta > 0$, $p_i \geq 2$ verifying

$$\left(f_0\left(x,u\right), u\right) \geq \alpha \left|u\right|^2 - C_0\left(x\right), \tag{4.104}$$

$$(f_1(x,u), u) \geq \beta \sum_{i=1}^{d} \left|u^i\right|^{p_i} - C_1(x). \tag{4.105}$$

$(H4)$ There exist positive functions $C_2(x) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), C_3(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and constants $\gamma, \eta > 0$ verifying

$$|f_0(x,u)| \leq C_2(x) + \eta|u|, \tag{4.106}$$

$$\sum_{i=1}^{d} |f_1^i(x,u)|^{\frac{p_i}{p_i - 1}} \leq C_3(x) + \gamma \sum_{i=1}^{d} |u^i|^{p_i}. \tag{4.107}$$

Here, $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^m$ for $m \geq 1$, $(\cdot,\cdot)$ the scalar product in $\mathbb{R}^d$.

The Kneser property for this system was studied before in [64] but considering $C_2(x) \in L^2(\mathbb{R}^N)$, $C_3(x) \in L^1(\mathbb{R}^N)$ and assuming an additional condition on the derivatives of f_0, f_1. Our aim is to apply the technique of

Section 4.4 in order to avoid such condition. Instead, we have to assume that $C_2(x), C_3(x) \in L^\infty(\mathbb{R}^N)$.

First, we shall state the equivalent statements of Lemmas 4.18 and 4.19.

We take a sequence $0 < \epsilon_k < 1$ converging to 0 as $k \to \infty$ and define a sequence of smooth functions $\psi_k : \mathbb{R}^+ \longrightarrow [0,1]$ satisfying

$$\psi_k(s) := \begin{cases} 1, \text{ if } 0 \le s \le \sqrt{\epsilon_k}, \\ 0 \le \psi_k \le 1, \text{ if } \sqrt{\epsilon_k} \le s \le 2\sqrt{\epsilon_k}, \\ 0, \text{ if } 2\sqrt{\epsilon_k} \le s \le 1/\epsilon_k, \\ 0 \le \psi_k \le 1, \text{ if } 1/\epsilon_k \le s \le 1/\epsilon_k + 1, \\ 1, \text{ if } s \ge 1/\epsilon_k + 1. \end{cases}$$

Let $\rho_{\epsilon_k} : \mathbb{R}^d \to \mathbb{R}^+$ be a mollifier, that is, $\rho_{\epsilon_k} \in \mathbb{C}_0^\infty(\mathbb{R}^d; \mathbb{R})$, $\operatorname{supp}\rho_{\epsilon_k} \subset B_{\epsilon_k}$, $\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)ds = 1$ and $\rho_{\epsilon_k}(s) \ge 0$ for all $s \in \mathbb{R}^d$, where $B_{\epsilon_k} = \{u \in \mathbb{R}^d : |u| \le \epsilon_k\}$.

We define the following approximating functions

$$f_{0k}^i(x,u) := \psi_k(|u|)\left(C_0^0 u^i + f_0^i(x,0)\right) + (1 - \psi_k(|u|)) \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) f_0^i(x, u-s)\, ds,$$

$$f_{1k}^i(x,u) := \psi_k\left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right)\left(C_0^1 \left|u^i\right|^{p_i-2} u^i + f_1^i(x,0)\right) + \left(1 - \psi_k\left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right)\right) \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) f_1^i(x, u-s)\, ds,$$

where $k \ge 1$, $p_i \ge 2$, and C_0^0, C_0^1 are positive constants.

Let $f^k = f_{0k} + f_{1k}$. Then for a.a. $x \in \mathbb{R}^N$ we have

$$\sup_{|u| \le A} \left|f^k(x,u) - f(x,u)\right| \longrightarrow 0, \text{ as } k \longrightarrow \infty, \text{ for any } A > 0.$$

Lemma 4.27 *Assume (4.104)-(4.107). Then the functions f_{0k}, f_{1k} also satisfy conditions (4.104)-(4.107) with constants and functions not depending on k, i.e. there exist constants $\widehat{\alpha}, \widehat{\beta}, \widehat{\eta}, \widehat{\gamma} > 0$, and positive functions $\widehat{C}_0(x)$, $\widehat{C}_1(x) \in L^1(\mathbb{R}^N)$, $\widehat{C}_2(x) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\widehat{C}_3(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $f_{0,k}$, $f_{1,k}$ satisfy (4.104)-(4.107), for k large enough.*

Proof We check first (4.106) and (4.107). Indeed, for f_{0k} we have the following cases.

1) If $0 \leq |u| \leq \sqrt{\epsilon_k}$ or $|u| \geq 1/\epsilon_k + 1$, then we have
$$|f_{0k}(x,u)| = C_0^0 |u| + |f_0(x,0)|,$$
so (4.106) yields
$$|f_{0k}(x,u)| \leq C_0^0 |u| + C_2(x).$$

2) If $\sqrt{\epsilon_k} \leq |u| \leq 2\sqrt{\epsilon_k}$ or $1/\epsilon_k \leq |u| \leq 1/\epsilon_k + 1$, then using (4.106) we have
$$\begin{aligned} |f_{0k}(x,u)| &\leq C_0^0 |u| + |f_0(x,0)| + C_2(x) + \eta \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) |u-s| ds \\ &\leq C_0^0 |u| + 2C_2(x) + \eta \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) (|u| + |s|) ds \\ &\leq 2C_2(x) + \left(C_0^0 + 2\eta\right) |u|. \end{aligned}$$

3) If $2\sqrt{\epsilon_k} \leq |u| \leq 1/\epsilon_k$, then arguing as in the previous case we have
$$|f_{0k}(x,u)| \leq C_2(x) + 2\eta |u|.$$

Finally, we obtain
$$|f_{0k}(x,u)| \leq \widehat{C}_2(x) + \widehat{\eta} |u|,$$
where
$$\widehat{C}_2(x) := 2C_2(x) \in L^2\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right),$$
and
$$\widehat{\eta} := C_0^0 + 2\eta > 0.$$

On the other hand, for f_{1k}, in a similar way we have the following cases.

1) If $0 \leq \sqrt{\sum_{i=1}^d |u^i|^{p_i}} \leq \sqrt{\epsilon_k}$ or $\sqrt{\sum_{i=1}^d |u^i|^{p_i}} \geq 1/\epsilon_k + 1$, then we have
$$\begin{aligned} \sum_{i=1}^d |f_{1k}^i(x,u)|^{\frac{p_i}{p_i-1}} &= \sum_{i=1}^d \left\{ \left| C_0^1 \left| u^i \right|^{p_i-2} u^i + f_1^i(x,0) \right|^{\frac{p_i}{p_i-1}} \right\} \\ &\leq \sum_{i=1}^d \left\{ 2^{\frac{1}{p_i-1}} \left| C_0^1 \right|^{\frac{p_i}{p_i-1}} \left| u^i \right|^{p_i} + 2^{\frac{1}{p_i-1}} \left| f_1^i(x,0) \right|^{\frac{p_i}{p_i-1}} \right\} \\ &\leq \left(\sum_{i=1}^d 2^{\frac{1}{p_i-1}} \left| C_0^1 \right|^{\frac{p_i}{p_i-1}} \right) \sum_{i=1}^d \left| u^i \right|^{p_i} \\ &+ \left(\sum_{i=1}^d 2^{\frac{1}{p_i-1}} \right) \sum_{i=1}^d \left| f_1^i(x,0) \right|^{\frac{p_i}{p_i-1}}, \end{aligned}$$

so (4.107) yields

$$\sum_{i=1}^{d}|f_{1k}^i(x,u)|^{\frac{p_i}{p_i-1}} \leq \left(\sum_{i=1}^{d} 2^{\frac{1}{p_i-1}} \left|C_0^1\right|^{\frac{p_i}{p_i-1}}\right)\sum_{i=1}^{d} \left|u^i\right|^{p_i} + \left(\sum_{i=1}^{d} 2^{\frac{1}{p_i-1}}\right) C_3(x).$$

2) If $\sqrt{\epsilon_k} \leq \sqrt{\sum_{i=1}^{d}|u^i|^{p_i}} \leq 2\sqrt{\epsilon_k}$ or $1/\epsilon_k \leq \sqrt{\sum_{i=1}^{d}|u^i|^{p_i}} \leq 1/\epsilon_k + 1$, we have

$$\begin{aligned}\sum_{i=1}^{d}|f_{1k}^i(x,u)|^{\frac{p_i}{p_i-1}} &\leq \sum_{i=1}^{d}\left\{2^{\frac{2}{p_i-1}} \left|C_0^1\right|^{\frac{p_i}{p_i-1}} \left|u^i\right|^{p_i} + 2^{\frac{2}{p_i-1}} \left|f_1^i(x,0)\right|^{\frac{p_i}{p_i-1}}\right\} \\ &\quad + \sum_{i=1}^{d} 2^{\frac{1}{p_i-1}} \left(\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \left|f_1^i(x,u-s)\right| ds\right)^{\frac{p_i}{p_i-1}}.\end{aligned} \tag{4.108}$$

We observe that using Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \left|f_1^i(x,u-s)\right| ds &= \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)^{1/p_i'} \left|f_1^i(x,u-s)\right| \rho_{\epsilon_k}(s)^{1/p_i} ds \\ &\leq \left(\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \left|f_1^i(x,u-s)\right|^{p_i'} ds\right)^{1/p_i'},\end{aligned}$$

where $p_i' = \frac{p_i}{p_i-1}$ is the conjugate exponent of p_i.

Then

$$\left(\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \left|f_1^i(x,u-s)\right| ds\right)^{\frac{p_i}{p_i-1}} \leq \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \left|f_1^i(x,u-s)\right|^{\frac{p_i}{p_i-1}} ds. \tag{4.109}$$

Taking into account (4.108), we get

$$\begin{aligned}\sum_{i=1}^{d}|f_{1k}^i(x,u)|^{\frac{p_i}{p_i-1}} &\leq \sum_{i=1}^{d}\left\{2^{\frac{2}{p_i-1}} \left|C_0^1\right|^{\frac{p_i}{p_i-1}} \left|u^i\right|^{p_i} + 2^{\frac{2}{p_i-1}} \left|f_1^i(x,0)\right|^{\frac{p_i}{p_i-1}}\right\} \\ &\quad + \sum_{i=1}^{d} 2^{\frac{1}{p_i-1}} \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \left|f_1^i(x,u-s)\right|^{\frac{p_i}{p_i-1}} ds \\ &\leq \left(\sum_{i=1}^{d} 2^{\frac{2}{p_i-1}} \left|C_0^1\right|^{\frac{p_i}{p_i-1}}\right) \sum_{i=1}^{d} \left|u^i\right|^{p_i} \\ &\quad + \left(\sum_{i=1}^{d} 2^{\frac{2}{p_i-1}}\right) \sum_{i=1}^{d} \left|f_1^i(x,0)\right|^{\frac{p_i}{p_i-1}} \\ &\quad + \left(\sum_{i=1}^{d} 2^{\frac{1}{p_i-1}}\right) \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \sum_{i=1}^{d} \left|f_1^i(x,u-s)\right|^{\frac{p_i}{p_i-1}} ds,\end{aligned}$$

and using (4.107), we have

$$\sum_{i=1}^{d}|f_{1k}^i(x,u)|^{\frac{p_i}{p_i-1}} \leq \left(\sum_{i=1}^{d} 2^{\frac{2}{p_i-1}} \left|C_0^1\right|^{\frac{p_i}{p_i-1}}\right) \sum_{i=1}^{d} \left|u^i\right|^{p_i} + \left(\sum_{i=1}^{d} 2^{\frac{2}{p_i-1}} + \sum_{i=1}^{d} 2^{\frac{1}{p_i-1}}\right) C_3(x) + \left(\sum_{i=1}^{d} 2^{\frac{1}{p_i-1}}\right) \gamma \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \sum_{i=1}^{d} |u^i - s^i|^{p_i} ds. \tag{4.110}$$

We observe that

$$\sqrt{\epsilon_k} \leq \sqrt{\sum_{i=1}^{d} |u^i|^{p_i}},$$

so

$$\left|s^i\right|^{p_i} \leq \epsilon_k^{p_i} \leq \epsilon_k^{p_i-1} \sum_{i=1}^{d} \left|u^i\right|^{p_i},$$

and

$$\sum_{i=1}^{d} \left|s^i\right|^{p_i} \leq \sum_{i=1}^{d} \left|u^i\right|^{p_i} \sum_{i=1}^{d} \epsilon_k^{p_i-1} \leq \frac{1}{2} \sum_{i=1}^{d} \left|u^i\right|^{p_i}.$$

Then, we can deduce that

$$\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \sum_{i=1}^{d} |u^i - s^i|^{p_i} ds \leq \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \sum_{i=1}^{d} 2^{p_i-1} \left(\left|u^i\right|^{p_i} + \left|s^i\right|^{p_i}\right) ds \leq \left(\sum_{i=1}^{d} 2^{p_i-1} + \sum_{i=1}^{d} 2^{p_i-2}\right) \sum_{i=1}^{d} \left|u^i\right|^{p_i},$$

and taking into account (4.110), we have

$$\sum_{i=1}^{d}|f_{1k}^i(x,u)|^{\frac{p_i}{p_i-1}} \leq \left(\sum_{i=1}^{d} 2^{\frac{2}{p_i-1}} \left|C_0^1\right|^{\frac{p_i}{p_i-1}}\right) \sum_{i=1}^{d} \left|u^i\right|^{p_i} + \gamma \left(\sum_{i=1}^{d} 2^{\frac{1}{p_i-1}}\right) \left(\sum_{i=1}^{d} 2^{p_i-1} + \sum_{i=1}^{d} 2^{p_i-2}\right) \sum_{i=1}^{d} \left|u^i\right|^{p_i} + \left(\sum_{i=1}^{d} 2^{\frac{2}{p_i-1}} + \sum_{i=1}^{d} 2^{\frac{1}{p_i-1}}\right) C_3(x).$$

3) If $2\sqrt{\epsilon_k} \leq \sqrt{\sum_{i=1}^{d} |u^i|^{p_i}} \leq 1/\epsilon_k$, arguing as in the previous case, we have

$$\sum_{i=1}^{d} |f_{1k}^i(x,u)|^{\frac{p_i}{p_i-1}} \leq C_3(x) + \gamma \left(\sum_{i=1}^{d} 2^{p_i-1} + \sum_{i=1}^{d} 2^{p_i-2} \right) \sum_{i=1}^{d} \left| u^i \right|^{p_i}.$$

Finally, we obtain

$$\sum_{i=1}^{d} |f_{1k}^i(x,u)|^{\frac{p_i}{p_i-1}} \leq \widehat{C}_3 + \widehat{\gamma} \sum_{i=1}^{d} \left| u^i \right|^{p_i},$$

where

$$\widehat{C}_3 := \left(\sum_{i=1}^{d} 2^{\frac{2}{p_i-1}} + \sum_{i=1}^{d} 2^{\frac{1}{p_i-1}} \right) C_3(x) \in L^1\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right),$$

and

$$\widehat{\gamma} := \left(\sum_{i=1}^{d} 2^{\frac{2}{p_i-1}} \left| C_0^1 \right|^{\frac{p_i}{p_i-1}} + \gamma \left(\sum_{i=1}^{d} 2^{\frac{1}{p_i-1}} \right) \left(\sum_{i=1}^{d} 2^{p_i-1} + \sum_{i=1}^{d} 2^{p_i-2} \right) \right) > 0.$$

On the other hand, we note that

$$\begin{aligned} (f_{0k}(x,u),u) &= \psi_k\left(|u|\right) C_0^0(u,u) + \psi_k\left(|u|\right)\left(f_0(x,0),u\right) \\ &\quad + \left(1-\psi_k\left(|u|\right)\right) \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \left(f_0\left(x,u-s\right),u\right) ds \\ &= \psi_k\left(|u|\right) C_0^0 |u|^2 + \psi_k\left(|u|\right)\left(f_0(x,0),u\right) \\ &\quad + \left(1-\psi_k\left(|u|\right)\right) \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \left(f_0\left(x,u-s\right),u\right) ds, \end{aligned} \tag{4.111}$$

and

$$\begin{aligned} (f_{1k}(x,u),u) &= \psi_k\left(\sqrt{\sum_{i=1}^{d} |u^i|^{p_i}} \right) C_0^1 \sum_{i=1}^{d} \left| u^i \right|^{p_i} \\ &\quad + \psi_k\left(\sqrt{\sum_{i=1}^{d} |u^i|^{p_i}} \right) \left(f_1(x,0),u\right) \\ &\quad + \left(1 - \psi_k\left(\sqrt{\sum_{i=1}^{d} |u^i|^{p_i}} \right) \right) \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \left(f_1\left(x,u-s\right),u\right) ds. \end{aligned} \tag{4.112}$$

Hence, for property (4.104) we have the following cases.

1) If $2\sqrt{\epsilon_k} \leq |u| \leq 1/\epsilon_k$, then using (4.104), we have

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_0(x,u-s),u\right)ds &= \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_0(x,u-s),u-s\right)ds \\ &+ \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_0(x,u-s),s\right)ds \\ &\geq \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(\alpha|u-s|^2 - C_0(x)\right)ds \\ &+ \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_0(x,u-s),s\right)ds,\end{aligned}$$

and taking into account the Young inequality $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_0(x,u-s),u\right)ds &\geq \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(\alpha|u-s|^2 - C_0(x)\right)ds \\ &- \frac{1}{2}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\frac{\alpha}{2\eta^2}|f_0(x,u-s)|^2 ds \\ &- \frac{1}{2}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\frac{2\eta^2}{\alpha}|s|^2 ds.\end{aligned}$$

By (4.106), we have

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_0(x,u-s),u\right)ds &\geq \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(\alpha|u-s|^2 - C_0(x)\right)ds \\ &- \frac{\alpha}{2\eta^2}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(\eta^2|u-s|^2 + C_2^2(x)\right)ds \\ &- \frac{\eta^2}{\alpha}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)|s|^2 ds,\end{aligned}$$

so

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_0(x,u-s),u\right)ds &\geq \frac{\alpha}{2}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)|u-s|^2 ds \\ &- \frac{\eta^2}{\alpha}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)|s|^2 ds - C_{02}(x) \\ &\geq \frac{\alpha}{2}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(\frac{1}{2}|u|^2 - |s|^2\right)ds \\ &- \frac{\eta^2}{\alpha}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)|s|^2 ds - C_{02}(x), \quad (4.113)\end{aligned}$$

where

$$C_{02}(x) := C_0(x) + \frac{\alpha}{2\eta^2}C_2^2(x) \in L^1\left(\mathbb{R}^N\right),$$

and where in the last inequality we have used that

$$|u|^2 = |u - s + s|^2 \leq 2\left(|u - s|^2 + |s|^2\right).$$

We observe that

$$|u| \geq 2\sqrt{\epsilon_k} \geq \sqrt{\epsilon_k},$$

so that for k large enough,

$$|s|^2 \leq \epsilon_k^2 \leq \frac{1}{4}\epsilon_k \leq \frac{1}{4}|u|^2.$$

Then from (4.113) we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_0(x, u - s), u\right)ds \geq{}& \frac{\alpha}{2}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\frac{1}{4}|u|^2 ds \\ &- \frac{\eta^2}{\alpha}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)|s|^2 ds - C_{02}(x).\end{aligned}$$

Since for k large enough, we have

$$|s|^2 \leq \epsilon_k^2 \leq \frac{\alpha^2}{16\eta^2}\epsilon_k \leq |u|^2\frac{\alpha^2}{16\eta^2},$$

we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_0(x, u - s), u\right)ds &\geq \frac{\alpha}{8}|u|^2 - \frac{\alpha}{16}|u|^2 - C_{02}(x) \qquad (4.114)\\ &= \alpha_0|u|^2 - C_{02}(x),\end{aligned}$$

where $\alpha_0 := \frac{\alpha}{16} > 0$ and $C_{02}(x) \in L^1\left(\mathbb{R}^N\right)$.

2) If $0 \leq |u| \leq \sqrt{\epsilon_k}$ or $|u| \geq 1/\epsilon_k + 1$, then using the Young inequality and (4.106), we have

$$\begin{aligned}(f_{0k}(x, u), u) &= C_0^0|u|^2 + (f_0(x, 0), u)\\ &\geq C_0^0|u|^2 - \frac{C_0^0}{2}|u|^2 - \frac{1}{2C_0^0}|f(x, 0)|^2\\ &\geq \frac{C_0^0}{2}|u|^2 - \frac{1}{C_0^0}C_2^2(x),\end{aligned}$$

where $\frac{C_0^0}{2}$ is a positive constant and $\frac{1}{C_0^0}C_2^2(x) \in L^1\left(\mathbb{R}^N\right)$.

3) If $\sqrt{\epsilon_k} \leq |u| \leq 2\sqrt{\epsilon_k}$ or $1/\epsilon_k \leq |u| \leq 1/\epsilon_k + 1$, we argue as the first case to obtain (4.114).

From (4.111), using the Young inequality and (4.106), we have

$$\begin{aligned}
(f_{0k}(x,u),u) \geq\ & \psi_k(|u|)\, C_0^0 |u|^2 + \psi_k(|u|)\,(f_0(x,0),u) \\
& + (1-\psi_k(|u|))\left(\alpha_0 |u|^2 - C_{02}(x)\right) \\
\geq\ & \psi_k(|u|)\, C_0^0 |u|^2 + (1-\psi_k(|u|))\,\alpha_0 |u|^2 \\
& - C_{02}(x) + \psi_k(|u|)\,(f_0(x,0),u) \\
\geq\ & \psi_k(|u|)\, C_0^0 |u|^2 + (1-\psi_k(|u|))\,\alpha_0 |u|^2 \\
& - C_{02}(x) - \psi_k(|u|)\,\frac{C_0^0}{2}|u|^2 - \psi_k(|u|)\,\frac{1}{C_0^0}C_2^2(x),
\end{aligned}$$

and then, we can deduce that

$$\begin{aligned}
(f_{0k}(x,u),u) \geq\ & \psi_k(|u|)\,\frac{C_0^0}{2}|u|^2 + (1-\psi_k(|u|))\,\alpha_0 |u|^2 \\
& - \left(C_{02}(x) + \frac{1}{C_0^0}C_2^2(x)\right) \\
\geq\ & \widetilde{\alpha}_0 |u|^2 - \widetilde{C}_{02}(x),
\end{aligned}$$

where

$$\widetilde{\alpha}_0 = \min\left\{\alpha_0, \frac{C_0^0}{2}\right\} > 0, \tag{4.115}$$

and

$$\widetilde{C}_{02}(x) := C_{02}(x) + \frac{1}{C_0^0}C_2^2(x) \in L^1\left(\mathbb{R}^N\right). \tag{4.116}$$

Finally, we have

$$(f_{0k}(x,u),u) \geq \widehat{\alpha}\,|u|^2 - \widehat{C}_0(x),$$

where $\widehat{\alpha} := \widetilde{\alpha}_0 > 0$ is given by (4.115) and $\widehat{C}_0(x) := \widetilde{C}_{02}(x)$ is given by (4.116).

Now, for the property (4.105), we have the following cases.

1) If $2\sqrt{\epsilon_k} \le \sqrt{\sum_{i=1}^{d} |u^i|^{p_i}} \le 1/\epsilon_k$, then using (4.105), we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_1(x,u-s),u\right)ds = &\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_1(x,u-s),u-s\right)ds\\ &+\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_1(x,u-s),s\right)ds\\ \ge&\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(\beta\sum_{i=1}^{d}\left|u^i-s^i\right|^{p_i}-C_1(x)\right)ds\\ &+\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_1(x,u-s),s\right)ds,\end{aligned}$$

and then using the Young inequality $ab \le \frac{a^p}{\varepsilon^{p-1}p} + \frac{\varepsilon b^{p'}}{p'}$, we can deduce

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_1(x,u-s),u\right)ds \ge&\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(\beta\sum_{i=1}^{d}\left|u^i-s^i\right|^{p_i}-C_1(x)\right)ds\\ &-\frac{\beta}{\gamma}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\sum_{i=1}^{d}\frac{1}{p_i'}\left|f_1^i(x,u-s)\right|^{p_i'}ds\\ &-\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\sum_{i=1}^{d}\frac{1}{p_i}\left(\frac{\gamma}{\beta}\right)^{p_i-1}\left|s^i\right|^{p_i}ds,\end{aligned}$$

where $p_i' = \frac{p_i}{p_i-1}$ is the conjugate exponent of p_i.
Taking into account (4.107), we have

$$\begin{aligned}&\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_1(x,u-s),u\right)ds\\ &\ge \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(\beta\sum_{i=1}^{d}\left|u^i-s^i\right|^{p_i}-C_1(x)\right)ds\\ &-\frac{1}{p'}\frac{\beta}{\gamma}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(\gamma\sum_{i=1}^{d}\left|u^i-s^i\right|^{p_i}\right)ds\\ &-\frac{1}{p'}\frac{\beta}{\gamma}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\,C_3(x)ds\\ &-\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\sum_{i=1}^{d}\frac{1}{p_i}\left(\frac{\gamma}{\beta}\right)^{p_i-1}\left|s^i\right|^{p_i}ds,\end{aligned}$$

where $p' = \min\{p_1', .., p_d'\}$.

Then, we can deduce

$$\begin{aligned}
&\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\,(f_1(x, u-s), u)ds \\
&\geq \left(\beta - \frac{\beta}{p'}\right) \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \sum_{i=1}^{d} \left|u^i - s^i\right|^{p_i} ds \\
&- \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \sum_{i=1}^{d} \frac{1}{p_i} \left(\frac{\gamma}{\beta}\right)^{p_i - 1} \left|s^i\right|^{p_i} ds - C_{13}(x) \\
&\geq \left(\beta - \frac{\beta}{p'}\right) \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \sum_{i=1}^{d} \left(\frac{|u^i|^{p_i}}{2^{p-1}} - \left|s^i\right|^{p_i}\right) ds \\
&- \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \sum_{i=1}^{d} \frac{1}{p_i} \left(\frac{\gamma}{\beta}\right)^{p_i - 1} \left|s^i\right|^{p_i} ds - C_{13}(x),
\end{aligned} \tag{4.117}$$

where

$$C_{13}(x) := C_1(x) + \frac{1}{p'}\frac{\beta}{\gamma} C_3(x) \in L^1\left(\mathbb{R}^N\right), \tag{4.118}$$

and where in the last inequality we have used

$$\begin{aligned}
\sum_{i=1}^{d} \frac{|u^i|^{p_i}}{2^{p-1}} &\leq \sum_{i=1}^{d} \frac{|u^i|^{p_i}}{2^{p_i-1}} = \sum_{i=1}^{d} \frac{|u^i - s^i + s^i|^{p_i}}{2^{p_i-1}} \\
&\leq \sum_{i=1}^{d} \left(\left|u^i - s^i\right|^{p_i} + \left|s^i\right|^{p_i}\right),
\end{aligned}$$

with $p = \max\{p_1, .., p_d\}$.

We observe that

$$\sqrt{\epsilon_k} \leq 2\sqrt{\epsilon_k} \leq \sqrt{\sum_{i=1}^{d} |u^i|^{p_i}},$$

so

$$\left|s^i\right|^{p_i} \leq \epsilon_k^{p_i} \leq \epsilon_k^{p_i - 1} \sum_{i=1}^{d} \left|u^i\right|^{p_i},$$

and

$$\begin{aligned}
\sum_{i=1}^{d} \left|s^i\right|^{p_i} &\leq \sum_{i=1}^{d} \left|u^i\right|^{p_i} \sum_{i=1}^{d} \epsilon_k^{p_i - 1} \\
&\leq \frac{1}{2^p} \sum_{i=1}^{d} \left|u^i\right|^{p_i}.
\end{aligned}$$

Then from (4.117) we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_1(x,u-s),u\right)ds \geq & \left(\beta-\frac{\beta}{p'}\right)\frac{1}{2^p}\sum_{i=1}^{d}\left|u^i\right|^{p_i} \\ & -\int_{\mathbb{R}^d}\rho_{\epsilon_k}(s)\sum_{i=1}^{d}\frac{1}{p_i}\left(\frac{\gamma}{\beta}\right)^{p_i-1}\sum_{i=1}^{d}\left|s^i\right|^{p_i}ds \\ & - C_{13}(x).\end{aligned}$$

Since for k large enough, we have

$$\begin{aligned}\sum_{i=1}^{d}\left|s^i\right|^{p_i} &\leq \sum_{i=1}^{d}\epsilon_k^{p_i-1}\sum_{i=1}^{d}\left|u^i\right|^{p_i} \\ &\leq \left(\sum_{i=1}^{d}\frac{1}{p_i}\left(\frac{\gamma}{\beta}\right)^{p_i-1}\right)^{-1}\left(\beta-\frac{\beta}{p'}\right)\frac{1}{2^{p+1}}\sum_{i=1}^{d}\left|u^i\right|^{p_i},\end{aligned}$$

then we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\left(f_1(x,u-s),u\right)ds \geq & \left(\beta-\frac{\beta}{p'}\right)\frac{1}{2^p}\sum_{i=1}^{d}\left|u^i\right|^{p_i} \\ & -\left(\beta-\frac{\beta}{p'}\right)\frac{1}{2^{p+1}}\sum_{i=1}^{d}\left|u^i\right|^{p_i}-C_{13}(x) \\ & =\left(\beta-\frac{\beta}{p'}\right)\frac{1}{2^{p+1}}\sum_{i=1}^{d}\left|u^i\right|^{p_i}-C_{13}(x) \\ & =\beta_1\sum_{i=1}^{d}\left|u^i\right|^{p_i}-C_{13}(x), \qquad (4.119)\end{aligned}$$

where

$$\beta_1 := \left(\beta-\frac{\beta}{p'}\right)\frac{1}{2^{p+1}} > 0,$$

and $C_{13}(x)$ is given by (4.118).

2) If $0 \leq \sqrt{\sum_{i=1}^{d}\left|u^i\right|^{p_i}} \leq \sqrt{\epsilon_k}$ or $\sqrt{\sum_{i=1}^{d}\left|u^i\right|^{p_i}} \geq 1/\epsilon_k + 1$, then using the

Young inequality $ab \leq \frac{\varepsilon a^p}{p} + \frac{b^{p'}}{p'\varepsilon^{1/(p-1)}}$ and (4.107) we obtain

$$
\begin{aligned}
(f_{1k}(x,u),u) &= C_0^1 \sum_{i=1}^{d} \left|u^i\right|^{p_i} + (f_1(x,0),u) \\
&\geq C_0^1 \sum_{i=1}^{d} \left|u^i\right|^{p_i} - \frac{C_0^1}{2} \sum_{i=1}^{d} \left|u^i\right|^{p_i} \\
&- \sum_{i=1}^{d} \frac{1}{p_i'} \frac{1}{\left(C_0^1\right)^{1/(p_i-1)}} \left|f_1^i(x,0)\right|^{p_i'} \\
&\geq \frac{C_0^1}{2} \sum_{i=1}^{d} \left|u^i\right|^{p_i} - \left(\sum_{i=1}^{d} \frac{1}{p_i'} \frac{1}{\left(C_0^1\right)^{1/(p_i-1)}} \right) C_3(x),
\end{aligned}
$$

where $\frac{C_0^1}{2}$ is a positive constant and $\sum_{i=1}^{d} \frac{1}{p_i'} \frac{1}{\left(C_0^1\right)^{1/(p_i-1)}} C_3(x) \in L^1\left(\mathbb{R}^N\right)$ is a positive function.

3) If $\sqrt{\epsilon_k} \leq \sqrt{\sum_{i=1}^{d} |u^i|^{p_i}} \leq 2\sqrt{\epsilon_k}$ or $1/\epsilon_k \leq \sqrt{\sum_{i=1}^{d} |u^i|^{p_i}} \leq 1/\epsilon_k + 1$, we argue as the first case to obtain (4.119). From (4.112), using the Young inequality $ab \leq \frac{\varepsilon a^p}{p} + \frac{b^{p'}}{p'\varepsilon^{1/(p-1)}}$, we obtain

$$
\begin{aligned}
(f_{1k}(x,u),u) \geq \psi_k &\left(\sqrt{\sum_{i=1}^{d} |u^i|^{p_i}} \right) C_0^1 \sum_{i=1}^{d} \left|u^i\right|^{p_i} \\
- \psi_k &\left(\sqrt{\sum_{i=1}^{d} |u^i|^{p_i}} \right) \frac{C_0^1}{2} \sum_{i=1}^{d} \left|u^i\right|^{p_i} \\
- \psi_k &\left(\sqrt{\sum_{i=1}^{d} |u^i|^{p_i}} \right) \sum_{i=1}^{d} \frac{1}{p_i'} \frac{1}{\left(C_0^1\right)^{1/(p_i-1)}} \left|f_1^i(x,0)\right|^{p_i'} \\
+ &\left(1 - \psi_k \left(\sqrt{\sum_{i=1}^{d} |u^i|^{p_i}} \right) \right) \left(\beta_1 \sum_{i=1}^{d} \left|u^i\right|^{p_i} - C_{13}(x) \right).
\end{aligned}
$$

Using (4.107) we have

$$
\begin{aligned}
(f_{1k}(x,u),u) \geq\ & \psi_k\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)\frac{C_0^1}{2}\sum_{i=1}^{d}\left|u^i\right|^{p_i} \\
& -\psi_k\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)\left(\sum_{i=1}^{d}\frac{1}{p_i'}\frac{1}{\left(C_0^1\right)^{1/(p_i-1)}}\right)C_3(x) \\
& +\left(1-\psi_k\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)\right)\left(\beta_1\sum_{i=1}^{d}\left|u^i\right|^{p_i}-C_{13}(x)\right),
\end{aligned}
$$

so, we obtain

$$
\begin{aligned}
(f_{1k}(x,u),u) \geq\ & \psi_k\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)\frac{C_0^1}{2}\sum_{i=1}^{d}\left|u^i\right|^{p_i} \\
& +\left(1-\psi_k\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)\right)\beta_1\sum_{i=1}^{d}\left|u^i\right|^{p_i} \\
& -\left(C_{13}(x)+\sum_{i=1}^{d}\frac{1}{p_i'}\frac{1}{\left(C_0^1\right)^{1/(p_i-1)}}C_3(x)\right).
\end{aligned}
$$

Then, we can deduce

$$
(f_{1k}(x,u),u) \geq \widetilde{\beta}_1\sum_{i=1}^{d}\left|u^i\right|^{p_i}-\widetilde{C}_{13}(x),
$$

where

$$
\widetilde{\beta}_1 := \min\left\{\beta_1,\frac{C_0^1}{2}\right\} > 0, \tag{4.120}
$$

and $\widetilde{C}_{13}(x)$ is a positive function and is given by

$$
\widetilde{C}_{13}(x) := C_{13}(x)+\sum_{i=1}^{d}\frac{1}{p_i'}\frac{1}{\left(C_0^1\right)^{1/(p_i-1)}}C_3(x) \in L^1\left(\mathbb{R}^N\right), \tag{4.121}
$$

where $C_{13}(x)$ is given by (4.118).

Finally, we have

$$(f_{1k}(x,u),u) \geq \widehat{\beta}\sum_{i=1}^{d}\left|u^i\right|^{p_i} - \widetilde{C}_{13}(x),$$

where $\widehat{\beta} := \widetilde{\beta}_1$ is given by (4.120) and $\widehat{C}_1(x) := \widetilde{C}_{13}(x)$ is given by (4.121).

□

Lemma 4.28 *Assume (4.104)-(4.107). Then, f_{0k}, f_{1k} are continuously differentiable on u and there exist $D_{0\epsilon_k}$, $D_{1\epsilon_k}$ such that*

$$(f_{0ku}(x,u)w,w) \geq -D_{0\epsilon_k}\left|w\right|^2, \tag{4.122}$$

$$(f_{1ku}(x,u)w,w) \geq -D_{1\epsilon_k}\left|w\right|^2,\ \forall w,u \in \mathbb{R}^d,\ \textit{for a.a. } x \in \mathbb{R}^N, \tag{4.123}$$

where f_{0ku}, f_{1ku} denote the Jacobian matrixes of f_{0k} and f_{1k}, respectively.

Proof We note that the partial derivatives $\frac{\partial}{\partial u_i}\psi_k\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)$ and $\frac{\partial}{\partial u_i}\psi_k\left(|u|\right)$ are uniformly bounded on $\mathbb{R}^d$. Denote

$$v = \begin{pmatrix} \left|u^1\right|^{p_1-2}u^1 \\ \left|u^2\right|^{p_2-2}u^2 \\ \vdots \\ \left|u^d\right|^{p_d-2}u^d \end{pmatrix},$$

$$\psi_{ku}\left(|u|\right) = \left(\frac{\partial}{\partial u_1}\psi_k\left(|u|\right),...,\frac{\partial}{\partial u_d}\psi_k\left(|u|\right)\right),$$

$$\psi_{ku}\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right) = \left(\frac{\partial}{\partial u_1}\psi_k\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right),...,\frac{\partial}{\partial u_d}\psi_k\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)\right),$$

and

$$\rho_{\epsilon_k u}\left(u\right) = \left(\frac{\partial}{\partial u_1}\rho_{\epsilon_k}\left(u\right),...,\frac{\partial}{\partial u_d}\rho_{\epsilon_k}\left(u\right)\right).$$

Hence,

$$\begin{aligned} f_{0ku}(x,u) &= C_0^0\psi_k\left(|u|\right)I + C_0^0 u\psi_{ku}\left(|u|\right) \\ &+ f_0(x,0)\psi_{ku}\left(|u|\right) \\ &+ \left(1-\psi_k\left(|u|\right)\right)\int_{\mathbb{R}^d} f_0\left(x,s\right)\rho_{\epsilon_k u}\left(u-s\right)ds \\ &- \left(\int_{\mathbb{R}^d}\rho_{\epsilon_k}\left(s\right)f_0\left(x,u-s\right)ds\right)\psi_{ku}\left(|u|\right), \end{aligned} \tag{4.124}$$

$$
\begin{aligned}
f_{1ku}(x,u) &= C_0^1\psi_k\left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right) I_u + C_0^1 v\psi_{ku}\left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right)\\
&+ f_1(x,0)\psi_{ku}\left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right)\\
&+ \left(1-\psi_k\left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right)\right)\int_{\mathbb{R}^d} f_1(x,s)\,\rho_{\epsilon_k u}(u-s)\,ds\\
&- \left(\int_{\mathbb{R}^d}\rho_{\epsilon_k}(s)\,f_1(x,u-s)\,ds\right)\psi_{ku}\left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right), \qquad (4.125)
\end{aligned}
$$

where $u\psi_{ku}(|u|)$, $f_0(x,0)\psi_{ku}(|u|)$, $f_i(x,s)\,\rho_{\epsilon_k u}(u-s)$ with $i=0,1$,

$$\left(\int_{\mathbb{R}^d}\rho_{\epsilon_k}(s)\,f_0(x,u-s)\,ds\right)\psi_{ku}(|u|),$$

$$v\psi_{ku}\left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right),$$

$$f_1(x,0)\psi_{ku}\left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right),$$

and

$$\left(\int_{\mathbb{R}^d}\rho_{\epsilon_k}(s)\,f_1(x,u-s)\,ds\right)\psi_{ku}\left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right),$$

are $d\times d$ matrixes (a product of a column vector with a row vector), I is the identity matrix, I_u is a diagonal matrix such that $(I_u)_{ii} = (p_i-1)\,|u^i|^{p_i-2}$.

We consider each term in (4.124),

- As C_0^0 is a positive constant, for the first term, we have

$$(C_0^0\psi_k(|u|)\,Iw,w)\geq 0.$$

- For the second term, we get

$$\left|(C_0^0 u\psi_{ku}(|u|)\,w,w)\right| \leq C_0^0\left(\frac{1}{\epsilon_k}+1\right)C_{\psi_k}\,|w|^2,$$

and then

$$\left(C_0^0 u\psi_{ku}\left(|u|\right) w, w\right) \geq -C_0^0\left(\frac{1}{\epsilon_k}+1\right) C_{\psi_k}\left|w\right|^2.$$

- For the third term, using (4.106), we obtain

$$\begin{aligned}\left(f_0(x,0)\psi_{ku}\left(|u|\right) w, w\right) &\geq -\left|f_0(x,0)\right| C_{\psi_k}\left|w\right|^2\\ &\geq -C_2(x) C_{\psi_k}\left|w\right|^2\\ &\geq -\left\|C_2\right\|_\infty C_{\psi_k}\left|w\right|^2.\end{aligned}$$

- For the fourth term, we have to consider several cases.

  If $0 \leq |u| \leq \sqrt{\epsilon_k}$ or $|u| \geq 1/\epsilon_k + 1$, we obtain

$$\left(\left(1-\psi_k\left(|u|\right)\right)\left(\int_{\mathbb{R}^d} f_0\left(x,s\right)\rho_{\epsilon_k u}\left(u-s\right) ds\right) w, w\right) = 0.$$

  If $\sqrt{\epsilon_k} < |u| < 1/\epsilon_k + 1$, then using (4.106) we have

$$\begin{aligned}&\left|\left(\left(1-\psi_k\left(|u|\right)\right)\left(\int_{\mathbb{R}^d} f_0\left(x,s\right)\rho_{\epsilon_k u}\left(u-s\right) ds\right) w, w\right)\right|\\ &\leq \left|w\right|^2 \int_{\overline{B}(0,\epsilon_k)} \left|f_0\left(x,u-s\right)\right| \left|\rho_{\epsilon_k u}\left(s\right)\right| ds\\ &\leq \left|w\right|^2 \int_{\overline{B}(0,\epsilon_k)} \left(C_2(x)+\eta\left|u-s\right|\right) \left|\rho_{\epsilon_k u}\left(s\right)\right| ds\\ &\leq \left|w\right|^2 \left(\left\|C_2\right\|_\infty + 2\eta(1/\epsilon_k+1)\right) \int_{\mathbb{R}^d} \left|\rho_{\epsilon_k u}\left(s\right)\right| ds\\ &\leq \left|w\right|^2 D^1_{0\epsilon_k},\end{aligned}$$

  as $\rho_{\epsilon_k} \in \mathbb{C}_0^\infty\left(\mathbb{R}^d;\mathbb{R}\right)$ and then

$$\left(\left(1-\psi_k\left(|u|\right)\right)\left(\int_{\mathbb{R}^d} f_0\left(x,s\right)\rho_{\epsilon_k u}\left(u-s\right) ds\right) w, w\right) \geq -D^1_{0\epsilon_k}\left|w\right|^2.$$

- For the last term, if $\sqrt{\epsilon_k} < |u| < 1/\epsilon_k + 1$, using (4.106) we have

$$\begin{aligned}
&\left|\left(-\left(\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\, f_0(x, u-s)\, ds\right)\psi_{ku}(|u|)\, w, w\right)\right| \\
&\leq |w|^2 \left(C_2(x) + \eta \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\, |u-s|\, ds\right) |\psi_{ku}(|u|)| \\
&\leq |w|^2 \left(\|C_2\|_\infty + \eta \int_{\overline{B}(0,\epsilon_k)} \rho_{\epsilon_k}(s)\,(|u| + |s|)\, ds\right) |\psi_{ku}(|u|)| \qquad (4.126) \\
&\leq |w|^2 \left(\|C_2\|_\infty + 2\eta\, |u|\right) |\psi_{ku}(|u|)| \\
&\leq |w|^2 \left(\|C_2\|_\infty + 2\eta\,(1/\epsilon_k + 1)\right) C_{\psi_k} = D_{0\epsilon_k}^2 |w|^2,
\end{aligned}$$

  and then

$$\left(-\left(\int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)\, f_0(x, u-s)\, ds\right)\psi_{ku}(|u|)\, w, w\right) \geq -D_{0\epsilon_k}^2 |w|^2.$$

  In other case $\psi_{ku}(|u|) = 0$. Then (4.122) holds.

Now, we consider each term in (4.125),

- As C_0^1 is a positive constant, for the first term, we have

$$(C_0^1 \psi_k \left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right) I_u w, w) \geq 0.$$

- For the second term, we get

$$\left|(C_0^1 v \psi_{ku} \left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right) w, w)\right| \leq |w|^2 C_0^1 |v|\, C_{\psi_k}.$$

  We observe that

$$\left|u^i\right| \leq \left(\frac{1}{\epsilon_k} + 1\right)^{\frac{2}{p_i}},$$

  so

$$|v| = \sqrt{\sum_{i=1}^d \left(|u^i|^{p_i - 1}\right)^2} \leq \sqrt{\sum_{i=1}^d \left(\frac{1}{\epsilon_k} + 1\right)^{\frac{4}{p_i}(p_i - 1)}}.$$

  Then, we can deduce that

$$\left|(C_0^1 v \psi_{ku} \left(\sqrt{\sum_{i=1}^d |u^i|^{p_i}}\right) w, w)\right| \leq |w|^2 C_0^1 C_{\psi_k} \sqrt{\sum_{i=1}^d \left(\frac{1}{\epsilon_k} + 1\right)^{\frac{4}{p_i}(p_i - 1)}},$$

and then

$$(C_0^1 v\psi_{ku}\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)w,w)\geq -|w|^2\,C_0^1C_{\psi_k}\sqrt{\sum_{i=1}^{d}\left(\frac{1}{\epsilon_k}+1\right)^{\frac{4}{p_i}(p_i-1)}}.$$

- For the third term, we obtain

$$\left|\left(f_1(x,0)\psi_{ku}\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)w,w\right)\right|\leq |f_1\left(x,0\right)|\left|\psi_{ku}\right||w|^2.$$

Using (4.107), we have

$$|f_1\left(x,0\right)|=\sqrt{\sum_{i=1}^{d}|f_1^i\left(x,0\right)|^2}\leq\sqrt{\sum_{i=1}^{d}C_3^{2\frac{p_i-1}{p_i}}\left(x\right)}\leq\sqrt{\sum_{i=1}^{d}\|C_3\|_\infty^{2\frac{p_i-1}{p_i}}},$$

then we can deduce that

$$\left(f_1(x,0)\psi_{ku}\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)w,w\right)$$
$$\geq -C_{\psi_k}\sqrt{\sum_{i=1}^{d}\|C_3\|_\infty^{2\frac{p_i-1}{p_i}}}\,|w|^2.$$

- For the fourth term, we have to consider several cases.
If $0\leq\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\leq\sqrt{\epsilon_k}$ or $\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\geq 1/\epsilon_k+1$, we obtain

$$\left(\left(1-\psi_k\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)\right)\left(\int_{\mathbb{R}^d}f_1\left(x,s\right)\rho_{\epsilon_k u}\left(u-s\right)ds\right)w,w\right)=0.$$

If $\sqrt{\epsilon_k}<\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}<1/\epsilon_k+1$, then we have

$$\left|\left(\left(1-\psi_k\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)\right)\left(\int_{\mathbb{R}^d}f_1\left(x,s\right)\rho_{\epsilon_k u}\left(u-s\right)ds\right)w,w\right)\right|$$
$$\leq |w|^2\int_{\mathbb{R}^d}|f_1(x,u-s)|\,|\rho_{\epsilon_k u}(s)|\,ds.$$

Using (4.107), we obtain

$$\sum_{i=1}^{d}\left|f_1^i(x,u^i-s^i)\right|^2 \leq \sum_{i=1}^{d} 2^{\frac{p_i-2}{p_i}} C_3^{2^{\frac{p_i-1}{p_i}}}(x) + \sum_{i=1}^{d} 2^{\frac{p_i-2}{p_i}} \gamma^{2^{\frac{p_i-1}{p_i}}} \left(\sum_{i=1}^{d} 2^{p_i-1}\left(\left|u^i\right|^{p_i}+\left|s^i\right|^{p_i}\right)\right)^{2^{\frac{p_i-1}{p_i}}}. \tag{4.127}$$

We observe that thanks to $\sqrt{\epsilon_k} < \sqrt{\sum_{i=1}^{d}|u^i|^{p_i}} < 1/\epsilon_k + 1$, we can deduce that

$$\left|s^i\right| \leq \epsilon_k \leq \sum_{i=1}^{d}\left|u^i\right|^{p_i},$$

so

$$\left|s^i\right|^{p_i} \leq \epsilon_k^{p_i-1}\sum_{i=1}^{d}\left|u^i\right|^{p_i},$$

and then

$$\begin{aligned}\sum_{i=1}^{d}\left|s^i\right|^{p_i} &\leq \sum_{i=1}^{d}\epsilon_k^{p_i-1}\sum_{i=1}^{d}\left|u^i\right|^{p_i}\\ &\leq \frac{1}{2}\sum_{i=1}^{d}\left|u^i\right|^{p_i}.\end{aligned}$$

Using (4.127), we obtain

$$\begin{aligned}\sum_{i=1}^{d}\left|f_1^i(x,u^i-s^i)\right|^2 \leq& \sum_{i=1}^{d} 2^{\frac{p_i-2}{p_i}} \|C_3\|_\infty^{2^{\frac{p_i-1}{p_i}}}\\ &+\sum_{i=1}^{d} 2^{\frac{p_i-2}{p_i}} \gamma^{2^{\frac{p_i-1}{p_i}}}\left(\left(\sum_{i=1}^{d}2^{p_i-1}+\sum_{i=1}^{d}2^{p_i-2}\right)(1/\epsilon_k+1)^2\right)^{2^{\frac{p_i-1}{p_i}}}\\ &:= C_{\epsilon_k}.\end{aligned}$$

Then, we have

$$\begin{aligned}|f_1(x,u-s)| &= \sqrt{\sum_{i=1}^{d}\left|f_1^i(x,u^i-s^i)\right|^2}\\ &\leq \sqrt{C_{\epsilon_k}}.\end{aligned} \tag{4.128}$$

Then, we deduce that

$$\left|\left(\left(1-\psi_k\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)\right)\left(\int_{\mathbb{R}^d} f_1(x,s)\,\rho_{\epsilon_k u}(u-s)\,ds\right)w,w\right)\right|$$
$$\leq |w|^2\sqrt{C_{\epsilon_k}}\int_{\mathbb{R}^d}|\rho_{\epsilon_k u}(s)|\,ds \leq |w|^2 D^1_{1\epsilon_k},$$

as $\rho_{\epsilon_k} \in \mathbb{C}_0^\infty\left(\mathbb{R}^d;\mathbb{R}\right)$ and $|\rho_{\epsilon_k u}(s)| = \sqrt{\sum_{i=1}^{d}\left|\frac{\partial}{\partial u_i}\rho_{\epsilon_k}(s)\right|^2}$, and therefore

$$\left(\left(1-\psi_k\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)\right)\left(\int_{\mathbb{R}^d} f_1(x,s)\,\rho_{\epsilon_k u}(u-s)\,ds\right)w,w\right)\geq -D^1_{1\epsilon_k}|w|^2.$$

- For the last term, we have

$$\left|\left(-\left(\int_{\mathbb{R}^d}\rho_{\epsilon_k}(s)\,f_1(x,u-s)\,ds\right)\psi_{ku}\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)w,w\right)\right|$$
$$\leq |w|^2\left|C_{\psi_k}\right|\int_{\mathbb{R}^d}|\rho_{\epsilon_k}(s)|\,|f_1(x,u-s)|\,ds.$$

We argue as the previous case to obtain (4.128). Then, we deduce that

$$\left|\left(-\left(\int_{\mathbb{R}^d}\rho_{\epsilon_k}(s)\,f_1(x,u-s)\,ds\right)\psi_{ku}\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)w,w\right)\right|$$
$$\leq |w|^2\left|C_{\psi_k}\right|\sqrt{C_{\epsilon_k}} = |w|^2 D^2_{1\epsilon_k},$$

then

$$\left(-\left(\int_{\mathbb{R}^d}\rho_{\epsilon_k}(s)\,f_1(x,u-s)\,ds\right)\psi_{ku}\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right)w,w\right)\geq -D^2_{1\epsilon_k}|w|^2.$$

In other case $\psi_{ku}\left(\sqrt{\sum_{i=1}^{d}|u^i|^{p_i}}\right) = 0.$

Then (4.84) holds.

□

Remark 4.29 *We note that the class of multi-valued non-autonomous dynamical system (MNDS) for which the evolution depends only on the elapsed time, that is, a MNDS for which* $U(t,\tau,\cdot) = U(t-\tau,0,\cdot)$ *for* $t \geq \tau$ *is called a multi-valued semiflow and the family of operators* $\{G(t,\cdot) : t \geq 0\}$ *given by* $G(t,\cdot) = U(t,0,\cdot)$, $t \geq 0$ *satisfies* $G(0,\cdot) = Id$ *and* $G(t+s,\cdot) = G(t, G(s,\cdot))$ *for all* $t, s \geq 0$.

For each $u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d$ we denote by $S(u_0)$ the set of all weak solutions of (4.102)-(4.103) defined for all $t \geq 0$. Such a set is non-empty as in [63] it is proved that at least one weak global solution exists for any $u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d$.

We define a multi-valued map $G : \mathbb{R}^+ \times \left[L^2\left(\mathbb{R}^N\right)\right]^d \to \mathcal{P}(\left[L^2\left(\mathbb{R}^N\right)\right]^d)$ by

$$G(t,u_0) = \{u(t) : \ u \in S(u_0)\}, \quad t \geq 0, \quad u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d. \tag{4.129}$$

In [63] it is shown that the multi-valued mapping G defined by (4.129) is a strict multivalued semiflow on $L^2(\Omega)$, that is, $G(0,\cdot) = Id$ and $G(t+s,u_0) = G(t, G(s,u_0))$ for all $t, s \geq 0, u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d$.

In [63] it is also proved that the set $G(t,u_0)$ is compact.

Our aim is to prove the connectedness of the set $G(t,u_0) \subset \left[L^2\left(\mathbb{R}^N\right)\right]^d$ for any $t \geq 0$, $u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d$. Then we obtain the Kneser property.

Using lemmas 4.27, 4.28 and the same proof of Theorem 7 in [64] we have the following result.

Theorem 4.30 *Assume (4.104)-(4.107). Then* $G(t,u_0)$ *is connected for any* $t \geq 0$, $u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d$.

Remark 4.31 *The same result is true, with slight changes in the proofs, for the following system*

$$u_t = a\Delta u - f(x,u) + h(x), \ x \in \mathbb{R}^N, t > 0,$$
$$u(0) = u_0 \in \left[L^2\left(\mathbb{R}^N\right)\right]^d,$$

where $h \in \left[L^2\left(\mathbb{R}^N\right)\right]^d$ *and* $(H1) - (H4)$ *hold.*

Finally, we observe that in [63] it is proved the existence of a global compact invariant attractor $\mathcal{A}$ for the multivalued semiflow G. Using Theorem 4.30 and arguing as in [64, Section 4] we obtain the following.

Theorem 4.32 *The global attractor* $\mathcal{A}$ *of* G *is connected.*

4.7 A generalized logistic equation

In this section we consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + c(x)\,|u|^r - u^{p-1} + h(t)\,, \\ \quad u = 0 \ \text{ on } \ \partial\Omega \times (\tau, +\infty), \\ \quad u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{4.130}$$

where $\Omega \subset \mathbb{R}^N$ satisfies the Poincaré inequality, p is and even natural number, $0 < r < p-1$, $c(x) \in L^{\frac{p}{p-r-1}}(\Omega) \cap L^\infty(\Omega)\,, c(x) \geq 0$, and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ satisfies (4.22). This kind of nonlinearities for the logistic equation (instead of the classical $(1-u)\,u$) has been considered in [65, Chapter 11].

We note that

$$f(x,u) = c(x)\,|u|^r - u^{p-1}$$

and

$$f(x,u)\,u = c(x)\,|u|^r\,u - u^p \leq -\frac{1}{2}\,|u|^p + K_1 c(x)^{\frac{p}{p-r-1}}\,, \tag{4.131}$$

$$|f(x,u)|^{\frac{p}{p-1}} \leq K_2\left(c(x)^{\frac{p}{p-1}}\,|u|^{\frac{pr}{p-1}} + |u|^p\right) \leq K_3\left(c(x)^{\frac{p}{p-r-1}} + |u|^p\right), \tag{4.132}$$

so that conditions (4.3)-(4.4) hold.

In view of Theorems 4.22, 4.25 we obtain the following result.

Theorem 4.33 *Problem (4.130) generates a MNDS U such that:*

1. *$U(t,\tau,u_\tau)$ is connected in $L^2(\Omega)$ for any $t \geq \tau$ and $u_\tau \in L^2(\Omega)$.*
2. *The MNDS U has a unique pullback $\mathcal{D}_{\lambda_1}$-attractor $\mathcal{A}_{\mathcal{D}_{\lambda_1}}$ belonging to $\mathcal{D}_{\lambda_1}$, which is strictly invariant and connected, where λ_1 is the constant in (4.1).*

When $\Omega = \mathbb{R}^N$ (so that the Poincaré inequality is not satisfied) and $h \equiv 0$ we can obtain the following result.

Theorem 4.34 *If $p = 2$ and $\Omega = \mathbb{R}^N$, then problem (4.130) generates a multivalued semiflow G such that:*

1. *$G(t,u_0)$ is connected in $L^2(\Omega)$ for any $t \geq 0$ and $u_0 \in L^2(\Omega)$.*
2. *G possesses a global compact invariant connected attractor $\mathcal{A}$.*

Proof We take $f_0(x,u) = f_1(x,u) = \frac{1}{2}(-c(x)|u|^r + u)$, $d = 1$, $p_1 = 2$. Then, in view of (4.131)-(4.132) conditions (4.104)-(4.107) hold. The results follow from Theorems 4.30 and 4.32.

Remark 4.35 *The results given in [64] are not applicable to problem (4.130), as the condition on the derivative used in that paper is not satisfied. On the other hand, the results of the previous section can be applied also to the complex Ginzburg-Landau equation (as done in [64]). More precisely, it follows from Theorems 4.30, 4.32 and Remark 4.31 that Theorem 14 in [64] is true.*

Chapter 5
Pullback attractors for non-autonomous reaction-diffusion equations with dynamical boundary conditions

In this chapter we study the existence of pullback attractors for the process associated to a non-autonomous reaction-diffusion model with dynamical boundary conditions. As we mentioned in Introduction, we only have references in the literature of this approach in the stochastic context, with the help of random dynamical systems.

The structure of the chapter is as follows. In Section 5.2 we give a weak formulation of the problem, the concept of weak solution, and establish the existence and uniqueness of solution using the monotonicity method. A continuous dependence result with respect to initial data, which is the main key for the asymptotic compactness we will require later, is addressed in Section 5.3. There we use an energy method that strengthens the energy equality satisfied by the solutions. In Section 5.4, the main goal of proving the existence of different families of pullback attractors for different universes, and the relation among them under certain suitable assumption, are finally established. These results can be found in [8].

5.1 Setting of the problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$.

We consider the following problem for a non-autonomous reaction-diffusion equation with dynamical boundary condition,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \kappa u + f(u) = h(t) & \text{in } \Omega \times (\tau, \infty), \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial \vec{n}} + g(u) = \rho(t) & \text{on } \partial\Omega \times (\tau, \infty), \\ u(x,\tau) = u_\tau(x), & \text{for } x \in \Omega, \\ u(x,\tau) = \psi_\tau(x), & \text{for } x \in \partial\Omega, \end{cases} \tag{5.1}$$

where $\vec{n}$ is the outer normal to $\partial\Omega$, $\tau \in \mathbb{R}$ is an initial time, and

$$\kappa > 0, \quad u_\tau \in L^2(\Omega), \quad \psi_\tau \in L^2(\partial\Omega), \tag{5.2}$$

$$h \in L^2_{loc}\left(\mathbb{R}; L^2(\Omega)\right), \quad \rho \in L^2_{loc}\left(\mathbb{R}; L^2(\partial\Omega)\right), \tag{5.3}$$

are given.

We also assume that the functions f and $g \in C(\mathbb{R})$ are given, and satisfy that there exist constants $p \geq 2$, $q \geq 2$, $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta > 0$, and $l > 0$, such that

$$\alpha_1 |s|^p - \beta \leq f(s)s \leq \alpha_2 |s|^p + \beta, \quad \text{for all } s \in \mathbb{R}, \tag{5.4}$$

$$\alpha_1 |s|^q - \beta \leq g(s)s \leq \alpha_2 |s|^q + \beta, \quad \text{for all } s \in \mathbb{R}, \tag{5.5}$$

$$\left(f(s) - f(r)\right)(s - r) \geq -l\,(s-r)^2, \quad \text{for all } s, r \in \mathbb{R}, \tag{5.6}$$

and

$$\left(g(s) - g(r)\right)(s - r) \geq -l\,(s-r)^2, \quad \text{for all } s, r \in \mathbb{R}. \tag{5.7}$$

It is easy to see from (5.4) and (5.5) that there exists a constant $C > 0$ such that

$$|f(s)| \leq C\left(1 + |s|^{p-1}\right), \quad |g(s)| \leq C\left(1 + |s|^{q-1}\right), \quad \text{for all } s \in \mathbb{R}. \tag{5.8}$$

Remark 5.1 *If u is regular enough, then a compatibility condition for problem (5.1) is that ψ_τ must coincide with the restriction to $\partial\Omega$ of u_τ, and therefore the fourth equation in (5.1) is omitted. Nevertheless, this equation seems necessary for the concept of weak solution (see below).*

Remark 5.2 *If $p > 2$, the assumption $\kappa > 0$ is not necessary. Indeed, if $\kappa \leq 0$, then $f(u) + \kappa u = \bar{f}(u) + u$, where $\bar{f}(s) := f(s) + (\kappa - 1)s$, satisfies*

$$\left(\bar{f}(s) - \bar{f}(r)\right)(s - r) \geq -(l - \kappa + 1)(s - r)^2, \quad \text{for all } s, r \in \mathbb{R},$$

and taking into account Young's inequality, if $p > 2$,

$$\frac{\alpha_1}{2}|s|^p - \beta - \frac{p-2}{p}\left(\frac{4}{p\alpha_1}\right)^{2/(p-2)}(1-\kappa)^{p/(p-2)} \leq s\bar{f}(s) \leq \alpha_2|s|^p + \beta,$$

for all $s \in \mathbb{R}$.

5.2 Existence and Uniqueness of Solution

In this section we prove the existence and uniqueness of solution for our model. We denote by $(\cdot,\cdot)_\Omega$ (respectively, $(\cdot,\cdot)_{\partial\Omega}$) the inner product in $L^2(\Omega)$ (respectively, in $L^2(\partial\Omega)$), and by $|\cdot|_\Omega$ (respectively, $|\cdot|_{\partial\Omega}$) the associated norm. We will also denote $(\cdot,\cdot)_\Omega$ (respectively, $(\cdot,\cdot)_{\partial\Omega}$) the inner product in $(L^2(\Omega))^N$, and the duality product between $L^{p'}(\Omega)$ and $L^p(\Omega)$ (respectively, the duality product between $L^{q'}(\partial\Omega)$ and $L^q(\partial\Omega)$). If $r \neq 2$, we will denote $|\cdot|_{r,\Omega}$ (respectively $|\cdot|_{r,\partial\Omega}$) the norm in $L^r(\Omega)$ (respectively in $L^r(\partial\Omega)$). By $\|\cdot\|_\Omega$ we denote the norm in $H^1(\Omega)$, which is associated to the inner product $((\cdot,\cdot))_\Omega := (\nabla\cdot, \nabla\cdot)_\Omega + (\cdot,\cdot)_\Omega$.

We use the notation γ_0 for the trace operator $u \mapsto u|_{\partial\Omega}$. The trace operator belongs to $\mathcal{L}(H^1(\Omega), H^{1/2}(\partial\Omega))$, and we will use $\|\gamma_0\|$ to denote the norm of γ_0 in this space.

Finally, we will use $\|\cdot\|_{\partial\Omega}$ to denote the norm in $H^{1/2}(\partial\Omega)$, which is given by $\|\phi\|_{\partial\Omega} = \inf\{\|v\|_\Omega : \gamma_0(v) = \phi\}$. We remember that with this norm, $H^{1/2}(\partial\Omega)$ is a Hilbert space.

Definition 5.3 *A weak solution of (5.1) is a pair of functions (u, ψ), satisfying*

$$u \in C([\tau,\infty); L^2(\Omega)), \quad \psi \in C([\tau,\infty); L^2(\partial\Omega)), \tag{5.9}$$

$$u \in L^2(\tau,T; H^1(\Omega)) \cap L^p(\tau,T; L^p(\Omega)), \quad \text{for all } T > \tau, \tag{5.10}$$

$$\psi \in L^2(\tau,T; H^{1/2}(\partial\Omega)) \cap L^q(\tau,T; L^q(\partial\Omega)), \quad \text{for all } T > \tau, \tag{5.11}$$

$$\gamma_0(u(t)) = \psi(t), \quad a.e.\ t \in (\tau,\infty), \tag{5.12}$$

$$\begin{cases} \frac{d}{dt}(u(t),v)_\Omega + \frac{d}{dt}(\psi(t),\gamma_0(v))_{\partial\Omega} + (\nabla u(t),\nabla v)_\Omega + \kappa(u(t),v)_\Omega \\ +(f(u(t)),v)_\Omega + (g(\gamma_0(u(t))),\gamma_0(v))_{\partial\Omega} = (h(t),v)_\Omega + (\rho(t),\gamma_0(v))_{\partial\Omega} \\ \text{in } \mathcal{D}'(\tau,\infty), \text{ for all } v \in H^1(\Omega)\cap L^p(\Omega) \text{ such that } \gamma_0(v)\in L^q(\partial\Omega), \end{cases} \tag{5.13}$$

$$u(\tau) = u_\tau, \quad \text{and} \quad \psi(\tau) = \psi_\tau. \tag{5.14}$$

Remark 5.4 *If a pair of functions (u,ψ) satisfies (5.10)–(5.13), then there exists a version of these functions satisfying (5.9). The function ψ is the $L^2(\partial\Omega)$-continuous version of $\gamma_0(u)$ (see* (5.17)–(5.19) *below).*

We have the following result.

Theorem 5.5 *Under the assumptions (5.2)–(5.7), there exists a unique solution $(u,\psi) = (u(\cdot;\tau,u_\tau,\psi_\tau),\psi(\cdot;\tau,u_\tau,\psi_\tau))$ of the problem (5.1). Moreover, this solution satisfies the energy equality*

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left(|u(t)|_\Omega^2 + |\psi(t)|_{\partial\Omega}^2\right) + |\nabla u(t)|_\Omega^2 + \kappa|u(t)|_\Omega^2 \\ & +(f(u(t)),u(t))_\Omega + (g(\psi(t)),\psi(t))_{\partial\Omega} \\ = \ & (h(t),u(t))_\Omega + (\rho(t),\psi(t))_{\partial\Omega}, \quad a.e.\ t>\tau. \end{aligned} \tag{5.15}$$

Proof The proof of this result is standard (see for example [33]). For the sake of completeness, we give a sketch of a proof.

Let us consider the Hilbert space

$$H := L^2(\Omega)\times L^2(\partial\Omega),$$

with the natural inner product $((v,\phi),(w,\varphi))_H = (v,w)_\Omega + (\phi,\varphi)_{\partial\Omega}$, which in particular induces the norm $|(\cdot,\cdot)|_H$ given by

$$|(v,\phi)|_H^2 = |v|_\Omega^2 + |\phi|_{\partial\Omega}^2, \quad (v,\phi)\in H.$$

Let us also consider the space

$$V_1 := \left\{(v,\gamma_0(v)) : v\in H^1(\Omega)\right\}.$$

We note that V_1 is a closed vector subspace of $H^1(\Omega)\times H^{1/2}(\partial\Omega)$, and therefore, with the norm $\|(\cdot,\cdot)\|_{V_1}$ given by

$$\|(v,\gamma_0(v))\|_{V_1}^2 = \|v\|_\Omega^2 + \|\gamma_0(v)\|_{\partial\Omega}^2, \quad (v,\gamma_0(v))\in V_1,$$

V_1 is a Hilbert space.

On the other hand, V_1 is densely embedded in H. In fact, if we consider $(w, \phi) \in H$ such that

$$(v, w)_\Omega + (\gamma_0(v), \phi)_{\partial\Omega} = 0, \quad \text{for all } v \in H^1(\Omega),$$

in particular, we have

$$(v, w)_\Omega = 0, \quad \text{for all } v \in H_0^1(\Omega),$$

and therefore $w = 0$. Consequently,

$$(\gamma_0(v), \phi)_{\partial\Omega} = 0, \quad \text{for all } v \in H^1(\Omega),$$

and then, as $H^{1/2}(\partial\Omega) = \gamma_0(H^1(\Omega))$ is dense in $L^2(\partial\Omega)$, we have that $\phi = 0$.

Now, on the space V_1 we define a continuous symmetric linear operator $A_1 : V_1 \to V_1'$, given by

$$\langle A_1((v, \gamma_0(v))), (w, \gamma_0(w))\rangle = (\nabla v, \nabla w)_\Omega + \kappa(v, w)_\Omega, \ \ \forall v, w \in H^1(\Omega).$$

We observe that A_1 is coercive. In fact, we have

$$\begin{aligned} \langle A_1((v, \gamma_0(v)), (v, \gamma_0(v)))\rangle &\geq \min\{1, \kappa\} \|v\|_\Omega^2 \\ &= \frac{1}{1 + \|\gamma_0\|^2} \min\{1, \kappa\} \|v\|_\Omega^2 \\ &+ \frac{\|\gamma_0\|^2}{1 + \|\gamma_0\|^2} \min\{1, \kappa\} \|v\|_\Omega^2 \\ &\geq \frac{1}{1 + \|\gamma_0\|^2} \min\{1, \kappa\} \|(v, \gamma_0(v))\|_{V_1}^2, \end{aligned} \tag{5.16}$$

for all $v \in H^1(\Omega)$. Let us denote

$$V_2 = L^p(\Omega) \times L^2(\partial\Omega), \quad V_3 = L^2(\Omega) \times L^q(\partial\Omega),$$

$$A_2(v, \phi) = (f(v), 0), \quad A_3(v, \phi) = (0, g(\phi)), \quad \vec{h}(t) = (h(t), \rho(t)).$$

From (5.8) one deduces that $A_i : V_i \to V_i'$, for $i = 2, 3$.

Observe also that by (5.3),

$$\vec{h} \in L^2_{loc}(\mathbb{R}; H) \subset L^2_{loc}(\mathbb{R}; V_1').$$

With this notation, and denoting $V = \cap_{i=1}^3 V_i$, $p_1 = 2$, $p_2 = p$, $p_3 = q$, $\vec{u} = (u, \psi)$, one has that (5.9)–(5.14) is equivalent to

$$\vec{u} \in C([\tau, \infty); H), \quad \vec{u} \in \bigcap_{i=1}^{3} L^{p_i}(\tau, T; V_i), \quad \text{for all } T > \tau, \tag{5.17}$$

$$(\vec{u})'(t) + \sum_{i=1}^{3} A_i(\vec{u}(t)) = \vec{h}(t) \quad \text{in } \mathcal{D}'(\tau, \infty; V'), \tag{5.18}$$

$$\vec{u}(\tau) = (u_\tau, \psi_\tau). \tag{5.19}$$

Applying a slight modification of [56, Ch.2,Th.1.4], it is not difficult to see that problem (5.17)–(5.19) has a unique solution. Moreover, $\vec{u}$ satisfies the energy equality

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}(t)\|_H^2 + \sum_{i=1}^{3}\langle A_i(\vec{u}(t)), \vec{u}(t)\rangle_i = (\vec{h}(t), \vec{u}(t))_H \quad \text{a.e. } t > \tau,$$

where $\langle \cdot, \cdot \rangle_i$ denotes the duality product between V_i' and V_i.

This last equality turns out to be just (5.15).

□

Remark 5.6 *The assumption $\kappa > 0$ is not necessary for the existence and uniqueness of weak solution to (5.1).*

5.3 A continuous dependence result

In this section, we prove a result on continuous dependence of the solutions of (5.1) with respect to the initial datum (u_τ, ϕ_τ). This result will be crucial in the proof of the existence of pullback attractors for (5.1).

Theorem 5.7 *Under the assumptions (5.2)–(5.7), let $\{(u_\tau^{(n)}, \psi_\tau^{(n)})\}_{n\geq 1} \subset L^2(\Omega) \times L^2(\partial\Omega)$ be a sequence such that*

$$(u_\tau^{(n)}, \psi_\tau^{(n)}) \rightharpoonup (u_\tau, \psi_\tau) \text{ weakly in } L^2(\Omega) \times L^2(\partial\Omega). \tag{5.20}$$

Let us denote $\vec{u}^{(n)} = (u^{(n)}, \psi^{(n)}) = (u(\cdot; \tau, u_\tau^{(n)}, \psi_\tau^{(n)}), \psi(\cdot; \tau, u_\tau^{(n)}, \psi_\tau^{(n)}))$ and $\vec{u} = (u, \psi) = (u(\cdot; \tau, u_\tau, \psi_\tau), \psi(\cdot; \tau, u_\tau, \psi_\tau))$, the corresponding weak solutions of (5.1). Then, for all $T > \tau$,

$$\begin{aligned}
\vec{u}^{(n)} &\rightharpoonup \vec{u} && \text{weakly in } L^2(\tau, T; H^1(\Omega)) \times L^2(\tau, T; H^{1/2}(\partial\Omega)),\\
\vec{u}^{(n)} &\overset{*}{\rightharpoonup} \vec{u} && \text{weakly-star in } L^\infty(\tau, T; L^2(\Omega)) \times L^\infty(\tau, T; L^2(\partial\Omega)),\\
\vec{u}^{(n)} &\rightharpoonup \vec{u} && \text{weakly in } L^p(\tau, T; L^p(\Omega)) \times L^q(\tau, T; L^q(\partial\Omega)),\\
f(u^{(n)}) &\rightharpoonup f(u) && \text{weakly in } L^{p'}(\tau, T; L^{p'}(\Omega)),\\
g(\psi^{(n)}) &\rightharpoonup g(\psi) && \text{weakly in } L^{q'}(\tau, T; L^{q'}(\partial\Omega)),\\
\vec{u}^{(n)} &\to \vec{u} && \text{strongly in } L^2(\tau, T; L^2(\Omega)) \times L^2(\tau, T; L^2(\partial\Omega)), \qquad (5.21)\\
\vec{u}^{(n)}(t) &\to \vec{u}(t) && \text{strongly in } L^2(\Omega) \times L^2(\partial\Omega), \text{ for all } t > \tau. \qquad (5.22)
\end{aligned}$$

Proof For the sake of clarity, we split the proof in two parts. Firstly, for all but last of the above convergences we only require to obtain suitable a priori estimates and well-known compactness results; secondly, for the last convergence, we use an energy method that strengthens the energy equality satisfied by the solutions.

Step 1: All but last of the convergences in the above statement hold.

By (5.15) applied to $\vec{u}^{(n)}$, and taking into account (5.4), (5.5) and (5.16), we have

$$\begin{aligned} &\frac{d}{dt}\left(|u^{(n)}(t)|_\Omega^2 + |\psi^{(n)}(t)|_{\partial\Omega}^2\right) \\ &+\frac{2\min\{1,\kappa\}}{1+\|\gamma_0\|^2}(\|u^{(n)}(t)\|_\Omega^2 + \|\psi^{(n)}(t)\|_{\partial\Omega}^2) + 2\alpha_1(|u^{(n)}(t)|_{p,\Omega}^p + |\psi^{(n)}(t)|_{q,\partial\Omega}^q) \\ \leq\ & 2\beta(|\Omega| + |\partial\Omega|) + |h(t)|_\Omega^2 + |\rho(t)|_{\partial\Omega}^2 + |u^{(n)}(t)|_\Omega^2 + |\psi^{(n)}(t)|_{\partial\Omega}^2, \end{aligned} \tag{5.23}$$

a.e. $t > \tau$.

By (5.20) in particular we know that there exists a constant $C > 0$ such that

$$|u_\tau^{(n)}|_\Omega^2 + |\psi_\tau^{(n)}|_{\partial\Omega}^2 \leq C \quad \text{for all } n \geq 1.$$

Thus, integrating (5.23) between τ and t, and applying Gronwall lemma, we see that the sequence $\{u^{(n)}\}$ is bounded in $L^2(\tau,T;H^1(\Omega))\cap C([\tau,T];L^2(\Omega))\cap L^p(\tau,T;L^p(\Omega))$, and the sequence $\{\psi^{(n)}\}$ is bounded in $L^2(\tau,T;H^{1/2}(\partial\Omega))\cap C([\tau,T];L^2(\partial\Omega))\cap L^q(\tau,T;L^q(\partial\Omega))$, for all $T > \tau$.

Then, taking into account (5.8) and (5.13) for $(u^{(n)},\psi^{(n)})$, we deduce that the sequence $\{f(u^{(n)})\}$ is bounded in $L^{p'}(\tau,T;L^{p'}(\Omega))$ and the sequence $\{g(\psi^{(n)})\}$ is bounded in $L^{q'}(\tau,T;L^{q'}(\partial\Omega))$. Moreover, the sequence of time derivatives $\{(u^{(n)})'\}$ is bounded in $L^2(\tau,T;(H^1(\Omega))') + L^{p'}(\tau,T;L^{p'}(\Omega)) \subset L^{p'}(\tau,T;(H^1(\Omega)\cap L^p(\Omega))')$, and finally, the sequence of time derivatives $\{(\psi^{(n)})'\}$ is bounded in the space $L^2(\tau,T;(H^{1/2}(\partial\Omega))')+L^{q'}(\tau,T;L^{q'}(\partial\Omega)) \subset L^{q'}(\tau,T;(H^{1/2}(\partial\Omega)\cap L^q(\partial\Omega))')$, for all $T > \tau$.

Let us fix $T > \tau$. Taking into account the compactness of the injection of $H^1(\Omega)$ into $L^2(\Omega)$, and the compactness of the injection of $H^{1/2}(\partial\Omega)$ into $L^2(\partial\Omega)$, from the boundedness results above and the Aubin-Lions compactness lemma (e.g. cf. [56]), we deduce that there exist a subsequence $\{(u^{(n')},\psi^{(n')})\}_{n'\geq 1} \subset \{(u^{(n)},\psi^{(n)})\}_{n\geq 1}$ and functions $\hat{u} \in L^2(\tau,T;H^1(\Omega))\cap L^\infty(\tau,T;L^2(\Omega))\cap L^p(\tau,T;L^p(\Omega))$, $\hat{\psi} \in L^2(\tau,T;H^{1/2}(\partial\Omega))\cap L^\infty(\tau,T;L^2(\partial\Omega)) \cap L^q(\tau,T;L^q(\partial\Omega))$, $\hat{f} \in L^{p'}(\tau,T;L^{p'}(\Omega))$, $\hat{g} \in L^{q'}(\tau,T;L^{q'}(\partial\Omega))$, $\xi_T \in L^2(\Omega)$, and $\eta_T \in L^2(\partial\Omega)$, such that

$$\begin{aligned}
\vec{u}^{(n')} &\rightharpoonup (\hat{u},\hat{\psi}) && \text{weakly in } L^2(\tau,T;H^1(\Omega))\times L^2(\tau,T;H^{1/2}(\partial\Omega)),\\
\vec{u}^{(n')} &\overset{*}{\rightharpoonup} (\hat{u},\hat{\psi}) && \text{weakly-star in } L^\infty(\tau,T;L^2(\Omega))\times L^\infty(\tau,T;L^2(\partial\Omega)),\\
\vec{u}^{(n')} &\rightharpoonup (\hat{u},\hat{\psi}) && \text{weakly in } L^p(\tau,T;L^p(\Omega))\times L^q(\tau,T;L^q(\partial\Omega)),\\
f(u^{(n')}) &\rightharpoonup \hat{f} && \text{weakly in } L^{p'}(\tau,T;L^{p'}(\Omega)), && (5.24)\\
g(\psi^{(n')}) &\rightharpoonup \hat{g} && \text{weakly in } L^{q'}(\tau,T;L^{q'}(\partial\Omega)), && (5.25)\\
\vec{u}^{(n')} &\to (\hat{u},\hat{\psi}) && \text{strongly in } L^2(\tau,T;L^2(\Omega))\times L^2(\tau,T;L^2(\partial\Omega)),\\
u^{(n')} &\to \hat{u} && \text{a.e. in } \Omega\times(\tau,T), && (5.26)\\
\vec{u}^{(n')}(T) &\rightharpoonup (\xi_T,\eta_T) && \text{weakly in } L^2(\Omega)\times L^2(\partial\Omega).
\end{aligned}$$

By the continuity of f and g, from (5.24), (5.25) and (5.26), one deduces (see [56, Ch.1,Lem.1.3]) that $\hat{f} = f(\hat{u})$ and $\hat{g} = g(\hat{\psi})$. Now, it is a standard matter to deduce from (5.20) and the above convergences, that

$$\gamma_0(\hat{u}(t)) = \hat{\psi}(t), \qquad \text{a.e. } t \in (\tau,T), \tag{5.27}$$

$$\begin{cases} \dfrac{d}{dt}(\hat{u}(t),v)_\Omega + \dfrac{d}{dt}(\hat{\psi}(t),\gamma_0(v))_{\partial\Omega} + (\nabla\hat{u}(t),\nabla v)_\Omega + \kappa(\hat{u}(t),v)_\Omega \\ +(f(\hat{u}(t)),v)_\Omega+(g(\gamma_0(\hat{u}(t))),\gamma_0(v))_{\partial\Omega}=(h(t),v)_\Omega+(\rho(t),\gamma_0(v))_{\partial\Omega} \\ \text{in } \mathcal{D}'(\tau,T), \text{ for all } v\in H^1(\Omega)\cap L^p(\Omega), \text{ such that } \gamma_0(v)\in L^q(\partial\Omega), \end{cases} \tag{5.28}$$

$$\hat{u}(\tau) = u_\tau, \qquad \hat{\psi}(\tau) = \psi_\tau, \tag{5.29}$$

and

$$(\hat{u}(T),\hat{\psi}(T)) = (\xi_T,\eta_T). \tag{5.30}$$

Consequently, by uniqueness of solution to (5.27)–(5.29), we deduce that $(\hat{u},\hat{\psi})$ coincides with the restriction to $[\tau,T]$ of $\vec{u}=(u,\psi)$, the above convergences hold for the whole sequence $\{(u^{(n)},\psi^{(n)})\}_{n\geq 1}$, and therefore, by the arbitrariness of $T>\tau$, all but last convergences in the statement are satisfied, as we wanted to prove.

Step 2: We prove now that (5.22) holds.

From above, and by (5.30), we also deduce that

$$(u^{(n)}(t),\psi^{(n)}(t)) \rightharpoonup (u(t),\psi(t)) \text{ weakly in } L^2(\Omega)\times L^2(\partial\Omega), \tag{5.31}$$

for all $t>\tau$.

Now, we will prove that

$$|u^{(n)}(t)|_\Omega^2 + |\psi^{(n)}(t)|_{\partial\Omega}^2 \to |u(t)|_\Omega^2 + |\psi(t)|_{\partial\Omega}^2, \quad \text{for all } t > \tau, \tag{5.32}$$

which jointly with (5.31) will imply (5.22).

In order to prove (5.32), observe that from (5.21) we deduce in particular that for any subsequence $\{(u^{(n')}, \psi^{(n')})\}_{n' \geq 1} \subset \{(u^{(n)}, \psi^{(n)})\}_{n \geq 1}$ there exists another subsequence $\{(u^{(n'')}, \psi^{(n'')})\}_{n'' \geq 1} \subset \{(u^{(n')}, \psi^{(n')})\}_{n' \geq 1}$ such that

$$|u^{(n'')}(t)|_\Omega^2 + |\psi^{(n'')}(t)|_{\partial\Omega}^2 \to |u(t)|_\Omega^2 + |\psi(t)|_{\partial\Omega}^2, \quad \text{a.e. } t > \tau. \tag{5.33}$$

Let us define

$$\begin{aligned} J(t) \;&:=\; \frac{1}{2}(|u(t)|_\Omega^2 + |\psi(t)|_{\partial\Omega}^2) - \beta(|\Omega| + |\partial\Omega|)t \\ &\quad - \int_\tau^t [(h(s), u(s))_\Omega + (\rho(s), \psi(s))_{\partial\Omega}]\, ds, \end{aligned}$$

and

$$\begin{aligned} J_{n''}(t) \;&:=\; \frac{1}{2}(|u^{(n'')}(t)|_\Omega^2 + |\psi^{(n'')}(t)|_{\partial\Omega}^2) - \beta(|\Omega| + |\partial\Omega|)t \\ &\quad - \int_\tau^t [(h(s), u^{(n'')}(s))_\Omega + (\rho(s), \psi^{(n'')}(s))_{\partial\Omega}]\, ds, \end{aligned}$$

for all $t \geq \tau$.

It is clear that J and $J_{n''}$ are well defined continuous functions on $[\tau, \infty)$, and by (5.21), if we prove that

$$J_{n''}(t) \to J(t) \quad \text{for all } t > \tau, \tag{5.34}$$

then (5.32) will hold.

From (5.21) and (5.33), we have that

$$J_{n''}(t) \to J(t) \quad \text{a.e. } t \in (\tau, \infty). \tag{5.35}$$

On the other hand, from the energy equality, (5.4), and (5.5), we obtain that J and $J_{n''}$ are non-increasing functions of t.

Let us fix $t \in (\tau, \infty)$, and $\varepsilon > 0$. From (5.35) and the continuity of J, we can take $t_2 < t < t_1$ such that

$$J_{n''}(t_i) \to J(t_i), \quad \text{as } n'' \to \infty,\, i = 1, 2, \tag{5.36}$$

and

$$J(t_2) - J(t_1) = |J(t_2) - J(t)| + |J(t) - J(t_1)| \leq \varepsilon.$$

From this inequality and the non-increasing character of $J_{n''}$, we have

$$\begin{aligned} J_{n''}(t) - J(t) &= J_{n''}(t) - J_{n''}(t_2) + J_{n''}(t_2) - J(t_2) + J(t_2) - J(t) \\ &\leq |J_{n''}(t_2) - J(t_2)| + |J(t_2) - J(t)| \\ &\leq |J_{n''}(t_2) - J(t_2)| + \varepsilon. \end{aligned} \tag{5.37}$$

Analogously, we have

$$\begin{aligned} J(t) - J_{n''}(t) &= J(t) - J(t_1) + J(t_1) - J_{n''}(t_1) + J_{n''}(t_1) - J_{n''}(t) \\ &\leq |J(t) - J(t_1)| + |J(t_1) - J_{n''}(t_1)| \\ &\leq \varepsilon + |J(t_1) - J_{n''}(t_1)|. \end{aligned} \tag{5.38}$$

From (5.36)–(5.38), we deduce that

$$\limsup_{n''\to\infty} |J(t) - J_{n''}(t)| \leq \varepsilon,$$

and therefore, as $\varepsilon > 0$ is arbitrary, we obtain (5.34).

□

5.4 Existence of Pullback Attractors

Now, by the previous results, we are able to define correctly a process U on $H = L^2(\Omega) \times L^2(\partial\Omega)$ associated to (5.1), and to obtain the existence of minimal pullback attractors.

Proposition 5.8 *Assume that $\kappa > 0$, and the assumptions (5.3)–(5.7), are satisfied. Then, the bi-parametric family of maps $U(t,\tau) : H \to H$, with $\tau \leq t$, given by*

$$U(t,\tau)(u_\tau, \psi_\tau) = (u(t), \psi(t)), \tag{5.39}$$

where $(u, \psi) = (u(\cdot;\tau, u_\tau, \psi_\tau), \psi(\cdot;\tau, u_\tau, \psi_\tau))$ is the unique weak solution of (5.1), defines a continuous process on H.

Proof It is a consequence of Theorem 5.5 and (5.22) in Theorem 5.7.

□

For the obtention of a pullback absorbing family for the process U, let us observe that the space $H^1(\Omega) \times H^{1/2}(\partial\Omega)$ is compactly imbedded in H, and therefore, for the symmetric and coercive linear continuous operator $A_1 : V_1 \to V_1'$, defined in the proof of Theorem 5.5, there exists a non-decreasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ of eigenvalues associated to the operator A_1. In particular, one has for the first eigenvalue

$$\lambda_1 = \min_{v\in H^1(\Omega),\, v\neq 0} \frac{|\nabla v|_\Omega^2 + \kappa |v|_\Omega^2}{|v|_\Omega^2 + |\gamma_0(v)|_{\partial\Omega}^2} > 0. \tag{5.40}$$

We have the following result.

Lemma 5.9 *Under the assumptions of Theorem 5.5, for any $\mu \in (0, 2\lambda_1)$ the solution (u, ψ) of (5.1) satisfies*

$$\begin{aligned} |u(t)|_\Omega^2 + |\psi(t)|_{\partial\Omega}^2 \;\leq\; & e^{-\mu(t-\tau)}(|u_\tau|_\Omega^2 + |\psi_\tau|_{\partial\Omega}^2) + \frac{2\beta}{\mu}(|\Omega| + |\partial\Omega|) \\ & + \frac{e^{-\mu t}}{2\lambda_1 - \mu} \int_\tau^t e^{\mu s}(|h(s)|_\Omega^2 + |\rho(s)|_{\partial\Omega}^2)\, ds, \end{aligned} \tag{5.41}$$

for all $t \geq \tau$.

Proof From (5.15), and taking into account (5.4), (5.5) and (5.40), we obtain

$$\begin{aligned} & \frac{d}{dt}\left[e^{\mu t}(|u(t)|_\Omega^2 + |\psi(t)|_{\partial\Omega}^2)\right] + (2\lambda_1 - \mu)e^{\mu t}(|u(t)|_\Omega^2 + |\psi(t)|_{\partial\Omega}^2) \\ & + 2\alpha_1 e^{\mu t}(|u(t)|_{p,\Omega}^p + |\psi(t)|_{q,\partial\Omega}^q) \\ \leq\; & 2\beta e^{\mu t}(|\Omega| + |\partial\Omega|) + 2e^{\mu t}[(h(t), u(t))_\Omega + (\rho(t), \psi(t))_{\partial\Omega}], \end{aligned}$$

a.e. $t > \tau$, and then, observing that

$$\begin{aligned} 2e^{\mu t}[(h(t), u(t))_\Omega + (\rho(t), \psi(t))_{\partial\Omega}] \;\leq\; & (2\lambda_1 - \mu)e^{\mu t}(|u(t)|_\Omega^2 + |\psi(t)|_{\partial\Omega}^2) \\ & + \frac{e^{\mu t}}{2\lambda_1 - \mu}(|h(t)|_\Omega^2 + |\rho(t)|_{\partial\Omega}^2), \end{aligned}$$

we have in particular

$$\frac{d}{dt}\left[e^{\mu t}(|u(t)|_\Omega^2 + |\psi(t)|_{\partial\Omega}^2)\right] \leq 2\beta e^{\mu t}(|\Omega| + |\partial\Omega|) + \frac{e^{\mu t}}{2\lambda_1 - \mu}(|h(t)|_\Omega^2 + |\rho(t)|_{\partial\Omega}^2),$$

a.e. $t > \tau$.

Integrating in this last inequality, we obtain (5.41).

□

Taking into account the estimate (5.41), we define the following universe.

Definition 5.10 *For any $\mu \in (0, 2\lambda_1)$, we will denote by $\mathcal{D}_\mu$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H)$ such that*

$$\lim_{\tau \to -\infty} \left(e^{\mu\tau} \sup_{(v,\phi) \in D(\tau)} (|v|_\Omega^2 + |\phi|_{\partial\Omega}^2) \right) = 0.$$

Accordingly to the notation introduced in the Chapter 1, $\mathcal{D}_F^H$ will denote the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of H.

Remark 5.11 *Observe that $\mathcal{D}_F^H \subset \mathcal{D}_\mu$ and that both are inclusion-closed.*

As an evident consequence of Lemma 5.9, we have the following result.

Corollary 5.12 *Assume that $\kappa > 0$, and the assumptions (5.3)–(5.7), are satisfied. Suppose moreover that there exists some $\mu \in (0, 2\lambda_1)$ such that*

$$\int_{-\infty}^{0} e^{\mu s}[|h(s)|_\Omega^2 + |\rho(s)|_{\partial\Omega}^2]\, ds < +\infty. \tag{5.42}$$

Then, the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_H(0, R_H^{1/2}(t))$, the closed ball in H of center zero and radius $R_H^{1/2}(t)$, where

$$R_H(t) = 1 + \frac{2\beta}{\mu}(|\Omega| + |\partial\Omega|) + \frac{e^{-\mu t}}{2\lambda_1 - \mu} \int_{-\infty}^{t} e^{\mu s}[|h(s)|_\Omega^2 + |\rho(s)|_{\partial\Omega}^2]\, ds,$$

is pullback $\mathcal{D}_\mu$−absorbing for the process $U : \mathbb{R}_d^2 \times H \to H$ given by (5.39) (and therefore $\mathcal{D}_F^H$−absorbing too), and $\widehat{D}_0 \in \mathcal{D}_\mu$.

We also have the character $\mathcal{D}_\mu$−pullback asymptotically compact of the process U.

Lemma 5.13 *Under the assumptions of Corollary 5.12, the process U defined by (5.39) is pullback $\mathcal{D}_\mu$−asymptotically compact.*

Proof Let us consider $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu$, $t \in \mathbb{R}$, and sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{(u_{\tau_n}, \psi_{\tau_n})\} \subset H$ satisfying $\tau_n \to -\infty$ and $(u_{\tau_n}, \psi_{\tau_n}) \in D(\tau_n)$ for all n. We must prove that the sequence $\{U(t, \tau_n)(u_{\tau_n}, \psi_{\tau_n})\}$ is relatively compact in H.

As $\tau_n \to -\infty$ and $(u_{\tau_n}, \psi_{\tau_n}) \in D(\tau_n)$ for all n, by Corollary 5.12, there exists n_0 such that $\tau_n < t-1$, and

$$U(t-1,\tau_n)(u_{\tau_n}, \psi_{\tau_n}) \in D_0(t-1) = \overline{B}_H(0, R_H^{1/2}(t-1)),$$

for all $n \geq n_0$.

Thus, the sequence $\{U(t-1,\tau_n)(u_{\tau_n}, \psi_{\tau_n}) : \ n \geq n_0\}$ is bounded in H, and therefore, there exist $(u_{t-1}, \psi_{t-1}) \in H$, and a subsequence

$$\{U(t-1,\tau_\nu)(u_{\tau_\nu}, \psi_{\tau_\nu})\} \subset \{U(t-1,\tau_n)(u_{\tau_n}, \psi_{\tau_n}) : \ n \geq n_0\},$$

such that

$$U(t-1,\tau_\nu)(u_{\tau_\nu}, \psi_{\tau_\nu}) \rightharpoonup (u_{t-1}, \psi_{t-1}) \quad \text{weakly in } H, \text{ as } \nu \to \infty.$$

But then, from (5.22) in Theorem 5.7, we deduce that

$$U(t,\tau_\nu)(u_{\tau_\nu}, \psi_{\tau_\nu}) = U(t,t-1)(U(t-1,\tau_\nu)(u_{\tau_\nu}, \psi_{\tau_\nu})) \to U(t,t-1)(u_{t-1}, \psi_{t-1})$$

strongly in H, as $\nu \to \infty$.

□

As a consequence of the above results, we obtain the existence of minimal pullback attractors for the process $U : \mathbb{R}^2_d \times H \to H$ defined by (5.39).

Theorem 5.14 *Assume that $\kappa > 0$ and the assumptions (5.3)–(5.7) are satisfied. Suppose moreover that there exists some $\mu \in (0, 2\lambda_1)$ such that the condition (5.42) holds. Then, there exist the minimal pullback $\mathcal{D}_F^H$-attractor*

$$\mathcal{A}_{\mathcal{D}_F^H} = \{\mathcal{A}_{\mathcal{D}_F^H}(t) : t \in \mathbb{R}\}$$

and the minimal pullback $\mathcal{D}_\mu$-attractor

$$\mathcal{A}_{\mathcal{D}_\mu} = \{\mathcal{A}_{\mathcal{D}_\mu}(t) : t \in \mathbb{R}\},$$

for the process U defined by (5.39). The family $\mathcal{A}_{\mathcal{D}_\mu}$ belongs to $\mathcal{D}_\mu$, and the following relation holds:

$$\mathcal{A}_{\mathcal{D}_F^H}(t) \subset \mathcal{A}_{\mathcal{D}_\mu}(t) \subset \overline{B}_H(0, R_H^{1/2}(t)) \quad \forall t \in \mathbb{R}.$$

If moreover the pair (h, ρ) satisfies

$$\sup_{s \leq 0} \left(e^{-\mu s} \int_{-\infty}^{s} e^{\mu\theta} [|h(\theta)|^2_\Omega + |\rho(\theta)|^2_{\partial\Omega}] \, d\theta \right) < +\infty, \tag{5.43}$$

then

$$\mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_\mu}(t) \quad \textit{for all } t \in \mathbb{R}. \tag{5.44}$$

Proof All but the last results are consequence of Theorem 1.11 and Corollary 1.13. Finally, (5.44) follows from (5.43) and Remark 1.14, taking into account the expression $R_H(t)$ given in Corollary 5.12.

□

Remark 5.15 *Observe that if the pair* (h, ρ) *satisfies (5.3) and (5.42) for some* $\mu \in (0, 2\lambda_1)$*, then it also satisfies*

$$\int_{-\infty}^{0} e^{\sigma s}[|h(s)|_{\Omega}^{2} + |\rho(s)|_{\partial\Omega}^{2}]\, ds < \infty, \quad \text{for all } \sigma \in (\mu, 2\lambda_1).$$

Thus, for any $\sigma \in (\mu, 2\lambda_1)$ *there exists the corresponding minimal* $\mathcal{D}_\sigma$*-pullback attractor,* $\mathcal{A}_{\mathcal{D}_\sigma}$*.*

Since $\mathcal{D}_\mu \subset \mathcal{D}_\sigma$*, it is evident that, for any* $t \in \mathbb{R}$*,*

$$\mathcal{A}_{\mathcal{D}_\mu}(t) \subset \mathcal{A}_{\mathcal{D}_\sigma}(t) \quad \textit{for all } \sigma \in (\mu, 2\lambda_1).$$

Moreover, if the pair (h, ρ) *satisfies (5.43), then, by (5.44),*

$$\mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_\mu}(t) = \mathcal{A}_{\mathcal{D}_\sigma}(t) \quad \textit{for all } t \in \mathbb{R}, \textit{ and any } \sigma \in (\mu, 2\lambda_1).$$

Some final Remarks

In this thesis we have presented a theory about pullback attractors and set-valued non-autonomous dynamical systems. We also have studied three problems related to a non-autonomous reaction-diffusion equation. But, there are some open problems which are motivated by this study.

In Chapters 4 and 5, we have analyzed the existence of pullback attractors in $L^2(\Omega)$ for two problems related to a non-autonomous reaction-diffusion equation. However, the existence of pullback attractors in $H_0^1(\Omega)$ for these problems still remains as an open question.

Other interesting study is to prove some regularity and exponential growth results for the pullback attractors of the model considered in Chapter 5.

Taking into account that in Chapter 5 we have studied the existence of pullback attractors for non-autonomous reaction-diffusion equations with dynamical boundary conditions, it is natural to extend these results to the set-valued case, which has not been considered in the literature yet, as far as we know.

Other interesting objective is to extend the results of Chapters 2, 4 and 5 to a stochastic context, with the help of the theory of random dynamical systems.

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