Some new thin sets of integers in Harmonic Analysis

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Abstract. *We randomly construct various subsets Λ of the integers which have both smallness and largeness properties. They are small since they are very close, in various meanings, to Sidon sets: the continuous functions with spectrum in Λ have uniformly convergent series, and their Fourier coefficients are in ℓ_p for all $p > 1$; moreover, all the Lebesgue spaces L^q_Λ are equal for $q < +\infty$. On the other hand, they are large in the sense that they are dense in the Bohr group and that the space of the bounded functions with spectrum in Λ is non separable. So these sets are very different from the thin sets of integers previously known.*

Résumé. *On construit aléatoirement des ensembles Λ d'entiers positifs jouissant simultanément de propriétés qui les font apparaître à la fois comme petits et comme grands. Ils sont petits car très proches à plus d'un égard des ensembles de Sidon: les fonctions continues à spectre dans Λ ont une série de Fourier uniformément convergente, et ont des coefficients de Fourier dans ℓ_p pour tout $p > 1$; de plus, tous les espaces de Lebesgue L^q_Λ coïncident pour $q < +\infty$. Mais ils sont par ailleurs grands au sens où ils sont denses dans le compactifié de Bohr et où l'espace des fonctions bornées à spectre dans Λ n'est pas séparable. Ces ensembles sont donc très différents des ensembles minces d'entiers connus auparavant.*

Key-words. ergodic set – lacunary set – $\Lambda(q)$-set – quasi-independent set – random set – p-Rider set – Rosenthal set – p-Sidon set – set of uniform convergence – uniformly distributed set.

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Introduction

It is well known that the Fourier series of an integrable function defined on the unit-circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ of the complex plane $\mathbb{C}$ can be badly behaved. For example, it is well known that there exist continuous functions whose Fourier series is not everywhere convergent (see [30], Th. 18.1, and Th. 19.5 for the optimal result), and integrable ones with everywhere divergent Fourier series (see [30], Th. 19.2 for instance; see also [29]).

The problem of thin sets of integers is the following: instead of considering all the integrable functions on $\mathbb{T}$, or all the continuous ones, we consider only those whose spectrum (the set where their Fourier coefficients do not vanish) is contained in a prescribed subset Λ of the integers $\mathbb{Z}$. This set Λ will be said "thin" if the Fourier series of these functions behaves better than in the general case. A typical example is $\Lambda = \{1, 3, 3^2, \ldots, 3^n, \ldots\}$. It is well known (see [62], for instance) that every integrable function f with spectrum in Λ ($f \in L^1_\Lambda$) is actually square integrable, and that every continuous function f with spectrum in Λ ($f \in \mathcal{C}_\Lambda$) has a normally convergent Fourier series (equivalently $\widehat{f} \in \ell_1$).

In his seminal paper [54], W. Rudin defined two notions of thinness for Λ: Λ is a Sidon set if $f \in \mathcal{C}_\Lambda$ implies that $\widehat{f} \in \ell_1$, and Λ is a $\Lambda(q)$-set for some $q > 1$, if $f \in L^1_\Lambda$ implies that $f \in L^q$. These concepts may as well be defined in the more general setting of a compact abelian group G equipped with its normalized Haar measure, and for a subset Λ of its discrete dual group Γ.

W. Rudin studied the general properties of those sets and the connection between the two notions. In particular, he showed that Sidon sets are $\Lambda(q)$-sets for all $q < +\infty$, and that, more precisely:

(0.1) Λ Sidon implies $\|f\|_q \leq C\sqrt{q}\|f\|_2$ for every Λ-polynomial f and for every $q \geq 2$, where C is a constant which depends only on the Sidon constant of Λ.

Since then, several new notions of thin sets emerged. These include p-Sidon sets (see [2], [3], [4], [5], [14], [18], [21], [23], [24], [34], [47], [60]), and sets of uniform convergence (see [1], [19], [20], [21], [26], [33], [44], [45], [56], [57]): every continuous function with spectrum in such a set has its Fourier seriesin ℓ_p or uniformly convergent, respectively. But the examples of such sets were always nearly the same: products (sometimes "fractional products": [3], [4], [5]), or sums of Sidon sets, which is a severe restriction for the geometry of the Banach space $\mathcal{C}_\Lambda$. For example, F. Lust–Piquard ([40]) proved that:

(0.2) The injective tensor product $\ell_1 \widehat{\otimes}_\varepsilon \cdots \widehat{\otimes}_\varepsilon \ell_1$ has the Schur property (*i.e.* weakly null sequences converge in norm to zero).

It follows easily that:

(0.3) If $\Lambda = E_1 \times \cdots \times E_k$, where the E_j's are Sidon sets, then $\mathcal{C}_\Lambda$ has the Schur property; in particular, $\mathcal{C}_\Lambda$ does not contain c_0, the space of sequences going to zero at infinity.

Since these sets were essentially the only known examples of p-Sidon sets (they are exactly $2N/(N+1)$-Sidon), one could believe that all p-Sidon sets have this property. It should be mentionned that in [3], R. Blei constructed for each $p \in]1, 2[$, exactly p-Sidon sets, using fractional products, so of a different type, but the corresponding space $\mathcal{C}_\Lambda$ appears as an ℓ_1-sum of finite dimensional spaces, and so does have the Schur property (we thank R. Blei for this remark).

Because of this lack of examples, the comparison between two classes of thin sets proved to be very difficult: whether a p-Sidon, or a set of uniform convergence is a $\Lambda(q)$-set for some $q > 1$ is still an open problem. On the other hand, considerable progress concerning the Sidon sets or $\Lambda(q)$-sets has been made: for example, G. Pisier ([47], Th. 6.2) proved that the converse of (0.1) is true, and J. Bourgain ([12]) proved that for each $q > 2$ there exist "exactly" $\Lambda(q)$-sets, *i.e.* sets which are $\Lambda(q)$, but $\Lambda(q')$ for no $q' > q$. Both authors used random methods, and more specifically, J. Bourgain popularized the "method of selectors" to produce several thin sets Λ with unusual properties, such as being "uniformly distributed", which implies, by a result of F. Lust–Piquard ([42]), that $\mathcal{C}_\Lambda$ contains c_0 and therefore is not a Rosenthal set (*i.e.* there are bounded measurable functions with spectrum in Λ wich are not almost everywhere equal to a continuous function), and which also implies that Λ is dense in the Bohr group (see [6], Theorem 1). This allowed the first named author to see that there are sets of integers which are $\Lambda(q)$ for all $q < +\infty$ but not Rosenthal ([37]; see also [43]).

The aim of this paper is the construction of random sets Λ of integers which have thinness properties, but wich are not Rosenthal sets (*i.e.* $\mathcal{C}_\Lambda$ is not the whole L^∞_Λ), actually such that $\mathcal{C}_\Lambda$ contains c_0, and are dense in the Bohr group. In view of (0.3), these sets will necessarily be very exotic compared to the previously known examples. This shows that replacing absolute convergence of the Fourier series by uniform convergence (sets of uniform convergence) or by ℓ_p convergence for $p > 1$ (p-Sidon sets) gives sets which are very far from Sidon sets. This constrasts with Pisier's result saying that Λ is necessarily a Sidon set whenever $\widehat{f} \in \ell_{1,\infty}$ for every $f \in \mathcal{C}_\Lambda$ (from [48], Théorème 2.3 (vi), and the top of page 688). On the other hand, though non-Sidon Rosenthal sets do exist ([53]), it follows from Bourgain-Milman's cotype theorem ([13]) that, for every non-Sidon set Λ, $\mathcal{C}_\Lambda$ does contain ℓ_∞^n uniformly, so that the presence of c_0 inside $\mathcal{C}_\Lambda$ for non-Sidon Λ may appear not so surprising. Although it is not known whether Sidon sets may be dense in the Bohr group, we obtain in this paper, as mentioned above, sets which are dense in the Bohr group, and are of uniform convergence and p-Sidon for every $p > 1$.

We construct essentially four types of sets. Each of them will be a non Rosenthal set, but a set of uniform convergence, $\Lambda(q)$ for all $q < +\infty$, and with moreover additional properties of p-Sidonicity.

The first one (Theorem 2.2) is a very lacunary set Λ with the nicest properties: it is p-Sidon for all $p > 1$. The second and third ones (Theorem 2.5 and Theorem 2.6) are medium lacunary sets: for each p with $1 < p < 4/3$, they are, in Theorem 2.5, p-Rider (a weaker property than being p-Sidon, see the definition below), but not q-Rider for $q < p$, and are q-Sidon for every $q > p/(2-p)$; and in Theorem 2.6, they are q-Rider for every $q > p$, but not p-Rider, and they are q-Sidon for every $q > p/(2-p)$. Finally, the fourth type (Theorem 2.7) is a set Λ which is, in some sense as little lacunary as possible if we want its trace on each interval $[N, 2N[$ to have a bounded Sidon constant. It leads to sets which are 4/3-Rider, but not q-Rider for $q < 4/3$.

We construct these sets by using various choices of selectors, and adding arithmetical, functional or probabilistic arguments. The treatment of the last case requires a different probabilistic approach, taken from [8].

It should be noted that in the two first cases the sets are uniformly distributed; in the fourth case , however, the sets Λ only have positive upper density in uniformly distributed sets. Nevertheless, $\mathcal{C}_\Lambda$ still contains c_0, by a result of F. Lust-Piquard ([42], Th. 5).

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1 Notation, definitions and preliminary results

We denote by $\mathbb{T}$ the compact abelian group of complex numbers of modulus one, equipped with its normalized Haar measure m. $\mathcal{C}(\mathbb{T})$ denotes the space of continuous complex functions defined on $\mathbb{T}$, equipped with its sup norm $\| \|_\infty$ and identified as usual with the space of continuous 2π-periodic complex functions defined on $\mathbb{R}$. If Λ is a subset of the dual group $\mathbb{Z}$, $\mathcal{C}_\Lambda$ will denote the subspace of $\mathcal{C}(\mathbb{T})$ consisting of functions whose spectrum lies in Λ:

$$\widehat{f}(n) \equiv \int_{\mathbb{T}} f e_{-n}\, dm = 0 \quad \text{ if } n \in \mathbb{Z} \setminus \Lambda,$$

where $e_n(z) = z^n$, or equivalently, $e_n(t) = \mathrm{e}^{int}$.

$\mathcal{C}_\Lambda$ is the uniform closure of the space $\mathcal{P}_\Lambda$ of trigonometric polynomials with spectrum in Λ, *i.e.* the uniform closure of the subspace $\mathcal{P}_\Lambda$ generated by the characters e_n, with $n \in \Lambda$.

For $f \in \mathcal{C}(\mathbb{T})$, $1 \leq q < +\infty$, M and N positive integers, we shall denote the Fourier sums of f by:

$$S_{M,N}(f) = \sum_{-M}^{N} \widehat{f}(n) e_n$$

and the symmetric Fourier sums of f by:

$$S_N(f) = S_{N,N}(f) = \sum_{-N}^{N} \widehat{f}(n) e_n\,.$$

$|A|$ denotes the cardinality of the finite set A.

A *relation* in $\Lambda \subseteq \mathbb{Z}^* \equiv \mathbb{Z} \setminus \{0\}$ is a $(+1, -1, 0)$-valued sequence $(\theta_k)_{k \in \Lambda}$ such that $\sum |\theta_k| < +\infty$ and $\sum \theta_k k = 0$. The set $S = \{k\,;\ \theta_k \neq 0\}$ is called the *support* of the the relation, and $|S| = \sum |\theta_k|$ is called its *length*.

The relation $(\theta'_k)_{k \in \Lambda}$ is said to be *longer* than the relation $(\theta_k)_{k \in \Lambda}$ if $\theta_k \neq 0$ implies $\theta_k = \theta'_k$.

The set $\Lambda \subseteq \mathbb{Z}^*$ is *quasi-independent* if it contains no non-trivial relation (*i.e.* with non-empty support). Typically, $\Lambda = \{1, 2, 4, \ldots, 2^n, \ldots\}$ is quasi-independent. The quasi-independent sets are the prototype of Sidon sets, *i.e.*

of sets Λ for which: $\|\widehat{f}\|_1 \leq K\|f\|_\infty$ for all $f \in \mathcal{C}_\Lambda$. The best constant K in this inequality is called the Sidon constant of Λ and is denoted by $S(\Lambda)$. We will refer to [39] for standard notions on Sidon sets. It is known that quasi-independent sets are not only Sidon sets but their Sidon constant is bounded by an absolute constant: this follows from [54], Th. 2.4 and [49], Lemma 1.7. Other proofs can be found in [48], lemme 3.2, and in [9], Prop. 1. We shall use the fact that $S(\Lambda) \leq 8$ if Λ is quasi-independent.

Let us recall now some classical definitions and results.

A set $\Lambda \subseteq \mathbb{Z}$ is said to be a *$\Lambda(q)$-set* (where $q > 2$) if there exists a positive constant C_q such that $\|f\|_q \leq C_q\|f\|_2$ for every $f \in \mathcal{P}_\Lambda$.

The notion of a $\Lambda(q)$-set is, in some sense, local. That follows from the Littlewood-Paley theory. The next proposition is essentially well-known, except for the growth of the constant, for which we have found no reference. Accordingly, we offer a short proof.

Proposition 1.1 *Let $\Lambda \subseteq [2, +\infty[$. Then:*

(a) *Let $(M_n)_{n\geq 1}$ be a sequence of positive integers such that $M_1 \leq 2$ and $M_{n+1}/M_n \geq \alpha > 1$. If $\Lambda \cap [M_n, M_{n+1}[$, $n \geq 1$, has a uniformly bounded Sidon constant, then Λ is $\Lambda(q)$ for all $q \geq 2$; more precisely: $\|f\|_q \leq C(q, \alpha)\|f\|_2$ for every $f \in \mathcal{P}_\Lambda$.*

(b) *If $\Lambda \cap [2^n, 2^{n+1}[$, $n \geq 1$, has a uniformly bounded Sidon constant, Λ is $\Lambda(q)$ for every $q \geq 2$ and, more precisely: $\|f\|_q \leq Cq^2\|f\|_2$ for every $f \in \mathcal{P}_\Lambda$ and for some numerical constant C.*

Proof. (a) Set

$$f_k = \sum_{M_k \leq n < M_{k+1}} \widehat{f}(n)e_n \qquad \text{and} \qquad Sf = \Big(\sum_{k=1}^{+\infty} |f_k|^2\Big)^{1/2}.$$

Since $M_{k+1}/M_k \geq \alpha > 1$ and $\Lambda \subseteq [M_1, +\infty[$, we have ([62], Chap. XV, Th. 2.1):

$$\|f\|_q \leq C_0(q, \alpha)\|Sf\|_q\,.$$

Now, using the 2-convexity of the L^q-norm for $q \geq 2$, we obtain:

$$\|Sf\|_q \leq \Big(\sum_{k=1}^{+\infty} \|f_k\|_q^2\Big)^{1/2}.$$

But $f_k \in \mathcal{P}_{\Lambda_k}$, where $\Lambda_k = \Lambda \cap [M_k, M_{k+1}[$ has a uniformly bounded Sidon constant. Therefore $\|f_k\|_q \leq C_1\sqrt{q}\|f\|_2$, where C_1 is a numerical constant. The result follows.

(b) We now make use of the classical square function

$$Sg = \Big(\sum_{k\in\mathbb{Z}} |g_k|^2\Big)^{1/2},$$

where

$$g_k = \sum_{2^k \leq n < 2^{k+1}} \widehat{g}(n) e_n \,, \text{ if } k \geq 0 \quad \text{ and } \quad g_k = \sum_{-2^{|k|+1} < n \leq -2^{|k|}} \widehat{g}(n) e_n \text{ if } k < 0 \,.$$

For this classical square function, we have the following sharp inequality, due to J. Bourgain ([11], Th. 1):

$$\|Sg\|_p \leq C_0 (p-1)^{-3/2} \|g\|_p \quad \text{ for } 1 < p \leq 2 \,,$$

where C_0 is a numerical constant. We deduce by duality that:

$$\|f\|_q \leq C_0 q^{3/2} \|Sf\|_q \quad \text{ for } 2 \leq q < +\infty \,.$$

In fact, by orthogonality (recall that $f \in \mathcal{P}_\Lambda$ and that $\Lambda \subseteq [2, +\infty[$) and the Cauchy–Schwarz inequality, we have, for every $g \in L^p$ with $\|g\|_p = 1$ ($1/p + 1/q = 1$):

$$\begin{aligned} | < f, g > | = & \Big| \sum_{k=1}^{+\infty} < f_k, g_k > \Big| = \Big| \int_{\mathbb{T}} \sum_{k=1}^{+\infty} f_k(-t) g_k(t) \, dm(t) \Big| \\ & \leq \int_{\mathbb{T}} Sf(-t) Sg(t) \, dm(t) \\ & \leq \|Sf\|_q \|Sg\|_p \leq C_0 (p-1)^{-3/2} \|Sf\|_q \\ & \leq C_0 q^{3/2} \|Sf\|_q. \end{aligned}$$

This means that here we are allowed to take $C_0(q, 2) = C_0 q^{3/2}$ in part (a) of the proof. The rest is unchanged, and we can also take $C(q, 2) = C_1 \sqrt{q} C_0 \, q^{3/2} = Cq^2$. □

A set $\Lambda \subseteq \mathbb{Z}$ is called a *set of uniform convergence* (in short a *UC-set*) if, for any $f \in \mathcal{C}_\Lambda$, the symmetric Fourier sums $S_N(f)$ converge uniformly to f. Its constant of uniform convergence $U(\Lambda)$ is the smallest constant K such that, for any $f \in \mathcal{C}_\Lambda$:

$$\sup_N \|S_N(f)\|_\infty \leq K \|f\|_\infty.$$

The following variant turns out to be more tractable ([56]). Λ is called a *set of complete uniform convergence* (in short a CUC-set) if the translates $(\Lambda + a)$ are uniformly UC for $a \in \mathbb{Z}$, or equivalently, if the Fourier sums $S_{M,N}(f)$ converge uniformly to f as M, N go to $+\infty$, for every $f \in \mathcal{C}_\Lambda$.

The two notions turn out to be distinct ([20]), but clearly coincide if $\Lambda \subseteq \mathbb{N}$, which will always be the case in the sequel. The notion of CUC-set is also a local one as the following proposition shows.

Proposition 1.2 ([57], Th. 3) *Let* $\Lambda \subseteq \mathbb{N}^*$ *and* $\Lambda_N = \Lambda \cap [N, 2N[$.

(a) *If* $U(\Lambda_N)$ *is bounded by* K *for* $N = 1, 2, \ldots$, *then* Λ *is a* CUC*-set.*

(b) *Let* $(M_n)_{n \geq 1}$ *be a sequence of positive integers such that* $M_{n+1}/M_n \geq 2$. *Then, if* $\Lambda \cap [M_n, M_{n+1}[$ *are quasi-independent for each* n, *or more generally if they are Sidon sets with uniformly bounded Sidon constant, then* Λ *is a* CUC*-set.*

Remark. (b) is a useful criterion to produce sets that are CUC but not Sidon; for instance, if $\Lambda = \bigcup_{n=1}^{+\infty}\{2^n + 2^j\,;\ j = 0,\dots,n-1\}$, then $\Lambda \cap [2^n, 2^{n+1}[$ is quasi-independent, whereas $\Lambda\cap[1,N]$ has about $(\log N)^2$ elements, and therefore cannot be Sidon (the mesh condition for Sidon sets, see Proposition 1.6 below, is violated).

The random variables which we shall use will always be defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ which will play no explicit role, and the expectation with respect to $\mathbb{P}$ will always be denoted by $\mathbb{E}$:

$$\mathbb{E}(X) = \int_\Omega X(\omega)\, d\mathbb{P}(\omega)\ .$$

Recall the (more or less) classical deviation inequality (see [32], § 6.3):

Lemma 1.3 *Let $X_1,\dots,X_N$ be independent centered complex random variables such that $|X_k| \leq 1$, $k = 1,\dots,N$. Let $\sigma \geq \sum_{k=1}^{N} \mathbb{E}|X_k|^2$. Then, one has, for every $a \leq \sigma$:*

$$\mathbb{P}(|X_1 + \cdots + X_N| \geq a) \leq 4\exp(-a^2/8\sigma)\,.$$

Let $(r_n)_n$ be a Bernoulli sequence, *i.e.* a sequence of independent random variables such that:

$$\mathbb{P}(r_n = 1) = \mathbb{P}(r_n = -1) = 1/2\,.$$

For $f \in \mathcal{P}$, the space of trigonometric polynomials, $[\![f]\!]$ denotes the norm of f in the Pisier's space $\mathcal{C}^{a.s.}$:

$$[\![f]\!] = \mathbb{E}\Big\| \sum_n r_n \widehat{f}(n) e_n \Big\|_\infty\,.$$

See [25] and [47] for more information about this norm.

Definition 1.4 *A set $\Lambda \subseteq \mathbb{Z}$ is called a p-Sidon set ($1 \leq p < 2$) if there exists a constant K such that $\|\widehat{f}\|_p \leq K\|f\|_\infty$ for all $f \in \mathcal{P}_\Lambda$.*

It is said to be a p-Rider set if there exists a constant K such that $\|\widehat{f}\|_p \leq K[\![f]\!]$ for all $f \in \mathcal{P}_\Lambda$.

p-Rider sets were implicitely introduced, with different definition, in [18] (Th. 2.4), and in [23], p. 213, as class $\mathcal{T}_p$ (see also [47], Th. 6.3). They were explicitely defined and studied in [51] and [52] under the name "p-Sidon presque sûrs". We used "almost surely p-Sidon set" in the first version of this paper, but, following a suggestion of J.-P. Kahane, we now use the terminology "p-Rider".

Clearly, every p-Sidon set is p-Rider. The converse is true for $p = 1$: this is a remarkable result due to D. Rider ([50]), making clever use of Drury's convolution device (which proves that the union of two Sidon sets is Sidon [17]). Whether this converse is still true for $1 < p < 2$ is an open problem.

Definition 1.5 *We shall say that a finite set $B \subseteq \Lambda$ is M-pseudo-complemented in Λ if there exists a measure μ on $\mathbb{T}$ such that:*

$$|\widehat{\mu}| \geq 1 \text{ on } B\,; \quad \widehat{\mu} = 0 \text{ on } \Lambda \setminus B\,; \quad \|\mu\| \leq M\,.$$

The following proposition gives some necessary, sufficient, or necessary and sufficient conditions for a set Λ to be p-Sidon or p-Rider. Part (b) of this proposition seems to be new.

Proposition 1.6 *Let $\Lambda \subseteq \mathbb{Z}^*$ and $1 \leq p < 2$. Set $\varepsilon(p) = 2/p - 1$. Then:*

(a) *Λ is a p-Rider set if and only if there exists a constant $\delta > 0$ such that, for every finite set $A \subseteq \Lambda$, there exists a quasi-independent subset $B \subseteq A$ such that $|B| \geq \delta |A|^{\varepsilon(p)}$.*

(b) *Let $q_0 > 1$. If there exists a constant $\delta > 0$ such that, for every finite set $A \subseteq \Lambda$, there exists a quasi-independent subset $B \subseteq A$ such that $|B| \geq \delta |A|^{1/q_0}$ and if B can moreover be taken M-pseudo-complemented in Λ, for some fixed M, then Λ is a q-Sidon set for every $q > q_0$.*

(c) *If Λ is a p-Rider set, we have the following mesh condition:*

$$|\Lambda \cap [1, N]| \leq C(\log N)^{p/(2-p)}.$$

Proof. We refer to [51] for the proof of (a) and (c). To prove (b), let $f \in \mathcal{P}_\Lambda$, fix $t > 0$, and set $A = \{\, |\widehat{f}| > t\,\}$. Take $B \subseteq A$ and μ as in Definition 1.4. Then B is a Sidon set with Sidon constant ≤ 8, and since $f * \mu = \sum_{n \in B} \widehat{f}(n)\widehat{\mu}(n) e_n$,

$$\|f\|_\infty \geq M^{-1} \Big\| \sum_B \widehat{f}(n)\widehat{\mu}(n) e_n \Big\|_\infty \geq \frac{1}{8M} \sum_B |\widehat{f}(n)|\, |\widehat{\mu}(n)|$$
$$\geq \frac{1}{8M} \sum_B |\widehat{f}(n)| \geq \frac{t|B|}{8M} \geq \frac{t\delta |A|^{1/q_0}}{8M}.$$

In other words, for some constant $C > 0$, one has:

$$t\,.|\,\{\, |\widehat{f}| > t\,\}\,|^{1/q_0} \leq C\|f\|_\infty\,, \qquad \text{for every } t > 0\,,$$

which means that the Lorentz norm of $\widehat{f}$ in the Lorentz space $\ell_{q_0,\infty}$ is dominated by $\|f\|_\infty$.

Now, $\ell_{q_0,\infty}$ is continuously injected in ℓ_q for $q > q_0$ (see for instance [38], II p. 143), and this gives the desired result. □

We denote, as usual, by c_0 the classical space of sequences $x = (x_n)_{n \geq 0}$ tending to zero at infinity, equipped with the norm $\|x\| = \sup_n |x_n|$. We say, in the usual familiar way, that a Banach space X "contains c_0" if X has a closed subspace isomorphic to c_0. Our notation for Banach spaces is classical, as can be found in [16], [38] or [59] for instance.

A subset Λ of $\mathbb{Z}$ is said to be a *Rosenthal set* if every bounded measurable function on $\mathbb{T}$ with spectrum in Λ is almost everywhere equal to a continuous function (in short $L^\infty_\Lambda = \mathcal{C}_\Lambda$). Λ is not Rosenthal if and only if L^∞_Λ is not separable, so such a set can be thought as being a big set.

Every Sidon set is clearly Rosenthal, but H.P. Rosenthal gave examples of non-Sidon sets which are Rosenthal ([53]). We shall make use of the following well known negative criterion (see [41], § 3), which follows from the classical theorem of C. Bessaga and A. Pełczyński ([38], I.2.e.8), saying that a dual space which contains c_0 has to contain also ℓ_∞.

Proposition 1.7 *If $\mathcal{C}_\Lambda$ contains c_0, then Λ is not a Rosenthal set.*

Definition 1.8 *Let $\Lambda \subseteq \mathbb{N}^* \equiv \mathbb{N} \setminus \{0\}$, and set*

$$\Lambda_N = \Lambda \cap [1, N] \qquad \textit{and} \qquad A_N(t) = \frac{1}{|\Lambda_N|} \sum_{n \in \Lambda_N} e_n(t)\,.$$

We say that Λ is:

- ergodic *if $(A_N(t))_{N\geq 1}$ converges to a limit $l_\Lambda(t) \in \mathbb{C}$ for each $t \in \mathbb{T}$.*

- strongly ergodic *if it is ergodic and moreover the limit function l_Λ defines an element of $c_0(\mathbb{T})$: for every $\varepsilon > 0$ the set $\{t \in \mathbb{T}\,;\ |l_\Lambda(t)| > \varepsilon\}$ is finite.*

- uniformly distributed *if it is (strongly) ergodic and, moreover, $l_\Lambda(t) = 0$ for $t \neq 0$ mod. 2π.*

The reason for this terminology is that the ergodic sets are those for which an ergodic theorem holds: $(1/|\Lambda_N|) \sum_{n\in\Lambda_N} T^n$ converges in the strong operator topology for every contraction T of a Hilbert space. Typically, the set of d^{th} perfect powers, or the set of prime numbers are strongly ergodic (according to the result of Vinogradov for t irrational mod. 2π, and to the Dirichlet's arithmetic progression theorem for t rational mod. 2π). The third name comes from H. Weyl's classical criterion for the equidistribution of a real sequence mod. 2π.

The relationship between these notions comes from:

Theorem 1.9 (F. Lust-Piquard [42]) *Let $\Lambda \subseteq [1, +\infty[$ be a set of positive integers. Then:*

(a) *If Λ is strongly ergodic, $\mathcal{C}_\Lambda$ contains c_0.*

(b) *More generally, if Λ is strongly ergodic and $D \subseteq \Lambda$ has a positive upper density with respect to Λ, then $\mathcal{C}_D$ contains c_0 as well.*

Here "positive upper density" means that:

$$\overline{\lim_{N\to+\infty}} \frac{|D \cap [1, N]|}{|\Lambda \cap [1, N]|} > 0\,.$$

Part (b) will be useful to us in the last theorem of Section 2.

See [42] for the proof of this theorem. The underlying idea for (a) is that if $A_N(t) \to l_\Lambda(t)$ for every $t \in \mathbb{T}$, l_Λ defines an element of the biorthogonal $\mathcal{C}_\Lambda^{\perp\perp}$, and the condition $l_\Lambda \in c_0(\mathbb{T})$ implies that it is the sum of a weakly unconditionally Cauchy series of continuous functions. By using a perturbation argument due to A. Pełczyński (see [55], lemma 15.7, p. 446) and the classical Bessaga-Pełczyński theorem, one obtains that $\mathcal{C}_\Lambda$ contains c_0.

This theorem allowed its author to prove that $\mathcal{C}_\Lambda$ contains c_0 when $\Lambda = \{1, 2^d, 3^d, \ldots\}$ is the set of the d^{th} perfect powers, and when $\Lambda = \{2, 3, 5, 7, \ldots\}$ is the set of the prime numbers. On the other hand, K. I. Oskolkov ([44]; see also [1]) showed that the set of the d^{th} powers is not a UC-set, and J. Fournier and L. Pigno ([21], Th. 4) proved that the set of prime numbers is not a UC-set either. This could be taken as an indication that containing c_0 is an obstruction to being UC. As we shall see in the next section, this is far from being the case: there do exist sets Λ which are UC and for which $\mathcal{C}_\Lambda$ contains c_0.

The last ingredient we require is a random procedure to produce ergodic sets.

Let $(\varepsilon_k)_{k\geq 1}$ be a sequence of independent $0-1$ valued random variables, called "*selectors*" according to the terminology coined by J. Bourgain. To those selectors is associated a random set Λ of positive integers

$$\Lambda = \Lambda(\omega) = \{k \geq 1\,;\ \varepsilon_k(\omega) = 1\,\}\,.$$

Theorem 1.10 (J. Bourgain [10], Prop. 8.2) *Let $\varepsilon_1, \ldots, \varepsilon_N, \ldots$ be selectors of respective expectations $\delta_1, \ldots, \delta_N, \ldots$ and assume that $\sigma_N / \log N \to +\infty$, where $\sigma_N = \delta_1 + \cdots + \delta_N$ (which is in particular the case when $k\delta_k \to +\infty$), and that $(\delta_n)_n$ decreases. Then the set $\Lambda = \Lambda(\omega)$ is almost surely uniformly distributed. In particular, it is almost surely strongly ergodic.*

2 Main results

In this section, we will always consider selectors ε_n, $n \geq 1$, with mean $\delta_n = \alpha_n/n$, with $(\alpha_n)_n$ tending to infinity and $(\delta_n)_n$ decreasing.

Moreover, except in the last theorem of this section, we will assume that $(\alpha_n)_n$ is increasing.

If $\Lambda = \Lambda(\omega) = \{n \geq 1\,;\ \varepsilon_n(\omega) = 1\}$ is the corresponding random set of integers, Λ is almost surely uniformly distributed by Bourgain's theorem. Moreover, it also has the nice almost sure property of being asymptotically independent; more precisely, there exists an increasing sequence $(M_n)_n$ of positive integers such that $\Lambda \cap [M_n, +\infty[$ is both large and without relations of length $\leq n$. A subset B of $\Lambda \cap [M_n, +\infty[$ with n elements is then automatically quasi-independent, and this allows us to use Propositions 1.1, 1.2, 1.6 to show that Λ has good additional properties: UC, p-Sidon, *etc.*... To obtain this asymptotic quasi-independence, the following half-combinatorial, half-probabilistic lemma plays a crucial role.

Recall that $\sigma_n = \delta_1 + \cdots + \delta_n$.

Lemma 2.1 *Let $s \geq 2$ and M be integers. Set*

$$\Omega_s(M) = \{\omega \in \Omega\,;\ \Lambda(\omega) \cap [M, +\infty[\ \textit{contains at least a relation of length } s\,\}\,.$$

Then:

$$\mathbb{P}\big(\Omega_s(M)\big) \le \frac{s\,2^{s-2}}{(s-2)!}\sum_{j>M}\delta_j^2\sigma_j^{s-2} \le \frac{(4\mathrm{e})^s}{s^s}\sum_{j>M}\delta_j^2\sigma_j^{s-2}.$$

The important fact in this lemma is the presence of the exponent 2 in the factor δ_j^2 and of the factorial in the denominator.

Proof. We thank the referee for suggesting the following proof.

We have $\Omega_s(M) = \bigcup_{l\ge M+s-1}\Delta_l$, where $\Delta_l = \Delta(l, M, s)$ is defined by:

$$\Delta_l = \{\omega\,;\,\Lambda(\omega)\cap[M,+\infty[\text{ contains at least a relation of length } s, \text{ with largest term } l\,\}.$$

In other words, $\omega\in\Delta_l$ if and only if $\Lambda(\omega)\cap[M,+\infty[$ has at least a relation of length s which contains l and which is contained in $\{M,\dots,l\}$.

We clearly have:

$$\Delta_l \subseteq \bigcup_{(i_1,\dots,i_{s-1})}\Delta(l, i_1,\dots,i_{s-1}),$$

where

$$\Delta(l, i_1,\dots,i_{s-1}) = \{\omega\,;\ \varepsilon_{i_1}(\omega) = \cdots = \varepsilon_{i_{s-1}}(\omega) = \varepsilon_l(\omega) = 1\},$$

and where $(i_1,\dots,i_{s-1})$ runs over the $(s-1)$-tuples of integers such that:

$(*)$ $M\le i_1<\cdots<i_{s-1}<l$,

$(**)$ $\theta_1 i_1+\theta_2 i_2+\cdots+\theta_{s-1}i_{s-1}+\theta_s l = 0,\quad \theta_1,\dots,\theta_s\in\{-1,+1\}$.

Observe that $\delta_{i_{s-1}}\le(s-1)\delta_l$ for such $(s-1)$-tuples. In fact, it follows from $(**)$ that $l\le i_1+\cdots+i_{s-1}\le(s-1)i_{s-1}$, so

$$\delta_{i_{s-1}} = \frac{\alpha_{i_{s-1}}}{i_{s-1}} \le \frac{\alpha_l}{i_{s-1}} = \frac{\alpha_l}{l}\,\frac{l}{i_{s-1}} \le (s-1)\delta_l.$$

Observe also that, when $i_1,\dots,i_{s-2}$ are fixed, $i_{s-1} = \pm l\pm i_{s-2}\pm\cdots\pm i_1$ can take at most 2^{s-1} values, so that

$$\begin{aligned}\mathbb{P}(\Delta_l) &\le \sum\mathbb{P}\big(\Delta(l,i_1,\dots,i_{s-1})\big) = \sum\delta_{i_1}\dots\delta_{i_{s-1}}\delta_l\\ &\le (s-1)2^{s-1}\delta_l^2\sum_{M\le i_1<\cdots<i_{s-2}\le l-1}\delta_{i_1}\dots\delta_{i_{s-2}}\\ &\le (s-1)2^{s-1}\delta_l^2\,\frac{(\delta_M+\cdots+\delta_{l-1})^{s-2}}{(s-2)!}\end{aligned}$$

by the multinomial formula.

Therefore, noting that $s\ge 2$, we have:

$$\mathbb{P}\big(\Omega_s(M)\big) \le \sum_{l\ge M+s-1}(s-1)2^{s-1}\delta_l^2\frac{\sigma_{l-1}^2}{(s-2)!} \le \sum_{j>M}\frac{(s-1)2^{s-1}}{(s-2)!}\delta_j^2\sigma_j^{s-2}. \quad\square$$

The following theorem is the main result of the paper. It states that subsets of integers can, in several ways, be very close to Sidon sets, but in the same time be rather large.

Theorem 2.2 *There exist sets Λ of integers which are:*

(1) *p-Sidon for all $p > 1$, $\Lambda(q)$ for all $q < +\infty$, and* CUC, *but which are also*

(2) *uniformly distributed; in particular, they are dense in the Bohr group, and $\mathcal{C}_\Lambda$ contains c_0, so Λ is not a Rosenthal set.*

Proof. We use selectors ε_k of mean

$$\delta_k = c\,\frac{\log\log k}{k} \qquad (k \geq 3),$$

where c is a constant to be specified latter. Since this constant plays no role in the beginning of the proof, for convenience, we first assume that $c = 1$.

The last assertion follows at once from Bourgain's and Lust-Piquard's theorems. The rest of the proof depends on the following lemma, where we set $\Lambda_n = \Lambda \cap [1, n]$ and $\Lambda'_n = \Lambda \cap [M_n, M_{n+1}[$.

Lemma 2.3 *If $M_n = n^n$, one has the following properties, where C_0 denotes a numerical constant:*

(1) $\sum_{n\geq 3} \mathbb{P}\big(\Omega_n(M_n)\big) < +\infty$.

(2) *Almost surely $|\Lambda_{M_n}| \leq C_0\, n(\log n)^2$ for n large enough.*

(3) *Almost surely $|\Lambda'_n| \leq C_0(\log n)^2$ for n large enough.*

Proof of Lemma 2.3. Note first that

$$\sigma_n = \sum_{3\leq k\leq n} \frac{\log\log k}{k} \leq (\log\log n) \sum_{3\leq k\leq n} \frac{1}{k} \leq (\log\log n)\int_1^n \frac{dt}{t} = \log n\,\log\log n.$$

Now, take $n \geq 64$, and use Lemma 2.1 to obtain (setting $C = 4\mathrm{e}$):

$$\begin{aligned}\mathbb{P}\big(\Omega_n(M_n)\big) &\leq \frac{C^n}{n^n} \sum_{j>M_n} \Big(\frac{\log\log j}{j}\Big)^2 (\log j\,\log\log j)^{n-2}\\ &\leq \frac{C^n}{n^n} \sum_{j>M_n} \frac{(\log j\,\log\log j)^n}{j^2}\,.\end{aligned}$$

But, for fixed n, the function

$$\frac{u}{v} = \frac{(\log x\,.\log\log x)^n}{x^2}$$

decreases on $[M_n, +\infty[$. Indeed, we have to check that $u'(x)v(x) \leq u(x)v'(x)$ on this interval, *i.e.* that

$$nx(1+\log\log x)(\log x\,\log\log x)^{n-1} \leq 2x(\log x\,\log\log x)^n$$

or, equivalently, that

$$n(1+\log\log x) \le 2\log x\,\log\log x\,.$$

Now, if $x \ge n^n$, we see that

$$n(1+\log\log x) \le 2n\log\log x \le 2n\log n\,\log\log x \le 2\log x\,\log\log x\,.$$

Therefore,

$$\mathbb{P}\big(\Omega_n(M_n)\big) \le \frac{C^n}{n^n}\int_{M_n}^{+\infty}\frac{(\log t\,\log\log t)^n}{t^2}\,dt\,.$$

Setting

$$f_n(t) = \frac{(\log t\,\log\log t)^n}{t^2} \qquad \text{and} \qquad I_n = \int_{M_n}^{+\infty} f_n(t)\,dt,$$

we have, by summation by parts:

$$I_n = \frac{(\log M_n\,\log\log M_n)^n}{M_n} + \int_{M_n}^{+\infty} n f_n(t)\Big(\frac{1}{\log t} + \frac{1}{\log t\,\log\log t}\Big)\,dt\,.$$

Since the function in the integrand is less than

$$\frac{2n}{\log t}f_n(t) \le \frac{2n}{n\log n}f_n(t) \le \frac{1}{2}f_n(t)$$

(recall that $n \ge 64$), this gives:

$$I_n \le \frac{(\log M_n\,\log\log M_n)^n}{M_n} + \frac{1}{2}I_n\,,$$

so:

$$\begin{aligned}\mathbb{P}\big(\Omega_n(M_n)\big) \le \frac{C^n}{n^n}I_n &\le 2\frac{C^n}{n^n}\frac{(\log M_n\,\log\log M_n)^n}{M_n}\\ &\le 2\frac{C^n}{n^n}\frac{(n\log n.2\log n)^n}{M_n} = 2\frac{(2C)^n(\log n)^{2n}}{n^n}\,,\end{aligned}$$

which proves (1).

To prove (2), first note that

$$\sigma_n \ge \sum_{16\le k\le n}\frac{1}{k} \ge \frac{1}{2}\log n \qquad \text{for } n \ge 256.$$

Now, using Lemma 1.3 with $X_k = \varepsilon_k - \delta_k$, we obtain:

$$\begin{aligned}\mathbb{P}\Big(\big|\,|\Lambda_n| - \sigma_n\big| \ge \frac{\sigma_n}{2}\Big) &= \mathbb{P}\Big(\Big|\sum_{3\le k\le n}X_k\Big| \ge \frac{\sigma_n}{2}\Big)\\ &\le 4\exp\Big(-\frac{\sigma_n}{32}\Big) \le 4\exp\Big(-\frac{\log n}{64}\Big).\end{aligned}$$

In particular, since $M_n = n^n$,

$$\mathbb{P}\Big(\big| \, |\Lambda_{M_n}| - \sigma_{M_n} \big| \geq \frac{\sigma_{M_n}}{2} \Big) \leq 4 \exp \Big(- \frac{n \log n}{64} \Big) ,$$

and the Borel-Cantelli lemma shows that, almost surely,

$$\big| \, |\Lambda_{M_n}| - \sigma_{M_n} \big| < \sigma_{M_n}/2$$

for n large enough (depending on ω). Thus:

$$|\Lambda_{M_n}| \leq 2\sigma_{M_n} \leq 2 \log M_n \, \log\log M_n \leq C_0 \, n (\log n)^2 ,$$

for some numerical constant C_0, and this gives (2).

The proof of (3) goes the same way. Set:

$$\sigma_n' = \sum_{M_n \leq k < M_{n+1}} \frac{\log\log k}{k}$$

and observe that (here, and in the remainder of the paper, the sign $\sim$ between two functions will mean that these two functions are equivalent up to a constant factor):

$$\sigma_n' \sim \log n \sum_{M_n \leq k < M_{n+1}} \frac{1}{k} \sim \log n \, \log \frac{M_{n+1}}{M_n} \sim (\log n)^2 ,$$

so that:

$$C_0^{-1} (\log n)^2 \leq \sigma_n' \leq C_0 (\log n)^2$$

for some numerical constant C_0.

Then, using again Lemma 1.3, we get:

$$\mathbb{P}\big(\big| \, |\Lambda_n'| - \sigma_n' \big| \geq \sigma_n'/2 \big) \leq 4 \exp \Big(- \frac{\sigma_n'}{32} \Big) \leq 4 \exp \Big(- \frac{(\log n)^2}{32 C_0} \Big) ;$$

so the Borel-Cantelli lemma shows that, almost surely,

$$|\Lambda_n'| \leq 2\sigma_n' \leq 2C_0 (\log n)^2$$

for n large enough, which gives (3), provided we enlarge C_0, and completes the proof of Lemma 2.3. □

We now conclude the proof of Theorem 2.2.

We first choose the constant c in order that, not only $\sum_{n \geq 3} \mathbb{P}\big(\Omega_n(M_n)\big) < +\infty$, but $\sum_{n \geq 3} \mathbb{P}\big(\Omega_n(M_n)\big) < 1$. So, using Lemma 2.3, we can find $\Omega_0 \subseteq \Omega$ such that $\mathbb{P}(\Omega_0) > 0$ and with the property:

$$\begin{aligned} &\text{If } \omega \in \Omega_0, \text{ then } \omega \notin \bigcup_{n \geq 3} \Omega_n(M_n). \quad \text{There exists } n_0 = n_0(\omega) \text{ such that} \\ &|\Lambda_{2M_n}| \leq C_0 \, n (\log n)^2 \text{ and } |\Lambda_n'| \leq C_0 (\log n)^2 \leq n \text{ for } n > n_0. \end{aligned} \tag{2.1}$$

Indeed, an inspection of the proof of Lemma 2.3 shows that we also have, almost surely, $|\Lambda_{2M_n}| \leq 2\sigma_{2M_n} \leq C_0\, n(\log n)^2$ for n large enough, and this gives (2.1). We have the following consequences, where $\omega \in \Omega_0$, and $\Lambda = \Lambda(\omega)$:

$$\Lambda \cap [M_n, +\infty[\text{ contains no relation of length } \leq n. \tag{2.2}$$

For, if $\Lambda \cap [M_n, +\infty[$ were to contain a relation R of support S with $|S| = s \leq n$, then necessarily $s \geq 3$, $S \subseteq \Lambda \cap [M_s, +\infty[$ and $\omega \in \Omega_s(M_s)$, which is not the case by (2.1). Now, (2.1) and (2.2) imply that, for $n > n_0$, Λ'_n is quasi-independent, and so is a Sidon set with bounded constant. So, we get that Λ is CUC and $\Lambda(q)$ for all $q < +\infty$ using Propositions 1.1 and 1.2, provided we notice that:

$$\frac{M_{n+1}}{M_n} = \frac{(n+1)^{n+1}}{n^n} \geq n + 1 \geq 2.$$

To end the proof, we first show that Λ is p-Rider, for every $p > 1$, and then, using Proposition 1.6 (c), prove that it is q-Sidon for every $q > 1$.

So, fix $p \in]1, 2[$, set $\varepsilon = 2/p - 1 \in]0, 1[$, and take $\omega \in \Omega_0$ and $n_1 = n_1(\varepsilon, \omega) \geq 2n_0(\omega)$ such that $C_0\, n(\log n)^2 \leq n^{1/\varepsilon}/2$ and $n^{1/\varepsilon}/2 \geq n$ for $n \geq n_1$.

Let $A \subseteq \Lambda$ be a finite subset, with $|A|^\varepsilon > n_1$. Set $n = [|A|^\varepsilon]$, where $[\ ]$ stands for integer part, so that $n \geq n_1$ and $|A| \geq n^{1/\varepsilon}$. Observe that:

$$\begin{aligned} \big|A \cap [M_n, +\infty[\big| &\geq |A| - |A \cap [1, M_n]| \geq |A| - |\Lambda_{M_n}| \\ &\geq n^{1/\varepsilon} - C_0\, n(\log n)^2 \geq n^{1/\varepsilon}/2 \geq n\,, \end{aligned}$$

and select $B \subseteq A \cap [M_n, +\infty[$ with $|B| = n$. It follows from (2.2) that B is quasi-independent, and $|B| = n \geq \frac{1}{2}|A|^\varepsilon$.

If now A is a subset of Λ with $1 \leq |A| \leq n_1$, simply take for B a singleton from A. Then B is quasi-independent, and $|B| = 1 \geq n_1^{-1}|A|^\varepsilon$.

The criterion of Proposition 1.6 (a) is verified with $\delta = n_1^{-1}$. Therefore Λ is p-Rider.

We shall verify that we are in position to apply part (b) of Proposition 1.6.

Take $p \in]1, 2[$ and $1/p < \varepsilon < 1$. Take $\omega \in \Omega_0$ and $n_1 = n_1(\varepsilon, \omega) \geq 2n_0(\omega)$ such that $C_0\, n(\log n)^2 \leq n^{1/\varepsilon}/2$ and $n^{1/\varepsilon}/2 \geq n$ for $n \geq n_1$.

Let $A \subseteq \Lambda$ be a finite subset with $|A|^\varepsilon > n_1$. Set $n = [|A|^\varepsilon]$, where $[\ ]$ stands for integer part, so that $n \geq n_1$ and $|A| \geq n^{1/\varepsilon}$. Observe that:

$$\begin{aligned} |A \cap [2M_n, +\infty[| &\geq |A| - |A \cap [1, 2M_n]| \geq |A| - |\Lambda_{2M_n}| \\ &\geq n^{1/\varepsilon} - C_0\, n(\log n)^2 \geq n^{1/\varepsilon}/2 \geq n \end{aligned}$$

in view of (2.1). We can thus select $B \subseteq A \cap [2M_n, +\infty[$ with $|B| = n - 1$ and have:

$$\text{If } k \in \Lambda \cap [M_n, +\infty[, \text{ then } B \cup \{k\} \text{ is quasi-independent.}$$

Indeed, $B \cup \{k\}$ is a set of cardinality less than n contained in $\Lambda \cap [M_n, +\infty[$, and is automatically quasi-independent, from (2.2).

We show now that B is 8-pseudo-complemented in Λ.

Put $\nu = \delta_0 - V_{M_n}$, where δ_0 is the Dirac point mass at 0, and V_{M_n} the de la Vallée-Poussin kernel of order M_n. Consider the Riesz product $R = \prod_{k\in B} (1 + \Re\mathrm{e}\, e_k)$, and set $\mu = 2\nu * R$. We claim that:

$$\|\mu\| \leq 8\,; \qquad \widehat{\mu} \geq 1 \quad \text{on } B\,; \qquad \widehat{\mu} = 0 \quad \text{on } \Lambda \setminus B.$$

Indeed, $\|\nu\| \leq 4$ and B is quasi-independent, so the Riesz product R verifies $\|R\| = \widehat{R}(0) = 1$. Therefore $\|\mu\| \leq 8$.

Take $l \in B$. Then $l > 2M_n$ and $\widehat{\nu}(l) = 1$. As $\widehat{R}(l) \geq 1/2$, we have $\widehat{\mu}(l) \geq 1$.

If $A \subseteq \Lambda$ and $|A|^\varepsilon \leq n_1$, any singleton B of A is quasi-independent, 1-complemented in Λ, and $|B| \geq n_1^{-1}|A|^\varepsilon$.

We have thus verified the hypothesis of part (b) of Proposition 1.6, and so Λ is q-Sidon for any $q > 1/\varepsilon$. In particular, it is p-Sidon, and this ends the proof of Theorem 2.2. □

Remark 1. The proof shows that we can actually extract from A, for every $\alpha > 0$, a quasi-independent set B such that $|B| \geq \delta|A|/(\log|A|)^{2+\alpha}$. Moreover, a slight modification leads to sets even closer to Sidon sets.

Proposition 2.4 *Let $\alpha > 1$ and φ_α be the Orlicz function $x \mapsto x\big(\log(1+x)\big)^\alpha$. Then, there exists a set Λ as in Theorem 2.2, and moreover such that $\widehat{f} \in \ell_{\varphi_\alpha,\infty}$ for every $f \in \mathcal{C}_\Lambda$.*

Recall that $\ell_{\varphi_\alpha,\infty}$ is the weak Orlicz-Lorentz space of sequences $(a_n)_n$ such that $\sup_n \varphi_\alpha^{-1}(n) a_n^* < +\infty$, where $(a_n^*)_n$ is the non-increasing rearrangement of $(|a_n|)_n$. Therefore, another way to phrase the proposition is, setting $a_n = \widehat{f}(n)$:

$$a_n^* \leq C_\alpha \|f\|_\infty (\log n)^\alpha / n \quad \text{for every } f \in \mathcal{C}_\Lambda.$$

The proof just consists in changing M_n. We take $M_n = \big[e^{n(\log\log n)^2}\big]$, where [ ] stands for the integer part. We still have $\sum_n \mathbb{P}\big(\Omega_n(M_n)\big) < +\infty$, since

$$\mathbb{P}\big(\Omega_n(M_n)\big) \leq \frac{2\,C^n}{n^n} \frac{(\log M_n\ \log\log M_n)^n}{M_n} \leq \exp\big(-n\,(\log\log n)^2/2\big)$$

for n large enough. Arguing as previously, we get for every finite subset A of Λ, a quasi-independent subset B of A such that $|B| \geq \delta|A|/(\log|A|)^\alpha$, and such that $\mathcal{C}_B$ is uniformly pseudo-complemented in $\mathcal{C}_A$. As in the proof of Proposition 1.6, we obtain

$$|\{|\widehat{f}| > t\}| \leq C\,\varphi_\alpha\Big(\frac{\|f\|_\infty}{t}\Big),$$

which gives the result (arguing as in [34] for instance).

We cannot eliminate a logarithmic factor, and replace $\alpha > 1$ by $\alpha > 0$ because, due to Bourgain's criterion, we have to assume that $\sigma_n/\log n$ goes to infinity in order that $\mathcal{C}_\Lambda$ contains c_0. However, for each $\alpha > 0$, there do exist non-Sidon sets Λ for which $\widehat{f} \in \ell_{\varphi_\alpha}$ when $f \in \mathcal{C}_\Lambda$ (as can be seen from [5], p. 69).

The set Λ is, in some sense, very close to be Sidon, whereas $\mathcal{C}_\Lambda$ contains c_0. However, it cannot be too close without being Sidon because if $\widehat{f} \in \ell_{1,\infty}$, the Lorentz space weak-ℓ_1, for every $f \in \mathcal{C}_\Lambda$, then Λ is Sidon. In fact, this condition implies an inequality of the type:

$$|\{|\widehat{f}| \geq t\}| \leq \frac{C}{t} \|f\|_\infty \qquad (*)$$

for every $f \in \mathcal{C}_\Lambda$. Let now A be a finite subset of Λ, and $f = \sum_{n \in A} e_n$ and $f_\omega = \sum_{n \in A} r_n(\omega) e_n$, where r_n, $n \geq 1$ are the Rademacher functions. Then, inequality ($(*)$) applied with $t = 1$ gives $\|f_\omega\|_\infty \geq (1/C)|A|$. Integrating in ω gives $[\![f]\!] \geq (1/C)|A|$, from which follows, by a result of G. Pisier ([48], Théorème 2.3 (vi)), that Λ is a Sidon set.

Remark 2. If one takes selectors of mean δ_n such that $n\delta_n$ is bounded, the corresponding random set $\Lambda(\omega)$ is almost surely a Sidon set. This is a well-known result of Y. Katznelson and P. Malliavin ([28], or [27]), and Lemma 2.1 gives another proof of this fact. It suffices to take $M_n = A^n$, where A is a given integer, large enough to have $\sum_{n=1}^{+\infty} \mathbb{P}\big(\Omega_n(M_n)\big) < 1$. Then, with positive probability $\Lambda(\omega) \cap [M_n, +\infty[$ contains no relation of length $\leq n$, whereas $|\Lambda(\omega) \cap [1, M_n]| \leq Cn$. Hence, for every finite subset A of $\Lambda(\omega)$, we can find a quasi-independent subset $B \subseteq A$ such that $|B| \geq \delta |A|$, for some fixed $\delta = \delta(\omega)$. It follows from Pisier's characterization ([48], Th. 2.3 (iv)) that, with positive probability, and hence almost surely by Kolmogorov's $0-1$ law, $\Lambda(\omega)$ is a Sidon set.

As is now well-known, Sidon sets are characterized by various properties (successively weaker) of the Banach space $\mathcal{C}_\Lambda$: Λ is a Sidon set *iff* $\mathcal{C}_\Lambda$ is isomorphic to ℓ_1 ([58]), *iff* $\mathcal{C}_\Lambda$ has cotype 2 ([31], Th. 3.1, [46]), and *iff* $\mathcal{C}_\Lambda$ has a finite cotype ([13]). This later property can be expressed by saying that $\mathcal{C}_\Lambda$ does not contain ℓ_∞^n uniformly. So, deterministically, one has the dichotomy:

(a) either Λ is a Sidon set, and so $\mathcal{C}_\Lambda$ is isomorphic to ℓ_1;

(b) or $\mathcal{C}_\Lambda$ contains ℓ_∞^n uniformly.

The probabilistic dichotomy is stronger: taking selectors of mean $\delta_1, \delta_2, \ldots$, with $(\delta_n)_n$ decreasing, one has:

(a) either almost surely Λ is a Sidon set (if $n\delta_n$ is bounded);

(b) or almost surely $\mathcal{C}_\Lambda$ contains c_0 (if $n\delta_n$ is not bounded), and Λ is even uniformly distributed.

Y. Katznelson ([27]) already noticed such a "dichotomy": he showed that (under a different choice of selectors from ours) either almost surely Λ is a Sidon set, or almost surely Λ is dense in the Bohr group. However, this is perhaps not a true dichotomy since it is a well-known open problem whether there can exist Sidon sets dense in the Bohr group (see [15], question 2, p. 14;

it is stated for the Bohr group of $\mathbb{R}$, but also makes sense for the Bohr group of $\mathbb{Z}$).

The dichotomy stated here strengthens Katznelson's result since every uniformly distributed set is dense in the Bohr group (see [6], Theorem 1); indeed, saying that $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ is uniformly distributed means that the measures $\mu_N = 1/N \sum_{n=1}^{+\infty} \delta_{\lambda_n}$ (δ_{λ_n} is there the Dirac measure at the point λ_n) converge weak-star to the Haar measure μ of the Bohr group $b\mathbb{Z}$; but these measures are carried by Λ, so the closed support of μ is contained in the Bohr closure of Λ; since the Haar measure is continuous, we get that this closure is the whole Bohr group.

Remark 3. The random sets Λ that we construct have an asymptotical quasi-independence: $\Lambda \cap [M_n, +\infty[$ contains no relation of length $\leq n$. This is reminiscent of the following result of J. Bourgain ([7]): if Λ is a Sidon set and $n \in \mathbb{N}^*$, there exists $l_n = l(\Lambda, n)$ such that Λ can be decomposed in l_n sets $\Lambda_1, \ldots, \Lambda_{l_n}$, each of which contains no relation of length $\leq n$.

We now investigate what happens when we let p increase away from 1.We get several different results, and $p = 4/3$ seems to play a special role.

We first state two very similar results.

Theorem 2.5 *For every $1 < p < 4/3$, there exists a set Λ of integers which is:*
(1) *uniformly distributed (so Λ is dense in the Bohr group, $\mathcal{C}_\Lambda$ contains c_0, and Λ is not a Rosenthal set), and which is:*
(2) *$\Lambda(q)$ for all $q < +\infty$, a* CUC*-set, and moreover is:*
(a) *p-Rider, but not q-Rider for $q < p$*
(b) *q-Sidon for all $q > p/(2-p)$.*

Theorem 2.6 *Same as Theorem 2.5, except that, instead of* (a)*, Λ is:*
(a') *q-Rider for every $q > p$, but is not p-Rider.*

Remark. After this paper was completed, P. Lefèvre and the third-named author proved ([36]) that every p-Rider set with $p < 4/3$ is a q-Sidon set, for all $q > p/(2-p)$. A weaker, unpublished, result, due to J. Bourgain, is quoted in [15], p. 41. Hence condition (b) always follows from condition (a), and is not specific to the construction. We do not know whether this gap between p and $p/(2-p)$ follows only from technical reasons. For $p > 1$, whether every p-Rider set is actually p-Sidon is an open question.

In Theorem 2.5, we obtain sets which are p-Rider but not q-Rider for $q < p$. We do not know if these sets are p-Sidon, so exactly p-Sidon, in the terminology of R. Blei. He constructed such sets using fractional products ([3], [4]). We may call the sets in Theorem 2.5 "*exactly p-Rider sets*". The sets appearing in Theorem 2.6 are of a different kind. We may call them "*exactly p^+-Rider sets*". Such sets were also obtained in [3], Corol. 1.7 d), where they were called "exactly non-p-Sidon", and were called "asymptotic p-Sidon" in [5].

Proof. It is similar to that of Theorem 2.2, so we shall be very sketchy.

Let $\alpha = 2(p-1)/(2-p) \in]\,0,1\,[$.

For Theorem 2.5, we use selectors ε_k of mean

$$\delta_k = c\,\frac{(\log k)^\alpha}{k(\log\log k)^{\alpha+1}} \qquad \text{for } k \geq 4.$$

As in Lemma 2.3, we have, with $M_n = n^n$, $\sum_{n\geq 1} \mathbb{P}\big(\Omega_n(M_n)\big) < +\infty$, and almost surely $C_0 n^{\alpha+1} \leq |\Lambda_{M_n}| \leq C_1\, n^{\alpha+1}$ and $|\Lambda'_n| \leq C_1\, n^\alpha$ for n large enough.

For Theorem 2.6, we increase the means δ_k slightly, replacing them by

$$\delta_k = c\,\frac{(\log k)^\alpha \log\log k}{k}\,\cdot \qquad \square$$

Remark. In order to prove our theorems, we used selectors with various means. They are smaller in Theorem 2.2 than in Theorem 2.5, for instance. We remark that selectors $(\varepsilon_k)_k$ of mean δ_k with $\delta_k \leq \delta'_k$ may be achieved as the product of two independent sequences of selectors $(\varepsilon'_k)_k$ and $(\varepsilon''_k)_k$ of mean δ'_k and $\delta''_k = \delta_k/\delta'_k$. It follows that, for example, the sets in Theorem 2.2 may be constructed inside the respective sets of Theorem 2.5.

In Theorem 2.5, the proof that Λ was *CUC* or $\Lambda(q)$ was based on the fact that $|\Lambda'_n| \subseteq \Lambda \cap [M_n, +\infty[$ is quasi-independent. For $\alpha \geq 1$ (*i.e.* $p \geq 4/3$), we no longer have $|\Lambda'_n| \leq n$, and therefore, *a priori*, must give up these properties. However, we can use another extraction procedure. This procedure was first introduced by J. Bourgain ([8]); later, a clear statement was given in [52], § III.2. Since this last reference is hardly available, we prefer to give a self-contained proof.

The corresponding set $\Lambda(\omega)$ of integers that we shall obtain in this manner satisfies $|\Lambda(\omega) \cap [2^n, 2^{n+1}[| \sim n \sim \log 2^n$, which is the limiting condition of mesh (on arithmetic progressions) for Sidon sets. This size is in some sense the largest possible if we want to obtain a set Λ with blocks having a uniformly bounded Sidon constant.

Theorem 2.7 *There exists a set Λ of integers which is uniformly distributed and contains a subset $E \subseteq \mathbb{N}^*$ which is:*

(1) *4/3-Rider, and not q-Rider for $q < 4/3$; a* CUC*-set; a $\Lambda(q)$-set for all $q < +\infty$ (more precisely, for all $q > 2$, we have: $\|f\|_q \leq Cq^2\|f\|_2$ for all $f \in \mathcal{P}_E$, where $C > 0$ is a numerical constant), and nevertheless,*

(2) *has positive upper density in Λ, so, in particular, $\mathcal{C}_E$ contains c_0, and E is not a Rosenthal set.*

Let A be a finite subset of integers. For the proof, it will be convenient to define:

$$\psi_A = \sup_{p\geq 2} \frac{\|e_A\|_p}{\sqrt{p}}, \qquad \text{where } e_A = \sum_{k\in A} e_k\,.$$

We need the following simple estimate of ψ_A.

Lemma 2.8 *Let $I = [a+1, a+N]$ be an interval of integers of length N, $N \geq 3$. Then:*

$$\psi_I \leq \frac{N}{\sqrt{2\log N}} \,.$$

Proof. For $p \geq 2$, $|e_I|^p \leq N^{p-2}|e_I|^2$, so $\int |e_I|^p\,dm \leq N^{p-2}\int |e_I|^2\,dm = N^{p-1}$ and $\|e_I\|_p/\sqrt{p} \leq N^{1-1/p}/\sqrt{p}$. Optimizing gives $p = 2\log N$ (≥ 2), and the lemma. □

This estimate is essentially optimal. Indeed, it is well-known that ψ_I is uniformly equivalent to $\theta = \|e_I\|_\Psi$ ($\|\ \|_\Psi$ being the norm associated to the Orlicz function $\Psi(x) = \mathrm{e}^{x^2} - 1$). But, for some constant γ, $|e_I(t)| \geq \gamma N$ for t in an interval J of length $\geq \gamma N^{-1}$ around 0, so one has:

$$2 \geq \int_J \exp\Big(\frac{|e_I|^2}{\theta^2}\Big)\,dm \geq \gamma N^{-1}\exp\Big(\frac{\gamma^2 N^2}{\theta^2}\Big)\,,$$

whence $\theta \geq \gamma^{-1}N/\sqrt{\log 2\gamma^{-1}N}$.

We now use selectors ε_k of mean $\delta_k = c\,n/2^n$ for $2^n \leq k < 2^{n+1}$, where $c > 0$ is a given constant.

Set

$$I_n = [2^n, 2^{n+1}[,\ n \geq 2\,; \qquad \delta_k = c\,\frac{n}{2^n} \text{ if } k \in I_n\,.$$

Note that $(\delta_k)_k$ decreases, and δ_k is of the form α_k/k, where $(\alpha_k)_k$ goes to $+\infty$.

If $\Lambda = \Lambda(\omega)$ is the corresponding set of integers, it will be convenient to set:

$$\Lambda_n = \Lambda \cap I_n\,; \qquad \sigma_n = \mathbb{E}|\Lambda_n| = \sum_{k\in I_n}\delta_k = cn\,.$$

For this proof, the value of ψ_{I_n} is somewhat large, and requires c be sufficiently small, say $c \leq 1/576$. We prefer to follow another route, which could be useful in other contexts, by choosing also a random set in I_n for which the ψ constant is small enough. We make the two random choices at the same time. Namely, we consider $(\varepsilon'_n)_{n\geq 1}$, a second sequence of selectors, independent of $(\varepsilon_n)_{n\geq 1}$, with fixed mean τ, and set $\Lambda'_n(\omega) = \{k \in \Lambda_n(\omega)\,; \varepsilon'_k(\omega) = 1\}$. In short:

$$\Lambda'_n = \{k \in \Lambda_n\,;\ \varepsilon'_k = 1\}\,; \qquad \Lambda' = \bigcup_{n=1}^{+\infty}\Lambda'_n\,.$$

The following lemma, which is a slight modification of Bourgain's construction in [8], is really the heart of the proof.

Lemma 2.9 *Almost surely, for n large enough, one has:*

(1) $(c/2)\,n \leq |\Lambda_n| \leq (2c)\,n$ *and* $(c\tau/2)\,n \leq |\Lambda'_n| \leq (2c\tau)\,n$

(2) Λ'_n *contains at most relations of length* $\leq l_n$*, where* $l_n = [144\,c^2\tau^2 n]$.

Proof of Lemma 2.9. We have already seen that:

$$\mathbb{P}\Big(\big|\,|\Lambda_n| - \sigma_n\big| \geq \frac{\sigma_n}{2}\Big) \leq \exp\Big(-\frac{\sigma_n}{32}\Big) = \exp\Big(-\frac{c\,n}{32}\Big)\,,$$

so, by the Borel-Cantelli lemma, $|\Lambda_n|$ is almost surely between $(c/2)\,n$ and $(2c)\,n$ for n large enough; and this proves the first half of (1). The second half holds for the same reason, since Λ'_n corresponds to selectors $\varepsilon_k\varepsilon'_k$ with mean $(c\tau)\,n/2^n$ for $k \in I_n$.

The proof of (2) is more elaborate.

Fix n, and consider the random trigonometric polynomial:

$$F_\omega = \sum_{j=l_n+1}^{|I_n|} \sum_{\substack{R\subseteq I_n\\ |R|=j}} \prod_{k\in R} \varepsilon_k(\omega)\varepsilon'_k(\omega)\big(e_k + e_{-k}\big)\,.$$

Set:

$$N_n(\omega) = \int_{\mathbb{T}} F_\omega(t)\,dm(t)\,.$$

Expanding F_ω, we see that:

$$\begin{aligned} F_\omega(t) &= \sum_{j=l_n+1}^{|I_n|} \sum_{\substack{R\subseteq I_n\\ |R|=j}} \sum_{\theta_k\in\{-1,+1\}^R} \prod_{k\in R} \varepsilon_k(\omega)\varepsilon'_k(\omega)e_k^{\theta_k}(t) \\ &= \sum_{j=l_n+1}^{|I_n|} \sum_{\substack{R\subseteq I_n\\ |R|=j}} \sum_{\theta_k\in\{-1,+1\}^R} \mathrm{e}^{it\left(\sum_{k\in R}\theta_k k\right)}. \end{aligned}$$

The contribution to $N_n(\omega)$ of an exponential of this sum is 0 if $\sum\limits_{k\in R} \theta_k k \neq 0$, and is 1 if $\sum\limits_{k\in R} \theta_k k = 0$. Therefore, $N_n(\omega)$ is exactly the number of relations of length $> l_n$ in Λ'_n.

We claim that $N_n(\omega)$ is almost surely zero for n large enough. To that effect, we majorize the expectation J of $N_n(\omega)$, using Fubini's theorem. Indeed, $J = \int_{\mathbb{T}} H(t)\,dm(t)$, where:

$$H(t) = \int_\Omega F_\omega(t)\,d\mathbb{P}(\omega) = \sum_{j=l_n+1}^{|I_n|} \sum_{\substack{R\subseteq I_n\\ |R|=j}} \delta^j \prod_{k\in R} (e_k + e_{-k})\ .$$

and $\delta = c\tau\,n/2^n$. Hence:

$$J = \sum_{j=l_n+1}^{|I_n|} \sum_{\substack{R\subseteq I_n\\ |R|=j}} \delta^j \int_{\mathbb{T}} \prod_{k\in R} \big((e_k(t) + e_{-k}(t)\big)\,dm(t)\,.$$

At this stage, it is useful to observe that:

$$\sum_{\substack{R\subseteq I_n\\ |R|=j}} \int_{\mathbb{T}} \prod_{k\in R} \big(e_k(t)+e_{-k}(t)\big)\,dm(t) \\ \leq \frac{1}{j!}\int_{\mathbb{T}} \Big(\sum_{k\in I_n} \big((e_k(t)+e_{-k}(t)\big)\Big)^j\,dm(t)\,. \tag{3}$$

Indeed, when we expand

$$\Big(\sum_{k\in I_n} \big((e_k(t)+e_{-k}(t)\big)\Big)^j\,,$$

each term $\prod_{k\in R}\big(e_k(t)+e_{-k}(t)\big)$ appears $j!$ times, whereas the other terms on the right hand side of (3) are positive. It now follows from (3) that:

$$J \leq \sum_{j=l_n+1}^{|I_n|} \frac{\delta^j}{j!}\int_{\mathbb{T}} \Big(\sum_{k\in I_n} \big((e_k(t)+e_{-k}(t)\big)\Big)^j\,dm(t) \\ \leq \sum_{j=l_n+1}^{|I_n|} \frac{\delta^j}{j!}2^j \Big\|\sum_{k\in I_n} e_k\Big\|_j^j \leq \sum_{j=l_n+1}^{|I_n|} \frac{2^j\delta^j}{j!}(\psi_{I_n}\sqrt{j})^j.$$

Since $j!\geq (j/\mathrm{e})^j \geq (j/3)^j$, this gives

$$J \leq \sum_{j=l_n+1}^{+\infty} \Big(\frac{6\delta\psi_{I_n}}{\sqrt{j}}\Big)^j \leq \sum_{j=l_n+1}^{+\infty} \Big(\frac{6\delta\psi_{I_n}}{\sqrt{l_n+1}}\Big)^j.$$

Therefore,

$$J\leq 2^{-l_n} \qquad \text{if} \quad \frac{6\delta\psi_{I_n}}{\sqrt{l_n+1}} \leq \frac{1}{2}\,,$$

i.e. if $l_n+1\geq 144(\delta\psi_{I_n})^2$. But, it follows from Lemma 2.8 that:

$$\psi_{I_n} \leq \frac{2^n}{\sqrt{(2\log 2)\,n}} \leq \frac{2^n}{\sqrt{n}}\,\cdot$$

Therefore

$$144(\delta\psi_{I_n})^2 \leq 144\Big(c\tau\frac{n}{2^n}\cdot\frac{2^n}{\sqrt{n}}\Big)^2 = 144\,c^2\tau^2 n,$$

and the choice of l_n just fits to obtain $J\leq 2^{-l_n}$. Of course, we have assumed n large enough to have $l_n\geq 1$ in that proof.

Finally, Markov's inequality implies:

$$\sum_{n\geq 2}\mathbb{P}(N_n\geq 1) \leq \sum_{n\geq 2}\mathbb{E}N_n \leq \sum_{n\geq 2} 2^{-l_n} < +\infty\,,$$

and by the Borel-Cantelli lemma, the integer N_n is almost surely zero for n large enough, and that ends the proof of Lemma 2.9. $\square$

Now, using Bourgain's Theorem 1.10 and Lemma 2.9, one can find $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for $\omega \in \Omega_0$, there exists $n_0 = n_0(\omega)$ such that $\Lambda = \Lambda(\omega)$ and $\Lambda' = \Lambda'(\omega)$ satisfy:

(4) Λ and Λ' are uniformly distributed

(5) $(c/2)\, n \leq |\Lambda_n| \leq (2c)\, n$ and $(c\tau/2)\, n \leq |\Lambda'_n| \leq (2c\tau)\, n$ for $n > n_0$

(6) Λ'_n contains at most relations of length less than $\leq l_n = [144c^2\tau^2 n]$ for $n > n_0$.

Λ'_n is not quite quasi-independent, so we shall modify it slightly. We adjust once and for all τ, depending on c, such that $144c^2\tau^2 \leq c\tau/4$ (*e.g.* taking $c\tau = 1/576$), so that $l_n \leq c\tau\, n/4 \leq |\Lambda'_n|/2$ for $n > n_0$, in view of (5). Select then in Λ'_n a relation R with support S_n of maximal cardinality. Then $|S_n| \leq l_n$ from (6), and $E_n = \Lambda'_n \setminus S_n$ is quasi-independent. Moreover:

$$|E_n| = |\Lambda'_n| - |S_n| \geq |\Lambda'_n| - l_n \geq |\Lambda'_n|/2$$

for $n > n_0$. Hence, if we set $E = \bigcup_{n>n_0} E_n$, we have $E_n = E \cap I_n$, and, moreover:

(7) E has positive upper density in Λ

(note that Λ' has upper density $\geq \tau/4$ in Λ by (5)),

(8) E_n is quasi-independent,

(9) $|E_n| \geq (c\tau/4)\, n$,

(10) If $A \subseteq E$ is a finite subset, then A contains a quasi-independent subset B with $|B| \geq (1/2)\, |A|^{1/2}$.

The last property is proved in the following way. Set $Z = \{n\,;\ A \cap E_n \neq \emptyset\}$ and $h = |Z|$. We distinguish two cases.

Case 1: there exists $n \in Z$ such that $|A \cap E_n| \geq |A|^{1/2}$.

Then, just take $B = A \cap E_n$ to have a quasi-independent set B such that $|B| \geq |A|^{1/2}$.

Case 2: $|A \cap E_n| < |A|^{1/2}$ for any $n \in Z$.

Then $h \geq |A|^{1/2}$. Write $Z = \{n_1 < \cdots < n_h\}$, and pick an integer $m_j \in A \cap E_{n_j}$ for each $j = 1, \ldots, h$. Then $B = \{m_1, m_3, \ldots\} =: \{\mu_1, \mu_2, \ldots\}$ is quasi-independent because we have $\mu_{j+1}/\mu_j \geq 2$. Moreover $|B| \geq h/2 \geq (1/2)\, |A|^{1/2}$.

It is now easy to see that E has the required properties. Indeed, it follows from (4), (7), and from F. Lust-Piquard's Theorem 1.9 that E has a positive upper density in Λ. That it is CUC follows from (8) and from Proposition 1.2. That it is $\Lambda(q)$ for all $q < +\infty$ follows from (8) and from Proposition 1.1.

The fact that E is 4/3-Rider follows from (a) in Proposition 1.6. Indeed, if $\varepsilon(p) = 2/p - 1$, then $\varepsilon(4/3) = 1/2$.

Finally, let N be a large integer, and n such that $2^n \leq N < 2^{n+1}$. Then

$$\begin{aligned}|E \cap [1, N]| &\geq |E_{n_0+1}| + \cdots + |E_{n-1}| \geq \frac{c\tau}{4}\big[(n_0 + 1) + \cdots + (n - 1)\big] \\ &\geq d_0 n^2 \geq d_1 (\log N)^2\end{aligned}$$

where d_0, d_1 are positive constants. If now E is a p-Rider set, we have the mesh condition $|E \cap [1, N]| = O\left((\log N)^{p/(2-p)}\right)$. This requires $2 \leq p/(2 - p)$, that is $p \geq 4/3$. And this ends the proof of Theorem 2.7. □

Remark. The third-named author proved the following ([52], Lema 2.4) (which is actually implicitly already contained in [48], Lemme 7.2, Théorème 7.1, and Théorème 2.3 (iv)):

(∗) For every finite subset $A \subseteq \mathbb{Z}$, there exists a quasi-independent subset $B \subseteq A$ such that $|B| \geq \delta(|A|/\psi_A)^2$, where $\delta > 0$ is a numerical constant.

On the other hand, G. Pisier ([47], Lemme 5.2) proved:

$$\mathbb{E}\Big\| \sum_k a_k r_k e_k \Big\|_\Psi \leq C\Big(\sum_k |a_k|^2\Big)^{1/2} \tag{1}$$

where C is a numerical constant, $(r_k)_k$ is the Rademacher sequence, and $\|\ \|_\Psi$ is the Orlicz space associated to $\Psi(x) = \mathrm{e}^{x^2} - 1$.

Taking our selectors ε_k with mean $\delta_k = c\,n/2^n$ for $k \in I_n$, standard symmetrization and centering arguments give:

$$\mathbb{E}\Big\| \sum_{k \in I_n} \varepsilon_k e_k \Big\|_\Psi \leq C\sqrt{n}\,. \tag{2}$$

In other terms, we have, in view of Lemma 2.9:

$$\mathbb{E}(\psi_{\Lambda_n}) \leq C\sqrt{n} \leq C'|\Lambda_n|^{1/2}. \tag{3}$$

If we could prove a concentration inequality, variant of Lemma 1.3, then this variant and the Borel-Cantelli lemma would imply from (3) that:

$$\text{Almost surely } \psi_{\Lambda_n} \leq C''|\Lambda_n|^{1/2} \text{ for } n \text{ large enough.} \tag{4}$$

We could then combine (∗) and (4) directly to obtain the following alternative proof of Theorem 2.7. Select $\omega \in \Omega$ such that Λ is strongly ergodic, with $|\Lambda_n| \geq c\,n/2$, and $\psi_{\Lambda_n} \leq C''|\Lambda_n|^{1/2}$; take then a quasi-independent set $E_n \subseteq \Lambda_n$ of size

$$|E_n| \geq \delta\left(\frac{|\Lambda_n|}{\psi_{\Lambda_n}}\right)^2 \geq \delta C''^{-2}|\Lambda_n| \geq \delta' n\,;$$

the set $E = \bigcup_n E_n$ then has the required properties.

To end this section, we consider the case $p > 4/3$. We cannot keep the property of uniform convergence (CUC), nor that of being q-Sidon stated in Theorem 2.5. We do not know whether this is only due to the method. But being p-Rider with $p > 4/3$ might be a rather weak condition (see [35] and [36]).

Theorem 2.10 *For every $4/3 \leq p < 2$ there exists a set Λ of integers which is p-Rider, but is not q-Rider for $q < p$ and which is $\Lambda(q)$ for every $q < +\infty$, but which is uniformly distributed (so in particular dense in the Bohr group, and $\mathcal{C}_\Lambda$ contains c_0).*

The proof is essentially the same as in Theorem 2.2, except that we take selectors ε_k of mean

$$\delta_k = c\,\frac{(\log k)^\alpha}{k(\log\log k)^{\alpha+1}} \quad \text{for } k \geq 1,$$

where $\alpha = 2(p-1)/(2-p) \geq 1$, and replace $M_n = n^n$ by the smallest integer $\geq n^{\beta n}$, with β any number $> \alpha$ (for instance $\beta = \alpha + 1$), which we call again M_n. The estimate:

$$\mathbb{P}\big(\Omega_n(M_n)\big) \leq 2\,\frac{c^n}{n^n}\,\frac{(\log M_n)^{n(\alpha+1)}}{M_n}$$

still holds, and now gives:

$$\mathbb{P}\big(\Omega_n(M_n)\big) \leq C'^{\,n}\,\frac{(\log n)^{n(\alpha+1)}}{n^{(\beta-\alpha)n}}\,\cdot$$

Then easy computations show that:

($*$) Almost surely $|\Lambda_{M_n}| \sim (n\log n)^{\alpha+1}$ for n sufficiently large;

($**$) Almost surely $|\Lambda'_n| \sim n^\alpha(\log n)^{\alpha+1}$ for n sufficiently large.

Property ($*$) guaranties that $\Lambda(\omega)$ will still be almost surely p-Rider, and ($**$) with the mesh condition implies that Λ is not q-Rider for $q < p$.

The $\Lambda(q)$ property cannot be obtained by the Littlewood-Paley method, but follows from [43], Theorem 4.7. □

3 Large thin sets in prescribed sets of integers

In this section, we start from a prescribed set $\Lambda_0 = \{\lambda_1 < \lambda_2 < \ldots < \lambda_N < \ldots\}$ of positive integers, and randomly construct a thin set Λ inside Λ_0 in the following way. We still have our selectors $\varepsilon_1, \ldots, \varepsilon_N, \ldots$ of respective means $\delta_1, \ldots, \delta_N, \ldots$. This time, however, we set

$$\Lambda = \Lambda(\omega) = \{\lambda_j \in \Lambda_0\,;\ \varepsilon_j(\omega) = 1\},$$

i.e. we select randomly some of the λ_j's, and ignore the other integers. Such constructions have been made previously by S. Neuwirth ([43]).

We always assume that Λ_0 is ergodic, namely that

$$A_{\Lambda_0,N}(t) = N^{-1}\big(\mathrm{e}^{i\lambda_1 t} + \cdots + \mathrm{e}^{i\lambda_N t}\big) \underset{N\to+\infty}{\longrightarrow} l(t)\,, \qquad \forall t \in \mathbb{T}.$$

In this context, we have the following theorem, which extends Bourgain's Theorem 1.10.

Theorem 3.1 ([43], Th. 5.4) *Let Λ_0 be an ergodic (resp. strongly ergodic, resp. uniformly distributed) set of positive integers, and let $\varepsilon_1, \ldots, \varepsilon_N, \ldots$ be selectors with respective expectation $\delta_1, \ldots, \delta_N, \ldots$ with $(\delta_n)_{n\geq 1}$ decreasing. Assume that $\sigma_N / \log \lambda_N \underset{N\to+\infty}{\longrightarrow} +\infty$, where $\sigma_N = \delta_1 + \cdots + \delta_N$. Then, almost surely, the set Λ is ergodic (resp. strongly ergodic, resp. uniformly distributed). More precisely, if $A_{\Lambda_0,N}(t) \underset{N\to+\infty}{\longrightarrow} l(t)$, we have, almost surely, with $\Lambda_N = \Lambda \cap \{\lambda_1, \ldots, \lambda_N\}$,*

$$A_N(t) = \frac{1}{|\Lambda_N|} \sum_{n\in\Lambda_N} e_n(t) \underset{N\to+\infty}{\longrightarrow} l(t)\,, \qquad \forall t \in \mathbb{T}.$$

We sketch the proof. First, we require

Lemma 3.2 *Let $\varepsilon_1, \ldots, \varepsilon_N$ be selectors of respective expectations $\delta_1, \ldots, \delta_N$. Setting $\sigma_N = \delta_1 + \cdots + \delta_N$, one has the following inequality:*

$$\mathbb{P}\big(\big\|\sum_{k=1}^{N}(\varepsilon_k - \delta_k)e_{\lambda_k}\big\|_\infty > 15\sqrt{\sigma_N \log \lambda_N}\big) \leq 8/N^2\,,$$

provided that $\sigma_N \geq 25 \log \lambda_N$.

Proof. Set $Q = \sum_{k=1}^{N}(\varepsilon_k - \delta_k)e_{\lambda_k}$. For fixed $t \in \mathbb{R}$, one has $Q(t) = \sum_{k=1}^{N} X_k$, where $X_k = e_{\lambda_k}(t)(\varepsilon_k - \delta_k)$. The X_k's are independent, bounded by 1, and centered complex random variables; so, letting $t_N = 5\sqrt{\sigma_N \log \lambda_N}$, and using Lemma 1.3, we get

$$\begin{aligned}\mathbb{P}(\|Q\|_\infty > 3t_N) &\leq \mathbb{P}(\sup_{t\in F_N} |Q(t)| > t_N) \leq \sum_{t\in F_N} \mathbb{P}(|Q(t)| > t_N) \\ &\leq 4|F_N| \exp\big(-\frac{t_N^2}{8\sigma_N}\big) = 8\lambda_N^{1-25/8} \leq N^{1-25/8} \leq 8N^{-2},\end{aligned}$$

where $F_N = \big\{ j\pi/\lambda_N\,;\ 0 \leq j \leq 2\lambda_N - 1 \big\}$ is the set of the $(2\lambda_N)^{th}$ roots of unity, and where the first inequality follows from Bernstein inequality (see [22]). □

Proof of Theorem 3.1. Notice first that

$$\frac{1}{\sigma_N} \sum_{n=1}^{N} \delta_n e_{\lambda_n}(t) \underset{N\to+\infty}{\longrightarrow} l(t)\,, \qquad \forall t \in \mathbb{T}\,.$$

In fact, set $E_n = e_{\lambda_1}(t) + \cdots + e_{\lambda_n}(t)$ and $l = l(t)$. Since $(\delta_n)_n$ is nonincreasing, two Abel's partial summations give:

$$\begin{aligned}\sum_{n=1}^{N} \delta_n e_{\lambda_n}(t) &= \sum_{n=1}^{N-1} (\delta_n - \delta_{n+1}) E_n + \delta_N E_N \\ &= \sum_{n=1}^{N-1} nl(\delta_n - \delta_{n+1}) + Nl\delta_N + o\Big(\sum_{n=1}^{N-1} n(\delta_n - \delta_{n+1}) + N\delta_N\Big) \\ &= l\sigma_N + o(\sigma_N).\end{aligned}$$

Setting $Q_N = \sum_{n=1}^{N} (\varepsilon_n - \delta_n) e_{\lambda_n}$, we have:

$$\Big\| A_N - \frac{1}{\sigma_N} \sum_{n=1}^{N} \delta_n e_{\lambda_n} \Big\|_\infty \leq \frac{2\|Q_N\|_\infty}{\sigma_N},$$

since

$$\begin{aligned}&\Big\| \frac{1}{S_N} \sum_{n=1}^{N} \varepsilon_n e_{\lambda_n} - \frac{1}{\sigma_N} \sum_{n=1}^{N} \delta_n e_{\lambda_n} \Big\|_\infty \\ &\qquad \leq \Big| \frac{1}{S_N} - \frac{1}{\sigma_N} \Big| \, \Big\| \sum_{n=1}^{N} \varepsilon_n e_{\lambda_n} \Big\|_\infty + \frac{1}{\sigma_N} \Big\| \sum_{n=1}^{N} (\varepsilon_n - \delta_n) e_{\lambda_n} \Big\|_\infty \\ &\qquad \leq \frac{|S_N - \sigma_N|}{\sigma_N} + \frac{\|Q_N\|_\infty}{\sigma_N} = \frac{|Q_N(0)| + \|Q_N\|_\infty}{\sigma_N} \leq \frac{2\|Q_N\|_\infty}{\sigma_N}.\end{aligned}$$

Now, Lemma 3.2 gives:

$$\mathbb{P}\big(\|Q_N\|_\infty > 15\sqrt{\sigma_N \log \lambda_N}\big) \leq 8\lambda_N^{-2} \leq 8N^{-2}$$

if $\sigma_N \geq 25 \log \lambda_N$; so we get, by the Borel-Cantelli lemma,

$$\frac{\|Q_N\|_\infty}{\sigma_N} = O\Big(\sqrt{\frac{\log \lambda_N}{\sigma_N}}\Big)$$

almost surely. In view of the hypothesis, we have:

$$\big\| A_N - \frac{1}{\sigma_N} \sum_{n=1}^{N} \delta_n e_{\lambda_n} \big\|_\infty \underset{N \to +\infty}{\longrightarrow} 0 \quad \text{almost surely;}$$

and so, almost surely $A_N(t) \underset{N \to +\infty}{\longrightarrow} l(t)$ for each t, which is the desired conclusion. □

3.2 Regularity

Let I be a finite interval of $\mathbb{N}^*$ and $\nu(I) = |\Lambda_0 \cap I|$ be the number of indices n for which $\lambda_n \in I$. In the sequel, we assume that Λ_0 has the following regularity property:

There exists a continuous eventually strictly increasing function $\varphi\colon \,]0,+\infty[\,\to\,]0,+\infty[$ *such that*:

$$\frac{\nu([N,2N[)}{\varphi(N)} \underset{N\to+\infty}{\longrightarrow} 1 \tag{3.1}$$

and:

$$\frac{\varphi(2x)}{\varphi(x)} \underset{x\to+\infty}{\longrightarrow} l > 1. \tag{3.2}$$

Note that $l \leq 2$, since $\nu([2^k, 2^{k+1}[\leq 2^k$ implies that $(1-\varepsilon)^{k-k_0} l^{k-k_0} \varphi(2^{k-k_0}) \leq \varphi(2^k) \leq (1+\varepsilon)\, 2^k$.

We say that Λ_0 is *regular* if these properties hold.

They are obviously verified when $\lambda_n = n^s$, and also, by the Prime Number Theorem, when $\lambda_n = p_n$, with $\varphi(x) = x/\log x$.

It is easy to see that (3.1) and (3.2) imply that Λ_0 has a polynomial growth, namely that there exist two constants, $a, d > 0$ such that:

$$\nu([1,k]) \geq a\, k^d. \tag{3.3}$$

(or, equivalently, $\lambda_N \leq a' N^{1/d}$).

It follows that the condition $\sigma_N/\log \lambda_N \underset{N\to+\infty}{\longrightarrow} +\infty$ of Theorem 3.1 reduces then to the previous condition $\sigma_N/\log N \underset{N\to+\infty}{\longrightarrow} +\infty$ of Theorem 1.10.

Moreover Λ_0 satisfies:

$$\lambda_{8n} \geq 2\lambda_n \qquad \textit{for } n \geq 1 \textit{ large enough.} \tag{3.4}$$

Indeed, if $\nu([1, 2^{k-1}[) < n \leq \nu([1, 2^k[)$, then $2\lambda_n \leq 2^{k+1}$ and it suffices to show that $\nu([2^{k-1}, 2^{k+1}[) \leq 7n$. But

$$\begin{aligned}
\nu([2^{k-1}, 2^{k+1}[) &\leq (1+\varepsilon)\big(\varphi(2^{k-1}) + \varphi(2^k)\big) \leq (1+\varepsilon)^2 (l^2 + l) \varphi(2^{k-2}) \\
&\leq (1+\varepsilon)^3 (l^2 + l) \nu([2^{k-2}, 2^{k-1}[) \leq (1+\varepsilon)^3 (l^2 + l) \nu([1, 2^{k-1}[) \\
&\leq 7\nu([1, 2^{k-1}[) \leq 7n
\end{aligned}$$

for $\varepsilon > 0$ small enough, and n large enough.

As in Section 2, we restrict ourselves to selectors with mean $\delta_n = \alpha_n/n$, where $(\delta_n)_n$ decreases to 0, and $(\alpha_n)_n$ tends to infinity, and moreover, except in the last theorem, $(\alpha_n)_n$ increases.

The following lemma is quite similar to Lemma 2.1. We indicate some changes which are needed, and how the regularity occurs.

Lemma 3.3 *Let* $s \geq 2$ *and* M *be integers and let*

$\Omega_s(M) = \{\omega \in \Omega\,;\ \Lambda(\omega) \cap [\lambda_M, +\infty[$ *contains at least a relation of length* s $\}$.

We have, for s *large enough,*

$$\mathbb{P}\big(\Omega_s(M)\big) \leq \frac{(16\mathrm{e})^s}{s^s} \sum_{j>M} \delta_j^2 \sigma_j^{s-2}.$$

Proof. As in the proof of Lemma 2.1, we write $\Omega_s(M) = \bigcup_{l \geq M+s-1} \Delta_l$, where Δ_l is defined by

$$\Delta_l = \{\omega\,;\ \Lambda(\omega) \cap [\lambda_M, +\infty[\text{ contains at least a relation of length } s \text{ and with greatest term } \lambda_l\}.$$

It suffices to show that

$$\mathbb{P}(\Delta_l) \leq \frac{8^s 2^{s-2}}{(s-2)!}\,\delta_l^2 \sigma_l^{s-2}.$$

The proof proceeds as in Lemma 2.1, replacing $i_1, \ldots, i_{s-1}$ and l by $\lambda_{i_1}, \ldots, \lambda_{i_{s-1}}$ and λ_l respectively. The relation $(**)$ gives $\lambda_{i_{s-1}} \geq \lambda_l / s$. The regularity appears now to say that $i_{s-1} \geq l/8^s$. Indeed, otherwise, by (3.4), we should have, for s large enough,

$$\lambda_l > \lambda_{8^s i_{s-1}} \geq 2^s \lambda_{i_{s-1}} \geq 2^s \frac{\lambda_l}{s} > \lambda_l.$$

This gives the lemma since $(\alpha_n)_n$ increases:

$$\delta_{i_{s-1}} = \frac{\alpha_{i_{s-1}}}{i_{s-1}} \leq \frac{\alpha_l}{i_{s-1}} = \frac{\alpha_l}{l}\,\frac{l}{i_{s-1}} \leq 8^s \delta_l\,. \qquad \square$$

Since this basic lemma still holds for random subsets of prescribed sets Λ_0, the first main theorems of Section 2 still hold and their proofs requires only minor modifications because of Theorem 3.1. We therefore content ourselves with stating them.

Theorem 3.4 *Let Λ_0 be a regular, strongly ergodic set of positive integers. There exists a set $\Lambda \subseteq \Lambda_0$ which is:*

(1) *p-Sidon for all $p > 1$, $\Lambda(q)$ for all $q < +\infty$,* CUC, *but which is:*

(2) *strongly ergodic (in particular, $\mathcal{C}_\Lambda$ contains c_0 and Λ is not a Rosenthal set).*

Theorem 3.5 *Let Λ_0 be as in the previous theorem, and let $1 < p < 4/3$. Then, there exists a set $\Lambda \subseteq \Lambda_0$ which is:*

(1) *strongly ergodic (in particular, $\mathcal{C}_\Lambda$ contains c_0 and so Λ is not a Rosenthal set), but which is:*

(2) *a* CUC*-set, $\Lambda(q)$ for all $q < +\infty$, and*
   - (a) *is p-Rider, but is not q-Rider for $q < p$,*
   - (b) *is q-Sidon for all $q > p/(2-p)$.*

Theorem 3.6 *Same as in the previous theorem, but instead of property* (a)*:*
(a') *Λ is q-Rider for every $q > p$, but is not p-Rider.*

Theorem 3.7 *Let $\Lambda_0 = \{\lambda_1, \ldots\}$ be a regular, strongly ergodic set. Then, there exists a set $\Lambda \subseteq \Lambda_0$ which is strongly ergodic and contains a set E which*

(1) *has a positive upper density in $\Lambda(\omega)$ (so in particular, $\mathcal{C}_E$ contains c_0 and E is not a Rosenthal set), and*

(2) *is a* CUC*-set, is* $4/3$*-Rider, but not* q*-Rider for* $q < 4/3$*, and is a* $\Lambda(q)$*-set for all* $q < +\infty$*; more precisely, for all* $q > 2$*, we have* $\|f\|_q \leq Cq^2\|f\|_2$ *for all* $f \in \mathcal{P}_E$*, where* $C > 0$ *is a numerical constant.*

The proof is the same as that of Theorem 2.7, so we omit it. We merily note the following facts.

The sequence $(\delta_k)_k$ is eventually decreasing. Indeed, for $n \geq n_\varepsilon$, we have, by the regularity conditions (3.1) and (3.2), if $\varepsilon > 0$ is chosen so that $(1-\varepsilon)^2 l \geq 1$,

$$\nu_{n+1} \geq (1-\varepsilon)\varphi(2^{n+1}) \geq (1-\varepsilon)^2 l\, \varphi(2^n) \geq (1-\varepsilon)^3 l\, \nu_n \geq \nu_n\,.$$

Next, (3.1) implies that, for some constant $\alpha > 0$, and for $2^q < N \leq 2^{q+1}$,

$$\begin{aligned}\sigma_N = \delta_1 + \cdots + \delta_N &\geq \sum_{n=1}^{q} \Big(\sum_{k \in I_n} \delta_k \Big) = c \sum_{n=1}^{q} \log \nu_n \\ &\geq c\alpha \sum_{n=1}^{q} n \geq c(\alpha/2) q^2.\end{aligned}$$

Since Λ_0 has polynomial growth: $\lambda_N = O\,(N^{1/d})$, we have $\log \lambda_N \leq \lambda_{2^{q+1}} = O\,(q)$. It follows that:

$$\sigma_N / \log \lambda_N \underset{N \to +\infty}{\longrightarrow} +\infty\,.$$

Finally, we have to replace the parameter ψ_A in the proof of Theorem 2.7 by:

$$\psi'_A = \sup_{p \geq 2} \frac{\|e'_A\|_p}{\sqrt{p}}, \qquad \text{where } e'_A = \sum_{\lambda_k \in A} e_{\lambda_k}\,.$$

Since we have, for any interval I : $\psi'_I \leq C\,\nu(I)/\sqrt{\log \nu(I)}$, the rest of the proof will then work with no essential change. □

Remark. Consider the ψ-parameter associated to the squares, that is:

$$\psi'_N = \sup_{q \geq 2} \frac{\|S_N\|_q}{\sqrt{q}}\,,$$

where $S_N(x) = \sum\limits_{n=1}^{N} \mathrm{e}^{in^2 x}$. It follows from results of Zalcwasser ([61]), that we have very precise estimates on $\|S_N\|_q$: there exist numerical constants $C_1, C_2 > 0$ such that:

$$C_1 N^{1-2/q} \leq \|S_N\|_q \leq C_2 N^{1-2/q}$$

whenever $q \geq 5$ and $N \geq 1$ (when q is near 4, a logarithmic factor $(\log N)^{1/q}$ should be added in the upper estimate). Therefore, the *a priori* crude estimate used in the proof of Theorem 2.7 is, at least for the squares, optimal, as it is for the set of all the positive integers.

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