Some revisited results about composition operators on Hardy spaces

Pascal Lefèvre, Daniel Li,
Hervé Queffélec, Luis Rodríguez-Piazza

January 19, 2010

Abstract. *We generalize, on one hand, some results known for composition operators on Hardy spaces to the case of Hardy-Orlicz spaces H^Ψ: construction of a "slow" Blaschke product giving a non-compact composition operator on H^Ψ; construction of a surjective symbol whose composition operator is compact on H^Ψ and, moreover, is in all the Schatten classes $S_p(H^2)$, $p > 0$. On the other hand, we revisit the classical case of composition operators on H^2, giving first a new, and simplier, characterization of closed range composition operators, and then showing directly the equivalence of the two characterizations of membership in the Schatten classes of Luecking and Luecking and Zhu.*

Mathematics Subject Classification. Primary: 47B33 – Secondary: 47B10

Key-words. Blaschke product – Carleson function – Carleson measure – composition operator – Hardy-Orlicz space – Nevanlinna counting function – Schatten classes

1 Introduction

The study of composition operators on Hardy spaces is now a classical subject (see [18], [3] for example). In [8] (see also [7]), we considered a more general setting and studied composition operators on Hardy-Orlicz spaces; we gave there a characterization of their compactness in terms of the Carleson function of their symbol (and in terms of the Nevanlinna counting function in [11]). This work was continued in [10]: we compared the compactness on Hardy spaces versus the compactness on Hardy-Orlicz spaces. For instance, we showed that there is, for every $1 \leq p < \infty$, an Orlicz function Ψ such that $H^{p+\varepsilon} \subseteq H^\Psi \subseteq H^p$ for every $\varepsilon > 0$, and a composition operator C_φ such that C_φ is compact on H^p and $H^{p+\varepsilon}$, but which is not compact on H^Ψ.

We carry on this study in the present work. In a first part (Section 3 and Section 4), we shall improve, and extend to the Hardy-Orlicz case, results known for Hardy spaces; in a second part (Section 5 and Section 6), we shall give new lights on some results concerning Hardy spaces. More precisely, the content of this paper is as following.

B. McCluer and J. Shapiro ([14], Theorem 3.10; see also [18], § 3.2) proved that, when their symbol φ is finitely-valent, compactness of composition operators C_φ on the Hardy space H^2 can be characterized by the behaviour of the modulus of φ near the frontier of $\mathbb{D}$: compactness is equivalent to $1-|z| = 0\,\big(1-|\varphi(z)|\big)$ as $|z|\to 1$, but that is not equivalent in general ([14], Example 3.8; see also [18], § 10.2). In [11], Theorem 5.3, we gave such a characterization for composition operators, with finitely-valent symbol, on Hardy-Orlicz spaces. In Section 3, we construct a "slow" Blaschke product (generalizing [18], § 10.2 and [8], Proposition 5.5) showing that this condition is not sufficient in general.

In Section 4, we construct a compact composition operator $C_\varphi\colon H^\Psi \to H^\Psi$ with surjective symbol φ and such that $C_\varphi\colon H^2 \to H^2$ is in all the Schatten classes $S_p(H^2)$, $p > 0$. This generalizes and improves a result of B. McCluer and J. Shapiro ([14], Example 3.12; see also the survey [16], § 2).

In Section 5, we give a characterization of composition operators $C_\varphi\colon H^p \to H^p$, $1 \le p < \infty$, with a closed range, simpler than the former ones (see [1] and [20]).

Finally, based on the main result of [11], we show directly, in Section 6, the equivalence of Luecking's and Luecking-Zhu's criteria ([12], [13]) for the membership of $C_\varphi\colon H^2 \to H^2$ in the Schatten classes.

Acknowledgement. Part of this work was made during the fourth-named author visited the University of Lille 1 and the University of Artois (Lens) in June 2009. This fourth-named author is partially supported by a Spanish research project MTM2006-05622.

2 Notation

The open unit disk is denoted by $\mathbb{D} = \{z \in \mathbb{C}\,;\ |z| < 1\}$ and its boundary, the unit circle, by $\mathbb{T} = \{z \in \mathbb{C}\,;\ |z| = 1\}$. The normalized Lebesgue measure $dt/2\pi$ on $\mathbb{T}$ is denoted by m. The normalized area measure $dx\,dy/\pi$ is denoted by A.

The Hardy space H^1 is the space of analytic functions $f\colon \mathbb{D}\to\mathbb{C}$ such that $\sup_{r<1}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta < \infty$. Every $f \in H^1$ has almost everywhere boundary values on $\mathbb{T}$, which are denoted by f^*.

An Orlicz function is a convex nondecreasing function $\Psi\colon [0,\infty)\to[0,\infty)$ such that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. If μ is a positive measure on some measurable space S, the Orlicz space $L^\Psi(\mu)$ is the set of all (classes of) measurable functions $f\colon S\to\mathbb{C}$ such that $\int_S \Psi(|f|/C)\,d\mu < \infty$ for some $C > 0$; the norm $\|f\|_\Psi$ is defined as the infimum of the positive numbers C for which $\int_S \Psi(|f|/C)\,d\mu \le 1$.

The Hardy-Orlicz space H^Ψ is the linear subspace of $f \in H^1$ such that $f^* \in L^\Psi(m)$ (see [8]).

Every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ defines a bounded composition operator $C_\varphi\colon f \in H^\Psi \mapsto f \circ \varphi \in H^\Psi$ (see [8]).

For every $\xi \in \mathbb{T}$ and $0 < h < 1$, the Carleson window is the set $W(\xi, h) = \{z \in \mathbb{D}\,;\ |z| \geq 1-h \text{ and } |\arg(z\,\bar\xi| \leq h\}$. The Carleson function ρ_φ of the analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ is defined, for $0 < h < 1$, by:

$$\rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} m\big(\{\mathrm{e}^{i\theta} \in \mathbb{T}\,;\ \varphi^*(\mathrm{e}^{i\theta}) \in W(\xi, h)\}\big)\,.$$

Alternatively, $\rho_\varphi(h) = \sup_{\xi\in\mathbb{T}} m_\varphi[W(\xi, h)]$, where m_φ is the pull-back measure of m by φ. We shall also use, instead of $W(\xi, h)$, the set $S(\xi, h) = \{z \in \mathbb{D}\,;\ |z - \xi| \leq h\}$, which has an equivalent size.

The Nevanlinna counting function N_φ is defined, for $w \in \varphi(\mathbb{D}) \setminus \{\varphi(0)\}$, by

$$N_\varphi(w) = \sum_{\varphi(z)=w} \log\frac{1}{|z|}\,,$$

each term $\log\frac{1}{|z|}$ being repeated according to the multiplicity of z, and $N_\varphi(w) = 0$ for the other $w \in \mathbb{D}$.

3 Slow Blaschke products

B. McCluer and J. Shapiro ([14], Theorem 3.10; see also [18], § 3.2) proved that, when φ is finitely-valent (meaning that, for some $s \geq 1$, the equation $\varphi(z) = w$ has at most s solutions), the composition operators $C_\varphi\colon H^p \to H^p$ is compact, $1 \leq p < \infty$, if and only if φ has an angular derivative at no point of $\mathbb{T}$; that means that:

$$\lim_{|z|\to 1} \frac{1-|z|}{1-|\varphi(z)|} = 0\,. \tag{3.1}$$

In [11], Theorem 5.3, we generalized this result to Hardy-Orlicz spaces and proved that if φ is finitely-valent, the composition operator $C_\varphi\colon H^\Psi \to\colon H^\Psi$ is compact if and only if:

$$\lim_{|z|\to 1} \frac{\Psi^{-1}\left[\dfrac{1}{1-|\varphi(z)|}\right]}{\Psi^{-1}\left[\dfrac{1}{1-|z|}\right]} = 0\,. \tag{3.2}$$

Without the assumption that φ is finitely-valent, condition (3.2) is no longer sufficient to ensure the compactness of $C_\varphi\colon H^\Psi \to H^\Psi$. Indeed, we are going to construct a Blaschke product satisfying (3.2), but whose associated composition operator is of course not compact on H^Ψ, as this is the case for every inner function. A Blaschke product satisfying (3.1) is constructed in [18], § 10.2; that construction uses Frostman's Theorem. Our construction, which is more general, is entirely elementary.

Theorem 3.1 *Let $\delta\colon (0,1) \to (0,1/2]$ be any function such that $\lim_{t\to 0} \delta(t) = 0$. Then, there exists a Blaschke product B such that:*

$$1 - |B(z)| \geq \delta(1 - |z|), \qquad \text{for all } z \in \mathbb{D}. \tag{3.3}$$

Corollary 3.2 *For every Orlicz function Ψ there exists a Blaschke product B which satisfies:*

$$\lim_{|z|\to 1} \frac{\Psi^{-1}\left[\dfrac{1}{1 - |B(z)|}\right]}{\Psi^{-1}\left[\dfrac{1}{1 - |z|}\right]} = 0\,.$$

though the composition operator $C_B\colon H^\Psi \to H^\Psi$ is not compact.

Proof. C_B is not compact since every compact composition operator should satisfy $|\varphi^*| < 1$ *a.e.* (see [8], Lemma 4.8). It suffices then to chose $\delta(t) = 1/\Psi\big(\sqrt{\Psi^{-1}(1/t)}\big)$, which satisfies the hypothesis of Theorem 3.1. Moreover:

$$\frac{\Psi^{-1}\big(1/\delta(t)\big)}{\Psi^{-1}(1/t)} = \frac{1}{\sqrt{\Psi^{-1}(1/t)}} \underset{t\to 0}{\longrightarrow} 0\,,$$

and condition (3.3) gives the result. □

Proof of Theorem 3.1. We shall essentially construct our Blaschke product B as an infinite product of finite Blaschke products

$$\prod_n B_n\,,$$

where each finite Blaschke product B_n has p_n zeros equidistributed in the circumference of radius r_n. That is, we will have, writing $\theta_k = 2\pi k/p_n$ and $z_k = r_n\,\mathrm{e}^{i\theta_k}$, for $k = 1, 2, \ldots, p_n$:

$$B_n(z) = \prod_{k=1}^{p_n} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z}_k z} = \prod_{k=1}^{p_n} \frac{r_n - \mathrm{e}^{-i\theta_k} z}{1 - r_n \mathrm{e}^{-i\theta_k} z}\,. \tag{3.4}$$

We shall need the following estimate for the finite Blaschke product in (3.4).

Lemma 3.3 *Let $p \in \mathbb{N}$, and $0 < r < 1$. Consider the finite Blaschke product*

$$G(z) = \prod_{k=1}^{p} \frac{r - \mathrm{e}^{-i\theta_k} z}{1 - r\mathrm{e}^{-i\theta_k} z}\,, \tag{3.5}$$

where $\theta_k = \frac{2k\pi}{p}$, for $k = 1, 2, \ldots, p$.

(*a*) *Then, for every $z \in \mathbb{D}$ with $|z| = r$,*

$$|G(z)| \leq \frac{2r^p}{1 + r^{2p}} = 1 - \frac{(1 - r^p)^2}{1 + r^{2p}}\,. \tag{3.6}$$

(*b*) *If besides we have* $p\,h \le 1/2$*, where* $h = 1 - r$*, we also have, for every* $z \in \mathbb{D}$ *with* $|z| = r$*,*

$$|G(z)| \le 1 - \frac{(p\,h)^2}{2\mathrm{e}}\,\cdot \tag{3.7}$$

Let us continue the proof of the theorem. Define $\chi\colon (0,1) \to (0,1]$ by:

$$\chi(x) = \sup_{t \le x} \big[\, \max\{2\delta(t), \sqrt{t}\}\big]\,. \tag{3.8}$$

Then χ is non-decreasing, $\lim_{x\to 0}\chi(x) = 0$ and $\lim_{x\to 1}\chi(x) = 1$. We can find a decreasing sequence $(h_n)_{n\ge 0}$ of point $h_n \in (0,1)$, such that $\chi(h_n) \le 2^{-n}$. This sequence converges to 0; in fact, $\sqrt{h_n} \le \chi(h_n) \le 2^{-n}$, by (3.8), and hence:

$$h_n \le 2^{-2n}\,. \tag{3.9}$$

We now define, for every $n \in \mathbb{N}$, a positive integer p_n, by:

$$p_n = \min\{p \in \mathbb{N}\,;\ \frac{p^2 h_n^2}{2\mathrm{e}} > 2^{-n}\ \}. \tag{3.10}$$

We have $p_n > 1$ because $h_n^2/2\mathrm{e} < h_n^2 \le 2^{-4n}$. So, for every n, we have $4(p_n - 1)^2 \ge p_n^2$, and then:

$$4 \cdot 2^{-n} \ge \frac{4(p_n-1)^2 h_n^2}{2\mathrm{e}} \ge \frac{p_n^2 h_n^2}{2\mathrm{e}}\,\cdot \tag{3.11}$$

This yields, for $n \ge 7$, that $(p_n h_n)^2 \le 8\mathrm{e}\,2^{-n} \le 1/4$. Therefore $p_n h_n \le 1/2$, and we can use the estimate in part (*b*) of Lemma 3.3.

Now, for $n \ge 7$, let B_n be the finite Blaschke product defined by (3.4), where $r_n = 1 - h_n$. Using (*b*) in Lemma 3.3, the Maximum Modulus Principle and the definition of p_n in (3.10), we have:

$$|B_n(z)| \le 1 - \frac{p_n^2 h_n^2}{2\mathrm{e}} < 1 - 2^{-n}, \quad \text{for } |z| \le r_n. \tag{3.12}$$

Consider then the Blaschke product D defined by:

$$D(z) = \prod_{n=7}^{\infty} B_n(z). \tag{3.13}$$

This product is convergent since, by (3.11), we have:

$$\sum p_n(1 - r_n) = \sum p_n h_n \le \sum \sqrt{8\mathrm{e}\,2^{-n}} < +\infty\,.$$

Finally, take $N \in \mathbb{N}$ big enough to have $r_6^N < 1/2$, and define:

$$B(z) = z^N D(z). \tag{3.14}$$

Thus B is a Blaschke product, and, if $|z| \leq r_6$, we have, since $\delta(t) \leq 1/2$:

$$|B(z)| \leq |z^N| \leq r_6^N < 1/2 \leq 1 - \delta(1 - |z|). \tag{3.15}$$

If $1 > |z| > r_6$, there exists $k \geq 7$ such that $r_k \geq |z| > r_{k-1}$. Therefore, thanks to (3.12),

$$|B(z)| \leq |D(z)| \leq |B_k(z)| \leq 1 - 2^{-k}. \tag{3.16}$$

On the other hand $r_k \geq |z| > r_{k-1}$ implies $h_k \leq 1 - |z| < h_{k-1}$, and so:

$$\delta(1 - |z|) \leq \frac{1}{2}\chi(1 - |z|) \leq \frac{1}{2}\chi(h_{k-1}) \leq 2^{-k}\,. \tag{3.17}$$

Combining (3.16) and (3.17) we get $|B(z)| \leq 1 - \delta(1 - |z|)$, when $1 > |z| > r_6$. From this and (3.15), Theorem 3.1 follows. □

Proof of Lemma 3.3. It is obvious that, for all $a, z \in \mathbb{C}$,

$$\prod_{k=1}^{p} (z - a\mathrm{e}^{i\theta_k}) = z^p - a^p\,.$$

Using this we have:

$$G(z) = \prod_{k=1}^{p} \frac{r - \mathrm{e}^{-i\theta_k} z}{1 - r\mathrm{e}^{-i\theta_k} z} = \prod_{k=1}^{p} \frac{z - r\mathrm{e}^{i\theta_k}}{rz - \mathrm{e}^{i\theta_k}} = \frac{z^p - r^p}{(rz)^p - 1}\,\cdot \tag{3.18}$$

Now, if $|z| = r$, we can write $z^p = r^p u$, for some u with $|u| = 1$. Then $|G(z)| = |T(u)|$, where T is the Moebius transformation

$$T(u) = \frac{r^p(u - 1)}{r^{2p}u - 1}\,\cdot$$

This transformation T maps the unit circle $\partial\mathbb{D}$ onto a circumference C. As T maps the extended real line $\mathbb{R}_\infty$ to itself, and $\partial\mathbb{D}$ is orthogonal to $\mathbb{R}_\infty$ at the intersection points 1 and -1, C is the circumference orthogonal to $\mathbb{R}_\infty$ crossing through the points $T(1) = 0$ and $T(-1) = \alpha$. It is easy to see that $|w| \leq |\alpha|$, for every $w \in C$; consequently:

$$|G(z)| \leq \sup_{u \in \partial\mathbb{D}} |T(u)| = |T(-1)| = \frac{2r^p}{1 + r^{2p}}\,\cdot$$

This finishes the proof of the statement (a).

To prove part (b), observe that, $1 + r^{2p} \leq 2$, and so, for $|z| = r$,

$$|G(z)| \leq 1 - \frac{(1 - r^p)^2}{1 + r^{2p}} \leq 1 - \frac{(1 - r^p)^2}{2}\,\cdot \tag{3.19}$$

Remember that $r = 1 - h$, so $r \leq \mathrm{e}^{-h}$, and $r^p \leq \mathrm{e}^{-ph}$. Thus $1 - r^p \geq 1 - \mathrm{e}^{-ph}$. Now, if $x \in [0, 1/2]$, we have, by the Mean Value theorem:

$$1 - \mathrm{e}^{-x} \geq \frac{x}{\sqrt{\mathrm{e}}}\,\cdot$$

Since $p\,h \leq 1/2$, we can apply this last estimate to (3.19) to get, as promised,

$$|G(z)| \leq 1 - \frac{(1-\mathrm{e}^{-ph})^2}{2} \leq 1 - \frac{p^2h^2}{2\mathrm{e}}\,,$$

and ending the proof of Lemma 3.3. □

Remark. The key point in the proof of Theorem 3.1 is the inequality (3.6) in Lemma 3.3. This inequality may be viewed as a consequence of the strong triangle inequality (applied to $a = z^p$, $b = r^p$ and $c = 0$):

$$d(a,b) \leq \frac{d(a,c) + d(c,b)}{1 + d(a,c)\,d(c,b)} \tag{3.20}$$

for the pseudo-hyperbolic distance $d(u,v) = \frac{|u-v|}{|1-\bar{u}v|}$ on $\mathbb{D}$. Let us recall a proof for the convenience of the reader: by conformal invariance, we may assume that $c = 0$; then:

$$1 - [d(a,b)]^2 = \frac{(1-|a|^2)(1-|b|^2)}{|1-\bar{a}b|^2} \geq \frac{(1-|a|^2)(1-|b|^2)}{(1+|a|\,|b|)^2} = 1 - [d(|a|,-|b|)]^2\,,$$

so that:

$$d(a,b) \leq d(|a|,-|b|) = \frac{|a|+|b|}{1+|a|\,|b|}\,,$$

proving (3.20), since $d(a,0) = |a|$ and $d(0,b) = |b|$.

4 A compact composition operator with a surjective symbol

A well-known result of J. H. Schwartz ([17], Theorem 2.8) asserts that the composition operator $C_\varphi\colon H^\infty \to H^\infty$ is compact if and only if $\|\varphi\|_\infty < 1$. In particular, the compactness of $C_\varphi\colon H^\infty \to H^\infty$ prevents the surjectivity of φ. It may be therefore to be expected that, the bigger Ψ, the more difficult it will be to obtain both the compactness of $C_\varphi\colon H^\Psi \to H^\Psi$ and the surjectivity of φ. Nevertheless, this is possible, as says the following theorem, and the case H^∞ appears really as a singular case (corresponding to an "Orlicz function" which is discontinuous and can take the value infinity).

Theorem 4.1 *For every Orlicz function Ψ, there exists a symbol $\varphi\colon \mathbb{D} \to \mathbb{D}$ which is 4-valent and surjective and such that $C_\varphi\colon H^\Psi \to H^\Psi$ is compact. Moreover, φ can be taken so as $C_\varphi\colon H^2 \to H^2$ is in all the Schatten classes $S_p(H^2)$, $p > 0$.*

In the case of H^2 ($\Psi(x) = x^2$), B. McCluer and J. Shapiro ([14], Example 3.12) gave an example based on the Riemann mapping theorem and on the fact that, for a finitely valent symbol φ, we have the equivalence:

$$C_\varphi\colon H^2 \to H^2 \text{ compact} \quad \Longleftrightarrow \quad \lim_{|z| \overset{<}{\to} 1} \frac{1-|\varphi(z)|}{1-|z|} = \infty. \tag{4.1}$$

A specific example is as follows. Take

$$R = \Big\{ z = x + iy \in \mathbb{C}\,;\ x > 0 \text{ and } \frac{1}{x} < y < \frac{1}{x} + 4\pi \Big\}, \tag{4.2}$$

let $g\colon \mathbb{D} \to R$ be a Riemann map and set $\varphi = \mathrm{e}^{-g}$. Then, φ is 2-valent, $\varphi(\mathbb{D}) = \mathbb{D}^*$ (where $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$), and the validity of (4.1) is tested through the use of the Julia-Carathéodory theorem (see [16] for details). To get a fully surjective mapping φ_1, just compose φ with the square of a Blaschke product:

$$\varphi_1(z) = B \circ \varphi, \qquad \text{with} \quad B(z) = \Big(\frac{z - \alpha}{1 - \overline{\alpha} z} \Big)^2, \quad \alpha \in D^* = \mathbb{D} \setminus \{0\}$$

(note that $B(0) = B(2\alpha/1 + |\alpha|^2)$. Since $C_{\varphi_1} = C_\varphi \circ C_B$, we see that C_{φ_1} is compact as well and we are done.

Here, we can no longer rely on the Julia-Carathéodory theorem. But we shall use the following necessary and sufficient condition, in terms of the maximal Carleson function ρ_φ, which is valid for any symbol, finitely-valent or not (see [8], Theorem 4.18 – or [7], Théorème 4.2, where a different, but equivalent, formulation is given):

$$C_\varphi\colon H^\Psi \to H^\Psi \text{ compact} \quad \Longleftrightarrow \quad \lim_{h \overset{>}{\to} 0} \frac{\Psi^{-1}(1/h)}{\Psi^{-1}\big(1/\rho_\varphi(h)\big)} = 0\,. \tag{4.3}$$

For the sequel, we shall set:

$$\Delta(h) = \frac{\Psi^{-1}(1/h)}{\Psi^{-1}\big(1/\rho_\varphi(h)\big)}\,\cdot \tag{4.4}$$

Our strategy will be to elaborate on the previous example to produce a (nearly) surjective φ such that $\rho_\varphi(h)$ is very small (depending on Ψ) for small h. The tool will be the notion of harmonic measure for certain open sets of the extended plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, called *hyperbolic* (see [2], Definition 19.9.3); for example, every conformal image of $\mathbb{D}$ is hyperbolic (see [2], Proposition 19.9.2 (d) and Theorem 19.9.7). If G is a hyperbolic domain and $a \in G$, the *harmonic measure* of G at a is the probability measure $\omega_G(a,\,.\,)$ supported by ∂G (here, and throughout the rest of this section, boundaries and closures will be taken in $\hat{\mathbb{C}}$) such that:

$$u(a) = \int_{\partial G} u(z)\, d\omega_G(a, z)$$

for each bounded and continuous function u on $\overline{G}$, which is harmonic in G (see [2], Definition 21.1.3). The harmonic measure at a of a Borel set $A \subseteq \partial G$ will be denoted by $\omega_G(a, A)$. Clearly,

$$\omega_{\mathbb{D}}(0,\,.\,) = m,$$

the Haar measure (*i.e.* normalized Lebesgue measure) of $\partial\mathbb{D}$.

R. Nevanlinna (see [2], Proposition 21.1.6) showed that harmonic measures share a *conformal invariance property*. Namely, assume that G is a simply connected domain, in which the Dirichlet problem can be solved (a *Dirichlet domain*), and $\tau\colon \overline{\mathbb{D}} \to \overline{G}$ is a continuous function which maps conformally $\mathbb{D}$ onto G; then τ maps $\partial\mathbb{D}$ onto ∂G, and, if $\tau(0) = a$:

$$\omega_G(a, A) = m\big(\tau^{-1}(A)\big) \tag{4.5}$$

for every Borel set $A \subseteq \partial G$. This explains why harmonic measures enter the matter when we consider composition operators C_φ: such an operator induces a map $H^\Psi \to L^\Psi(m_\varphi)$, where $m_\varphi = \varphi^*(m)$ appears as an image measure of m, as it happens for the harmonic measure of G at a in (4.5).

A useful alternative way of defining the harmonic measure, due to S. Kakutani, and completed by J. Doob (see [19], page 454, and [6], Appendix F, page 477) is the following: Let $(B_t)_{t>0}$ be the 2-dimensional Brownian motion starting at $a \in G$ (*i.e.* $B_0 = a$), and τ be the stopping time defined by:

$$\tau = \inf\{t > 0\,;\ B_t \notin G\}\,; \tag{4.6}$$

we have:

$$\omega_G(a, A) = \mathbb{P}_a(B_\tau \in A), \tag{4.7}$$

i.e. the harmonic measure of A at a is the probability that the Brownian motion starting at a exits from G through the Borel set $A \subseteq \partial G$. The following lemma will be basic for the construction of our example. We shall provide two proofs, the second one being more illuminating.

Lemma 4.2 (Hole principle) *Let G_0 and G_1 be two hyperbolic open sets and $H \subseteq \partial G_0$ a Borel set such that*

$$G_0 \subseteq G_1 \quad \textit{and} \quad \partial G_0 \subseteq \partial G_1 \cup H.$$

Then, for every $a \in G_0$, we have the following inequality:

$$\omega_{G_1}(a, \partial G_1 \setminus \partial G_0) \leq \omega_{G_0}(a, H). \tag{4.8}$$

Proof 1. From [2], Corollary 21.1.14, with $\Delta = \partial G_0 \cap \partial G_1$, one has $\omega_{G_0}(a, \Delta) \leq \omega_{G_1}(a, \Delta)$. But $\partial G_1 \setminus \Delta = \partial G_1 \setminus \partial G_0$, and hence, since harmonic measures are probability measures,

$$\omega_{G_1}(a, \partial G_1 \setminus \partial G_0) = \omega_{G_1}(a, \partial G_1 \setminus \Delta) = 1 - \omega_{G_1}(a, \Delta) \leq 1 - \omega_{G_0}(a, \Delta);$$

we get the result since $\partial G_0 = H \cup \Delta$, which implies $1 \leq \omega_{G_0}(a, H) + \omega_{G_0}(\Delta)$. □

Proof 2. Let us define

$$\tau_0 = \inf\{t > 0\,;\ B_t \notin G_0\}, \quad \tau_1 = \inf\{t > 0\,;\ B_t \notin G_1\} \tag{4.9}$$

and

$$E = \{B_{\tau_1} \in \partial G_1 \setminus \partial G_0\}, \qquad F = \{B_{\tau_0} \in H\}. \tag{4.10}$$

Inequality (4.8) amounts to proving that $\mathbb{P}_a(E) \leq \mathbb{P}_a(F)$, which will follow from the inclusion $E \subseteq F$. Suppose that the event E holds. Since $G_0 \subseteq G_1$, one has $\tau_0 \leq \tau_1$. The Brownian path $(B_s)_{0\leq s\leq \tau_1}$ being continuous with $B_0 = a \in G_0$, one has $B_{\tau_0} \in \partial G_0 \subseteq \partial G_1 \cup H$. If we had $B_{\tau_0} \in \partial G_1$, we should have $B_{\tau_0} \notin G_1$, since G_1 is open, and hence $\tau_0 = \tau_1$, since we know that $\tau_0 \leq \tau_1$. But then $B_{\tau_1} = B_{\tau_0} \in \partial G_0$, contrary to the definition of E. Therefore, $B_{\tau_0} \in H$ and F holds. □

We also shall need the following result (see [2], Proposition 21.1.17).

Proposition 4.3 (Continuity principle) *If G is a hyperbolic open set and $a \in G$, then the harmonic measure $\omega_G(a, .)$ is atomless.*

Proof of Theorem 4.1. It will be enough to construct a 2-valent mapping $\varphi\colon \mathbb{D} \to \mathbb{D}$ such that $\varphi(\mathbb{D}) = \mathbb{D}^*$ and $C_\varphi\colon H^\Psi \to H^\Psi$ is compact. We can then modify φ by the same trick as the one used by B. McCluer and J. Shapiro. Note that every point in $\mathbb{D}^*$ is the image by e^{-z} of two distinct points of R, except those which are the image of points of the hyperbola $y = (1/x)+2\pi$, which have only one pre-image.

For a positive integer n, set:

$$b_n = \frac{1}{4n\pi}\,, \tag{4.11}$$

and let $\varepsilon_n > 0$ such that:

$$\frac{\Psi^{-1}(2/b_{n+1})}{\Psi^{-1}(1/\varepsilon_n)} \leq \frac{1}{n}\,\cdot \tag{4.12}$$

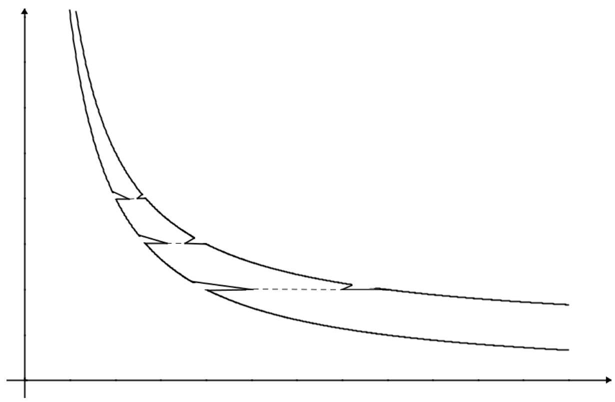
We now modify the domain R, including "barriers" in it (not in the sense of potential theory, nor of Perron!) in the following way.

Let, for every $n \geq 1$, M_n be the intersection point of the horizontal line $y = 4\pi n$ and of the hyperbola $y = (1/x) + 2\pi$, that is $M_n = \frac{1}{4\pi n-2\pi} + 4\pi n i$.

Define inductively closed sets P_n^+ and P_n^-, which are like small points of swords (two segments and a piece of hyperbola), in the following way:

- The lower part of P_n^+ and P_n^- are horizontal segments of altitude $4n\pi$.
- Those two horizontal segments are separated by a small open horizontal segment H_n whose middle is M_n.
- The upper part of P_n^+ is a slant segment whose upper extremity c_n^+ lies on the hyperbola $y = 1/x$.
- The upper part of P_n^- is a slant segment whose upper extremity c_n^- lies on the hyperbola $y = (1/x) + 4\pi$.

- The curvilinear part of P_n^+ is supported by the hyperbola $y = 1/x$.
- The curvilinear part of P_n^- is supported by the hyperbola $y = (1/x) + 4\pi$.
- One has $4(n+1)\pi - \mathfrak{Im}\, c_n^{\pm} > 2\pi$.



The size of the small horizontal holes will be determined inductively in the following way. Fix once and for all $a \in R$ such that $\mathfrak{Im}\, a < 4\pi$. Suppose that $H_1, H_2, \ldots, H_{n-1}$ have already been determined. Set:

$$\Omega_n = \big\{ z \in R \setminus \bigcup_{j<n} (P_j^+ \cup P_j^-)\,;\ \mathfrak{Im}\, z < 4n\pi \big\}. \tag{4.13}$$

We can adjust H_n so small that:

$$\omega_{\Omega_n}(a, H_n) \le \varepsilon_n. \tag{4.14}$$

Indeed, Ω_n is bounded above by the horizontal segment $[b_n + 4in\pi, b_{n-1} + 4in\pi]$, where the point M_n lies. If $H_n = [M_n - \delta, M_n + \delta]$, we see that H_n decreases to the singleton $\{M_n\}$ as δ decreases to zero. Therefore, by Proposition 4.3, we can adjust δ so as to realize (4.14).

We now define our modified open set Ω by the formula

$$\Omega = R \setminus \bigcup_{n\ge 1} (P_n^+ \cup P_n^-) = \bigcup_{n \ge 1} \Omega_n. \tag{4.15}$$

It is useful to observe that:

$$\inf_{w \in \partial\Omega_n} \mathfrak{Re}\, w = b_n\,. \tag{4.16}$$

This is obvious by the way we defined the upper part of $\partial\Omega_n$.

Now, we can easily finish the proof. Fix $h \le b_1/2$ and let n be the integer such that:

$$b_{n+1} < 2h \le b_n\,. \tag{4.17}$$

Let $g\colon \mathbb{D} \to \Omega$ be a conformal mapping such that $g(0) = a$. Since $\partial_\infty \Omega$ is connected, Caratheodory's Theorem (see [15]) ensures that g can be continuously extended from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. More explicitly, using the Moebius transformation $T(z) = 1/z$, we see that there exists an automorphism of the extended complex plane such that $\overline{\Omega}$ is sended onto a compact subset of $\mathbb{C}$; so, we can apply to Ω many results stated for bounded domains. For instance, the boundary of Ω is a continuous path in the extended plane; so, by [2], Theorem 14.5.5, g can be extended to a continuous function (for the extended plane topology) $g\colon \overline{\mathbb{D}} \to \overline{G}$. In particular, g has boundary values g^*.

We define $\varphi = \mathrm{e}^{-g}$.

As in the proof of B. McCluer and J. Shapiro ([14]), we have that φ is 2-valent (see the remark made at the beginnig of this proof), and we still have $\varphi(\mathbb{D}) = \mathbb{D}^*$, since, in the process for constructing Ω from R, for every point of $\mathbb{D}^*$, at least one of the preimages by e^{-z} in R has not been removed. Observe that, in particular, we did not remove any point in the hyperbola $y = (1/x) + 2\pi$, thanks to the choice of M_n.

Moreover, Ω is a Dirichlet domain (because each component of $\partial\Omega$ has more than one point: see the comment after Definition 19.7.1 in [2]), so we can use the conformal invariance. Then by (4.5), (4.14), (4.16) and by the hole principle, we see that, if $A = \{\Re\mathrm{e}\, g^*(\mathrm{e}^{it}) < 2h\}$:

$$
\begin{aligned}
\rho_\varphi(h) &\leq m_\varphi(\{|z| > 1 - h\}) = m(\{\mathrm{e}^{-\Re\mathrm{e}\, g^*(\mathrm{e}^{it})} > 1 - h\}) \\
&= m(\{\Re\mathrm{e}\, g^*(\mathrm{e}^{it}) < \log(1/1 - h)\}) \\
&\leq m(\{\Re\mathrm{e}\, g^*(\mathrm{e}^{it}) < 2h\}) = \omega_{\mathbb{D}}(0, A) \\
&= \omega_{g(\mathbb{D})}\big(g(0), g(A)\big) = \omega_\Omega(a, \{\Re\mathrm{e}\, w < 2h\}) \\
&\leq \omega_\Omega(a, \{\Re\mathrm{e}\, w \leq b_n\}) \\
&\leq \omega_\Omega(a, \partial\Omega \setminus \partial\Omega_n) \leq \omega_{\Omega_n}(a, H_n) \leq \varepsilon_n.
\end{aligned} \tag{4.18}
$$

It remains to observe that:

$$
\Delta(h) = \frac{\Psi^{-1}(1/h)}{\Psi^{-1}(1/\rho_\varphi(h))} \leq \frac{\Psi^{-1}(2/b_{n+1})}{\Psi^{-1}(1/\varepsilon_n)} \leq \frac{1}{n} \leq Ch\,,
$$

in view of (4.12) and of the choice of n, C being a numerical constant. We should point out the fact that we applied the hole principle to the domains $G_0 = \Omega_n$ and $G_1 = \Omega$ and that this was licit because the assumptions of the hole principle (in particular the inclusion $\partial\Omega_n \subseteq \partial\Omega \cup H_n$) are satisfied. We have therefore proved that:

$$
\lim_{h \overset{>}{\to} 0} \Delta(h) = 0\,,
$$

and this ends, as we already explained, the first part of the proof of Theorem 4.1.

To prove the last part, let us remark that in (4.12) we may take ε_n arbitrarily small. If one takes $\varepsilon_n \leq \mathrm{e}^{-n}$, one has, for some constant $c > 0$, $\rho_\varphi(h) \leq \mathrm{e}^{-c/h}$, by using (4.17) and (4.18). In particular, $\rho_\varphi(h) \leq C\, h^\alpha$ for every $\alpha > 1$. By Luecking's criterion, that implies that $C_\varphi \in S_p(H^2)$ for every $p > 0$ (see [9], Corollary 3.2). □

Remark. Let us note that our result is stronger than McCluer-Shapiro's, since our C_φ is in all the Schatten classes $S_p(H^2)$, $p > 0$. Though our construction follows McCluer-Shapiro's, it is the introduction of the "barriers" P_n^+ and P_n^- which allows to get this improvement.

5 Composition operators with closed range

In [1], J. Cima, J. Thomson and W. Wogen gave a characterization of composition operators $C_\varphi\colon H^p \to H^p$ with closed range. This characterization involves the Radon-Nikodym derivative of the restriction to $\partial\mathbb{D}$ of m_φ. They found it not satisfactory, and asked a characterization with the range of φ itself. N. Zorboska ([20]) gave such a characterization, but her statement is somewhat complicated. We shall give here more explicit characterizations, either in terms of the Nevanlinna counting function N_φ, or in terms of the Carleson measure m_φ.

Theorem 5.1 *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a non-constant analytic self map. Then the composition operator $C_\varphi\colon H^p \to H^p$, $1 \leq p < \infty$, has a closed range if and only if there is a constant $c > 0$ such that, for $0 < h < 1$,*

$$(5.1) \qquad \frac{1}{A\big(S(\xi,h)\big)} \int_{S(\xi,h)} N_\varphi(z)\, dA(z) \geq c\, h\,, \qquad \forall \xi \in \partial\mathbb{D}\,.$$

Theorem 5.1 will follow immediately from the next theorem, applied to $\mu = m_\varphi$, and from [11], Theorem 4.2.

Theorem 5.2 *Let μ be a finite positive measure on $\overline{\mathbb{D}}$. Assume that the canonical map $J\colon H^p \to L^p(\mu)$ is continuous, $1 \leq p < \infty$. Then J is one-to-one and has a closed range if and only if there is a constant $c > 0$ such that, for $0 < h < 1$,*

$$(5.2) \qquad \mu\big[W(\xi,h)\big] \geq c\, h\,, \qquad \forall \xi \in \partial\mathbb{D}\,.$$

Proof. 1) Assume that J has a closed range. By making a rotation on the variable z, we only have to find a constant $c > 0$ such that

$$(5.3) \qquad \mu(S_h) \geq c\, h\,,$$

for $h > 0$ small enough, where $S_h = S(1,h)$.

Since J is one-to-one, there is a constant $C > 0$ such that:

$$(5.4) \qquad \|f\|_{L^p(\mu)}^p \geq C^p\, \|f\|_p^p\,, \qquad \forall f \in H^p.$$

We are going to test (5.4) on

$$(5.5) \qquad f_N(z) = \left(\frac{1+z}{2}\right)^N .$$

It is classical that there is a constant $c_p > 0$ such that:

$$\|f_N\|_p^p = \int_{-\pi}^{\pi} \Big|\cos\frac{t}{2}\Big|^{pN} dt \geq \frac{c_p}{\sqrt{N}} \cdot \tag{5.6}$$

Now, since $|z+1|^2 + |z-1|^2 = 2(|z|^2+1) \leq 4$ for every $z \in \overline{\mathbb{D}}$, one has:

$$|f_N(z)| \leq \Big(1 - \frac{|z-1|^2}{4}\Big)^{N/2} \leq \mathrm{e}^{-\frac{N}{8}\,|z-1|^2} \,.$$

Hence, using $|f_N(z)| \leq 1$ when $|z-1| \leq h$, one has:

$$\begin{aligned}\|f_N\|_{L^p(\mu)}^p &\leq \mu(S_h) + \int_{|z-1|>h} \mathrm{e}^{-p\frac{N}{8}\,|z-1|^2}\, d\mu \\ &= \mu(S_h) + \int_0^{\mathrm{e}^{-pNh^2/8}} \mu\big(\{\mathrm{e}^{-p\frac{N}{8}\,|z-1|^2} > u\}\big)\, du \,,\end{aligned}$$

that is, making the change of variable $u = \mathrm{e}^{-p\frac{N}{8}\,x^2}$,

$$\|f_N\|_{L^p(\mu)}^p \leq \mu(S_h) + \int_h^{\infty} \mu(\{|z-1| \leq x\})\, \frac{pN}{4}\, x\, \mathrm{e}^{-p\frac{N}{8}\,x^2}\, dx \,.$$

Now, the continuity of J means, by Carleson's Theorem see [4], Theorem 9.3), that there is a constant $K > 0$ such that:

$$\sup_{|\xi|=1} \mu\big(S(\xi, x)\big) \leq K\, x \,, \qquad 0 \leq x < 1 \,. \tag{5.7}$$

We get hence:

$$\begin{aligned}\|f_N\|_{L^p(\mu)}^p &\leq \mu(S_h) + \int_h^{\infty} K\, x\, \frac{pN}{4}\, x\, \mathrm{e}^{-p\frac{N}{8}\,x^2}\, dx \\ &= \mu(S_h) + \frac{K\sqrt{8}}{\sqrt{p}}\, \frac{1}{\sqrt{N}} \int_{h\sqrt{\frac{pN}{8}}}^{\infty} y^2\, \mathrm{e}^{-y^2}\, dy \,.\end{aligned}$$

We take now for N the smaller integer $> 1/h^2$, multiplied by some constant integer a_p, large enough to have:

$$\frac{K\sqrt{8}}{\sqrt{p}} \int_{\sqrt{\frac{p\,a_p}{8}}}^{\infty} y^2\, \mathrm{e}^{-y^2}\, dy \leq \frac{c_p\, C^p}{2} \cdot$$

We get then, from (5.4) and (5.6):

$$\mu(S_h) \geq \frac{C^p\, c_p}{2}\, \frac{1}{\sqrt{N}} \,,$$

which gives (5.3).

2) Conversely, assume that (5.2) holds. Since the disk algebra $A(\mathbb{D})$ is dense in H^p, it suffices to show that there exists a constant $C > 0$ such that $\|f\|_{L^p(\mu)} \geq C\,\|f\|_p$ for every $f \in A(\mathbb{D})$.

Let $f \in A(\mathbb{D})$ such that $\|f\|_p = 1$. Choose an integer N such that:

$$\frac{1}{N}\sum_{n=1}^{N} |f(\mathrm{e}^{2\pi i n/N})|^p \geq \frac{1}{2}\int_{\partial\mathbb{D}} |f(\xi)|^p \, dm(\xi) = \frac{1}{2}\,,$$

and such that, due to the uniform continuity of f,

$$z, z' \in \overline{\mathbb{D}} \quad \text{and} \quad |z - z'| \leq \frac{2\pi}{N} \qquad \Longrightarrow \qquad |f(z) - f(z')| \leq \frac{1}{2^{(p+1)/p}}\,.$$

Then, setting $W_n = W(\mathrm{e}^{2\pi i n/N}, \pi/N)$, $1 \leq n \leq N$, one has:

$$\|f\|_{L^p(\mu)}^p = \int_{\mathbb{D}} |f|^p \, d\mu \geq \sum_{n=1}^{N} \int_{W_n} |f|^p \, d\mu\,.$$

If we choose $z_n \in W_n$ such that $|f(z_n)| = \min_{z \in W_n} |f(z)|$, we get, using (5.2):

$$\|f\|_{L^p(\mu)}^p \geq \sum_{n=1}^{N} |f(z_n)|^p \, \mu(W_n) \geq \frac{c\pi}{N} \sum_{n=1}^{N} |f(z_n)|^p\,.$$

Since $A^p \leq 2^{p-1}[(A-B)^p + B^p]$, by Hölder's inequality, one has:

$$|f(z_n)|^p \geq \frac{1}{2^{p-1}}\, |f(\mathrm{e}^{2\pi i n/N})|^p - |f(z_n) - f(\mathrm{e}^{2\pi i n/N})|^p$$

and hence:

$$\|f\|_{L^p(\mu)}^p \geq \frac{c\pi}{N} \sum_{n=1}^{N} \left[\frac{1}{2^{p-1}}\, |f(\mathrm{e}^{2\pi i n/N})|^p - |f(z_n) - f(\mathrm{e}^{2\pi i n/N})|^p\right].$$

Now, since $z_n \in W_n$, one has:

$$|z_n - \mathrm{e}^{2\pi i n/N}| \leq \left|z_n - \frac{z_n}{|z_n|}\right| + \left|\frac{z_n}{|z_n|} - \mathrm{e}^{2\pi i n/N}\right| \leq \frac{\pi}{N} + \frac{\pi}{N} = \frac{2\pi}{N}\,;$$

therefore $|f(z_n) - f(\mathrm{e}^{2\pi i n/N})| \leq 1/2^{p+1}$ and we get:

$$\begin{aligned}\|f\|_{L^p(\mu)}^p &\geq c\pi \left[\frac{1}{N}\sum_{n=1}^{N} \frac{1}{2^{p-1}}\, |f(\mathrm{e}^{2\pi i n/N})|^p - \frac{1}{2^{p+1}}\right] \\ &\geq c\pi \Big(\frac{1}{2^{p-1}}\,\frac{1}{2} - \frac{1}{2^{p+1}}\Big) = \frac{c\pi}{2^{p+1}}\,.\end{aligned}$$

That ends the proof of Theorem 5.2. □

Remark. To make the link with Cima-Thomson-Wogen's criterion, we shall see that condition 5.2 implies that the restriction of μ to the boundary $\mathbb{T} = \partial\mathbb{D}$ of the disk dominates the Lebesgue measure m. In fact, let I be an arc of $\mathbb{T}$. If $m(I) = h$, we can write:

$$I = \bigcap_{n\geq 1} \bigcup_{j=1}^{n} W(\xi_{n,j}, h/2n)\,,$$

with disjoint windows $W(\xi_{n,1}, h/2n), \ldots, W(\xi_{n,n}, h/2n)$; hence:

$$\mu(I) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu[W(\xi_{n,j}, h/2n)] \geq c \sum_{j=1}^{n} \frac{h}{2n} = \frac{c}{2}\, h\,.$$

6 Composition operators in Schatten classes

In [12], D. Luecking characterized composition operators $C_\varphi\colon H^2 \to H^2$ which are in the Schatten classes, by using, essentially, the m_φ-measure of Carleson windows. Five years later, D. Luecking and K. Zhu ([13]) characterized them by using the Nevanlinna counting function of φ. We shall see in this section how the result of [11] makes these two characterizations directly equivalent.

It will be convenient here to work with *modified* Carleson windows, namely:

$$W_{n,j} = \left\{ z \in \overline{\mathbb{D}}\,;\ 1 - 2^{-n} \leq |z| \leq 1 \text{ and } \frac{(2j-1)\pi}{2^n} \leq \arg z < \frac{(2j+1)\pi}{2^n} \right\}$$

($j = 0, 1, \ldots, 2^n - 1,\ n = 1, 2, \ldots$). We shall say that $W_{n,j}$ is the Carleson window centered at $\mathrm{e}^{2\pi i j/2^n}$ with size 2^{-n}.

Theorem 6.1 *For $p > 0$ the two following conditions are equivalent:*

a) $\dfrac{N_\varphi(z)}{\log(1/|z|)} \in L^{p/2}(\lambda)$, *where* $d\lambda(z) = (1 - |z|)^{-2}\, dA(z)$ *and A is the normalized area measure on $\mathbb{D}$;*

b) $\displaystyle\sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} \left[2^n\, m_\varphi(W_{n,j})\right]^{p/2} < \infty\,.$

Condition *b)* in the last theorem yields that $\lim_{n\to\infty} \max_j 2^n m_\varphi(W_{n,j}) = 0$, and it is not difficult to see that this implies that $m_\varphi(\partial\mathbb{D}) = 0$, or equivalently, that $|\varphi^*| < 1$ almost evereywhere on $\partial\mathbb{D}$. In this situation we know ([9], Proposition 3.3) that *b)* in Theorem 6.1 is equivalent to Luecking's condition in [12]. In fact the characterization of belonging to a Schatten class in [12] includes the requirement $m_\varphi(\partial\mathbb{D}) = 0$.

Proof. We may, and do, assume that $\varphi(0) = 0$.

1) Assume first that condition *b)* is satisfied. Let:

$$R_{n,j} = \left\{ z \in \mathbb{D}\,;\ 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1} \text{ and } \frac{(2j-1)\pi}{2^n} \leq \arg z < \frac{(2j+1)\pi}{2^n} \right\}$$

be the (disjoint) Luecking windows ($0 \leq j \leq 2^n - 1$, $n \geq 0$). One has $R_{n,j} \subseteq W_{n,j}$.

By [11], Theorem 3.1, there are a constant $C > 0$ and an integer K such that $N_\varphi(z) \leq C\, m_\varphi(\widetilde{W}_{n,j})$, for every $z \in R_{n,j}$, where $\widetilde{W}_{n,j}$ is the window centered at $\mathrm{e}^{2\pi i j/2^n}$, as $W_{n,j}$, but with size 2^{K-n}. The windows $W_{n-K,j}$, $j = 0, 1, \ldots, 2^{n-K} - 1$, have the same size as the windows $\widetilde{W}_{n,j}$, but may have a different center; nevertheless, each $\widetilde{W}_{n,j}$ can be covered with two windows $W_{n-K,l}$: for $n > K$, $\widetilde{W}_{n,j} \subseteq W_{n-K,l} \cup W_{n-K,l+1}$, for some $l = 1, 2, \ldots, 2^{n-K}$ (where $l+1$ is understood as 0 if $l = 2^{n-K} - 1$), we get (we shall use $\lesssim$ to mean $\leq$ up to a constant):

$$\begin{aligned}
\int_{\mathbb{D}} \frac{\big(N_\varphi(z)\big)^{p/2}}{(1-|z|)^{\frac{p}{2}+2}}\, dA(z) &\leq \sum_{n,j} \int_{R_{n,j}} (2^n)^{\frac{p}{2}+2} \big(N_\varphi(z)\big)^{p/2}\, dA(z) \\
&\lesssim \sum_{n,j} \int_{R_{n,j}} (2^n)^{\frac{p}{2}+2} \big(m_\varphi(\widetilde{W}_{n,j})\big)^{p/2}\, dA(z) \\
&\lesssim \sum_{n,j} (2^n)^{p/2} \big(m_\varphi(\widetilde{W}_{n,j})\big)^{p/2} \\
&\lesssim \sum_{\nu,l} (2^\nu)^{p/2} \big(m_\varphi(W_{\nu,l})\big)^{p/2} < \infty,
\end{aligned}$$

and $a)$ holds.

2) Conversely, assume that $a)$ is satisfied. We shall use the following inequality, whose proof will be postponed (for $p \geq 2$, (6.1) follows directly from [11], Theorem 4.2, and Hölder's inequality):

$$[m_\varphi(W_{n,j})]^{p/2} \lesssim \frac{1}{A(\widetilde{W}_{n,j})} \int_{\widetilde{W}_{n,j}} [N_\varphi(z)]^{p/2}\, dA(z)\,, \tag{6.1}$$

where $\widetilde{W}_{n,j}$ is a window with the same center as $W_{n,j}$ but with a bigger proportional size; say of size 2^{-n+L}. We get:

$$\begin{aligned}
\sum_{n,j} [2^n\, m_\varphi(W_{n,j})]^{p/2} &\lesssim \sum_{n,j} 2^{np/2}\, 2^{2n} \int_{\widetilde{W}_{n,j}} [N_\varphi(z)]^{p/2}\, dA(z) \\
&= \int_{\mathbb{D}} \bigg(\sum_n 2^{n(2+\frac{p}{2})} \Big[\sum_j \mathbb{1}_{\widetilde{W}_{n,j}}(z) \Big] \bigg) [N_\varphi(z)]^{p/2}\, dA(z)\,.
\end{aligned}$$

Let $k = 0, 1, \ldots$ such that $1 - 2^{-k+1} < |z| \leq 1 - 2^{-k}$. One has $z \in \widetilde{W}_{n,j}$ only if $n \leq k + L$, and then, for each such n, z is at most in 2^L windows $\widetilde{W}_{n,j}$. It follows that:

$$\sum_n 2^{n(2+\frac{p}{2})} \sum_j \mathbb{1}_{\widetilde{W}_{n,j}}(z) \leq 2^{(k+L+1)(2+\frac{p}{2})} \times 2^L\,.$$

But $|z| \geq 1 - 2^{-k+1}$ implies $2^{(k+L+1)(2+\frac{p}{2})} \leq C_p/(1-|z|)^{2+\frac{p}{2}}$; hence:

$$\sum_{n,j} [2^n \, m_\varphi(W_{n,j})]^{p/2} \lesssim \int_{\mathbb{D}} \frac{[N_\varphi(z)]^{p/2}}{(1-|z|)^{\frac{p}{2}+2}} \, dA(z) < \infty \,,$$

and $b)$ holds.

It remains to show (6.1).

By [11], Theorem 4.1, we can find a window W with the same center as $W_{n,j}$, but with greater size ch ($h = 2^{-n}$ is the size of the window $W_{n,j}$), such that:

$$m_\varphi(W_{n,j}) \lesssim \sup_{w \in W} N_\varphi(w).$$

There is hence some $w_0 \in W$ such that:

$$m_\varphi(W_{n,j}) \lesssim N_\varphi(w_0).$$

Take $R = |w_0| + ch$ (one has $R \geq 1$ since $w_0 \in W$ and W has size ch) and set $\varphi_0(z) = \varphi(z)/R$. One has $N_{\varphi_0}(z) = N_\varphi(Rz)$ for $|z| < 1/R$ and $N_{\varphi_0}(z) = 0$ if $|z| \geq 1/R$.

Let now u be the upper subharmonic regularization of N_{φ_0} ([13], Lemma 1, and its proof page 1140): u is a subharmonic function on $\mathbb{D} \setminus \{0\}$ such that $u \geq N_{\varphi_0}$ and $u = N_{\varphi_0}$ almost everywhere, with respect to dA.

A result of C. Fefferman and E. M. Stein ([5], Lemma 2), generously attributed by them to Hardy and Littlewood, asserts that for any $q > 0$, there exists a constant $C = C(q)$ such that

$$[u(a)]^q \leq \frac{C}{A\big(D(a,r)\big)} \int_{D(a,r)} [u(z)]^q \, dA(z) \tag{6.2}$$

for every nonnegative subharmonic function u on a domain G and every disk $D(a,r) \subseteq G$ (see also [13], Lemma 3).

If Δ is the disk centered at w_0/R and of radius $1-|w_0|/R$ (which is contained in $\mathbb{D} \setminus \{0\}$ since $R > |w_0|$), one has, by (6.2):

$$\begin{aligned} [N_\varphi(w_0)]^{p/2} &= [N_{\varphi_0}(w_0/R)]^{p/2} \leq [u(w_0/R)]^{p/2} \\ &\leq \frac{C}{A(\Delta)} \int_\Delta [u(z)]^{p/2} \, dA(z) \\ &= \frac{C}{A(\Delta)} \int_\Delta [N_{\varphi_0}(z)]^{p/2} \, dA(z) \\ &= \frac{C}{A(\Delta)} \int_{\Delta \cap D(0,1/R)} [N_\varphi(Rz)]^{p/2} \, dA(z) \\ &= \frac{C}{A(\tilde{\Delta})} \int_{\tilde{\Delta} \cap \mathbb{D}} [N_\varphi(w)]^{p/2} \, dA(w) \,, \end{aligned}$$

where $\tilde{\Delta} = D(w_0, R - |w_0|) = D(w_0, ch)$.

Since the center w_0 of $\tilde{\Delta}$ is in $\mathbb{D}$, $\tilde{\Delta} \cap \mathbb{D}$ contains more than a quarter of $\tilde{\Delta}$ (at least for $ch \leq 1$), and hence $A(\tilde{\Delta} \cap \mathbb{D}) \geq A(\tilde{\Delta})/4 = c^2h^2/4\pi$. Now, let $\tilde{W}_{n,j}$ be the window with the same center as $W_{n,j}$ and of size $2ch$. Since $2ch \geq ch + (1 - |w_0|)$, $\tilde{W}_{n,j}$ contains $\tilde{\Delta} \cap \mathbb{D}$ and $A(\tilde{W}_{n,j}) \approx h^2 \approx A(\tilde{\Delta})$ ($\approx$ meaning that the ratio is between two absolute constants). We therefore get:

$$[N_\varphi(w_0)]^{p/2} \lesssim \frac{1}{A(\tilde{W}_{n,j})} \int_{\tilde{W}_{n,j}} [N_\varphi(w)]^{p/2} \, dA(w) \,,$$

proving (6.1). □

References

[1] J. Cima, J. Thomson and W. Wogen, On some properties of composition operators, Indiana Univ. Math. J. 24 (3) (1974), 215–220.

[2] J. B. Conway, Functions of One Complex Variable II, Graduate Texts in Math. 159, Springer-Verlag (1995).

[3] C. C. Cowen and B. D. McCluer, Composition operators on spaces of analytic functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL (1995).

[4] P. L. Duren, Theory of H^p spaces, Second edition, Dover Publications (2000).

[5] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math. 129 (1972), 137–193.

[6] J. B. Garnett and D. E. Marshall, Harmonic measure, New Mathematical Monographs 2, Cambridge University Press, Cambridge (2005).

[7] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, Opérateurs de composition sur les espaces de Hardy-Orlicz, C. R. Math. Acad. Sci. Paris 344 (2007), no. 1, 5–10.

[8] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, Composition operators on Hardy-Orlicz spaces, *preprint*, math.FA/0610905, *to appear* in Memoirs Amer. Math. Soc.

[9] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Some examples of compact composition operators on H^2, J. Funct. Anal. 255, No. 11 (2008), 3098–3124.

[10] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Compact composition operators on H^2 and Hardy-Orlicz spaces, J. Math. Anal. Appl. 354 (2009), no. 1, 360–371.

[11] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Nevanlinna counting function and Carleson function of analytic maps, *preprint*, arXiv : 0904.2496.

[12] D. H. Luecking, Trace ideal criteria for Toeplitz operators, J. Funct. Anal. 73 (1987), 345–368.

[13] D. H. Luecking and K. Zhu, Composition operators belonging to the Schatten ideals, Amer. J. Math. 114 (1992), 878–906.

[14] B. McCluer and J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, Canad. J. Math. 38, no. 4 (1986), 878–906.

[15] C. Pommerenke, Boundary behaviour of conformal maps, Grundlehren der Mathematischen Wissenschaften 299, Springer-Verlag (1992).

[16] H. Queffélec, Carleson measures and composition operators, Proceed. 2008 OT Conference in Timisoara, *to appear*.

[17] H. J. Schwartz, Composition operators on H^p, Thesis, University of Toledo (1969).

[18] J. H. Shapiro, Composition Operators and Classical Function Theory, Universitext, Tracts in Mathematics, Springer-Verlag, New York (1993).

[19] D. W. Stroock, Probability Theory, An Analytic View, Cambridge University Press, Cambridge (1994).

[20] N. Zorboska, Composition operators with closed range, Trans. Amer. Math. Soc. 334 (2) (1994), 791–801.

Pascal Lefèvre, *Univ Lille Nord de France F-59 000 LILLE, FRANCE*
UArtois, Laboratoire de Mathématiques de Lens EA 2462,
Fédération CNRS Nord-Pas-de-Calais FR 2956,
F-62 300 LENS, FRANCE
pascal.lefevre@euler.univ-artois.fr

Daniel Li, *Univ Lille Nord de France F-59 000 LILLE, FRANCE*
UArtois, Laboratoire de Mathématiques de Lens EA 2462,
Fédération CNRS Nord-Pas-de-Calais FR 2956,
Faculté des Sciences Jean Perrin,
Rue Jean Souvraz, S.P. 18,
F-62 300 LENS, FRANCE
daniel.li@euler.univ-artois.fr

Hervé Queffélec, *Univ Lille Nord de France F-59 000 LILLE, FRANCE*
USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524,
F-59 655 VILLENEUVE D'ASCQ Cedex, FRANCE
queff@math.univ-lille1.fr

Luis Rodríguez-Piazza, *Universidad de Sevilla,*
Facultad de Matemáticas, Departamento de Análisis Matemático,
Apartado de Correos 1160,
41 080 SEVILLA, SPAIN
piazza@us.es