

The canonical injection of the Hardy-Orlicz space H^Ψ into the Bergman-Orlicz space \mathfrak{B}^Ψ

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Abstract. *We study the canonical injection from the Hardy-Orlicz space H^Ψ into the Bergman-Orlicz space \mathfrak{B}^Ψ .*

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1 Introduction and notation

1.1 Introduction

There are two natural Orlicz spaces of analytic functions on the unit disk \mathbb{D} of the complex plane: the Hardy-Orlicz space H^Ψ and the Bergman-Orlicz space \mathfrak{B}^Ψ . It is well-known that in the classical case $\Psi(x) = x^p$, $H^p \subseteq \mathfrak{B}^p$ and the canonical injection J_p from H^p to \mathfrak{B}^p is bounded, and even compact. In fact, for any Orlicz function Ψ , one has $H^\Psi \subseteq \mathfrak{B}^\Psi$ and the canonical injection $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$ is bounded, but we shall see in this paper that its compactness requires that Ψ does not grow too fast. We actually characterize in Section 2 the compactness: J_Ψ is compact if and only if $\lim_{x \rightarrow +\infty} \Psi(Ax)/[\Psi(x)]^2 = 0$ for every $A > 1$, and the weak compactness: J_Ψ is weakly compact if and only if $\limsup_{x \rightarrow +\infty} \Psi(Ax)/[\Psi(x)]^2 < +\infty$ for every $A > 1$. We show that, if these two properties are “often” equivalent (this happens for example if $\Psi(x)/x$ is non-decreasing for x large enough), it is not always the case. We actually show a stronger result in Section 4: there is an Orlicz function Ψ such that J_Ψ is weakly compact and is Dunford-Pettis, but such that J_Ψ is not compact.

1.2 Notation

An Orlicz function is a non-decreasing convex function $\Psi: [0, +\infty[\rightarrow [0, +\infty[$ such that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. One says that the Orlicz function Ψ has

property Δ_2 ($\Psi \in \Delta_2$) if $\Psi(2x) \leq C\Psi(x)$ for some constant $C > 0$ and x large enough. It is equivalent to say that, for every $\beta > 1$, $\Psi(\beta x) \leq C_\beta\Psi(x)$. It is known that if $\Psi \in \Delta_2$, then $\Psi(x) = O(x^p)$ for some $1 \leq p < +\infty$. One says (see [6], [7]) that Ψ satisfies the condition Δ^0 if, for some $\beta > 1$, one has $\lim_{x \rightarrow \infty} \Psi(\beta x)/\Psi(x) = +\infty$. If $\Psi \in \Delta^0$, then $\Psi(x)/x^p \xrightarrow{x \rightarrow \infty} +\infty$ for every $1 \leq p < \infty$. Indeed, let $1 \leq p < \infty$. For every $\beta > 1$ one can find $x_0 > 0$ such that $\Psi(\beta x)/\Psi(x) \geq \beta^p$ for $x \geq x_0$; then $\Psi(\beta^n x_0) \geq \beta^{np}\Psi(x_0)$ for every $n \geq 1$. That implies that $\Psi(x) \geq C_p x^p$ for every $x > 0$ large enough. Since $p \geq 1$ is arbitrary, we get $x^p = o[\Psi(x)]$.

We say that $\Psi \in \nabla_0(1)$ if, for every $A > 1$, $\Psi(Ax)/\Psi(x)$ is non-decreasing for x large enough. This is equivalent to say (see [7], Proposition 4.7) that $\log \Psi(e^x)$ is convex. When $\Psi \in \nabla_0(1)$, one has either $\Psi \in \Delta_2$, or $\Psi \in \Delta^0$.

If (S, \mathcal{S}, μ) is a finite measure space, one defines the Orlicz space $L^\Psi(\mu)$ as the set of all (classes of) measurable functions $f: S \rightarrow \mathbb{C}$ for which there is a $C > 0$ such that $\int_S \Psi(|f|/C) d\mu$ is finite. The norm $\|f\|_\Psi$ is the infimum of all $C > 0$ for which the above integral is ≤ 1 . The Morse-Transue space $M^\Psi(\mu)$ is the subspace of $f \in L^\Psi(\mu)$ for which $\int_S \Psi(|f|/C) d\mu$ is finite for all $C > 0$; it is the closure of $L^\infty(\mu)$ in $L^\Psi(\mu)$. One has $M^\Psi(\mu) = L^\Psi(\mu)$ if and only if $\Psi \in \Delta_2$.

If $\Psi(x)/x \xrightarrow{x \rightarrow +\infty} +\infty$, the conjugate function Φ of Ψ is defined by $\Phi(y) = \sup_{x>0} (xy - \Psi(x))$. It is an Orlicz function and $[M^\Psi(\mu)]^* = L^\Phi(\mu)$, isomorphically.

We may note that if $\Psi(x)/x$ does not converges to infinity, we must have $\Psi(x) \leq ax$ for some $a \geq 1$ and x large enough. Then $L^\Psi(\mu) = L^1(\mu)$ isomorphically and then $\Phi(y) = +\infty$ for $y > a$ (giving $L^\Phi(\mu) = L^\infty(\mu)$ isomorphically).

We denote by \mathbb{D} the open unit disk of \mathbb{C} and by $\mathbb{T} = \partial\mathbb{D}$ the unit circle. The normalized area-measure on \mathbb{D} is denoted by \mathcal{A} and the normalized Lebesgue measure on \mathbb{T} is denoted by m .

The Hardy-Orlicz space H^Ψ is defined as $\{f \in H^1; f^* \in L^\Psi(m)\}$, where f^* is the boundary values function of f , and $HM^\Psi = H^\Psi \cap M^\Psi(m)$ is the closure of H^∞ in H^Ψ . The Bergman-Orlicz space \mathfrak{B}^Ψ is the subspace of analytic $f \in L^\Psi(\mathcal{A})$, and $\mathfrak{B}M^\Psi = \mathfrak{B}^\Psi \cap M^\Psi(\mathcal{A})$ is the closure of H^∞ in \mathfrak{B}^Ψ . Since, for $f \in H^\Psi$, $\|f\|_{H^\Psi} = \sup_{0 < r < 1} \|f_r\|_{H^\Psi}$ (see [7], Proposition 3.1), where $f_r(z) = f(rz)$, one has:

$$\int_0^{2\pi} \Psi\left(\frac{|f(re^{it})|}{\|f\|_{H^\Psi}}\right) \frac{dt}{2\pi} \leq \int_0^{2\pi} \Psi\left(\frac{|f(re^{it})|}{\|f_r\|_{H^\Psi}}\right) \frac{dt}{2\pi} \leq 1;$$

hence:

$$\int_{\mathbb{D}} \Psi\left(\frac{|f(re^{it})|}{\|f\|_{H^\Psi}}\right) d\mathcal{A} = \int_0^1 \left[\int_0^{2\pi} \Psi\left(\frac{|f(re^{it})|}{\|f\|_{H^\Psi}}\right) \frac{dt}{2\pi} \right] 2r dr \leq 1,$$

so $f \in \mathfrak{B}^\Psi$ and $\|f\|_{\mathfrak{B}^\Psi} \leq \|f\|_{H^\Psi}$. It follows that $H^\Psi \subseteq \mathfrak{B}^\Psi$ and the canonical injection $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$ is bounded, and has norm 1. Let us point out that

the boundedness also follows from [7], Theorem 4.10, 2), since J_Ψ is a Carleson embedding $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi \subseteq L^\Psi(\mathcal{A})$.

This injection is not onto, since there are functions $f \in \mathfrak{B}^\Psi$ with no radial limit on a subset of \mathbb{T} of positive measure (the proof is the same as in \mathfrak{B}^p : see [4], § 3.2, Lemma 2, page 81). Note that J_Ψ is not an into-isomorphism: take $f_n(z) = z^n$, for every $n \in \mathbb{N}$; it is easy to see that $\{f_n\}_n$ tends to 0 in \mathfrak{B}^Ψ , but not in H^Ψ .

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2 Compactness and weak-compactness

In order to characterize the compactness and the weak-compactness of J_Ψ , we introduce the following quantity Q_A , $A > 1$:

$$(2.1) \quad Q_A = \limsup_{x \rightarrow +\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2},$$

which will turn out to be essential.

We are going to start with the compactness.

Theorem 2.1 *The canonical injection $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$ is compact if and only if*

$$(2.2) \quad \lim_{x \rightarrow +\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2} = 0 \quad \text{for every } A > 1.$$

Remarks. 1) Condition (2.2) means that $Q_A = 0$ for every $A > 1$. It is equivalent to say that:

$$(2.3) \quad \sup_{A>1} Q_A < +\infty.$$

Indeed, assume that $M := \sup_{A>1} Q_A < +\infty$. Let $0 < \varepsilon \leq 1$ and $A > 1$; we can find $x_A = x_A(\varepsilon) > 0$ such that $\Psi(Ax/\varepsilon)/[\Psi(x)]^2 \leq 2M$ for $x \geq x_A$. By convexity, one has $\Psi(Ax) \leq \varepsilon \Psi(Ax/\varepsilon)$, and hence $\Psi(Ax)/[\Psi(x)]^2 \leq 2\varepsilon M$ for $x \geq x_A$. We get $Q_A = 0$.

2) It is clear that condition (2.2) is satisfied whenever $\Psi \in \Delta_2$, but $\Psi(x) = e^{[\log(x+1)]^2} - 1$ satisfies (2.2) without being in Δ_2 . However, condition (2.2) implies that Ψ cannot grow too fast. More precisely, we must have

$$\Psi(x) = o(e^{x^\alpha}) \quad \text{for every } \alpha > 0.$$

Indeed, one has $\Psi(At) \leq [\Psi(t)]^2$ for $t \geq t_A$, and, by iteration, $\Psi(A^n t_A) \leq [\Psi(t_A)]^{2^n}$ for every $n \geq 1$. For every $x > 0$ large enough, taking $n \geq 1$ such that $A^n t_A \leq x < A^{n+1} t_A$, we get $\Psi(x) \leq C_1 e^{C_2 x^\alpha}$, with $\alpha = \log 2 / \log A$. Since $A > 1$ is arbitrary, α may be any positive number. The little-oh condition follows from the fact that the inequality is true for all $\alpha > 0$.

Proof of Theorem 2.1. By definition, \mathfrak{B}^Ψ is a subspace of $L^\Psi(\mathbb{D}, \mathcal{A})$; hence we can see J_Ψ as a Carleson embedding $J_\Psi: H^\Psi \rightarrow L^\Psi(\mathbb{D}, \mathcal{A})$. If $S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| < h\}$, the compactness of J_Ψ implies, by [7], Theorem 4.11, that, for every $A > 1$, every $\varepsilon > 0$, and $h > 0$ small enough:

$$h^2 \leq 4 \mathcal{A}[S(\xi, h)] \leq \frac{4\varepsilon}{\Psi[A\Psi^{-1}(1/h)]},$$

that is, setting $x = \Psi^{-1}(1/h)$, $\Psi(Ax) \leq 4\varepsilon [\Psi(x)]^2$, and (2.2) is satisfied.

Conversely, one has:

$$\sup_{0 < t \leq h} \sup_{|\xi|=1} \frac{\mathcal{A}[S(\xi, t)]}{t} \leq \sup_{0 < t \leq h} \frac{t^2}{t} = h,$$

which is $o((1/h)/\Psi[A\Psi^{-1}(1/h)])$ for every $A > 1$, if (2.2) holds; hence, by [7], Theorem 4.11, again, J_Ψ is compact. \square

We now turn ourself to the weak compactness.

Theorem 2.2 *The following assertions are equivalent:*

- (a) $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$ is weakly compact;
- (b) J_Ψ fixes no copy of c_0 ;
- (c) J_Ψ fixes no copy of ℓ_∞ ;
- (d) $Q_A < +\infty$, for every $A > 1$;
- (e) $H^\Psi \subseteq \mathfrak{BM}^\Psi$;
- (f) J_Ψ is strictly singular.

Recall that an operator $T: X \rightarrow Y$ between two Banach spaces is said to be strictly singular if there is no infinite-dimensional subspace X_0 of X on which T is an into-isomorphism.

The proof will be somewhat long, and before beginning it, we shall remark that if $\Psi \in \Delta^0$, then condition

$$(2.4) \quad Q_A < +\infty \quad \text{for every } A > 1$$

implies condition (2.2). Indeed, if $\lim_{x \rightarrow +\infty} \frac{\Psi(\beta x)}{\Psi(x)} = +\infty$, we get, for every $A > 1$:

$$\limsup_{x \rightarrow +\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2} = \limsup_{x \rightarrow +\infty} \frac{\Psi(Ax)}{\Psi(\beta Ax)} \frac{\Psi(\beta Ax)}{[\Psi(x)]^2} \leq \limsup_{x \rightarrow +\infty} \frac{\Psi(Ax)}{\Psi(\beta Ax)} Q_{\beta A} = 0.$$

Now, if, for some $A > 1$, $\Psi(Ax)/\Psi(x)$ is non-decreasing for x large enough (in particular if $\Psi \in \nabla_0(1)$), one has the dichotomy: either $\Psi \in \Delta_2$, and then J_Ψ is compact; or $\Psi \in \Delta^0$ and hence the weak compactness of J_Ψ implies, by the two above theorems, its compactness. Hence:

Proposition 2.3 *If, for some $A > 1$, $\Psi(Ax)/\Psi(x)$ is non-decreasing, for x large enough, then the weak compactness of J_Ψ is equivalent to its compactness.*

However, it is easy to construct an Orlicz function Ψ which satisfies condition (2.4), but not condition (2.2). We do not give an example here because we have a stronger result in Section 4.

In order to prove Theorem 2.2, we shall need several lemmas.

Lemma 2.4 *Let Ψ be any Orlicz function. If we define $\Psi_1(t) = [\Psi(t)]^2$, $t \geq 0$, then Ψ_1 is an Orlicz function for which $H^\Psi \subseteq \mathfrak{B}^{\Psi_1}$ and the canonical injection of H^Ψ into \mathfrak{B}^{Ψ_1} is continuous.*

Proof. It is enough to see that H^Ψ continuously embeds into $L^{\Psi_1}(\mathcal{A})$, and for this we can use Theorem 4.10 in [7]. Following the notation of that theorem for the measure $\mu = \mathcal{A}$, it is easy to see that, as $h \rightarrow 0^+$, $\rho_{\mathcal{A}}(h) \approx h^2$, and $K_{\mathcal{A}}(h) \approx h$. Observe that, for $t > 1$, we have $\Psi_1[\Psi^{-1}(t)] = t^2$, and so, for $h \in (0, 1)$,

$$\frac{1/h}{\Psi_1[\Psi^{-1}(1/h)]} = \frac{1/h}{1/h^2} = h \succeq K_{\mathcal{A}}(h).$$

Using part 2) of Theorem 4.10 in [7], the lemma follows. \square

Lemma 2.5 *Let $M > \delta > 0$ and $\{f_n\}_n$ be a sequence in $H^\Psi \cap \mathfrak{B}M^\Psi$ such that:*

- (a) $\{f_n\}_n$ tends to 0 uniformly on compact subsets of \mathbb{D} ;
- (b) $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$, for every $n \geq 1$;
- (c) $\|f_n\|_{H^\Psi} \leq M$, for every $n \geq 1$.

Then there exists a subsequence $\{f_{n_k}\}_k$ such that $\sum_k |f_{n_k}(z)| < +\infty$, for every $z \in \mathbb{D}$, and for every $\alpha = (\alpha_k)_k \in \ell_\infty$, one has, writing $T\alpha(z) = \sum_{k=1}^\infty \alpha_k f_{n_k}(z)$:

$$(2.5) \quad T\alpha \in \mathfrak{B}^\Psi \quad \text{and} \quad (\delta/2)\|\alpha\|_\infty \leq \|T\alpha\|_{\mathfrak{B}^\Psi} \leq 2M\|\alpha\|_\infty.$$

Remark. It is clear that, by (2.5), we are defining an operator T from ℓ_∞ into \mathfrak{B}^Ψ which is an isomorphism between ℓ_∞ and its image. In particular, the subsequence $\{f_{n_k}\}_k$ is equivalent, in \mathfrak{B}^Ψ , to the canonical basis of c_0 .

Proof. First we are going to construct, inductively, a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}$, and an increasing sequence $\{r_k\}_k$ in $(0, 1)$, such that $\lim_{k \rightarrow \infty} r_k = 1$ and, setting

$$D_k = \{z \in \mathbb{D}; |z| \leq r_k\}, \quad \text{for } k \geq 1,$$

and

$$C_1 = D_1, \quad C_k = D_k \setminus D_{k-1} = \{z \in \mathbb{D}; r_{k-1} < |z| \leq r_k\}, \quad k \geq 2,$$

we have:

$$(2.6) \quad |f_{n_k}(z)| \leq 2^{-k}, \quad \text{for every } z \in D_{k-1}, \text{ and every } k \geq 2;$$

and

$$(2.7) \quad \|f_{n_k} \mathbb{1}_{\mathbb{D} \setminus C_k}\|_{L^\Psi} < \delta 2^{-k-2}, \quad \text{for every } k \geq 1.$$

Start the construction by taking $n_1 = 1$. It is a known fact that, for every function f in the Morse-Transue space $M^\Psi(\mathcal{A})$, we have

$$(2.8) \quad \lim_{\mathcal{A}(A) \rightarrow 0} \|f \mathbb{1}_A\|_{L^\Psi} = 0.$$

Now, using (2.8), with $f = f_{n_1}$ and considering sets A of the form $A = \{z \in \mathbb{D}; r < |z| < 1\}$, we get $r_1 \in (0, 1)$ so that, for $C_1 = D_1 = \{z \in \mathbb{D}; |z| \leq r_1\}$, we have

$$\|f_1 \mathbb{1}_{\mathbb{D} \setminus C_1}\|_{L^\Psi} < \delta 2^{-3}.$$

By the uniform convergence of $\{f_n\}_n$ to 0 on D_1 , we can find $n_2 > n_1$ such that

$$|f_{n_2}(z)| \leq 1/4, \quad \text{for every } z \in D_1, \quad \text{and} \quad \|f_{n_2} \mathbb{1}_{D_1}\|_{L^\Psi} < \delta 2^{-5}.$$

Using this last inequality and (2.8) again (for $f = f_{n_2}$), we get $r_2 \in (r_1, 1)$, $r_2 > 1 - 1/2$, such that, setting $C_2 = \{z \in \mathbb{D}; r_1 < |z| \leq r_2\}$, we have

$$\|f_{n_2} \mathbb{1}_{\mathbb{D} \setminus C_2}\|_{L^\Psi} < \delta 2^{-4}.$$

Now that we have (2.6) and (2.7) for $k = 1$ and $k = 2$, it is clear how we must iterate the inductive construction. At the time of choosing $r_k \in (r_{k-1}, 1)$, we also impose the condition $r_k > 1 - 1/k$ in order to get $\lim_{k \rightarrow \infty} r_k = 1$.

Once the construction is achieved, let us see why the subsequence $\{f_{n_k}\}_k$ works. The condition (2.6) and the fact that $\lim_{k \rightarrow \infty} r_k = 1$ imply that, for every compact set K in \mathbb{D} and $z \in \mathbb{D}$, there exists $l_K \in \mathbb{N}$ such that:

$$|f_{n_k}(z)| \leq 2^{-k}, \quad \text{for every } z \in K, \quad \text{and every } k \geq l_K.$$

This yields two facts. First, $\sum_k |f_{n_k}(z)| < +\infty$, for every $z \in \mathbb{D}$, and secondly: for every bounded complex sequence $\alpha = (\alpha_k)_k \in \ell_\infty$, the series $\sum_k \alpha_k f_{n_k}$ converges uniformly on compact subsets of \mathbb{D} , and its sum, the function $T\alpha$, is analytic on \mathbb{D} .

It remains to prove the estimates in (2.5) about the norm of $T\alpha$ in $L^\Psi(\mathcal{A})$. By homogeneity, we may assume that $\|\alpha\|_\infty = 1$. Let us write $g_k = f_{n_k} \mathbb{1}_{C_k}$ and $h_k = f_{n_k} \mathbb{1}_{\mathbb{D} \setminus C_k}$, for every $k \geq 1$,

$$g = \sum_{k=1}^{\infty} \alpha_k g_k \quad \text{and} \quad h = \sum_{k=1}^{\infty} \alpha_k h_k.$$

We have $T\alpha = g + h$. By (2.7) and the fact that $|\alpha_k| \leq 1$, we have that $h \in L^\Psi(\mathcal{A})$ and $\|h\|_{L^\Psi} \leq \delta/4$.

By the condition (c) in the statement and the definition of the norm in H^Ψ we have, for every n and every $r \in (0, 1)$:

$$(2.9) \quad \frac{1}{2\pi} \int_0^{2\pi} \Psi(|f_n(re^{it})|/M) dt \leq 1.$$

The function g_k is 0 outside of C_k , and the sequence $\{C_k\}_k$ is a partition of \mathbb{D} . Therefore:

$$\begin{aligned} \int_{\mathbb{D}} \Psi(|g|/M) d\mathcal{A} &= \sum_{k=1}^{\infty} \int_{C_k} \Psi(|g|/M) d\mathcal{A} = \sum_{k=1}^{\infty} \int_{C_k} \Psi(|\alpha_k| |f_{n_k}|/M) d\mathcal{A} \\ &\leq \sum_{k=1}^{\infty} \int_{C_k} \Psi(|f_{n_k}|/M) d\mathcal{A}. \end{aligned}$$

Integrating in polar coordinates, setting $r_0 = 0$, and using (2.9), we get:

$$\begin{aligned} \int_{\mathbb{D}} \Psi(|g|/M) d\mathcal{A} &\leq \sum_{k=1}^{\infty} \int_{r_{k-1}}^{r_k} 2r \frac{1}{2\pi} \int_0^{2\pi} \Psi(|f_{n_k}(re^{it})|/M) dt dr \\ &\leq \sum_{k=1}^{\infty} \int_{r_{k-1}}^{r_k} 2r dr = 1, \end{aligned}$$

and therefore $\|g\|_{L^\Psi} \leq M$, and $\|T\alpha\|_{L^\Psi} \leq \delta/4 + M \leq 2M$.

On the other hand, for every k , we have:

$$\|g\|_{L^\Psi} \geq \|g \mathbf{1}_{C_k}\|_{L^\Psi} = |\alpha_k| \|f_{n_k} - h_k\|_{L^\Psi} \geq |\alpha_k| (\delta - \delta/2^{2+k}) \geq \frac{3\delta}{4} |\alpha_k|.$$

Taking the supremum on k , we get $\|g\|_{L^\Psi} \geq (3\delta/4) \|\alpha\|_\infty = 3\delta/4$. Consequently,

$$\|T\alpha\|_{L^\Psi} \geq \|g\|_{L^\Psi} - \|h\|_{L^\Psi} \geq (3\delta/4) - \delta/4 \geq \delta/2,$$

and Lemma 2.5 is fully proved. \square

In the following lemma we isolate the proof of the implication (c) \implies (d) in the statement of Theorem 2.2.

Lemma 2.6 *Assume that the Orlicz function Ψ is such that, for some $A > 1$,*

$$(2.10) \quad \limsup_{x \rightarrow +\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2} = +\infty$$

Then the injection $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$ fixes a copy of ℓ_∞ .

Proof. Let us take a sequence of positive numbers $\{d_n\}_n$, and a sequence $\{\xi_n\}_n$ in \mathbb{T} , such that the disks $\{D(\xi_n, d_n)\}_n$ are pairwise disjoint in \mathbb{D} . In particular, we should have $\lim_{n \rightarrow \infty} d_n = 0$.

The convexity of Ψ implies the existence of some $c > 0$ such that $\Psi(x) \geq cx$ for every $x \geq 1$. Given a sequence $\{\beta_n\}_n$ in $(4, +\infty)$ to be fixed later, we can find, thanks to (2.10), an increasing sequence $\{x_n\}$ satisfying:

$$(2.11) \quad x_n > 1, \quad \Psi(x_n) > 1, \quad \Psi(Ax_n) > \beta_n [\Psi(x_n)]^2, \quad \text{for every } n \in \mathbb{N}.$$

Define y_n as the point in the interval (x_n, Ax_n) such that

$$(2.12) \quad [\Psi(y_n)]^2 = \Psi(Ax_n).$$

Put now $h_n = 1/\Psi(y_n)$ and $r_n = 1 - h_n$. By (2.11) and (2.12), we have $[\Psi(y_n)]^2 > \beta_n > 4$, and therefore $h_n \in (0, 1/2)$. Define

$$u_n(z) = \left(\frac{h_n}{1 - r_n \overline{\xi_n} z} \right)^2, \quad \text{and} \quad f_n(z) = y_n u_n(z).$$

It is easy to see that $\|u_n\|_\infty = 1$, and that $\|u_n\|_{H^1} \leq h_n$.

The first condition imposed to β_n is $\beta_n > 16/d_n^2$. That gives $[\Psi(y_n)]^2 > 16/d_n^2$ and $h_n < d_n/4$. Let us write D_n for the disk $D(\xi_n, d_n)$. Observe that, for $z \in \overline{\mathbb{D}} \setminus D_n$, we have

$$|1 - r_n \overline{\xi_n} z| = |1 - r_n + r_n \xi_n \overline{\xi_n} - r_n \overline{\xi_n} z| \geq r_n |\xi_n - z| - h_n \geq (1/2)d_n - h_n \geq d_n/4,$$

and therefore, since $[\Psi(x_n)]^2 \geq \Psi(x_n) \geq cx_n$,

$$|f_n(z)| \leq y_n \left(\frac{4h_n}{d_n} \right)^2 = \frac{16y_n}{d_n^2 [\Psi(y_n)]^2} \leq \frac{16Ax_n}{d_n^2 \beta_n [\Psi(x_n)]^2} \leq \frac{16A}{c d_n^2 \beta_n}.$$

We also impose the condition $\beta_n > 16An^2/cd_n^2$, and so we have:

$$(2.13) \quad |f_n(z)| \leq \frac{1}{n^2}, \quad \text{for } z \in \overline{\mathbb{D}} \setminus D_n.$$

From (2.13) we deduce that $\{f_n\}_n$ converges to 0 uniformly on compact subsets of \mathbb{D} . Moreover (2.13) yields that, for every bounded sequence $\{\alpha_n\}_n$ of complex numbers, the series $\sum_{n \geq 1} \alpha_n f_n$ is uniformly convergent on compact subsets of \mathbb{D} . Let us write f_n^* for the boundary value (on $\mathbb{T} = \partial\mathbb{D}$) of the function f_n . We claim that :

$$(2.14) \quad S = \sum_{n=1}^{\infty} |f_n^*| \in L^\Psi(\mathbb{T}, m).$$

From this, it is not difficult to deduce that, for every bounded sequence $\{\alpha_n\}_n$ of complex numbers, the function $\sum_{n=1}^{\infty} \alpha_n f_n$ is in H^Ψ and, for $M = \|S\|_{L^\Psi(\mathbb{T})}$,

$$(2.15) \quad \left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\|_{H^\Psi} \leq M \|\{\alpha_n\}_n\|_\infty.$$

On the other hand, taking $A_n = \{z \in \mathbb{D}; |z - \xi_n| \leq h_n\}$, there exists a constant $\gamma \in (0, 1)$ such that $\mathcal{A}(A_n) \geq \gamma h_n^2$, and, for every $z \in A_n$, we have:

$$|1 - r_n \overline{\xi_n} z| \leq |1 - r_n| + |r_n \xi_n \overline{\xi_n} - r_n \overline{\xi_n} z| = h_n + r_n |z - \xi_n| \leq 2h_n,$$

and consequently $|u_n(z)| \geq 1/4$. If $\delta = \gamma/4A$, we have, for every n ,

$$\begin{aligned} \int_{\mathbb{D}} \Psi\left(\frac{|f_n|}{\delta}\right) d\mathcal{A} &\geq \int_{A_n} \Psi\left(\frac{y_n}{4\delta}\right) d\mathcal{A} \geq \gamma h_n^2 \Psi\left(\frac{1}{\gamma} A y_n\right) \\ &\geq h_n^2 \Psi(A y_n) > h_n^2 \Psi(A x_n) = 1. \end{aligned}$$

Thus $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$, for every $n \in \mathbb{N}$. We can apply Lemma 2.5. Using this lemma and (2.15), we get a subsequence $\{f_{n_k}\}_k$ such that, for every $\alpha = (\alpha_k)_k \in \ell_\infty$, we have:

$$(\delta/2) \|\{\alpha_k\}_k\|_\infty \leq \left\| \sum_{k=1}^{\infty} \alpha_k f_{n_k} \right\|_{\mathfrak{B}^\Psi} \leq \left\| \sum_{k=1}^{\infty} \alpha_k f_{n_k} \right\|_{H^\Psi} \leq M \|\{\alpha_k\}_k\|_\infty.$$

This clearly says that J_Ψ fixes a copy of ℓ_∞ .

It remains to prove (2.14). For obtaining this we impose the last condition to the sequence $\{\beta_n\}_n$. We shall need:

$$(2.16) \quad \sum_{n=1}^{\infty} 1/\sqrt{\beta_n} \leq 1.$$

Let us set $g_n = |f_n^*| \mathbb{1}_{D_n}$. Thanks to (2.13), $S - \sum_{n=1}^{\infty} g_n$ is a bounded function. Thus we just need to prove that $G = \sum_{n=1}^{\infty} g_n$ is in $L^\Psi(\mathbb{T})$. We have $\|G\|_{L^\Psi(\mathbb{T})} \leq A$. Indeed, recalling that the D_n 's are pairwise disjoint, and that each g_n is 0 out of D_n , we have:

$$\begin{aligned} \int_{\mathbb{T}} \Psi\left(\frac{G}{A}\right) dm &= \sum_{n=1}^{\infty} \int_{D_n \cap \mathbb{T}} \Psi\left(\frac{G}{A}\right) dm = \sum_{n=1}^{\infty} \int_{D_n \cap \mathbb{T}} \Psi\left(\frac{|f_n^*|}{A}\right) dm \\ &\leq \sum_{n=1}^{\infty} \int_{\mathbb{T}} \Psi\left(\frac{y_n |u_n^*|}{A}\right) dm \end{aligned}$$

and by the convexity of Ψ , and the fact that $|u_n| \leq 1$,

$$\begin{aligned} &\leq \sum_{n=1}^{\infty} \int_{\mathbb{T}} |u_n^*| \Psi\left(\frac{y_n}{A}\right) dm = \sum_{n=1}^{\infty} \|u_n\|_{H_1} \Psi\left(\frac{y_n}{A}\right) \\ &\leq \sum_{n=1}^{\infty} \frac{\Psi(y_n/A)}{\Psi(y_n)} \leq \sum_{n=1}^{\infty} \frac{\Psi(x_n)}{\Psi(y_n)} = \sum_{n=1}^{\infty} \frac{\Psi(x_n)}{\sqrt{\Psi(Ax_n)}} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{\beta_n}} \leq 1, \end{aligned}$$

by the required condition (2.16), and that ends the proof of Lemma 2.6. \square

We are now in position to prove Theorem 2.2.

Proof of Theorem 2.2. We shall prove that:

$$(a) \implies (b) \implies (c) \implies (d) \implies (e) \implies (a),$$

and that (b) \iff (f).

The implications (a) \implies (b) \implies (c) and (f) \implies (b) are trivial, and we have seen in Lemma 2.6 that (c) \implies (d).

(d) \implies (e). By Lemma 2.4, there exists a constant $C > 0$ such that, for every f in the unit ball of H^Ψ , we have:

$$(2.17) \quad \int_{\mathbb{D}} [\Psi(|f|/C)]^2 d\mathcal{A} \leq 1.$$

For every $A > 0$, there exist x_A , such that $\Psi(Ax) \leq (Q_A + 1)[\Psi(x)]^2$, for every $x \geq x_A$. Thus for every $x \geq 0$ we have $\Psi(Ax) \leq (Q_A + 1)[\Psi(x)]^2 + \Psi(Ax_A)$. Then, by (2.17), we have

$$\int_{\mathbb{D}} \Psi(A|f|/C) d\mathcal{A} < +\infty, \quad \text{for every } A > 0.$$

Therefore $f \in \mathfrak{B}M^\Psi$, for every f in the unit ball of H^Ψ , and thus for every f in H^Ψ .

(e) \implies (a). Let $\{f_n\}_n$ be in the unit ball of H^Ψ . We have to prove that $\{f_n\}_n$ has a subsequence which converges in the weak topology of \mathfrak{B}^Ψ . By Montel's Theorem $\{f_n\}_n$ has a subsequence converging uniformly on compact subsets of \mathbb{D} , to a function g which, by Fatou's lemma, also belongs to the unit ball of H^Ψ . If this subsequence converges to g in the norm of \mathfrak{B}^Ψ we are done. If not, after perhaps a new extraction of subsequence, there exist $\delta > 0$ and a subsequence $\{f_{n_k}\}_k$, such that

$$\|f_{n_k} - g\|_{\mathfrak{B}^\Psi} \geq \delta, \quad \text{and} \quad \|f_{n_k} - g\|_{H^\Psi} \leq 2.$$

Since moreover $\{f_{n_k} - g\}_k$ converges to 0 uniformly on compact subsets of \mathbb{D} and, by condition (e), $f_{n_k} - g \in \mathfrak{B}M^\Psi$, we may apply Lemma 2.5 and we get that $\{f_{n_k} - g\}_k$ has a subsequence equivalent to the canonical basis of c_0 in \mathfrak{B}^Ψ , and is therefore weakly null. This yields that $\{f_n\}_n$ has a subsequence converging to g in the weak topology of \mathfrak{B}^Ψ .

(b) \implies (f). Suppose there exists an infinite-dimensional subspace X of H^Ψ on which the norms $\|\cdot\|_{\mathfrak{B}^\Psi}$ and $\|\cdot\|_{H^\Psi}$ are equivalent. We shall have finished if we prove that X contains a subspace isomorphic to c_0 because then J_Ψ will fix a copy of c_0 .

We can assume that X is contained in $\mathfrak{B}M^\Psi$ because we already know that (b) implies (e). X being infinite-dimensional, there exists, for every $n \in \mathbb{N}$, $f_n \in X$, such that $\|f_n\|_{H^\Psi} = 1$, and $\widehat{f}_n(k) = 0$, for $k = 0, 1, \dots, n$. By the equivalence of the norms in X , there exists $\delta > 0$ such that $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$, for every n . The unit ball of H^Ψ is compact in the topology of $\mathcal{H}(\mathbb{D})$. Since

$$\lim_{n \rightarrow \infty} \widehat{f}_n(k) = 0, \quad \text{for every } k \geq 0,$$

the only possible limit of a subsequence of $\{f_n\}_n$ is the function 0. So $\{f_n\}_n$ converges to 0 uniformly on compact subsets of \mathbb{D} . As $f_n \in X \subseteq \mathfrak{B}M^\Psi$, for

every n , we can apply Lemma 2.5, and we get that $\{f_n\}_n$ has a subsequence generating an space Y isomorphic to c_0 in \mathfrak{B}^Ψ . This space Y is contained in X , where the norms are equivalent, so Y is also isomorphic to c_0 for the norm of H^Ψ . \square

3 Other properties

3.1 Dunford-Pettis

Recall that an operator $T: X \rightarrow Y$ between two Banach spaces X and Y is said to be *Dunford-Pettis* if $\{Tx_n\}_n$ converges in norm whenever $\{x_n\}_n$ converges weakly. Every compact operator is Dunford-Pettis. The next proposition shows that, in “most” of the cases, these two properties are equivalent for J_Ψ .

Proposition 3.1 *If the conjugate function of Ψ satisfies condition Δ_2 , then $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$ is Dunford-Pettis if and only if it is compact.*

We shall see in Section 4 that without condition Δ_2 for the conjugate function, J_Ψ may be Dunford-Pettis without being compact.

Proof. Remark first that speaking of the conjugate function of Ψ implicitly assume that $\Psi(x)/x$ tends to $+\infty$ as x goes to $+\infty$.

Assume that J_Ψ is not compact. By Theorem 2.1, there are some $A > 1$ and a sequence $\{x_j\}_j$ going to $+\infty$ such that $\Psi(Ax_j) \geq [\Psi(x_j)]^2$. Setting $r_j = 1 - 1/\Psi(x_j)$, this is equivalent to say that $A\Psi^{-1}(1/(1-r_j)) \geq \Psi^{-1}(1/(1-r_j)^2)$. Define:

$$f_j(z) = x_j \left(\frac{1-r_j}{1-r_j z} \right)^2.$$

One has $f_j \in HM^\Psi$ and $\|f_j\|_{H^\Psi} \leq 1$ (see [7], Corollary 3.10). Since $\{f_j\}_j$ converges to 0 uniformly on compact subsets of \mathbb{D} , $\{f_j\}_j$ converges to 0 in the weak-star topology of H^Ψ ([7], Proposition 3.7). But, since the conjugate function of Ψ satisfies condition Δ_2 , H^Ψ is the bidual of HM^Ψ ([7], Corollary 3.3); hence $\{f_j\}_j$ converges weakly to 0 in HM^Ψ .

On the other hand, if $S_j = D(1, 1-r_j) \cap \mathbb{D}$, one has $|1-r_j z| \leq 2(1-r_j)$ for $z \in S_j$; hence, writing $K = \|f_j\|_{\mathfrak{B}^\Psi}$, one has:

$$1 = \int_{\mathbb{D}} \Psi(|f_j|/K) d\mathcal{A} \geq \int_{S_j} \Psi(|f_j|/K) d\mathcal{A} \geq \mathcal{A}(S_j)\Psi(x_j/4K).$$

Since $\mathcal{A}(S_j) \geq \alpha(1-r_j)^2$, with $0 < \alpha < 1$, we get (since $\Psi(\alpha x_j/4K) \leq \alpha\Psi(x_j/4K)$, by convexity):

$$\|f_j\|_{\mathfrak{B}^\Psi} \geq (\alpha/4) \frac{x_j}{\Psi^{-1}(1/(1-r_j)^2)} = (\alpha/4) \frac{\Psi^{-1}(1/(1-r_j))}{\Psi^{-1}(1/(1-r_j)^2)} \geq \frac{\alpha}{4A}.$$

Therefore J_Ψ is not Dunford-Pettis. \square

On the other hand, one has:

Proposition 3.2 *If J_Ψ is Dunford-Pettis, then J_Ψ is weakly compact.*

Proof. By Theorem 2.2, if J_Ψ is not weakly compact, there is a subspace X_0 of H^Ψ isomorphic to c_0 on which J_Ψ is an into-isomorphism; hence J_Ψ cannot be Dunford-Pettis. \square

We shall see in the next section that J_Ψ may be weakly compact without being Dunford-Pettis.

3.2 Absolutely summing

Every p -summing operator is weakly compact and Dunford-Pettis; so it may be expected that J_Ψ is p -summing for some $p < \infty$. The next results show that this is never the case as soon as Ψ grows faster than all the power functions.

Recall that an operator $T: X \rightarrow Y$ between two Banach spaces X and Y is called (p, q) -summing if there is a constant $C > 0$ such that

$$\left(\sum_{k=1}^n \|Tx_k\|^p \right)^{1/p} \leq C \sup_{\|x^*\|_{X^*} \leq 1} \left(\sum_{k=1}^n |x^*(x_k)|^q \right)^{1/q},$$

for every finite sequence (x_1, \dots, x_n) in X . If $q = p$, it is said p -summing. Every p -summing operator is (p, q) -summing for $q \leq p$.

Theorem 3.3 *If $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$ is p -summing, then, for every $q > p$, $\Psi(x) = O(x^q)$ for x large enough. Moreover, if $p < 2$, then J_Ψ is compact.*

In order to prove this, we need two lemmas.

Lemma 3.4 *If the canonical injection $I_\Psi: A \rightarrow \mathfrak{B}^\Psi$ is $(p, 1)$ -summing, where $A = A(\mathbb{D})$ is the disk algebra, then $\Psi(x) = O(x^{2p})$ for x large enough.*

In particular, $J_r: H^r \rightarrow \mathfrak{B}^r$ is $(p, 1)$ -summing for no $p < r/2$, and, if $\Psi \in \Delta^0$, then J_Ψ is $(p, 1)$ -summing for no $p < \infty$.

Recall that the disk algebra is the space of continuous functions on $\overline{\mathbb{D}}$ which are analytic in \mathbb{D} .

We refer to [9] for a detailed study of r -summing Carleson embeddings $H^r \rightarrow L^r(\mu)$. In particular, it follows from these results that $J_r: H^r \rightarrow \mathfrak{B}^r$ is 1-summing for $1 \leq r < 2$. On the other hand, it is easy to see that $J_2: H^2 \rightarrow \mathfrak{B}^2$ is not Hilbert-Schmidt (*i.e.* not 2-summing): for the canonical orthonormal basis $\{z^n\}_n$ and $\{\sqrt{n+1}z^n\}_n$ of H^2 and \mathfrak{B}^2 , J_2 is the diagonal operator of multiplication by $\{1/\sqrt{n+1}\}_n$. It also follows from [9] that, for $r \geq 2$, J_r is p -summing for no finite p .

Proof. Assume that we do not have $\Psi(x) = O(x^{2p})$ for x large enough. Then $\limsup_{x \rightarrow +\infty} \Psi(x)/x^{2p} = +\infty$. Given any $K > 0$, take $y > 0$ such that $\Psi(y)/y^{2p} \geq K$ and such that $h = 1/\sqrt{\Psi(y)} \leq 1/2$. Let N be the integer part of $(1/h) + 1$. Writing $\xi_j = e^{2\pi i j/N}$, we set:

$$u_j(z) = \frac{h^2}{[1 - (1-h)\overline{\xi_j}z]^2}.$$

We have $u_j \in A(\mathbb{D})$. By [7], Lemma 5.6, one has, since $h \geq 1/N$:

$$\sum_{j=0}^{N-1} |u_j(e^{it})| \leq N h^2 \frac{1 - (1-h)^{2N}}{[1 - (1-h)^2][1 - (1-h)^N]^2} \leq \frac{e^2}{(1-e)^2} := C.$$

Hence:

$$\sup_{\|x^*\|_{A^*} \leq 1} \sum_{j=0}^{N-1} |x^*(u_j)| \leq C.$$

On the other hand, it is easy to see that $|u_j(z)| \geq 1/9$ when $|z - (1-h)\xi_j| < h$; hence, if $S_j = \{z \in \mathbb{D}; |z - (1-h)\xi_j| < h\}$, one has, since $\mathcal{A}(S_j) = h^2$:

$$1 = \int_{\mathbb{D}} \Psi\left(\frac{|u_j(z)|}{\|u_j\|_{\mathfrak{B}^\Psi}}\right) d\mathcal{A}(z) \geq \int_{S_j} \Psi\left(\frac{1/9}{\|u_j\|_{\mathfrak{B}^\Psi}}\right) d\mathcal{A} \geq h^2 \Psi\left(\frac{1/9}{\|u_j\|_{\mathfrak{B}^\Psi}}\right),$$

so $\|u_j\|_{\mathfrak{B}^\Psi} \geq 1/9\Psi^{-1}(1/h^2)$. Since $y = \Psi^{-1}(1/h^2)$, one gets:

$$\sum_{j=0}^{N-1} \|u_j\|_{\mathfrak{B}^\Psi}^p \geq (1/9)^p \frac{N}{y^p} \geq (1/9)^p \left[\frac{\Psi(y)}{y^{2p}}\right]^{1/2} \geq \frac{K^{1/2}}{9^p}.$$

This yields that the $(p, 1)$ -summing norm of I_Ψ should be greater than $K^{1/2p}/9C$, and, as K is arbitrary, that I_Ψ is not $(p, 1)$ -summing. \square

Remark. When $I_\Psi: A \hookrightarrow \mathfrak{B}^\Psi$ is p -summing, we have this shorter argument. By Pietsch's factorization theorem, this I_Ψ factors through H^p . It follows from [7], Theorem 4.10, that $\alpha h^2 \leq \rho_{\mathcal{A}}(h) \leq 1/\Psi^{-1}(A/h^{1/p})$, for some constants $0 < \alpha < 1$ and $A > 0$, and h small enough. That means that $\Psi(x) \leq C x^{2p}$ for x large enough.

Lemma 3.5 *If the canonical injection $I_\Psi: A \rightarrow \mathfrak{B}^\Psi$ is 1-summing, then J_Ψ is compact.*

Proof. The canonical injection $J_1: H^1 \rightarrow \mathfrak{B}^1$ (as well as J_Ψ whenever $\Psi \in \Delta_2$) is compact. Hence we may assume that H^Ψ is not H^1 and hence that $\Psi(x)/x$ tends to $+\infty$ as x tends to $+\infty$.

Assume that J_Ψ is not compact. Then, as in the proof of Proposition 3.1, there are some $A > 1$ and a sequence $\{x_k\}_k$ going to $+\infty$ such that $\Psi(Ax_k) \geq [\Psi(x_k)]^2$. Setting $h_k = 1/\Psi(x_k)$, we define, as in the proof of Proposition 3.4:

$$u_{k,j}(z) = \frac{h_k^2}{[1 - (1-h_k)\bar{\xi}_{k,j}z]^2},$$

where $\xi_{k,j} = e^{2\pi i j/N_k}$, with N_k the integer part of $(1/h_k) + 1$. One has $u_{k,j} \in A$ and (see the proofs of the two quoted propositions):

$$\sum_{j=0}^{N_k-1} |u_{k,j}(e^{it})| \leq C \quad \text{and} \quad \|u_{k,j}\|_{\mathfrak{B}^\Psi} \geq \frac{\delta\alpha}{A} \frac{1}{\Psi^{-1}(1/h_k)}.$$

It follows that:

$$\sum_{j=0}^{N_k-1} \|u_{k,j}\|_{\mathfrak{B}^\Psi} \geq \frac{\delta\alpha}{A} \frac{N_k}{\Psi^{-1}(1/h_k)} \geq \frac{\delta\alpha}{A} \frac{1/h_k}{\Psi^{-1}(1/h_k)} = \frac{\delta\alpha}{A} \frac{\Psi(x_k)}{x_k} \xrightarrow{k \rightarrow \infty} +\infty.$$

Hence I_Ψ is not 1-summing. \square

Proof of Theorem 3.3. Since $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$ is p -summing and the canonical injection $I_\Psi: A \rightarrow \mathfrak{B}^\Psi$ factors as $I_\Psi: A \rightarrow H^\Psi \rightarrow \mathfrak{B}^\Psi$, this injection is p -summing. By Lemma 3.4, $\Psi(x) = O(x^{2p})$ for x large enough. Hence we have the factorization $A \rightarrow H^{2p} \rightarrow H^\Psi \rightarrow \mathfrak{B}^\Psi$. Since the first injection is $2p$ -summing and the last one is p -summing, the composition is $\max(1, p_1)$ -summing, with $\frac{1}{p_1} = \frac{1}{2p} + \frac{1}{p}$ (see [2], Theorem 2.22), *i. e.* $p_1 = \frac{2}{3}p$. If $p_1 > 1$, we can use again Lemma 3.4 with p_1 instead of $2p$; we get that $\Psi(x) = O(x^{2p_1})$, for x large enough, and that the factorization $I_\Psi: A \rightarrow H^{2p_1} \rightarrow H^\Psi \rightarrow \mathfrak{B}^\Psi$ is $\max(1, p_2)$ -summing, with $\frac{1}{p_2} = \frac{1}{2p_1} + \frac{1}{p}$. Going on the same way, we get a decreasing sequence $\{p_n\}_n$ such that the canonical injection $A \rightarrow \mathfrak{B}^\Psi$ is $\max(1, p_n)$ -summing and $\frac{1}{p_{n+1}} = \frac{1}{2p_n} + \frac{1}{p}$. Writing $p_n = \alpha_n p$, we get $\alpha_{n+1} = \frac{2\alpha_n}{2\alpha_n + 1}$; hence $p_n \xrightarrow{n \rightarrow \infty} p/2$. In particular, $\Psi(x) = O(x^q)$ for every $q > p$.

If $p < 2$, one has $\max(1, p_n) = 1$ for n large enough, and Lemma 3.4 implies that J_Ψ is compact. \square

Remark 1. It is not clear whether J_Ψ p -summing, with $p \geq 2$, implies that J_Ψ is compact. However, when $r \geq 2$, $J_r: H^r \rightarrow \mathfrak{B}^r$ is p -summing for no $p < \infty$ (see [9]).

Remark 2. An operator $T: X \rightarrow Y$ between two Banach spaces is said to be *finitely strictly singular* (or *superstrictly singular*) if for every $\varepsilon > 0$, there is an integer $N_\varepsilon \geq 1$ such that, for every subspace X_0 of X of dimension $\geq N_\varepsilon$, there is an $x \in X_0$ such that $\|Tx\| \leq \varepsilon \|x\|$. Every finitely strictly singular operator is strictly singular. It is not difficult to see that every compact operator is finitely strictly singular and it is shown in [10] (see also [5], Corollary 2.3) that every p -summing operator is finitely strictly singular. We do not know when J_Ψ is finitely strictly singular.

3.3 Order boundedness

Recall that an operator $T: X \rightarrow Y$ from a Banach space X into a Banach lattice Y is said to be *order bounded* if there is $y \in Y_+$ such that $|Tx| \leq y$ for every x in the unit ball of X . Since the Bergman-Orlicz space \mathfrak{B}^Ψ is a subspace of the Banach lattice $L^\Psi(\mathbb{D}, \mathcal{A})$, we may study the order boundedness of J_Ψ . Actually, we are going to see that the natural space for the order boundedness of J_Ψ is not $L^\Psi(\mathbb{D}, \mathcal{A})$, but the *weak Orlicz space* $L^{\Psi, \infty}(\mathbb{D}, \mathcal{A})$, the definition of which we are recalling below (see [7], Definition 3.16).

Definition 3.6 Let (S, \mathcal{S}, μ) be a measure space; the weak- L^Ψ space $L^{\Psi, \infty}$ is the set of the (classes of) measurable functions $f: S \rightarrow \mathbb{C}$ such that, for some constant $c > 0$, one has, for every $t > 0$:

$$\mu(|f| > t) \leq \frac{1}{\Psi(ct)}.$$

One has $L^\Psi \subseteq L^{\Psi, \infty}$ and ([7], Proposition 3.18) the equality $L^\Psi = L^{\Psi, \infty}$ implies that $\Psi \in \Delta^0$. On the other hand, this equality holds when Ψ grows sufficiently; for example, if Ψ satisfies the condition Δ^1 : $x\Psi(x) \leq \Psi(\alpha x)$, for some constant $\alpha > 1$ and x large enough.

Proposition 3.7 $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$ is always order bounded into $L^{\Psi, \infty}(\mathbb{D}, \mathcal{A})$.

Proof. Since (see [7], Lemma 3.11):

$$(3.1) \quad \frac{1}{4}\Psi^{-1}\left(\frac{1}{1-|z|}\right) \leq \sup_{\|f\|_{H^\Psi} \leq 1} |f(z)| \leq 4\Psi^{-1}\left(\frac{1}{1-|z|}\right),$$

one has, denoting by $S(z)$ the supremum in (3.1), for t large enough:

$$\mathcal{A}(|S| > t) \leq \mathcal{A}(\{z \in \mathbb{D}; |z| > 1 - 1/\Psi(t/4)\}) \leq \frac{2}{\Psi(t/4)} \leq \frac{1}{\Psi(t/8)},$$

and the result follows. \square

Since we also have, for t large enough:

$$\mathcal{A}(|S| > t) \geq \mathcal{A}(\{z \in \mathbb{D}; |z| > 1 - 1/\Psi(4t)\}) \geq \frac{1}{\Psi(4t)},$$

we get:

Corollary 3.8 J_Ψ is order bounded into $L^\Psi(\mathbb{D}, \mathcal{A})$ if and only if $L^\Psi = L^{\Psi, \infty}$. This is the case if $\Psi \in \Delta^1$.

Remark. Contrary to the compactness, or the weak compactness, which requires that Ψ does not grow too fast, the order boundedness of J_Ψ into $L^\Psi(\mathbb{D}, \mathcal{A})$ holds when Ψ grows fast enough. Nevertheless, for $\Psi(x) = \exp[(\log(x+1))^2] - 1$, J_Ψ is compact and order bounded into $L^\Psi(\mathbb{D}, \mathcal{A})$.

When J_Ψ is weakly compact, J_Ψ maps H^Ψ into $\mathfrak{B}M^\Psi$ (Theorem 2.2); hence, we may ask whether J_Ψ may be order bounded into $M^\Psi(\mathbb{D}, \mathcal{A})$; however, we have:

Proposition 3.9 J_Ψ is never order bounded into $M^\Psi(\mathbb{D}, \mathcal{A})$.

Proof. If it were the case, we should have $S \in M^\Psi(\mathbb{D}, \mathcal{A})$, and hence

$$\int_{\mathbb{D}} \Psi \left[4 \times \frac{1}{4} \Psi^{-1} \left(\frac{1}{1-|z|} \right) \right] d\mathcal{A}(z) < +\infty,$$

which is false. \square

4 An example

Theorem 4.1 *There exists an Orlicz function Ψ such that J_Ψ is weakly compact and Dunford-Pettis, but which is not compact.*

Note that such an Orlicz function is very irregular: $\Psi \notin \Delta_2$, $\Psi \notin \Delta^0$, so, for every $A > 1$, $\Psi(Ax)/\Psi(x)$ is not non-decreasing for x large enough, and the conjugate function of Ψ does not satisfies condition Δ_2 .

The following lemma is undoubtedly well-known, but we have found no reference, so we shall give a proof. Recall that a sublattice X of $L^0(\mu)$ is solid if $|f| \leq |g|$ and $g \in X$ implies $f \in X$ and $\|f\| \leq \|g\|$.

Lemma 4.2 *Let (S, \mathcal{S}, μ) be a measure space, and let X be a solid Banach sublattice of $L^0(\mu)$, the space of all measurable functions. Then, for every weakly null sequence $\{f_n\}_n$ in X and every sequence $\{A_n\}_n$ of disjoint measurable sets, the sequence $\{f_n \mathbb{1}_{A_n}\}_n$ converges weakly to 0 in X .*

Proof. If the conclusion does not hold, there are a continuous linear functional $\sigma: X \rightarrow \mathbb{C}$ and some $\delta > 0$ such that, up to taking a subsequence, $|\sigma(f_n \mathbb{1}_{A_n})| \geq \delta$. Set, for every measurable set $A \in \mathcal{S}$:

$$\mu_n(A) = \sigma(f_n \mathbb{1}_A).$$

Then μ_n is a finitely additive measure with bounded variation. By Rosenthal's lemma (see [3], Lemma I.4.1, page 18, or [1], Chapter VII, page 82), there is an increasing sequence of integers $\{n_k\}_k$ such that:

$$\left| \mu_{n_k} \left(\bigcup_{l \neq k} A_{n_l} \right) \right| \leq |\mu_{n_k}| \left(\bigcup_{l \neq k} A_{n_l} \right) \leq \delta/2.$$

Now, if $A = \bigcup_{l \geq 1} A_{n_l}$, $\{f_{n_k} \mathbb{1}_A\}_k$ is weakly null, but:

$$|\sigma(f_{n_k} \mathbb{1}_A)| \geq |\sigma(f_{n_k} \mathbb{1}_{A_{n_k}})| - |\mu_{n_k}| \left(\bigcup_{l \neq k} A_{n_l} \right) \geq \delta - \frac{\delta}{2} = \frac{\delta}{2},$$

so we get a contradiction. \square

Proof of Theorem 4.1. We begin by defining a sequence $\{x_n\}_n$ of positive numbers in the following way: set $x_1 = 4$ and, for every $n \geq 1$, $x_{n+1} > 2x_n$ is the abscissa of the second intersection point of the parabola $y = x^2$ with the straight line containing (x_n, x_n^2) and $(2x_n, x_n^4)$; we have $x_{n+1} = x_n^3 - 2x_n$ (for example, $x_2 = 56$). Define $\Psi: [0, +\infty) \rightarrow [0, +\infty)$ by $\Psi(x) = 4x$ for $0 \leq x \leq 4$, and, for $n \geq 1$:

$$(4.1) \quad \Psi(x_n) = x_n^2, \quad \Psi(2x_n) = x_n^4, \quad \Psi \text{ affine between } x_n \text{ and } x_{n+1}.$$

Then Ψ is an Orlicz function and

$$(4.2) \quad x^2 \leq \Psi(x) \leq x^4 \quad \text{for } x \geq 4.$$

For this Orlicz function Ψ , J_Ψ is not compact, since $\Psi(2x)/[\Psi(x)]^2$ does not tend to 0. However, J_Ψ is weakly compact, because one has the factorization $H^\Psi \hookrightarrow H^2 \hookrightarrow \mathfrak{B}^4 \hookrightarrow \mathfrak{B}^\Psi$ (by (4.2) and Lemma 2.4).

Assume that J_Ψ is not Dunford-Pettis: there exists a weakly null sequence $\{f_n\}_n$ in the unit ball of H^Ψ which does not converges for the norm in \mathfrak{B}^Ψ . Then $\{f_n\}_n$ converges uniformly to 0 on the compact subsets of \mathbb{D} (since it is weakly null) and we may assume that $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$ for some $\delta > 0$. We may also assume that $\|f_n\|_\infty \xrightarrow{n \rightarrow \infty} +\infty$ because if $\{f_n\}_n$ were uniformly bounded, we should have $\|f_n\|_{\mathfrak{B}^\Psi} \xrightarrow{n \rightarrow \infty} 0$, by dominated convergence.

We are going to show that there exist a subsequence $\{f_{n_k}\}_k$ and pairwise disjoint measurable sets $A_k \subseteq \mathbb{T}$ such that the sequence $\{f_{n_k} \mathbf{1}_{A_k}\}_k \subseteq L^\Psi(\mathbb{T}, m)$ is equivalent to the canonical basis of ℓ_1 , whence a contradiction with Lemma 4.2.

It is worth to note from now that the Poisson integral \mathcal{P} maps boundedly $L^2(\mathbb{T})$ into $L^4(\mathbb{D})$. Indeed, $L^2(\mathbb{T}) = H^2 \oplus \overline{H_0^2}$ and the canonical injection is bounded from H^2 into \mathfrak{B}^4 , by Lemma 2.4.

We have seen in the proof of Lemma 2.5 that there exist a subsequence $\{f_{n_k}\}_k$ and disjoint measurable annuli $C_1 = \{z \in \mathbb{D}; |z| \leq r_1\}$ and $C_k = \{z \in \mathbb{D}; r_{k-1} < |z| \leq r_k\}$, $k \geq 2$, with $0 < r_1 < r_2 < \dots < r_n < \dots < 1$, such that $\|f_{n_k} \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \geq \delta/2$. The assumptions of that lemma are satisfied here: $\|f_n\|_{H^\Psi} \leq 1$, $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$, $\{f_n\}_n$ converges uniformly to 0 on the compact subsets of \mathbb{D} , and $f_n \in \mathfrak{B}M^\Psi$ because $H^\Psi \subseteq \mathfrak{B}M^\Psi$, since J_Ψ is weakly compact. Then:

Fact 1. *There exist two sequences $\{\alpha_k\}_k$ and $\{\beta_k\}_k$, with $\beta_n > \alpha_n \xrightarrow{n \rightarrow \infty} +\infty$ such that, if $g_k = f_{n_k}^* \mathbf{1}_{\{\alpha_k \leq |f_{n_k}^*| \leq \beta_k\}}$, then:*

$$\|\mathcal{P}(g_k)\|_{L^\Psi(\mathbb{D})} \geq \delta/3,$$

where $f_{n_k}^*$ is the boundary value of f_{n_k} on \mathbb{T} .

Proof. 1) Let $\alpha_k = \frac{\delta}{12} \Psi^{-1}(1/\mathcal{A}(C_k))$ and $v_k = \mathcal{P}(f_{n_k}^* \mathbf{1}_{\{|f_{n_k}^*| < \alpha_k\}}) \mathbf{1}_{C_k}$. One has:

$$\int_{\mathbb{D}} \Psi(|v_k|/(\delta/12)) d\mathcal{A} = \int_{C_k} \Psi(|v_k|/(\delta/12)) d\mathcal{A} \leq \Psi(\alpha_k/(\delta/12)) \mathcal{A}(C_k) = 1,$$

so $\|v_k\|_{L^\Psi(\mathbb{D})} \leq \delta/12$. Since $\mathcal{P}(f_{n_k}^*) = f_{n_k}$, we have $\|\mathcal{P}(f_{n_k}^*) \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} = \|f_{n_k} \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \geq \delta/2$, and we get:

$$\|\mathcal{P}(f_{n_k}^* \mathbf{1}_{\{|f_{n_k}^*| \geq \alpha_k\}}) \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \geq \|f_{n_k} \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} - \|v_k\|_{L^\Psi(\mathbb{D})} \geq \frac{\delta}{2} - \frac{\delta}{12} = \frac{5\delta}{12}.$$

2) Let $w_k = f_{n_k}^* \mathbf{1}_{\{|f_{n_k}^*| \geq \alpha_k\}}$. Since $\mathcal{P}(w_k \mathbf{1}_{\{|w_k| > \beta\}})$ tends to 0 uniformly on C_k when β goes to infinity, Lebesgue's dominated convergence theorem gives:

$$\|\mathcal{P}(w_k \mathbf{1}_{\{|w_k| > \beta\}}) \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \leq \|\mathcal{P}(w_k \mathbf{1}_{\{|w_k| > \beta\}}) \mathbf{1}_{C_k}\|_{L^4(\mathbb{D})} \xrightarrow{\beta \rightarrow +\infty} 0,$$

so there is some $\beta_k > \alpha_k$ such that $\|\mathcal{P}(w_k \mathbb{1}_{\{|w_k| > \beta\}}) \mathbb{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \leq \delta/12$.

We then have, with $g_k = f_{n_k}^* \mathbb{1}_{\{\alpha_k \leq |f_{n_k}^*| \leq \beta_k\}}$:

$$\|\mathcal{P}(g_k)\|_{L^\Psi(\mathbb{D})} \geq \|\mathcal{P}(g_k) \mathbb{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \geq \frac{5\delta}{12} - \frac{\delta}{12} = \frac{\delta}{3},$$

and that ends the proof of Fact 1. \square

Fact 2. *There are a further subsequence, denoted yet by $\{f_{n_k}\}_k$, and pairwise disjoint measurable subsets $E_k \subseteq \{\alpha_k \leq |f_{n_k}^*| \leq \beta_k\}$, such that, if $h_k = f_{n_k}^* \mathbb{1}_{E_k}$, then:*

$$\|\mathcal{P}(h_k)\|_{L^\Psi(\mathbb{D})} \geq \delta/4.$$

Proof. First, since $g_k \in L^\infty(\mathbb{T}) \subseteq M^\Psi(\mathbb{T})$, there exists $\varepsilon_k > 0$ such that $m(A) \leq \varepsilon_k$ implies $\|g_k \mathbb{1}_A\|_{L^\Psi(\mathbb{T})} \leq \delta/(12 \|\mathcal{P}\|)$ (where $\|\mathcal{P}\|$ stands for the norm of $\mathcal{P}: L^2(\mathbb{T}) \rightarrow L^4(\mathbb{D})$). Now, $\mathcal{P}: L^\Psi(\mathbb{T}) \rightarrow L^\Psi(\mathbb{D})$ is bounded and its norm is $\leq \|\mathcal{P}\|$, thanks to the factorization $L^\Psi(\mathbb{T}) \hookrightarrow L^2(\mathbb{T}) \hookrightarrow L^4(\mathbb{D}) \hookrightarrow L^\Psi(\mathbb{D})$. Hence $\|\mathcal{P}(g_k \mathbb{1}_A)\|_{L^\Psi(\mathbb{D})} \leq \delta/12$ for $m(A) \leq \varepsilon_k$.

Let $B_k = \{\alpha_k \leq |f_{n_k}^*| \leq \beta_k\}$. Up to taking a subsequence, we may assume that $\sum_{l>k} m(B_l) \leq \varepsilon_k$. Let

$$E_k = B_k \setminus \bigcup_{l>k} B_l.$$

The sets E_k , $k \geq 1$, are pairwise disjoint, and

$$\|\mathcal{P}(g_k \mathbb{1}_{E_k})\|_{L^\Psi(\mathbb{D})} \geq \|\mathcal{P}(g_k \mathbb{1}_{B_k})\|_{L^\Psi(\mathbb{D})} - \|\mathcal{P}(g_k \mathbb{1}_{\bigcup_{l>k} B_l})\|_{L^\Psi(\mathbb{D})} \geq \frac{\delta}{3} - \frac{\delta}{12} = \frac{\delta}{4};$$

so we get the Fact 2 with $h_k = g_k \mathbb{1}_{E_k} = f_{n_k}^* \mathbb{1}_{E_k}$. \square

Set

$$F_k = \{z \in E_k; \Psi(|f_{n_k}^*(z)|) \leq M |f_{n_k}^*(z)|^2\}.$$

For $z \in E_k \setminus F_k$, one has:

$$\int_{E_k \setminus F_k} |f_{n_k}^*|^2 dm \leq \frac{1}{M} \int_{\mathbb{T}} \Psi(|f_{n_k}^*|) dm \leq \frac{1}{M},$$

so $\|f_{n_k}^* \mathbb{1}_{E_k \setminus F_k}\|_{L^2(\mathbb{T})} \leq 1/\sqrt{M}$ and:

$$\begin{aligned} \|\mathcal{P}(f_{n_k}^* \mathbb{1}_{E_k \setminus F_k})\|_{L^\Psi(\mathbb{D})} &\leq \|\mathcal{P}(f_{n_k}^* \mathbb{1}_{E_k \setminus F_k})\|_{L^4(\mathbb{D})} \\ &\leq \|\mathcal{P}\| \|f_{n_k}^* \mathbb{1}_{E_k \setminus F_k}\|_{L^2(\mathbb{T})} \leq \frac{\|\mathcal{P}\|}{\sqrt{M}} \leq \frac{\delta}{8}, \end{aligned}$$

for M large enough. It follows that, for M large enough, $\|\mathcal{P}(f_{n_k}^* \mathbb{1}_{F_k})\|_{L^\Psi(\mathbb{D})} \geq \delta/8$ and

$$(4.3) \quad \|f_{n_k}^* \mathbb{1}_{F_k}\|_{L^\Psi(\mathbb{D})} \geq \delta/(8 \|\mathcal{P}\|).$$

Now, we may assume that, for some $\alpha > 0$,

$$\int_{\mathbb{T}} |f_{n_k}^*|^2 \mathbb{1}_{F_k} dm \geq \alpha,$$

because, if not, there would be a subsequence $\{f_{n_{k_j}}^* \mathbb{1}_{F_{k_j}}\}_j$ converging to 0 in $L^2(\mathbb{T})$; but then $\{\mathcal{P}(f_{n_{k_j}} \mathbb{1}_{F_{k_j}})\}_j$ would converge to 0 in \mathfrak{B}^4 , and hence in \mathfrak{B}^Ψ , contrary to (4.3). It follows, using (4.2), that:

$$(4.4) \quad \int_{F_k} \Psi(|f_{n_k}^*|) dm \geq \alpha.$$

The following lemma is now the key of the proof.

Lemma 4.3 *Let $\delta_n = 2x_{n-1}/x_n = 2/(x_{n-1}^2 - 2)$. If $\Psi(x) \leq Mx^2$ and $x \geq x_n$, then, for n large enough ($n \geq N$), one has $\Psi(\varepsilon x) \geq C_M \varepsilon \Psi(x)$ for $\delta_n \leq \varepsilon \leq 1$.*

Proof. We may assume that $x_n \leq x < x_{n+1}$, because if $x_k \leq x < x_{k+1}$ with $k \geq n$, then $\varepsilon \geq \delta_n$ implies $\varepsilon \geq \delta_k$.

Now, remark that:

$$(4.5) \quad \frac{\Psi(y)}{\Psi(x)} \leq 4 \frac{y}{x}, \quad \text{for } 2x_n \leq x \leq y \leq x_{n+1}.$$

Indeed, on the one hand, $\frac{\Psi(y) - \Psi(x_n)}{\Psi(x) - \Psi(x_n)} = \frac{y - x_n}{x - x_n} \leq \frac{y}{x/2} = 2 \frac{y}{x}$; and, on the other hand, $\Psi(y) - \Psi(x_n) \geq \Psi(y) - \Psi(y/2) \geq \Psi(y) - \frac{1}{2} \Psi(y) = \frac{1}{2} \Psi(y)$, so $\frac{\Psi(y)}{\Psi(x)} \leq \frac{\Psi(y)}{\Psi(x) - \Psi(x_n)} \leq 2 \frac{\Psi(y) - \Psi(x_n)}{\Psi(x) - \Psi(x_n)} \leq 4 \frac{y}{x}$.

We shall separate three cases:

1) $\varepsilon x \leq x_n \leq x \leq 2x_n$. Then $\varepsilon x \geq \varepsilon x_n$ and hence $\Psi(\varepsilon x) \geq \Psi(\varepsilon x_n)$. But $2x_{n-1} \leq \varepsilon x_n \leq x_n$, since $\varepsilon \geq \delta_n$; hence (4.5) implies that $\Psi(\varepsilon x) \geq (\varepsilon/4) \Psi(x_n) = (\varepsilon/4) x_n^2$. On the other hand, one has, by hypothesis, $\Psi(x) \leq Mx^2 \leq M(2x_n)^2$, so we get $\Psi(\varepsilon x) \geq (\varepsilon/16M) \Psi(x)$.

2) $x_n \leq \varepsilon x \leq x \leq 2x_n$. Then, since $1 \leq 1/\varepsilon$:

$$\frac{\Psi(x)}{\Psi(\varepsilon x)} \leq \frac{Mx^2}{\Psi(x_n)} \leq \frac{M(2x_n)^2}{x_n^2} = 4M \leq \frac{4M}{\varepsilon}.$$

3) For $x \geq 2x_n$, remark that the conditions $\Psi(x) \leq Mx^2$ and $x \geq 2x_n$ imply that $x \geq x_n^2/\sqrt{M}$. Indeed, if $x \geq 2x_n$, then $\Psi(x) \geq \Psi(2x_n) = x_n^4$, and the condition $\Psi(x) \leq Mx^2$ implies $x_n^4 \leq Mx^2$, i.e. $x \geq x_n^2/\sqrt{M}$.

In this case, one has $\varepsilon x \geq \varepsilon x_n^2/\sqrt{M} \geq \delta_n x_n^2/\sqrt{M} = 2(x_{n-1}/x_n) x_n^2/\sqrt{M} = 2x_{n-1} x_n/\sqrt{M} \geq 2x_n$, if $x_{n-1} \geq \sqrt{M}$. Hence (4.5) gives, for $2x_n \leq x < x_{n+1}$ (since then $2x_n \leq \varepsilon x \leq x < x_{n+1}$):

$$\frac{\Psi(x)}{\Psi(\varepsilon x)} \leq 4 \frac{x}{\varepsilon x} = \frac{4}{\varepsilon}.$$

That ends the proof of Lemma 4.3. \square

Extract now a further subsequence of $\{f_{n_k}\}$, yet denoted by $\{f_{n_k}\}$, in order that (see Fact 1) $\alpha_k \geq x_{N+k}$. Lemma 4.3 holds, with $x = \Psi(|f_{n_k}^*(z)|)$, $z \in F_k$, for every $k \geq 1$; one has (since, by definition, $\Psi(|f_{n_k}|) \leq M |f_{n_k}|^2$ on F_k):

$$\int_{F_k} \Psi(\varepsilon |f_{n_k}^*|) dm \geq \varepsilon C/\alpha := c\varepsilon, \quad \text{for } \delta_{N+k} \leq \varepsilon \leq 1.$$

The proof of Theorem 4.1 reaches now its end: put $u_k = f_{n_k}^* \mathbb{1}_{F_k}$, and take an arbitrary sequence of complex numbers such that $\sum_{k \geq 1} |\lambda_k| = 1$. Let $\delta_0 = \sum_{k \geq N} \delta_k$. One has $\delta_0 < 1$, because we may assume that N had been taken large enough. One gets:

$$\begin{aligned} \int_{\mathbb{T}} \Psi\left(\left|\sum_{k \geq 1} \lambda_k u_k\right|\right) dm &= \sum_{k \geq 1} \int_{F_k} \Psi(|\lambda_k f_{n_k}|) dm \\ &\geq \sum_{|\lambda_k| \geq \delta_{N+k}} c|\lambda_k| + \sum_{|\lambda_k| < \delta_{N+k}} \int_{F_k} \Psi(|\lambda_k f_{n_k}|) dm \\ &\geq \sum_{|\lambda_k| \geq \delta_{N+k}} c|\lambda_k| = c\left(1 - \sum_{|\lambda_k| < \delta_{N+k}} |\lambda_k|\right) \\ &\geq c\left(1 - \sum_{k \geq N} \delta_k\right) = c(1 - \delta_0) := c_0. \end{aligned}$$

Since $c_0 < 1$, this implies, by convexity, that

$$\left\| \sum_{k \geq 1} \lambda_k u_k \right\|_{L^\Psi(\mathbb{T})} \geq c_0.$$

Hence $\{u_k\}_k$ is equivalent to the canonical basis of ℓ_1 , and that achieves the proof of Theorem 4.1. \square

Remarks. 1) It follows from Theorem 3.3 that, for this Ψ , J_Ψ is not p -summing for $p < 4$. By modifying the definition of Ψ (taking $\Psi(x_n) = x_n^{r/2}$ and $\Psi(2x_n) = x_n^r$), we get, for every $4 \leq r < \infty$, an Orlicz function Ψ such that J_Ψ is Dunford-Pettis and weakly compact, without being p -summing for $p < r$, and without being compact. We do not know whether it is possible to have J_Ψ p -summing for no finite p .

2) Let us point out that the fact that J_Ψ is Dunford-Pettis does not trivially follow from its weak compactness: H^Ψ does not have the Dunford-Pettis property. In fact, if it were the case, the weakly compact injection $H^\Psi \hookrightarrow H^2$ would be Dunford-Pettis, and hence also $H^4 \hookrightarrow H^2$ (since $H^4 \hookrightarrow H^\Psi \hookrightarrow H^2$). But it is not the case: the sequence $\{z^n\}_n$ converges weakly to 0 in H^4 , whereas it does not converge in norm to 0 in H^2 .

Proposition 4.4 *There is an Orlicz function Ψ for which J_Ψ is weakly compact, but not Dunford-Pettis.*

Proof. Let us call Ψ_0 the Orlicz function constructed in Theorem 4.1, and let $\Psi(x) = \Psi_0(x^2)$. Then, with $\beta = 2$, $\Psi(\beta x) = \Psi_0(4x^2) \geq 4\Psi_0(x^2) = (2\beta)\Psi(x)$; that means that the conjugate function of Ψ satisfies Δ_2 .

J_Ψ is weakly compact (since J_Ψ factors as $H^\Psi \hookrightarrow H^4 \hookrightarrow \mathfrak{B}^8 \hookrightarrow \mathfrak{B}^\Psi$), but is not compact, since $[\Psi(\sqrt{x_n})]^2 = \Psi(\sqrt{2}\sqrt{x_n})$. Since the conjugate function satisfies Δ_2 , J_Ψ is not Dunford-Pettis, by Proposition 3.1. \square

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