A spectral radius type formula for approximation numbers of composition operators

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Abstract. *For approximation numbers $a_n(C_\varphi)$ of composition operators C_φ on weighted analytic Hilbert spaces, including the Hardy, Bergman and Dirichlet cases, with symbol φ of uniform norm < 1, we prove that $\lim_{n\to\infty}[a_n(C_\varphi)]^{1/n} = \mathrm{e}^{-1/\mathrm{Cap}\,[\varphi(\mathbb{D})]}$, where $\mathrm{Cap}\,[\varphi(\mathbb{D})]$ is the Green capacity of $\varphi(\mathbb{D})$ in $\mathbb{D}$. This formula holds also for H^p with $1 \le p < \infty$.*

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1 Introduction

The determination of the approximation numbers of composition operators on Hilbert spaces of analytic functions on the unit disk (Hardy space, weighted Bergman space, Dirichlet space) is a difficult problem. Some partial results (see [18], [15], [19], [16], [22]) show that no simple answer may be expected. However, we proved in [18] and [16] that these approximation numbers cannot decay faster than geometrically: we always have $a_n(C_\varphi) \ge c\, r^n$ for some constant $c > 0$ and some $0 < r < 1$. Moreover, we showed in those papers that $\lim_{n\to\infty}[a_n(C_\varphi]^{1/n} = 1$ if and only if $\|\varphi\|_\infty = 1$.

The quantity $\lim_{n\to\infty}[a_n(C_\varphi)]^{1/n}$ looks like a spectral radius formula for the approximation numbers. Recall that if T is a bounded operator on a complex Hilbert space H, with spectrum $\sigma(T)$, the classical spectral radius formula tells that for the spectral radius $r(T) := \sup_{\lambda\in\sigma(T)} |\lambda|$, one has the formula:

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$$

(the existence of the limit being part of the conclusion).

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Now, if $a_n = a_n(T)$ is the n-th approximation number of a bounded operator T on a Hilbert space H, it was shown ([12], p. 133), by taking a rank-one perturbation of an n-dimensional shift, that, given $0 < \sigma < 1$, we can have $a_1 = \cdots = a_{n-1} = 1$, and $a_n = \sigma$. Using orthogonal blocks of such normalized operators, one easily builds examples of compact operators T for which the quantity $[a_n(T)]^{1/n}$ has no limit as n goes to infinity, and indeed satisfies:

$$\liminf_{n\to\infty} [a_n(T)]^{1/n} = 0\,, \qquad \limsup_{n\to\infty} [a_n(T)]^{1/n} = 1\,.$$

We might as well use a diagonal operator with non-increasing positive diagonal entries ε_n such that $\liminf_n \varepsilon_n^{1/n} = 0$ and $\limsup_n \varepsilon_n^{1/n} = 1$. Nevertheless, the parameters

$$\beta^-(T) = \liminf_{n\to\infty} [a_n(T)]^{1/n}, \qquad \beta^+(T) = \limsup_{n\to\infty} [a_n(T)]^{1/n} \tag{1.1}$$

which satisfy $0 \le \beta^-(T) \le \beta^+(T) \le 1$ are similar to the term $\lim_{n\to\infty} \|T^n\|^{1/n}$ in the spectral radius formula. When the limit exists we will denote it by:

$$\beta(T) = \lim_{n\to\infty} [a_n(T)]^{1/n}. \tag{1.2}$$

These parameters were shown to play an important role in the study of composition operators (see [18] and [16]). As said above, the following was proved in these papers.

Theorem 1.1 *Let H be a weighted Bergman space $\mathfrak{B}_\alpha$ (in particular the Hardy space H^2) or the Dirichlet space $\mathcal{D}$ and $\varphi\colon \mathbb{D} \to \mathbb{D}$ inducing a composition operator $C_\varphi\colon H \to H$. Then:*

1) *if $0 < \|\varphi\|_\infty < 1$, one has $0 < \beta^-(C_\varphi) \le \beta^+(C_\varphi) < 1$;*
2) *if $\|\varphi\|_\infty = 1$, one has $\beta(C_\varphi) = 1$.*

The aim of this work is to complete this result by showing that $\beta(C_\varphi)$ exists as well when $\|\varphi\|_\infty < 1$ and to give a closed formula for this $\beta(C_\varphi)$ in terms of a Green capacity, relying on a basic work of [24] (see also [9]). We thus get another proof of 2) in the above theorem.

We end the paper with some words on the H^p case for $1 \le p < \infty$.

We begin by giving notations, definitions and facts which will be used throughout this work.

2 Background, framework, and notations

Recall that if X and Y are two Banach spaces of analytic functions on the unit disk $\mathbb{D}$, and $\varphi\colon \mathbb{D} \to \mathbb{D}$ is an analytic self-map of $\mathbb{D}$, one says that φ induces a *composition operator* $C_\varphi\colon X \to Y$ if $f \circ \varphi \in Y$ for every $f \in X$; φ is then called the *symbol* of the composition operator. One also says that φ is a symbol for X and Y if it induces a composition operator $C_\varphi\colon X \to Y$.

2.1 Singular numbers

For an operator $T\colon X \to Y$ between Banach spaces X and Y, its *approximation numbers* are defined, for $n \geq 0$, as:

$$a_n(T) = \inf_{\operatorname{rank} R < n} \|T - R\| . \tag{2.1}$$

One has $\|T\| = a_1(T) \geq a_2(T) \geq \cdots \geq a_n(T) \geq a_{n+1}(T) \geq \cdots$, and (assuming that Y has the Approximation Property), T is compact if and only if $a_n(T) \underset{n\to\infty}{\longrightarrow} 0$.

The *n-th Kolmogorov number* $d_n(T)$ of T is defined as (see [3], p. 49):

$$d_n(T) = \inf_{\substack{E \subseteq Y \\ \dim E < n}} \Big[\sup_{x \in B_X} \operatorname{dist}(Tx, E) \Big] = \inf_{\substack{E \subseteq Y \\ \dim E < n}} \|Q_E T\|_{Y/E} , \tag{2.2}$$

where $Q_E\colon Y \to Y/E$ is the quotient map. One always has $a_n(T) \geq d_n(T)$ and, when X and Y are Hilbert spaces, one has $a_n(T) = d_n(T)$ (see [3], p. 51).

As usual, the notation $A \lesssim B$ means that there is a constant c such that $A \leq C\, B$.

2.2 Weighted analytic Hilbert spaces

An *analytic Hilbert space* H on $\mathbb{D}$ is a Hilbert space $H \subset \mathcal{H}\mathrm{ol}(\mathbb{D})$, the analytic functions on the unit disk $\mathbb{D}$, for which the evaluations $f \mapsto f(a)$ are continuous on H for all $a \in \mathbb{D}$ and therefore given by a scalar product:

$$f(a) = \langle f, K_a \rangle , \quad K_a \in H.$$

Since weakly convergent sequences of H are norm-bounded, the *reproducing kernels* K_a are automatically norm-bounded on compact subsets of $\mathbb{D}$, that is:

$$L_r := \sup_{|a| \leq r} \|K_a\| < \infty, \quad \text{for all } r < 1. \tag{2.3}$$

We will be slightly less general here, and adopt the framework of [11]. Let $\omega\colon [0,1) \to (0,\infty)$ be a continuous, positive, and Lebesgue-integrable function. We extend this function to a radial weight on $\mathbb{D}$ by setting $\omega(z) = \omega(|z|)$. We denote by H_ω the space of analytic functions on $\mathbb{D}$ such that

$$\|f\|_\omega^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2\, \omega(z)\, dA(z) < +\infty,$$

where dA stands for the normalized area measure on $\mathbb{D}$. We will often omit the subscript ω and write $\|\,.\,\|$ for $\|\,.\,\|_\omega$.

If $f(z) = \sum_{n=0}^{\infty} b_n z^n$, a computation in polar coordinates shows that:

$$\|f\|^2 = \sum_{n=0}^{\infty} |b_n|^2\, w_n \tag{2.4}$$

where:

$$w_0 = 1 \quad \text{and} \quad w_n = 2n^2 \int_0^1 r^{2n-1} \omega(r)\, dr\,, \quad n \geq 1. \tag{2.5}$$

Observe that there is a constant $C = C(\omega) \geq 1$ and, for each $\varepsilon > 0$, a $\delta_\varepsilon > 0$ such that:

$$\delta_\varepsilon\, \mathrm{e}^{-\varepsilon n} \leq w_n \leq C\, n^2\,, \qquad n \geq 1. \tag{2.6}$$

Indeed, in one side, one has $w_n \leq 2n^2 \int_0^1 \omega(r)\, dr$, and, on the other side, for each $0 < \delta < 1$, setting $c_\delta = \inf_{0 \leq r \leq \delta} \omega(r)$, we have $c_\delta > 0$ and:

$$w_n \geq 2n^2\, c_\delta \int_0^\delta r^{2n-1}\, dr = c_\delta\, n\, \delta^{2n},$$

giving (2.6). This shows in passing that H_ω is an analytic Hilbert space, and we call it a *weighted analytic Hilbert space*. This framework is sufficiently general for our purposes and includes for example the case of (weighted) Bergman, Hardy, and Dirichlet spaces, corresponding to $\omega(r) = (1 - r^2)^\alpha$, $\alpha > -1$, that is $w_n \approx n^{1-\alpha}$. The standard Bergman, Hardy, Dirichlet spaces correspond to the respective values $\alpha = 2, 1, 0$.

The following simple fact will be used. Let $a \in \mathbb{D}$ and $j \geq 0$; then:

$$f \mapsto f^{(j)}(a) \text{ is a continuous linear form on } H. \tag{2.7}$$

This holds for any analytic Hilbert space on $\mathbb{D}$, thanks to (2.3), and here can also be viewed as a consequence of (2.6).

An analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ which induces a composition operator $C_\varphi\colon H \to H$ will be called a *symbol* for $H = H_\omega$. In our space H, we have a quite easy case for deciding if some φ is a symbol.

Lemma 2.1 *If $\|\varphi\|_\infty < 1$, then φ is a symbol* if and only if *$\varphi \in H$. Equivalently, if and only if the* positive *measure $\mu = |\varphi'|^2 \omega\, dA$ is finite. In that case, we moreover have $\|\varphi^k\| \leq C\, k\, \|\varphi\|_\infty^k\, \|\varphi\|$ for every $k \geq 1$.*

Proof. If φ is a symbol, then $\varphi = C_\varphi(z) \in H$. Conversely, let $\rho = \|\varphi\|_\infty < 1$. We first note that, if $\varphi \in H$, we have for any integer $k \geq 1$:

$$\begin{aligned} \|\varphi^k\|^2 &= |\varphi(0)|^{2k} + \int_{\mathbb{D}} \omega(z)\, k^2\, |\varphi(z)|^{2(k-1)} |\varphi'(z)|^2\, dA(z) \\ &\leq \rho^{2k} (1 + k^2 \rho^{-2})\, \|\varphi\|^2. \end{aligned} \tag{2.8}$$

Now, let $\varepsilon > 0$ be such that $\rho\, \mathrm{e}^\varepsilon < 1$. If $f(z) = \sum b_k z^k \in B_H$, the unit ball of H, we have by (2.6): $|b_k| \leq w_k^{-1/2} \leq C_\varepsilon \mathrm{e}^{k\varepsilon}$, so that, using (2.8), we see that the series $\sum b_k\, \varphi^k = f \circ \varphi$ converges absolutely in H, which proves that C_φ is compact (and even nuclear). $\square$

2.3 Green capacity

The *Green function* $g\colon \mathbb{D}\times\mathbb{D}\to(0,\infty]$ of the unit disk $\mathbb{D}$ is defined as:

$$g(z,w)=\log\left|\frac{1-\overline{w}z}{z-w}\right|. \tag{2.9}$$

If μ is a finite positive Borel measure on $\mathbb{D}$ with compact support in $\mathbb{D}$, its Green potential is:

$$G_\mu(z)=\int_{\mathbb{D}} g(z,w)\,d\mu(w) \tag{2.10}$$

and its *energy integral* is:

$$I(\mu)=\iint_{\mathbb{D}\times\mathbb{D}} g(z,w)\,d\mu(z)\,d\mu(w)\,. \tag{2.11}$$

Of course,

$$I(\mu)=\int_{\mathbb{D}} G_\mu(z)\,d\mu(z)\,. \tag{2.12}$$

For any subset E of $\mathbb{D}$, one sets:

$$V(E)=\inf_{\mu} I(\mu)\,, \tag{2.13}$$

where the infimum is taken over all probability measures μ supported by a compact subset of E. Then the *Green capacity*[1] of E in $\mathbb{D}$ is:

$$\mathrm{Cap}\,(E)=1/V(E)\,. \tag{2.14}$$

If $K\subseteq\mathbb{D}$ is compact, the infimum in (2.13) is attained for a probability measure μ_0. If moreover $V(K)<\infty$ (i.e. $\mathrm{Cap}\,(K)>0$), this measure is *unique* and is called the *equilibrium measure* of K. One always has $V(K)<\infty$ when K has non-empty interior, since then $I(\lambda)<\infty$ where λ is the normalized planar measure on some open disk $\Delta\subseteq K$. It is clear that we have:

$$K\subseteq L\Rightarrow V(K)\geq V(L)\Rightarrow \mathrm{Cap}\,(K)\leq\mathrm{Cap}\,(L)\,,$$

i.e. $\mathrm{Cap}\,(K)$ increases with K and:

$$\mathrm{Cap}\,(E)=\sup_{K\subseteq E,K\ \text{compact}}\mathrm{Cap}\,(K)\,.$$

We refer to [2] and [7] and to the clear presentation of [20] for the definition of the Green capacity and of its basic properties. Actually, in [2], the capacity is defined by another way (see [2], Chapitre V, pp. 52–55), as follows.

[1] Actually the inner capacity, but for open and compact sets, it it is equal to the outer capacity and hence, *is* the capacity: see [2], Chapitre V, p. 63. Choquet's Theorem ([4]; see also [2], Chapitre V, p. 66), asserts that the inner capacity is equal to the outer capacity for all Borel sets.

Lemma 2.2 *For every compact set $K \subseteq \mathbb{D}$, one has:*

$$\begin{aligned}
&\mathrm{Cap}\,(K) \\
&\qquad = \sup\{\|\mu\|\,;\ \mu \textit{ positive Borel measure supported by } K \textit{ and } G_\mu \leq 1 \textit{ on } \mathbb{D}\}
\end{aligned}$$

This is the definition of de la Vallée-Poussin. Since our main result is based on H. Widom's paper [24], it must be specified that he also used this definition in [24].

Let us note, though we will not use that, that we also have:

$$\begin{aligned}
\mathrm{Cap}\,(K) &= \inf\{\|\mu\|\,;\ \mu \text{ positive Borel measure on } \mathbb{D} \text{ and } G_\mu \geq 1 \text{ on } K\} \\
&= \inf\{\|\mu\|\,;\ \mu \text{ positive Borel measure on } \mathbb{D} \text{ and } G_\mu \geq 1 \ q.e. \text{ on } K\}\,,
\end{aligned}$$

where *q.e.* means: out of a set of null capacity. The equivalence between these two definitions is shown in [20], Lemma 4.1 (see also [2], Chapitre XI, p. 140 and pp. 144–145).

An important fact for this paper is well-known to specialists on the (Green) capacity. This fact, kindly communicated to us with its proof by A. Ancona ([1]), is as follows.

Theorem 2.3 *For every* connected *Borel subset E of $\mathbb{D}$ whose closure $\overline{E}$ is contained in $\mathbb{D}$, one has:*

$$\mathrm{Cap}\,(E) = \mathrm{Cap}\,(\overline{E})\,. \tag{2.15}$$

For sake of completeness, we provide details for the reader. We begin with a definition: a subset E of $\mathbb{D}$ is said to be *thin* (in French: "*effilé*") at $u \in \overline{E}$ if there exists a function s which is superharmonic in a neighbourhood of u and such that

$$s(u) < \liminf_{\substack{v \to u \\ v \in E}} s(v)\,.$$

We denote by $\tilde{E}$ the union of E and of points in $\overline{E}$ at which E is *not* thin (it is known that $\tilde{E}$ is the closure of E for the fine topology: see [7], Proposition 21.13.10). Then:

Lemma 2.4 *If E is a* connected *Borel subset of $\mathbb{D}$ whose closure $\overline{E}$ is contained in $\mathbb{D}$, one has:*

$$\tilde{E} = \overline{E}\,.$$

Proof. Lemma 2.4 is an immediate consequence of the following result (see [2], Chapitre VII, Corollaire, p. 89).

Theorem 2.5 (Beurling-Brelot) *Let $E \subseteq \mathbb{D}$ and $u \in \overline{E}$. If E is thin at u, there exist* circles *with center u and arbitrarily small radius > 0 which do not intersect E.*

Indeed, taking the previous result for granted, suppose that E is thin at $u \in \overline{E}$, $u \notin E$, and let $v_0 \in E$, with $|v_0 - u| = d > 0$. The function $\rho\colon E \to \mathbb{R}$ defined by $\rho(v) = |v - u|$ takes the value d as well as arbitrarily small values since $u \in \overline{E}$. By the intermediate value theorem, it takes every value in $(0, d]$, contradicting Theorem 2.5. This contradiction shows that $\overline{E} \subseteq \tilde{E}$, thereby ending the proof of Theorem 2.3. $\square$

Now,

Lemma 2.6 *One has:*

$$\mathrm{Cap}\,(E) = \mathrm{Cap}\,(\tilde{E})\,.$$

Proof. We know (Cartan's Theorem) that $\mathrm{Cap}\,(\tilde{E} \setminus E) = 0$ (see [7], Theorem 21.12.14, and Proposition 21.13.10, or see [2], Chapitre VII, p. 86 and Chapitre V, p. 57, or [21], Proposition 8.2 and Proposition 8.3). Since the capacity of Borel sets is easily seen (see [2], Chapitre V, p. 62, or [13], Chap. II, § 1, p. 145) to be a subadditive set function, one gets $\mathrm{Cap}\,(E) \leq \mathrm{Cap}\,(\tilde{E}) \leq \mathrm{Cap}\,(E) + \mathrm{Cap}\,(\tilde{E} \setminus E) = \mathrm{Cap}\,(E)$. $\square$

Throughout this paper, for convenience, we sometimes use the notation:

$$M(E) := \mathrm{e}^{-1/\mathrm{Cap}\,(E)} = \mathrm{e}^{-V(E)}. \tag{2.16}$$

3 Main result

The goal of this paper is to prove the following result.

Theorem 3.1 *Let H be a weighted analytic Hilbert space with norm $\|\,.\,\|$. Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a symbol for H, with $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$. Then*

$$\lim_{n\to\infty} [a_n(C_\varphi)]^{1/n} =: \beta(C_\varphi)$$

exists and the value of this limit is:

$$\beta(C_\varphi) = \mathrm{e}^{-1/\mathrm{Cap}\,[\varphi(\mathbb{D})]}. \tag{3.1}$$

Note that, by Theorem 2.3, $\mathrm{Cap}\,[\varphi(\mathbb{D})] = \mathrm{Cap}\,[\overline{\varphi(\mathbb{D})}]$, so Theorem 3.1 will follow immediately from Theorem 3.8 and Theorem 3.11 below.

The proof is based on two results of H. Widom ([24]). Though those theorems are in the H^∞ setting, we will be able to transfer them to our Hilbertian setting. Before giving this proof, we will check the result "by hand" with an explicit example.

3.1 A very special test case

Before going into the proof of Theorem 3.1, we are going to illustrate it in a simple situation.

Let φ be a symbol acting on $H = H^2$ with $\|\varphi\|_\infty < 1$. We know from [18] that $\beta^+(C_\varphi) < 1$, and for very special φ's we will show directly, without appealing to Widom's results, that (3.1) holds.

Theorem 3.2 *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a fractional linear function mapping $\mathbb{D}$ into $\mathbb{D}$, i.e. :*

$$|a|^2 + |b|^2 + 2\,|\overline{a}b - \overline{c}d| \leq |c|^2 + |d|^2 \quad \textit{and} \quad |c| \leq |d|\,.$$

Then $\beta(C_\varphi) = \exp\Big[-\frac{1}{\mathrm{Cap}\,(K)}\Big]$.

The example $\varphi(z) = z/(2z+1)$ shows that one cannot omit the condition $|c| \leq |d|$.

Recall that the pseudo-hyperbolic distance on $\mathbb{D}$ is defined by:

$$\rho(z,w) = \Big|\frac{z-w}{1-\overline{z}w}\Big|\,, \qquad z, w \in \mathbb{D}\,. \tag{3.2}$$

We denote by $\Delta(w,r) = \{z \in \mathbb{D}\,;\ \rho(z,w) < r\}$ the open pseudo-hyperbolic disk of center w and radius r.

We have the following two facts ([20], p. 3173 for the first one).

Lemma 3.3 *Let $L = \overline{\Delta}(w,r)$ be a closed pseudo-hyperbolic disk of pseudo-hyperbolic radius r. Then:*

$$\mathrm{Cap}\,(L) = \frac{1}{\log(1/r)}\,\cdot \tag{3.3}$$

Lemma 3.4 *Let $u, v \colon \mathbb{D} \to \mathbb{D}$ be univalent analytic maps such that $u(\mathbb{D}) = v(\mathbb{D})$. Then, $u = v \circ \psi$ where $\psi \in \mathrm{Aut}\,(\mathbb{D})$.*

Indeed , by hypothesis $u = v\circ\psi$ with ψ well-defined and holomorphic for v is injective. Moreover, $u(\mathbb{D}) = v[\psi(\mathbb{D})] = v(\mathbb{D})$, whence $\psi(\mathbb{D}) = \mathbb{D}$, again because v is injective. Finally ψ is injective since u is. □

Proof of Theorem 3.2. We may assume $\|\varphi\|_\infty < 1$. We first consider the case $\varphi(z) = az$, with $|a| < 1$. In that case, it is clear that $a_n(C_\varphi) = |a|^{n-1}$, and hence $\beta(C_\varphi) = |a|$ and $\overline{\varphi(\mathbb{D})} = \overline{D}(0,|a|) = \overline{\Delta}(0,|a|)$. So that (3.1) holds in view of (3.3).

In the general case, one might say that the conformal invariance of Cap and β does the rest. Let us provide some details.

In general, $\varphi(\mathbb{D})$ is an euclidean disk, therefore a pseudo-hyperbolic disk $\Delta(w,r) := \{z\,;\ \rho\,(z,w) < r\} = \psi_1[\Delta(0,r)]$, where ρ is the pseudo-hyperbolic distance and $\psi_1 \in \mathrm{Aut}\,(\mathbb{D})$; one has the same radius since automorphisms preserve ρ. If $h(z) = rz$, one therefore has $\varphi(\mathbb{D}) = \psi_1[h(\mathbb{D})]$ (since $\overline{\Delta}(0,r)$ and the euclidean disk $\overline{D}(0,r)$ coincide). From Lemma 3.4, $\varphi = \psi_1 \circ h \circ \psi_2$ with $\psi_2 \in \mathrm{Aut}\,(\mathbb{D})$, and so $C_\varphi = C_{\psi_2} C_h C_{\psi_1}$, implying

$$\beta(C_\varphi) = \beta(C_h) = r\,,$$

by the ideal property. Moreover,

$$\mathrm{Cap}\,[\varphi(\mathbb{D})] = \mathrm{Cap}\,[h(\mathbb{D})]$$

by conformal invariance. Since we know that the desired equality between β and Cap holds for h, we get the result. □

Let us *numerically* test the claimed value of $\beta(C_\varphi)$ on the affine example $\varphi(z) = \varphi_{a,b}(z) = az + b$ with $a, b > 0$ and $a + b < 1$ (note that $C_{\varphi_{a,b}}$ and $C_{\varphi_{|a|,|b|}}$ are unitarily equivalent and have the same approximation numbers a_n, so that there is no loss of generality by assuming $a, b > 0$). In that case, the $a_n(C_\varphi) = a_n$ were computed *exactly* by Clifford and Dabkowski ([6]). Their result is as follows. One sets:

$$\Delta = (a^2 - b^2 - 1)^2 - 4b^2 \quad \text{and} \quad Q = \frac{1 + a^2 - b^2 - \sqrt{\Delta}}{2a^2}\,. \tag{3.4}$$

Then, one has $a_n = a^{n-1} Q^{n-1/2}$, and so:

$$\beta(C_\varphi) = aQ\,. \tag{3.5}$$

The result of the theorem can be tested on that example. Indeed, we have $K := \overline{\varphi(\mathbb{D})} = \overline{D}(b,a)$, so that ([13], p. 175–177):

$$\mathrm{Cap}\,(K) = \frac{1}{\log \lambda}\,,$$

where $\lambda > 1$ is the biggest root of the quadratic polynomial

$$P(z) = az^2 - (1 + a^2 - b^2)z + a\,.$$

In explicit terms:

$$\mathrm{e}^{-1/\mathrm{Cap}\,(K)} = \frac{1}{\lambda} = \frac{1 + a^2 - b^2 - \sqrt{\Delta_0}}{2a}\,,$$

with:

$$\Delta_0 = (1 + a^2 - b^2)^2 - 4a^2\,. \tag{3.6}$$

To get $\beta(C_\varphi) = \mathrm{e}^{-1/\mathrm{Cap}\,(K)}$, it remains to compare (3.5) and (3.1), using (3.4) and (3.6), and to observe that

$$\Delta = \Delta_0 = (1 + a + b)(1 + a - b)(1 - a + b)(1 - a - b)\,.$$

3.2 Widom's results reformulated

We are going to state Widom's results in a form suitable for us. We first quote the following lemma from [24].

Lemma 3.5 (Widom) *Let $K \subseteq \mathbb{D}$ be compact. Then, given $\varepsilon > 0$, there exists a cycle γ, which is a finite union of disjoint Jordan curves contained in $\mathbb{D}$, and whose interior U contains K, and a rational function R of degree $< n$, having no zero on γ and all poles on $\partial\mathbb{D}$, such that, for n large enough:*
1) $|R(z)| \geq \mathrm{e}^{-\varepsilon n}$ *for* $z \notin U$*;*
2) $|R(z)| \leq \mathrm{e}^{\varepsilon n}\, \mathrm{e}^{-n/\mathrm{Cap}\,(K)}$ *for* $z \in K$*.*

The first theorem of Widom ([24], Theorem 2, p. 348), in which $\mathcal{C}(K)$ denotes the space of complex, continuous functions on K with the sup-norm, can now be rephrased as follows.

Theorem 3.6 (Widom) *Let $K \subseteq \mathbb{D}$ be a compact set, and $\varepsilon > 0$. Then, there exist a constant $C_\varepsilon > 0$ and, for every integer n large enough, a rational function R with poles on $\partial\mathbb{D}$ and points $\zeta_i \in \mathbb{D} \setminus K$ such that for every $g \in H^\infty$, one has:*

$$\|g - h\|_{\mathcal{C}(K)} \leq C_\varepsilon\, \mathrm{e}^{\varepsilon n}\, \mathrm{e}^{-n/\mathrm{Cap}\,(K)}\, \|g\|_\infty\,, \tag{3.7}$$

where:

$$h(w) = R(w) \sum_{\substack{i,k\\ 1\leq k\leq m_i}} c_{i,k}(g)\, (w - \zeta_i)^{-k} \quad \textit{with} \quad \sum_i m_i < n$$

and the maps $g \in H^\infty \mapsto c_{i,k}(g)$ are linear.

Moreover, if H is a weighted analytic Hilbert space, these maps, restricted to $H^\infty \cap H$, extend to continuous linear forms on H.

Widom's theorem precisely says the following. If R and γ are the rational function and cycle of Lemma 3.5, let ζ_i be the zeros of R inside γ. Consider, for $w \in K$, the function

$$G(w) = R(w) \left[\frac{1}{2\pi i} \int_\gamma \frac{g(\zeta)}{R(\zeta)\,(\zeta - w)}\, d\zeta\right];$$

then, by the residues theorem,

$$G(w) = g(w) - R(w) \sum_{i,k} c_{i,k}(g)\, (w - \zeta_i)^{-k} = g(w) - h(w)\,,$$

and Widom's theorem says that $\|G\|_{\mathcal{C}(K)} \leq C_\varepsilon\, \mathrm{e}^{2\,\varepsilon n}\, [M(K)]^n\, \|g\|_\infty$.

The only additional remark made here is that the $c_{i,k}$ are of the form

$$c_{i,k}(g) = \sum_{j \leq m_i - k} \lambda_{i,j,k}\, g^{(j)}(\zeta_i)$$

where $\lambda_{i,j,k}$ are fixed scalars, so that by (2.7) they extend to continuous linear forms on H.

Observe that the linear forms $g \mapsto c_{i,k}(g')$ are also continuous on H since

$$c_{i,k}(g') = \sum_{j \leq m_i - k} \lambda_{i,j,k}\, g^{(j+1)}(\zeta_i)\,. \tag{3.8}$$

This observation will be useful later.

Remark. The rational function h above is analytic in $\mathbb{D}$. Indeed, since the ζ_i are zeros of R, the polar factors $(w - \zeta_i)^{-k}$ are compensated by $R(w)$ with the right multiplicity, so that the only poles of R have modulus ≥ 1. However (see [24], Lemma 1, p. 346), the poles of R are located on $\partial\mathbb{D}$, but we cannot ensure that $h \in H$. Fortunately, we will see that $h \circ \varphi \in H$, and this will be sufficient for our purposes.

We will need a second theorem of H. Widom ([24], Theorem 7, p. 353), which goes as follows.

Theorem 3.7 (Widom) *Let K be a compact subset of $\mathbb{D}$ and $\mathcal{C}(K)$ be the space of continuous functions on K with its natural norm. Set:*

$$\delta_n(K) = \inf_E \left[\sup_{f \in B_{H^\infty}} \operatorname{dist}(f, E) \right],$$

where E runs over all $(n-1)$-dimensional subspaces of $\mathcal{C}(K)$ and $\operatorname{dist}(f, E) = \inf_{h \in E} \|f - h\|_{\mathcal{C}(K)}$. *Then*

$$\delta_n(K) \geq \alpha\, \mathrm{e}^{-n/\mathrm{Cap}\,(K)} \tag{3.9}$$

for some positive constant α.

3.2.1 The upper bound

Theorem 3.8 *Let H be an analytic weighted Hilbert space with norm $\|\,.\,\|$. Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a symbol for H, such that $\|\varphi\|_\infty = \rho < 1$. Then:*

$$\beta^+(C_\varphi) := \limsup_{n\to\infty} [a_n(C_\varphi)]^{1/n} \leq \mathrm{e}^{-1/\mathrm{Cap}\,[\overline{\varphi(\mathbb{D})}]}.$$

Proof. Fix $\varepsilon > 0$ such that $\rho\, \mathrm{e}^{\varepsilon} < 1$.

If $f(z) = \sum_{k=0}^{\infty} b_k z^k \in H$, let $g(z) := S_l f(z) = \sum_{k=0}^{l-1} b_k z^k$, with $l = l(n)$ be an integer to be adjusted.

Lemma 3.9 *We have:*

$$\|f \circ \varphi - g \circ \varphi\| \leq K_\varepsilon \rho^l \mathrm{e}^{\varepsilon l}.$$

Proof. For $f(z) = \sum_k b_k z^k$, we have:

$$\begin{aligned}\|f\circ\varphi - g\circ\varphi\| &= \Big\|\sum_{k=l}^{\infty} b_k \varphi^k\Big\| \leq \sum_{k=l}^{\infty} |b_k|\,\|\varphi^k\| \\ &\leq \bigg(\sum_{k=l}^{\infty} |b_k|^2 w_k\bigg)^{1/2} \bigg(\sum_{k=l}^{\infty} \|\varphi^k\|^2 w_k^{-1}\bigg)^{1/2} \leq K_\varepsilon \rho^l \mathrm{e}^{\varepsilon l}\,,\end{aligned}$$

by using Cauchy-Schwarz inequality, the fact that $\|f\| \leq 1$, the inequalities (2.6), and a geometric progression. $\square$

Also, remark that we have, by the Cauchy-Schwarz inequality:

$$\begin{aligned}\|(S_l f)'\|_\infty &\leq \sum_{k=0}^{l-1} k\,|b_k| \leq \bigg(\sum_{k=0}^{l-1} |b_k|^2 w_k\bigg)^{1/2} \bigg(\sum_{k=0}^{l-1} k^2 w_k^{-1}\bigg)^{1/2} \\ &\leq \|f\| \bigg(\sum_{k=0}^{l-1} k^2 w_k^{-1}\bigg)^{1/2}.\end{aligned}$$

Therefore, using (2.6), we see that the linear map $S'_l : H \to H^\infty$, defined by $S'_l(f) = (S_l f)'$, is continuous with a norm less than $(\sum_{k=0}^{l-1} k^2 w_k^{-1})^{1/2} \leq K_\varepsilon\, \mathrm{e}^{\varepsilon l}$.

We now use Theorem 3.6, with $K = \overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$ (and for $n-1$ instead of n). Set, for $n \geq 2$, large enough:

$$h_1(w) = R(w) \sum_{\substack{i,k \\ 1\leq k\leq m_i}} c_{i,k}(g')(w-\zeta_i)^{-k} \quad \text{with} \quad \sum_i m_i < n-1\,.$$

Recall that h_1 is analytic in $\mathbb{D}$. Remark that h_1 depends linearly on f and the map $f \mapsto h_1$ has a rank $< n-1$. We denote by $I_1 \in \mathcal{H}\mathrm{ol}\,(\mathbb{D})$ the primitive of h_1 taking the value $g[\varphi(0)]$ at $\varphi(0)$:

$$I_1(z) = \int_{\varphi(0)}^{z} h_1(u)\,du + g[\varphi(0)]\,.$$

Next, define an operator A of rank $< n$ on H (the continuity of A being justified by (3.8)) by the formula:

$$A(f) = I_1 \circ \varphi\,. \tag{3.10}$$

Note that, even if $I_1 \notin H$, we easily check on the integral representation of the norm that $I_1 \circ \varphi \in H$ since we assumed $\varphi \in H$, i.e. (see Lemma 2.1) that φ is a symbol.

Assuming for the rest of the proof that $\|f\| \leq 1$, we have the following lemma.

Lemma 3.10 *We have:*

$$\|g\circ\varphi - I_1\circ\varphi\| \leq K_\varepsilon\, \mathrm{e}^{\varepsilon(n-1)}\, \mathrm{e}^{\varepsilon l}\, \mathrm{e}^{-(n-1)/\mathrm{Cap}\,(K)}.$$

Proof. Since $\varphi \in H$ and since $h_1 = I_1'$ approximates g' uniformly on K and $\|g'\|_\infty = \|(S_l f)'\|_\infty \leq K_\varepsilon \mathrm{e}^{\varepsilon l}$, we have, by Theorem 3.6:

$$\begin{aligned}\|g\circ\varphi - I_1\circ\varphi\|^2 &= \int_{\mathbb{D}} \big|g'[\varphi(z)] - h_1[\varphi(z)]\big|^2 |\varphi'(z)|^2 \omega(z)\, dA(z)\\ &\leq K_\varepsilon^2 \mathrm{e}^{2\varepsilon(n-1)}[M(K)]^{2(n-1)}\|g'\|_\infty^2 \int_{\mathbb{D}} |\varphi'(z)|^2\omega(z)\, dA(z)\\ &\leq C\, K_\varepsilon^3 \mathrm{e}^{2\varepsilon l}\, \mathrm{e}^{2\varepsilon(n-1)}[M(K)]^{2(n-1)}\,,\end{aligned}$$

(with $C = \|\varphi\|_\omega^2$), hence the lemma, provided that we increase K_ε. $\square$

We can now end the proof of Theorem 3.8.

Writing:

$$\begin{aligned}\|C_\varphi(f) - A(f)\| &= \|f\circ\varphi - I_1\circ\varphi\|\\ &\leq \|f\circ\varphi - g\circ\varphi\| + \|g\circ\varphi - I_1\circ\varphi\|\,,\end{aligned}$$

we have:

1) $\|f\circ\varphi - g\circ\varphi\| \leq K_\varepsilon\, \rho^l\, \mathrm{e}^{\varepsilon l}$ by Lemma 3.9;

2) $\|g\circ\varphi - I_1\circ\varphi\| \leq K_\varepsilon \mathrm{e}^{\varepsilon(n-1)}[M(K)]^{n-1}\mathrm{e}^{\varepsilon l}$ by Lemma 3.10.

We therefore get, since $a_n := a_n(C_\varphi) \leq \|C_\varphi - A\|$:

$$a_n \leq K_\varepsilon\, \rho^l\, \mathrm{e}^{\varepsilon l} + K_\varepsilon\, \mathrm{e}^{\varepsilon l}\mathrm{e}^{\varepsilon(n-1)}[M(K)]^{n-1}.$$

Next, since $(a+b)^{1/n} \leq a^{1/n} + b^{1/n}$, we infer that:

$$a_n^{1/n} \leq (K_\varepsilon)^{1/n}(\rho\, \mathrm{e}^\varepsilon)^{l/n} + K_\varepsilon^{1/n}\mathrm{e}^{\varepsilon l/n}\mathrm{e}^{\varepsilon(n-1)/n}M(K)^{(n-1)/n}. \tag{3.11}$$

We now adjust $l = Nn$, where N is a fixed positive integer, and pass to the upper limit with respect to n in (3.11). We get:

$$L := \limsup a_n^{1/n} \leq [\rho\, \mathrm{e}^\varepsilon]^N + \mathrm{e}^\varepsilon \mathrm{e}^{\varepsilon N} M(K).$$

Letting ε go to 0, we get $L \leq \rho^N + M(K)$. Finally, letting N tend to infinity, we get $L \leq M(K)$ as claimed, and that ends the proof of Theorem 3.8. $\square$

3.3 The lower bound

Theorem 3.11 *Let H be a weighted analytic Hilbert space and $\varphi \in H$ such that $\|\varphi\|_\infty < 1$. Then:*

$$\beta^-(C_\varphi) := \liminf_{n\to\infty} [a_n(C_\varphi)]^{1/n} \geq \mathrm{e}^{-1/\mathrm{Cap}\,[\varphi(\mathbb{D})]}\,.$$

It will be convenient to work with the Kolmogorov numbers $d_n(C_\varphi)$ instead of the approximation numbers $a_n(C_\varphi)$. Recall that, for Hilbert spaces, one has $d_n(C_\varphi) = a_n(C_\varphi)$. We begin with a simple lemma, undoubtedly well known to experts, on approximation numbers of an operator T on a Hilbert space H.

Lemma 3.12 *For every Hilbert space H and every compact operator $T\colon H \to H$, one has, B_H denoting the unit ball of H:*

$$d_n(T) = \inf_{\dim E < n} \left[\sup_{f \in B_H} \operatorname{dist}\big(Tf, T(E)\big) \right]. \tag{3.12}$$

Proof. Indeed, if $\varepsilon_n(T)$ denotes the right hand side in (3.12), we clearly have $d_n(T) \leq \varepsilon_n(T)$. Now, let:

$$Tf = \sum_{j=1}^{\infty} a_j(T) \, \langle f, v_j \rangle \, u_j ,$$

with (u_j) and (v_j) two orthonormal sequences, be the Schmidt decomposition of T. Let E_0 be the span of $v_1, \ldots, v_{n-1}$. Observe that $u_j = T(a_j^{-1} v_j) \in T(E_0)$ for $j < n$. Now, if $f \in B_H$, one has:

$$\begin{aligned} \big[\operatorname{dist}\big(Tf, T(E_0)\big)\big]^2 &= \Big\| \sum_{j=n}^{\infty} a_j(T) \, \langle f, v_j \rangle \, u_j \Big\|^2 = \sum_{j=n}^{\infty} [a_j(T)]^2 |\, \langle f, v_j \rangle|^2 \\ &\leq [a_n(T)]^2 \sum_{j=n}^{\infty} |\langle f, v_j \rangle|^2 \leq [a_n(T)]^2; \end{aligned}$$

so that $\varepsilon_n(T) \leq \sup_{f \in B_H} \operatorname{dist}\big(Tf, T(E_0)\big) \leq a_n(T) = d_n(T)$. $\square$

Proof of Theorem 3.11. Let $0 < r_j < 1$, $r_j \to 1$ and $\psi_j \colon \mathbb{D} \to \mathbb{D}$ be given by $\psi_j(z) = r_j \, z$. Set $K_j = \overline{\varphi \circ \psi_j(\mathbb{D})} = \overline{\varphi(r_j \mathbb{D})}$. Let E be a subspace of H of dimension $< n$. By restriction, E can be viewed as a subspace of $\mathcal{C}(K_j)$. By the second result of Widom (Theorem 3.7), we can find $f \in B_{H^\infty}$, $f(z) = \sum_{k \geq 0} b_k z^k$, such that:

$$\| f - h \|_{\mathcal{C}(K_j)} \geq 2\alpha \, [M(K_j)]^n , \qquad \forall h \in E ,$$

where $\alpha > 0$ is an absolute constant. If H^∞ contractively embeds into H, we can continue with this f. In the general case, we have to correct f in order to be in B_H, the unit ball of H. To that effect, we simply consider a partial sum:

$$g(z) = \sum_{k=0}^{l-1} b_k z^k$$

and we note that, setting $\rho_j = \sup_{w \in K_j} |w|$, one has $\rho_j < 1$ and:

$$\| f - g \|_{\mathcal{C}(K_j)} \leq \frac{\rho_j^l}{(1 - \rho_j^2)^{1/2}} \tag{3.13}$$

$$\| g \|_H \leq C \, l , \tag{3.14}$$

where $C = C(\omega) \geq 1$ is the constant appearing in (2.6).

Indeed, we have $\|f-g\|_{\mathcal{C}(K_j)} \leq \sum_{k=l}^{\infty} |b_k|\, \rho_j^k$ and then (3.13) follows from Cauchy-Schwarz's inequality and the fact that $\sum_{k\geq 0} |b_k|^2 \leq 1$ since $f \in B_{H^\infty}$. For (3.14), we simply use that, by (2.6), the weight w satisfies $w_k \leq C\,(k+1)^2$ and get:

$$\|g\|_H^2 = \sum_{k=0}^{l-1} |b_k|^2\, w_k \leq C\, l^2 \sum_{k=0}^{l-1} |b_k|^2 \leq C\, l^2 \leq C^2\, l^2.$$

We then notice that (3.13) gives, for every $h \in E$:

$$\begin{aligned} \|g-h\|_{\mathcal{C}(K_j)} &\geq \|f-h\|_{\mathcal{C}(K_j)} - \|f-g\|_{\mathcal{C}(K_j)} \\ &\geq 2\,\alpha\,[M(K_j)]^n - \frac{\rho_j^l}{(1-\rho_j^2)^{1/2}} \geq \alpha\,[M(K_j)]^n\,, \end{aligned} \tag{3.15}$$

if we take $l = A_j n$ where A_j is a large positive integer depending only on j. Explicitly:

$$A_j > \frac{\log\big[1/\big(\alpha\,(1-\rho_j^2)^{1/2}\big)\big]}{\log(1/\rho_j)} + \frac{\log[1/M(K_j)]}{\log(1/\rho_j)}\,\cdot$$

Finally, set $F = g/C\,l$. Then $F \in B_H$. Since E is a vector space, (3.14) and (3.15) imply:

$$\|F-h\|_{\mathcal{C}(K_j)} = \frac{1}{C\,l}\|g - C\,l\,h\|_{\mathcal{C}(K_j)} \geq \frac{1}{C\,l}\,\alpha\,[M(K_j)]^n.$$

But we also know that:

$$\|F-h\|_{\mathcal{C}(K_j)} = \|F\circ\varphi\circ\psi_j - h\circ\varphi\circ\psi_j\|_\infty \leq L_{r_j}\|F\circ\varphi - h\circ\varphi\|_H\,,$$

so we are left with (recall that $l = A_j n$):

$$\|C_\varphi F - C_\varphi h\|_H \geq \frac{\alpha}{C\,L_{r_j} A_j}\,\frac{M(K_j)^n}{n}\,, \quad \forall h \in E,$$

implying by Lemma 3.12:

$$a_n(C_\varphi) = d_n(C_\varphi) \geq \frac{\alpha}{C\,L_{r_j} A_j}\,\frac{[M(K_j)]^n}{n}\,\cdot$$

Now, taking n-th roots and passing to the lower limit, we get:

$$\beta^-(C_\varphi) \geq M(K_j). \tag{3.16}$$

It remains now to let $j \to \infty$. Observe that the compact subsets $K_j \subseteq \varphi(\mathbb{D})$ form an exhaustive sequence of compact subsets of $\varphi(\mathbb{D})$. Let then $L \subseteq \varphi(\mathbb{D})$ be compact; we have $L \subseteq K_{j_0}$ for some j_0, and using (3.16), we get $\beta^-(C_\varphi) \geq M(K_{j_0}) \geq M(L)$. Passing to the supremum on L, we get $\beta^-(C_\varphi) \geq M[\varphi(\mathbb{D})]$, and this ends the proof of Theorem 3.11. $\square$

3.4 The case $\|\varphi\|_\infty = 1$

As said in the Introduction, for weighted Bergman spaces (including the Hardy space), and for the Dirichlet space, we proved in [18] and [16], respectively, that $\beta(C_\varphi) = 1$ if $\|\varphi\|_\infty = 1$ for every φ inducing a composition operator on one of those spaces.

In this section, we use Theorem 3.1 to generalize this result to all composition operators C_φ on weighted analytic Hilbert spaces, with another, and simpler, proof.

For that, it suffices to use the following result, which is certainly well-known to specialists. The pseudo-hyperbolic metric ρ on $\mathbb{D}$ is defined in (3.2) and we denote by diam_ρ the diameter for this metric.

Theorem 3.13 *Let K be a compact and connected subset of $\mathbb{D}$. Then, for $0 < \varepsilon < 1$:*

$$\mathrm{diam}_\rho\, K > 1 - \varepsilon \quad \Longrightarrow \quad \mathrm{Cap}\,(K) \geq c \log 1/\varepsilon\,,$$

for some absolute positive constant c.

Hence, the Green capacity of K tends to ∞ as its pseudo-hyperbolic diameter tends to 1.

Before proving that, let us give two suggestive examples, borrowed from [13], p. 175–177.

1) Let $K = \overline{D}(0, r)$; then:

$$\mathrm{diam}_\rho\, K = \frac{2r}{1 + r^2} \quad \text{and} \quad \mathrm{Cap}\,(K) = \frac{1}{\log 1/r}\,\cdot$$

One sees that r goes to 1 when $\mathrm{diam}_\rho\, K$ goes to 1, and hence $\mathrm{Cap}\,(K)$ tends to infinity.

2) Let $K = [0, h]$, with $0 < h < 1$. Then:

$$\mathrm{diam}_\rho\, K = h \quad \text{and} \quad \mathrm{Cap}\,(K) = \frac{1}{\pi}\,\frac{I'}{I}\,,$$

where I and I' are the elliptic integrals:

$$I = \int_0^1 \frac{1}{\sqrt{(1 - t^2)(1 - k^2\, t^2)}}\, dt \quad \text{and} \quad I' = \int_0^1 \frac{1}{\sqrt{(1 - t^2)(1 - k'^2\, t^2)}}\, dt\,,$$

with $k = \frac{1-h}{1+h}$ and ${k'}^2 = 1 - k^2$.

If $0 \leq a < b \leq h$, then $b - a + hab \leq h - a + ah^2 = h - a(1 - h^2) \leq h$, so that $\rho(a, b) \leq h$. Therefore, in this example again, the assumption $\mathrm{diam}_\rho\, K \longrightarrow 1$ implies successively that $h \to 1$, $k \to 0$, $k' \to 1$, $I \to \pi/2$, $I' \to \infty$, and at last $\mathrm{Cap}\,(K) \to \infty$.

This example shows that Theorem 3.13 is optimal since

$$\int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k'^2 t^2)}} \approx \log \frac{1}{1 - {k'}^2} \approx \log \frac{1}{1 - h}$$

as h (and hence k') goes to 1.

The following proof of Theorem 3.13 was kindly shown to the second-named author by E. Saksman ([23]).

It make use of the following alternative definition of Green capacity, where $\mathcal{C}_0^\infty(\mathbb{D})$ is the space of infinitely differentiable functions on $\mathbb{D}$ which are null on $\partial\mathbb{D}$, and $dz = dxdy$ is the usual 2-dimensional Lebesgue measure.

Lemma 3.14 *For every compact subset K of $\mathbb{D}$, one has:*

$$\mathrm{Cap}\,(K) = \inf\left\{\frac{1}{2\pi}\int_{\mathbb{D}} |\nabla u(z)|^2\,dz\,;\ u \in \mathcal{C}_0^\infty(\mathbb{D}) \text{ and } u \geq 1 \text{ on } K\right\}.$$

Proof of Theorem 3.13. If $\mathrm{diam}_\rho\, K > 1-\varepsilon$ and K is connected, it contains two points z_1 and z_2 such that $\rho(z_1, z_2) = 1-\varepsilon$. By the invariance of the Green capacity and of ρ under automorphisms of the disk, we can assume that $z_1 = 0$ and $z_2 = 1-\varepsilon$. Take $\varepsilon < r < 1$. Denote by Δ_r the intersection of the closed disk with center 1 and radius r with the closed unit disk. We observe that K meets the exterior of Δ_r at 0 and its interior at $1-\varepsilon$. The connectedness of K implies that K meets the boundary of Δ_r: there is $b \in K$ such that $|b-1| = r$. Write $b = 1 + r\mathrm{e}^{i\vartheta}$. Take now $a = 1 + r\mathrm{e}^{i\theta}$ with $|a| = 1$ and $0 \leq \theta \leq \vartheta \leq 2\pi$. Since $u(a) = 0$ and $u(b) \geq 1$, we get, by the fundamental theorem of calculus, that:

$$\begin{aligned} 1 \leq u(b) - u(a) &= \int_\theta^\vartheta ir\,\mathrm{e}^{it}\nabla u(1 + r\mathrm{e}^{it})\,dt = \left|\int_\theta^\vartheta ir\,\mathrm{e}^{it}\nabla u(1 + r\mathrm{e}^{it})\,dt\right| \\ &\leq r\int_\theta^\vartheta |\nabla u(1 + r\mathrm{e}^{it})|\,dt \leq r\int_0^{2\pi} |\nabla u(1 + r\mathrm{e}^{it})|\,dt. \end{aligned}$$

Now, Cauchy-Schwarz inequality gives:

$$\int_0^{2\pi} |\nabla u(1 + r\mathrm{e}^{it})|^2\,dt \geq \frac{1}{2\pi r^2}\,\cdot$$

Integrating in polar coordinates centered at 1 and remembering that $u = 0$ outside $\mathbb{D}$, we get:

$$\begin{aligned} \int_{\mathbb{D}} |\nabla u(z)|^2\,dz &\geq \int_{\varepsilon < |z-1| < 1} |\nabla u(z)|^2\,dz \\ &= \int_\varepsilon^1 \left[\int_0^{2\pi} |\nabla u(1 + r\mathrm{e}^{it})|^2\,dt\right] r\,dr \\ &\geq \frac{1}{2\pi}\int_\varepsilon^1 \frac{dr}{r} = \frac{1}{2\pi}\log\frac{1}{\varepsilon}\,\cdot \end{aligned}$$

In view of (3.14), this ends the proof of Theorem 3.13. $\square$

Proof of Lemma 3.14. Though this result is often considered as "well-known", we were not able to find anywhere an explicit reference. Since the average reader (if any!) of this paper will not be a specialist in Potential theory, we give a proof here.

1) We first prove that the capacity of the compact K is less than the right-hand side (though we only need that it is greater). We shall use Lemma 2.2.

We know ([7], Corollary 21.4.7, or [13], p. 91) that for every measure μ on $\mathbb{D}$ supported by K, one has $\Delta G_\mu = -2\pi\mu$, where G_μ is seen as a distribution. Hence, for every function $u \in \mathcal{C}_0^\infty(\mathbb{D})$ such that $u \geq 1$ on K and every positive measure μ supported by K such that $G_\mu \leq 1$ on $\mathbb{D}$, one has:

$$\mu(K) = \int_K d\mu \leq \int_{\mathbb{D}} u\, d\mu = -\frac{1}{2\pi}\int_{\mathbb{D}} u(z)\,\Delta G_\mu(z)\,dz\,.$$

Then, by definition of the Laplacian of a distribution, we get:

$$\mu(K) \leq -\frac{1}{2\pi}\int_{\mathbb{D}} \Delta u(z)\,G_\mu(z)\,dz\,.$$

But (see [2], Chapitre XI, p. 132 and pp. 144–145, or [13], Chap. IV, § 1, p. 215), for every real Borel measures ν_1 and ν_2 with finite energy (meaning that their positive and negative parts have finite energy), this energy is positive and one has the Cauchy-Schwarz inequality for the Dirichlet space :

$$\left|\int_{\mathbb{D}} G_{\nu_1}\,d\nu_2\right| \leq \left(\int_{\mathbb{D}} G_{\nu_1}\,d\nu_1\right)^{1/2}\left(\int_{\mathbb{D}} G_{\nu_2} d\nu_2\right)^{1/2}.$$

Applying this to the measures $\nu_1 = \mu$ and $\nu_2 = \nu = \Delta u.dz$, we get, since $G_\mu \leq 1$:

$$\begin{aligned}
\mu(K) &\leq \frac{1}{2\pi}\left(\int_{\mathbb{D}} G_\mu(z)\,d\mu(z)\right)^{1/2}\left(\int_{\mathbb{D}} G_\nu(z)\Delta u(z)\,dz\right)^{1/2}\\
&\leq \frac{1}{2\pi}\,[\mu(K)]^{1/2}\left(\int_{\mathbb{D}} G_\nu(z)\Delta u(z)\,dz\right)^{1/2}\\
&= \frac{1}{2\pi}\,[\mu(K)]^{1/2}\left(\int_{\mathbb{D}} G_\nu\,d\nu\right)^{1/2}.
\end{aligned}$$

Now, since $u \in \mathcal{C}_0^\infty(\mathbb{D})$, one has G. C. Evans' theorem [8] (see [2], Chapitre XI, Lemme 1, p. 141, or [13], Theorem 1.20, p. 97):

$$\int_{\mathbb{D}} G_\nu\,d\nu = 2\pi\int_{\mathbb{D}} |\nabla u(z)|^2\,dz\,.$$

Therefore, we get:

$$\mu(K) \leq \frac{1}{2\pi}\int_{\mathbb{D}} |\nabla u(z)|^2\,dz\,.$$

Taking the supremum on μ of the left-hand side and the infimum on u of the right-hand side, we obtain:

$$\mathrm{Cap}\,(K) \leq \inf\left\{\frac{1}{2\pi}\int_{\mathbb{D}} |\nabla u(z)|^2\,dz\,;\ u \in \mathcal{C}_0^\infty(\mathbb{D}) \text{ and } u \geq 1 \text{ on } K\right\}.$$

2) Let $\varepsilon > 0$.

Let $K_j = \{z \in \mathbb{C}\,;\ \mathrm{dist}\,(z, K) \leq 1/j\}$, $j \geq 1$. Each K_j is compact and is contained in $\mathbb{D}$ for j large enough, say $j \geq j_0$. Since $K = \bigcap_{j \geq j_0} K_j$ (and the sequence is decreasing), one has $\mathrm{Cap}\,(K_j) \underset{j\to\infty}{\longrightarrow} \mathrm{Cap}\,(K)$ ([7], Proposition 21.7.15; note that though this proposition is stated for the logarithmic capacity, the proof clearly works also for the Green capacity). Hence, there is some $j \geq j_0$ such that, for $K' = K_j$, one has $(1+\varepsilon)\,\mathrm{Cap}\,(K) \geq \mathrm{Cap}\,(K')$.

Let μ_0 be an equilibrium measure of K'. One has $\mu_0(K') = 1$, $I(\mu_0) = V(K')$, $G_{\mu_0} \leq V(K')$ on $\mathbb{D}$. Moreover, by [7], Lemma 21.10.1 (based on Frostman's theorem: see [7], Theorem 21.7.12, whose proof works also for the Green capacity), one has $G_{\mu_0} = V(K')$ on $\mathrm{int}\,(K')$, hence on K. Let $\mu = \mathrm{Cap}\,(K')\,\mu_0$. Then $\mu(K') = \mathrm{Cap}\,(K')$, $I(\mu) = [\mathrm{Cap}\,(K')]^2 I(\mu_0) = \mathrm{Cap}\,(K')$, and, since $G_\mu = \mathrm{Cap}\,(K')\,G_{\mu_0}$, one has also $G_\mu \leq 1$ on $\mathbb{D}$ and $G_\mu = 1$ on K.

By a theorem of G. Choquet [5], we can find, by regularization ([2], p. 26 and Lemma, p. 135 and pp. 142–145, or [13], Theorem 1.9, p. 70, which applies since $G_\mu - U_2^\mu$ is a harmonic function) an increasing sequence of positive infinitely differentiable functions v_n on $\mathbb{D}$ which converges pointwise to G_μ and such that:

$$\int_{\mathbb{D}} |\nabla v_n(z)|^2\,dz \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{D}} |\nabla G_\mu(z)|^2\,dz\,.$$

Since $(v_n)_n$ is increasing and converges pointwise to 1 on the compact set K, Dini's theorem tells that one has uniform convergence. Hence, we can find some $v = v_n$ such that $v \geq (1+\varepsilon)^{-1}$ on K and

$$\int_{\mathbb{D}} |\nabla v(z)|^2\,dz \leq (1+\varepsilon) \int_{\mathbb{D}} |\nabla G_\mu(z)|^2\,dz\,.$$

Note that $v = 0$ on $\partial\mathbb{D}$ since $0 \leq v \leq G_\mu$, which is equal to 0 on $\partial\mathbb{D}$.

Putting $u = (1+\varepsilon)v$, one has $u \in \mathcal{C}_0^\infty(\mathbb{D})$, $u \geq 1$ on K and

$$\int_{\mathbb{D}} |\nabla u(z)|^2\,dz \leq (1+\varepsilon)^3 \int_{\mathbb{D}} |\nabla G_\mu(z)|^2\,dz\,.$$

But we know by G. C. Evans's theorem (see [21], Proposition 7.3, or [2], Chapitre XI, p. 142 and pp. 144–145, or [13], Theorem 1.20, p. 97) that:

$$I(\mu) = \frac{1}{2\pi} \int_{\mathbb{D}} |\nabla G_\mu(z)|^2\,dz\,.$$

We get hence:

$$\begin{aligned}(1+\varepsilon)\,\mathrm{Cap}\,(K) \geq \mathrm{Cap}\,(K') = I(\mu) &= \frac{1}{2\pi}\int_{\mathbb{D}} |\nabla G_\mu(z)|^2\,dz \\ &\geq \frac{1}{(1+\varepsilon)^3}\,\frac{1}{2\pi}\int_{\mathbb{D}} |\nabla u(z)|^2\,dz\,.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get:

$$\mathrm{Cap}\,(K) \geq \inf\Big\{\frac{1}{2\pi}\int_{\mathbb{D}} |\nabla u(z)|^2\,dz\,;\ u \in \mathcal{C}_0^\infty(\mathbb{D}) \text{ and } u \geq 1 \text{ on } K\Big\}\,,$$

and that ends the proof. □

Remark. After this paper was completed, we have found an alternative proof of Theorem 3.13. We sketch it here.

As in the above proof, we may assume that 0 and $1-\varepsilon$ belong to K. Consider $K^* = \{|z|\,;\ z \in K\}$. Since K is connected, the same holds for K^*. Hence the interval $[0, 1-\varepsilon]$ is contained in K^*. It follows that $\mathrm{Cap}\,([0,1-\varepsilon]) \leq \mathrm{Cap}\,(K^*)$. But we saw in Example 2 that $\mathrm{Cap}\,([0,1-\varepsilon]) \approx \log(1/\varepsilon)$; hence $\mathrm{Cap}\,(K^*) \gtrsim \log(1/\varepsilon)$. It remains to use that the map $\alpha\colon z \mapsto |z|$ is a contraction for the pseudo-hyperbolic metric and hence $\mathrm{Cap}\,(K^*) \leq \mathrm{Cap}\,(K)$ (see [13], Chap. II, Theorem 2.9, and the comment p. 175 for the Green capacity). In fact, if ν is any probability measure supported by K^*, there exists (see [10], Chap. III, Lemma 4.6) a probability measure μ on K such that $\alpha(\mu) = \nu$. Hence:

$$\begin{aligned} V(K) \leq I_K(\mu) &= \iint_{\mathbb{D}\times\mathbb{D}} g(z,w)\,d\mu(z)\,d\mu(w) = \iint_{\mathbb{D}\times\mathbb{D}} \log\frac{1}{\rho(z,w)}\,d\mu(z)\,d\mu(w) \\ &\leq \iint_{\mathbb{D}\times\mathbb{D}} \log\frac{1}{\rho(|z|,|w|)}\,d\mu(z)\,d\mu(w) \\ &= \iint_{\mathbb{D}\times\mathbb{D}} \log\frac{1}{\rho(z,w)}\,d\nu(z)\,d\nu(w) = I_{K^*}(\nu). \end{aligned}$$

Taking the infimum over all ν, we get $V(K) \leq V(K^*)$. $\square$

As a corollary of Theorem 3.13, we get a new proof of [18], Theorem 3.4 and of [16], Theorem 2.2.

Theorem 3.15 *There exists an absolute constant $c > 0$ such that, for any symbol φ on a weighted analytic space H, one has:*

$$\mathrm{diam}_\rho\,[\varphi(\mathbb{D})] > r \quad\Longrightarrow\quad \beta(C_\varphi) \geq \exp\left[-\frac{c}{\log 1/(1-r)}\right].$$

In particular:

$$\|\varphi\|_\infty = 1 \quad\Longrightarrow\quad \beta(C_\varphi) = 1\,.$$

Proof. The first statement is a direct consequence of Theorem 3.1, modulo Theorem 2.3 and Theorem 3.13, applied to $\varphi(\mathbb{D})$ and its closure.

One cannot replace $\mathrm{diam}_\rho\,[\varphi(\mathbb{D})] > r$ by $\|\varphi\|_\infty > r$ in this first statement as indicated by the following example:

$$\varphi(z) = \frac{a-(z/2)}{1-\overline{a}(z/2)} = \Phi_a[h(z)]\,,$$

where $\Phi_a(z) = \frac{a-z}{1-\overline{a}z}$ with $a \in \mathbb{D}$ and $h(z) = z/2$ is the dilation with ratio $1/2$. Then $\|\varphi\|_\infty \geq |\Phi_a(0)| = |a|$ and $\beta(C_\varphi) = \beta(C_h) = 1/2$.

However, one can do so if moreover $\varphi(0) = 0$ because then, clearly:

$$\|\varphi\|_\infty > r \quad\Longrightarrow\quad \mathrm{diam}_\rho\,[\varphi(\mathbb{D})] > r\,.$$

This is enough for the second statement since, putting $a = \varphi(0)$, we have, due to the fact that Φ_a is unimodular on the whole unit circle: $\|\Phi_a \circ \varphi\|_\infty = \|\varphi\|_\infty = 1$, $(\Phi_a \circ \varphi)(0) = 0$ and $\beta(C_\varphi) = \beta(C_{\Phi_a\circ\varphi})$. $\square$

3.5 A remark

We proved in [14] that every composition operator C_φ which is bounded on the Dirichlet space $\mathcal{D}$ is compact on the Hardy space H^2 (and hence on the Bergman space $\mathfrak{B}^2$), and even in all Schatten classes on H^2 and $\mathfrak{B}^2$. So one may expect that the approximation numbers of composition operators on the Dirichlet space are bigger than those on the Hardy space (and bigger than those on the Bergman space). Since Theorem 3.1 and Theorem 1.1 show that $\beta(C_\varphi)$ is the same for these three spaces, it follows that the answer will be certainly quite subtle and cannot only involve $\log a_n(C_\varphi)$.

4 The H^p case, $1 \leq p < \infty$

Here, we consider the case of composition operators on H^p for $1 \leq p < \infty$.

For every $a \in \mathbb{D}$, we denote by $e_a \in (H^p)^*$ the evaluation map at a, namely:

$$e_a(f) = f(a)\,, \quad f \in H^p. \tag{4.1}$$

We know that ([26], p. 253):

$$\|e_a\| = \left(\frac{1}{1-|a|^2}\right)^{1/p} \tag{4.2}$$

and the mapping equation

$$C_\varphi^*(e_a) = e_{\varphi(a)} \tag{4.3}$$

still holds.

Throughout this section we denote by $\|\,.\,\|$, without any subscript, the norm in the dual space $(H^p)^*$.

Let us stress that this dual norm of $(H^p)^*$ is, for $1 < p < \infty$, equivalent, but not equal, to the norm $\|\,.\,\|_q$ of H^q, and the equivalence constant tends to infinity when p goes to 1 or to ∞.

With this preliminaries, we are going to see that Theorem 3.1 remains true.

Theorem 4.1 *Let $1 \leq p < \infty$ and $C_\varphi\colon H^p \to H^p$.*

1) *If $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$, then:*

$$\beta(C_\varphi) = \mathrm{e}^{-1/\mathrm{Cap}\,[\varphi(\mathbb{D})]}\,.$$

2) *One has:*

$$\|\varphi\|_\infty = 1 \quad \Longrightarrow \quad \beta(C_\varphi) = 1\,.$$

We begin with the following lemma, which extends Lemma 3.12.

Lemma 4.2 *Let X be a Banach space, and $T\colon X \to X$ be a compact operator. Let us set:*

$$\varepsilon_n(T) = \inf_{\dim E < n} \left[\sup_{x \in B_X} \operatorname{dist}(Tx, TE) \right]. \tag{4.4}$$

Then $\varepsilon_n(T) \leq 2\sqrt{n}\, c_n(T)$.

Proof. Let $\varepsilon > 0$, and let F be a subspace of X of codimension $< n$ such that $\|T_{|F}\| \leq c_n(T) + \varepsilon$. Let $Q\colon X \to F$ be an onto projection of norm $\|Q\| \leq \sqrt{n} + 1 \leq 2\sqrt{n}$ (see [17], Chapitre 5, Théorème III. 4, 2), or [25], III.B.11) and let $R = T(I - Q)$. Then $E = (I - Q)X$ satisfies $\dim E < n$. If $x \in B_X$, the closed unit ball of X, then:

$$\operatorname{dist}(Tx, TE) \leq \|Tx - Rx\| = \|TQx\| \leq \|T_{|F}\|\, \|Qx\| \leq (c_n(T) + \varepsilon)\, 2\sqrt{n}\,.$$

This implies $\varepsilon_n(T) \leq 2\sqrt{n}\,(c_n(T) + \varepsilon)$.

The result follows since ε was arbitrary. □

Proof of Theorem 4.1. 1) a) We first prove that $\beta^-(C_\varphi) \geq \mathrm{e}^{-1/\mathrm{Cap}\,[\varphi(\mathbb{D})]}$.

Let $\tilde{L}_r = \sup_{|a| \leq r} \|e_a\| = \big(\frac{1}{1 - r^2}\big)^{1/p}$, for $0 < r < 1$. Using the same notations and estimations as in Theorem 3.11, up to the replacement of L_r by $\tilde{L}_r$, we get:

$$\varepsilon_n(T) \geq (1 - \varepsilon)\, \tilde{L}_{r_j}^{-1}\, \alpha\, [M(K_j)]^n\,.$$

Lemma 4.2 now implies:

$$a_n(T) \geq c_n(T) \geq \alpha\, \frac{1 - \varepsilon}{2\sqrt{n}}\, \tilde{L}_{r_j}^{-1} [M(K_j)]^n\,.$$

The rest of the proof is unchanged, since the presence of the factor $1/\sqrt{n}$ does not affect the result.

b) The upper bound is even simpler since $H^\infty \subseteq H^p$. For example, with the notations of Section 3.2.1, setting $A(f) = h \circ \varphi$ as in (3.10), we can replace Lemma 3.10 by

$$\|g \circ \varphi - h \circ \varphi\|_p \leq \|g \circ \varphi - h \circ \varphi\|_\infty = \|g - h\|_{\mathcal{C}(K)}\,,$$

where $K = \overline{\varphi(\mathbb{D})}$.

2) That follows from Theorem 3.13, as in Section 3.13. □

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