Research Article

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Approximation numbers of composition operators on H^p

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Abstract: We give estimates for the approximation numbers of composition operators on the H^p spaces, $1 \leq p < \infty$.

Keywords: approximation numbers; Blaschke product; composition operator; Hardy space; interpolation sequence

1 Introduction

Recently, the study of approximation numbers of composition operators on H^2 has been initiated (see [10], [11], [8], [18], [12]), and (upper and lower) estimates have been given. However, most of the techniques used there are specifically Hilbertian (in particular Weyl's inequality; see [10]). Here, we consider the case of composition operators on H^p for $1 \leq p < \infty$. We focus essentially on lower estimates, because the upper ones are similar, with similar proofs, as in the Hilbertian case. We give in Theorem 2 a minoration involving the uniform separation constant of finite sequences in the unit disk and the interpolation constant of their images by the symbol. We finish with some upper estimates.

1.1 Preliminary

Recall that if X and Y are two Banach spaces of analytic functions on the unit disk $\mathbb{D}$, and $\varphi\colon \mathbb{D} \to \mathbb{D}$ is an analytic self-map of $\mathbb{D}$, one says that φ induces a *composition operator* $C_\varphi\colon X \to Y$ if $f \circ \varphi \in Y$ for every $f \in X$; φ is then called the *symbol* of the composition operator. One also says that φ is a symbol for X and Y if it induces a composition operator $C_\varphi\colon X \to Y$.

For every $a \in \mathbb{D}$, we denote by $e_a \in (H^p)^*$ the evaluation map at a, namely:

$$e_a(f) = f(a)\,, \quad f \in H^p. \tag{1.1}$$

We know that ([22], p. 253):

$$||e_a|| = \left(\frac{1}{1 - |a|^2}\right)^{1/p} \tag{1.2}$$

and the mapping equation

$$C_\varphi^*(e_a) = e_{\varphi(a)} \tag{1.3}$$

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still holds.

Throughout this section we denote by $\|\,.\,\|$, without any subscript, the norm in the dual space $(H^p)^*$.

Let us stress that this dual norm of $(H^p)^*$ is, for $1 < p < \infty$, equivalent, but not equal, to the norm $\|\,.\,\|_q$ of H^q, and the equivalence constant tends to infinity when p goes to 1 or to ∞.

As usual, the notation $A \lesssim B$ means that there is a constant c such that $A \leq c\,B$ and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

1.2 Singular numbers

For an operator $T\colon X \to Y$ between Banach spaces X and Y, its *approximation numbers* are defined, for $n \geq 1$, as:

$$a_n(T) = \inf_{\operatorname{rank} R<n} \|T - R\|\,. \tag{1.4}$$

One has $\|T\| = a_1(T) \geq a_2(T) \geq \cdots \geq a_n(T) \geq a_{n+1}(T) \geq \cdots$, and (assuming that Y has the Approximation Property), T is compact if and only if $a_n(T) \underset{n\to\infty}{\longrightarrow} 0$.

We will also need other singular numbers (see [2], p. 49).

The *n-th Bernstein number* $b_n(T)$ of T, defined as:

$$b_n(T) = \sup_{\substack{E\subseteq X\\ \dim E=n}} \inf_{x\in S_E} \|Tx\|\,, \tag{1.5}$$

where $S_E = \{x \in E\,;\ \|x\| = 1\}$ is the unit sphere of E. When these numbers tend to 0, T is said to be super-strictly singular, or finitely strictly singular (see [17]).

The *n-th Gelfand number* of T, defined as:

$$c_n(T) = \inf_{\substack{L\subseteq Y\\ \operatorname{codim} L<n}} \|T_{|L}\|\,, \tag{1.6}$$

One always has:

$$a_n(T) \geq c_n(T) \quad \text{and} \quad a_n(T) \geq b_n(T)\,, \tag{1.7}$$

and, when X and Y are Hilbert spaces, one has $a_n(T) = b_n(T) = c_n(T)$ ([16], Theorem 2.1).

2 Lower bounds

2.1 Sub-geometrical decay

We first show that, as in the Hilbertian case H^2 ([10], Theorem 3.1), the approximation numbers of the composition operators on H^p cannot decrease faster than geometrically.

Though we cannot longer appeal to the Hilbertian techniques of [10], Weyl's inequality has the following generalization ([3], Proposition 2).

Proposition 1 (Carl-Triebel)**.** *Let T be a compact operator on a complex Banach space E and $\big(\lambda_n(T)\big)_{n\geq1}$ be the sequence of its eigenvalues, indexed such that $|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots$. Then, for $n = 1, 2, \ldots$ and $m = 0, 1, \ldots, n-1$, one has:*

$$\prod_{j=1}^{n} |\lambda_j(T)| \leq 16^n \|T\|^m a_{m+1}(T)^{n-m}\,. \tag{2.1}$$

(see [1] for an optimal result). Then, we can state:

Theorem 1. *For every non-constant analytic self-map* $\varphi\colon \mathbb{D} \to \mathbb{D}$*, there exist* $0 < r \leq 1$ *and* $c > 0$*, depending only on* φ*, such that the approximation numbers of the composition operator* $C_\varphi\colon H^p \to H^p$ *satisfy:*

$$a_n(C_\varphi) \geq c\, r^n\,, \qquad n = 1, 2, \ldots$$

In particular $\liminf_{n\to\infty} [a_n(C_\varphi)]^{1/n} \geq r > 0$.

Proof. If C_φ is not compact, the result is trivial, with $r = 1$; so we assume that C_φ is compact.

Before carrying on, we first recall some notation used in [10]. For every $z \in \mathbb{D}$, let

$$\varphi^\sharp(z) = \frac{|\varphi'(z)|\,(1 - |z|^2)}{1 - |\varphi(z)|^2}$$

be the pseudo-hyperbolic derivative of φ at z, and

$$[\varphi] = \sup_{z\in\mathbb{D}} \varphi^\sharp(z)\,.$$

By the Schwarz-Pick inequality, one has $[\varphi] \leq 1$. Moreover, since φ is not constant, one has $[\varphi] > 0$.

We also set, for every operator $T\colon H^p \to H^p$:

$$\beta^-(T) = \liminf_{n\to\infty} [a_n(T)]^{1/n}\,.$$

For every $a \in \mathbb{D}$, we are going to show that $\beta^-(C_\varphi) \geq \big(\varphi^\sharp(a)\big)^2$, which will give $\beta^-(C_\varphi) \geq [\varphi]^2$, by taking the supremum for $a \in \mathbb{D}$, and the stated result, with $0 < r < [\varphi]^2$.

If $\varphi^\sharp(a) = 0$, the result is obvious, so we assume that $\varphi^\sharp(a) > 0$.

We consider the automorphism Φ_a, defined by $\Phi_a(z) = \frac{a-z}{1-\bar{a}z}$, and set

$$\psi_a = \Phi_{\varphi(a)} \circ \varphi \circ \Phi_a\,.$$

One has $\psi_a(0) = 0$ and $|\psi_a'(0)| = \varphi^\sharp(a)$.

Since C_φ is compact on H^p, $C_{\psi_a} = C_{\Phi_a} \circ C_\varphi \circ C_{\Phi_{\varphi(a)}}$ is also compact on H^p. But we know that this is equivalent to say that it is compact on H^2. Since $\psi_a(0) = 0$ and $\psi_a'(0) = \varphi^\sharp(a) \neq 0$, we know, by the Eigenfunction Theorem ([19], p. 94), that the eigenvalues of $C_{\psi_a}\colon H^2 \to H^2$ are the numbers $\big(\psi_a'(0)\big)^j$, $j = 0, 1, \ldots$, and have multiplicity one. Moreover, the proof given in [19], § 6.2 shows that the eigenfunctions σ^j are not only in H^2, but in all H^q, $1 \leq q < \infty$. Hence $\lambda_j(C_{\psi_a}) = \big(\psi_a'(0)\big)^{j-1}$. We now use Proposition 1, with $2n$ instead of n and $m = n - 1$; we get:

$$\begin{aligned} |\psi_a'(0)|^{n(2n-1)} = \prod_{j=1}^{2n} |\lambda_j(C_{\psi_a})| &\leq 16^{2n} \|C_{\psi_a}\|^{n-1} a_n(C_{\psi_a})^{n+1} \\ &\leq 16^{2n} \|C_{\psi_a}\|^{n} a_n(C_{\psi_a})^{n}\,, \end{aligned}$$

since $a_n(C_{\psi_a}) \leq \|C_{\psi_a}\|$.

That implies that $\beta^-(C_{\psi_a}) \geq |\psi_a'(0)|^2 = \big(\varphi^\sharp(a)\big)^2$.

Since C_{Φ_a} and $C_{\Phi_{\varphi(a)}}$ are automorphisms, we have $\beta^-(C_\varphi) = \beta^-(C_{\psi_a})$, hence the result. □

2.2 Main result

In this section, we use the fortunate fact that, though the evaluation maps at well-chosen points of $\mathbb{D}$ can no longer be said to constitute a Riesz sequence, they will still constitute an unconditional sequence in H^p with good constants, as we are going to see, which will be sufficient for our purposes.

Recall (see [5], p. 276) that the *interpolation constant* κ_σ of a finite sequence $\sigma = (z_1, \dots, z_n)$ of points $z_1, \dots, z_n \in \mathbb{D}$ is defined by:

$$\kappa_\sigma = \sup_{|a_1|,\dots,|a_n|\le 1} \inf\{\|f\|_\infty \,;\, f \in H^\infty \text{ and } f(z_j) = a_j\,,\ 1 \le j \le n\}\,. \tag{2.2}$$

Then:

Lemma 1. *For every finite sequence $\sigma = (z_1, \dots, z_n)$ of distinct points $z_1, \dots, z_n \in \mathbb{D}$, one has:*

$$\kappa_\sigma^{-1} \Big\| \sum_{j=1}^n \lambda_j e_{z_j} \Big\| \le \Big\| \sum_{j=1}^n \omega_j \lambda_j e_{z_j} \Big\| \le \kappa_\sigma \Big\| \sum_{j=1}^n \lambda_j e_{z_j} \Big\| \tag{2.3}$$

for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and all complex numbers numbers $\omega_1, \dots, \omega_n$ such that $|\omega_1| = \cdots = |\omega_n| = 1$.

Proof. Set $L = \sum_{j=1}^n \lambda_j e_{z_j}$ and $L_\omega = \sum_{j=1}^n \omega_j \lambda_j e_{z_j}$. There exists $h \in H^\infty$ such that $\|h\|_\infty \le \kappa_\sigma$ and $h(z_j) = \omega_j$ for every $j = 1, \dots, n$. For every $g \in H^p$, one has $L_\omega(g) = \sum_{j=1}^n \omega_j \lambda_j g(z_j) = \sum_{j=1}^n h(z_j) \lambda_j g(z_j) = L(hg)$; hence:

$$|L_\omega(g)| \le \|L\|\,\|hg\|_p \le \|L\|\,\|h\|_\infty \|g\|_p \le \kappa_\sigma \|L\|\,\|g\|_p$$

and we get $\|L_\omega\| \le \kappa_\sigma \|L\|$, which is the right-hand side of (2.3). The left-hand side follows, by replacing $\lambda_1, \dots, \lambda_n$ by $\overline{\omega_1}\lambda_1, \dots, \overline{\omega_n}\lambda_n$. □

We now prove the following lower estimate.

Theorem 2. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ and $C_\varphi\colon H^p \to H^p$, with $1 \le p < \infty$. Let $u_1, \dots, u_n \in \mathbb{D}$ such that $v_1 = \varphi(u_1), \dots, v_n = \varphi(u_n)$ are distinct. Then, for some constant c_p depending only on p, we have:*

$$a_n(C_\varphi) \ge c_p\, \kappa_v^{-1} \Big(1 + \log \frac{1}{\delta_u}\Big)^{-1/\min(p,2)} \inf_{1 \le j \le n} \Big(\frac{1 - |u_j|^2}{1 - |v_j|^2}\Big)^{1/p}, \tag{2.4}$$

where δ_u is the uniform separation constant of the sequence $u = (u_1, \dots, u_n)$ and κ_v the interpolation constant of $v = (v_1, \dots, v_n)$.

For the proof, we need to know some precisions on the constant in Carleson's embedding theorem. Recall that the *uniform separation constant* δ_σ of a finite sequence $\sigma = (z_1, \dots, z_n)$ in the unit disk $\mathbb{D}$, is defined by:

$$\delta_\sigma = \inf_{1 \le j \le n} \prod_{k \ne j} \Big| \frac{z_j - z_k}{1 - \overline{z_j} z_k} \Big| \,. \tag{2.5}$$

Lemma 2. *Let $\sigma = (z_1, \dots, z_n)$ be a finite sequence of distinct points in $\mathbb{D}$ with uniform separation constant δ_σ. Then:*

$$\sum_{j=1}^n (1 - |z_j|^2)\, |f(z_j)|^p \le 12 \Big[1 + \log \frac{1}{\delta_\sigma}\Big] \|f\|_p^p \tag{2.6}$$

for all $f \in H^p$.

Proof. For $a \in \mathbb{D}$, let $k_a(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}$ be the normalized reproducing kernel. For every positive Borel measure μ on $\mathbb{D}$, let:

$$\gamma_\mu = \sup_{a \in \operatorname{supp}\mu} \int_{\mathbb{D}} |k_a(z)|^2 \, d\mu(z)\,.$$

The so-called Reproducing Kernel Thesis (see [14], Lecture VII, pp. 151–158) says that there is an absolute positive constant A_1 such that:

$$\int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \le A_1\, \gamma_\mu\, \|f\|_p^p$$

for every $f \in H^p$ (that follows from the case $p = 2$ in writing $f = Bh^{2/p}$ where B is a Blaschke product and $h \in H^2$). Actually, one can take $A_1 = 2\,\mathrm{e}$ (see [15], Theorem 0.2). But when μ is the discrete measure $\sum_{j=1}^n (1 - |z_j|^2)\,\delta_{z_j}$, it is not difficult to check (see [4], Lemma 1, p. 150, or [6], p. 201) that:

$$\gamma_\mu \leq 1 + 2\,\log\frac{1}{\delta_\sigma}\,\cdot$$

That gives the result since $4\,\mathrm{e} \leq 12$. □

Proof of Theorem 2. We will actually work with the Bernstein numbers of C_φ^*. Recall that they are defined in (1.5). That will suffice since $a_n(C_\varphi) \geq a_n(C_\varphi^*)$ (one has equality if C_φ is compact: see [7] or [2], pp. 89–91) and $a_n(C_\varphi^*) \geq b_n(C_\varphi^*)$.

Take $u_1, \dots, u_n \in \mathbb{D}$ such that $v_1 = \varphi(u_1), \dots, v_n = \varphi(u_n)$ are distinct. The points $u_1, \dots, u_n$ are then also distinct and the subspace $E = \operatorname{span}\{e_{u_1}, \dots, e_{u_n}\}$ of $(H^p)^*$ is n-dimensional. Let

$$L = \sum_{j=1}^n \lambda_j e_{u_j}$$

be in the unit sphere of E. We set, for $f \in H^p$ and for $j = 1, \dots, n$:

$$\Lambda_j = \lambda_j\,\|e_{u_j}\|\,, \quad \text{and} \quad F_j = \|e_{u_j}\|^{-1} f(u_j)\,,$$

and finally:

$$\Lambda = (\Lambda_1, \dots, \Lambda_n) \quad \text{and} \quad F = (F_1, \dots, F_n)\,.$$

We will separate three cases.

Case 1: $1 < p \leq 2$.

One has $\|C_\varphi^*(L)\| = \left\| \sum_{j=1}^n \lambda_j\, e_{v_j} \right\|$. Using Lemma 1, we obtain for any choice of complex signs $\omega_1, \dots, \omega_n$:

$$\|C_\varphi^*(L)\| \geq \kappa_V^{-1} \left\| \sum_{j=1}^n \omega_j \lambda_j e_{v_j} \right\|. \tag{2.7}$$

Let now q be the conjugate exponent of p. We know that the space H^p is of type p as a subspace of L^p ([9], p. 169) and therefore its dual $(H^p)^*$ is of cotype q ([9], p. 165), with cotype constant $\leq \tau_p$, the type p constant of L^p (let us note that we might use that $(H^p)^*$ is isomorphic to the subspace H^q of L^q, but we have then to introduce the constant of this isomorphism). Hence, by averaging (2.7) over all independent choices of signs and using the cotype q property of $(H^p)^*$, we get:

$$\|C_\varphi^*(L)\| \geq \tau_p^{-1}\,\kappa_V^{-1} \left(\sum_{j=1}^n |\lambda_j|^q \|e_{v_j}\|^q \right)^{1/q} \geq \tau_p^{-1}\,\kappa_V^{-1} \mu_n \left(\sum_{j=1}^n |\lambda_j|^q \|e_{u_j}\|^q \right)^{1/q},$$

so that

$$\|C_\varphi^*(L)\| \geq \tau_p^{-1}\,\kappa_V^{-1} \mu_n\, \|\Lambda\|_q\,, \tag{2.8}$$

where:

$$\mu_n = \inf_{1 \leq j \leq n} \frac{\|e_{v_j}\|}{\|e_{u_j}\|} = \inf_{1 \leq j \leq n} \left(\frac{1 - |u_j|^2}{1 - |v_j|^2} \right)^{1/p}.$$

It remains to give a lower bound for $\|\Lambda\|_q$.

But, by Hölder's inequality:

$$|L(f)| = \left| \sum_{j=1}^n \lambda_j f(u_j) \right| = \left| \sum_{j=1}^n \Lambda_j F_j \right| \leq \|\Lambda\|_q \|F\|_p\,.$$

Since

$$\|F\|_p^p = \sum_{j=1}^n \|e_{u_j}\|^{-p} |f(u_j)|^p = \sum_{j=1}^n (1 - |u_j|^2)\,|f(u_j)|^p\,,$$

Lemma 2 gives:

$$|L(f)| \le ||\Lambda||_q \left[12\left(1+\log\frac{1}{\delta_u}\right)\right]^{1/p} ||f||_p .$$

Taking the supremum over all f with $||f||_p \le 1$, we get, taking into account that $||L|| = 1$:

$$||\Lambda||_q \ge \left[12\left(1+\log\frac{1}{\delta_u}\right)\right]^{-1/p} . \tag{2.9}$$

By combining (2.8) and (2.9), we get:

$$||C_\varphi^\star(L)|| \ge (12)^{-1/p}\, \tau_p^{-1}\, \mu_n\, \kappa_v^{-1}\left(1+\log\frac{1}{\delta_u}\right)^{-1/p} .$$

Therefore:

$$b_n(C_\varphi^\star) \ge (12)^{-1/p}\, \tau_p^{-1}\, \mu_n\, \kappa_v^{-1}\left(1+\log\frac{1}{\delta_u}\right)^{-1/p} .$$

Case 2: $2 < p < \infty$.

We follow the same route, but in this case, H^p is of type 2 and hence $(H^p)^\star$ is of cotype 2. Therefore, we get:

$$||C_\varphi^\star(L)|| \ge \tau_2^{-1}\, \kappa_v^{-1}\, \mu_n\, ||\Lambda||_2 \tag{2.10}$$

and, using Cauchy-Schwarz inequality:

$$||\Lambda||_2 \ge \left[12\left(1+\log\frac{1}{\delta_u}\right)\right]^{-1/2} ; \tag{2.11}$$

so:

$$||C_\varphi^\star(L)|| \ge (12)^{-1/2}\, \tau_2^{-1}\, \mu_n\, \kappa_v^{-1}\left(1+\log\frac{1}{\delta_u}\right)^{-1/2} . \tag{2.12}$$

Case 3: $p = 1$.

In this case $(H^1)^\star$ (which is isomorphic to the space $BMOA$) has no finite cotype. But, for each $k = 1, \ldots, n$, one has, using Lemma 1:

$$\begin{aligned} |\lambda_k|\, ||e_{v_k}|| &= \frac{1}{2}\left\| \left(\sum_{j\neq k} \lambda_j e_{v_j} + \lambda_k e_{v_k}\right) - \left(\sum_{j\neq k} \lambda_j e_{v_j} - \lambda_k e_{v_k}\right)\right\| \\ &\le \frac{1}{2}\left(\left\| \sum_{j\neq k} \lambda_j e_{v_j} + \lambda_k e_{v_k}\right\| + \left\| \sum_{j\neq k} \lambda_j e_{v_j} - \lambda_k e_{v_k}\right\|\right) \\ &\le \kappa_v \left\| \sum_{j=1}^n \lambda_j e_{v_j}\right\| ; \end{aligned}$$

hence:

$$||C_\varphi^\star(L)|| \ge \kappa_v^{-1}\, \mu_n\, ||\Lambda||_\infty . \tag{2.13}$$

Since $|L(F)| \le ||\Lambda||_\infty ||F||_1$, we get, as above, using Lemma 2:

$$||\Lambda||_\infty \ge \left[12\left(1+\log\frac{1}{\delta_u}\right)\right]^{-1} , \tag{2.14}$$

and therefore:

$$||C_\varphi^\star(L)|| \ge (12)^{-1}\, \mu_n\, \kappa_v^{-1}\left(1+\log\frac{1}{\delta_u}\right)^{-1} \tag{2.15}$$

and that finishes the proof of Theorem 2. □

Example. We will now apply this result to lens maps. We refer to [19] or [8] for their definition. For $\theta \in (0, 1)$, we denote:

$$\lambda_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta}\,. \tag{2.16}$$

Proposition 2. *Let λ_θ be the lens map of parameter θ acting on H^p, with $1 \leq p < \infty$. Then, for positive constants a and b, depending only on θ and p:*

$$a_n(C_{\lambda_\theta}) \geq a\,\mathrm{e}^{-b\sqrt{n}}.$$

Actually, this estimate is valid for polygonal maps as well.

Proof. Let $0 < \sigma < 1$ and consider $u_j = 1 - \sigma^j$ and $v_j = \lambda_\theta(u_j)$, $1 \leq j \leq n$. We know from [10], Lemma 6.4 and Lemma 6.5, that, for $\alpha = \frac{\pi^2}{2}$ and $\beta = \beta_\theta = \frac{\pi^2}{2^\theta \theta}$:

$$\delta_u \geq \mathrm{e}^{-\alpha/(1-\sigma)} \quad \text{and} \quad \delta_v \geq \mathrm{e}^{-\beta/(1-\sigma)}.$$

But we know that the interpolation constant κ_s of the finite sequence s is related to its uniform separation constant δ_s by the following inequality ([5] page 278), in which Λ is a positive numerical constant:

$$\frac{1}{\delta_s} \leq \kappa_s \leq \frac{\Lambda}{\delta_s}\left(1 + \log\frac{1}{\delta_s}\right). \tag{2.17}$$

Actually, S. A. Vinogradov, E. A. Gorin and S. V. Hruščëv [21] (see [13], p. 505) proved that

$$\kappa_s \leq \frac{2\,\mathrm{e}}{\delta_s}\left(1 + 2\log\frac{1}{\delta_s}\right),$$

so we can take $\Lambda \leq 4\,\mathrm{e} \leq 12$.

It follows that

$$\kappa_v^{-1} \geq \frac{1-\sigma}{\Lambda(\beta+1)}\,\mathrm{e}^{-\beta/(1-\sigma)}. \tag{2.18}$$

Setting $\tilde{p} = \min(p, 2)$, we have:

$$\left(1 + \log\frac{1}{\delta_u}\right)^{-1/\tilde{p}} \geq \left(\frac{1-\sigma}{\alpha+1}\right)^{1/\tilde{p}}. \tag{2.19}$$

We now estimate $\inf_{1\leq j\leq n}\left(\frac{1-|u_j|^2}{1-|v_j|^2}\right)^{1/p} = \mu_n$.

Since $\lambda_\theta(0) = 0$, Schwarz's lemma says that $|\lambda_\theta(z)| \leq |z|$; hence $\frac{1-|z|^2}{1-|\lambda_\theta(z)|^2} \geq \frac{1-|z|}{1-|\lambda_\theta(z)|}$. But $1 - v_j = 1 - \lambda_\theta(u_j) = \frac{2\sigma^{j\theta}}{(2-\sigma^j)^\theta + \sigma^{j\theta}}$; hence (since u_j and v_j are real):

$$\frac{1-|u_j|^2}{1-|v_j|^2} \geq \frac{1-u_j}{1-v_j} = \frac{\sigma^j}{2\sigma^{j\theta}}\left[(2-\sigma^j)^\theta + \sigma^{j\theta}\right].$$

Since the function $f(x) = (2-x)^\theta + x^\theta$ increases on $[0, 1]$, one gets:

$$\frac{1-|u_j|^2}{1-|v_j|^2} \geq \left(\frac{1}{2}\sigma^j\right)^{1-\theta},$$

and therefore:

$$\mu_n \geq \left(\frac{1}{2}\sigma^n\right)^{(1-\theta)/p}. \tag{2.20}$$

Applying now Theorem 2 and using (2.18), (2.19) and (2.20), we get:

$$a_n(C_{\lambda_\theta}) \geq \alpha_{p,\theta}\,\mathrm{e}^{-\beta/(1-\sigma)}\,(1-\sigma)^{1/\tilde{p}}\,\sigma^{n(1-\theta)/p}$$

with $\alpha_{p,\theta} = \dfrac{c_p}{\Lambda(\beta+1)(\alpha+1)^{1/\tilde{p}}\,2^{(1-\theta)/p}}$.

Taking $\sigma = \mathrm{e}^{-\varepsilon}$ where $0 < \varepsilon < 1$, we get, since $1 - \mathrm{e}^{-\varepsilon} \geq \varepsilon/2$:

$$a_n(C_{\lambda_\theta}) \geq \alpha_{p,\theta}\, \mathrm{e}^{-2\beta/\varepsilon} \left(\frac{\varepsilon}{2}\right)^{1/\tilde{p}} \mathrm{e}^{-\varepsilon n(1-\theta)/p}.$$

Optimizing by taking $\varepsilon = \sqrt{\frac{3\beta p}{1-\theta}}\,\frac{1}{\sqrt{n}}$ gives, for n large enough (in order to have $\varepsilon < 1$):

$$a_n(C_{\lambda_\theta}) \geq \alpha'_{p,\theta}\, n^{-1/(2\tilde{p})}\, \mathrm{e}^{-\beta_{p,\theta}\sqrt{n}} \tag{2.21}$$

with $\alpha'_{p,\theta} = \alpha_{p,\theta}\Big(\frac{\beta p}{2(1-\theta)}\Big)^{1/(2\tilde{p})}$ and $\beta_{p,\theta} = \sqrt{\frac{2\beta(1-\theta)}{p}}$.

We get Theorem 2, with $b > \beta_{p,\theta}$. □

Let us note that $\beta_{p,\theta} = \frac{2^{\frac{1-\theta}{2}}\pi}{\sqrt{p}}\sqrt{\frac{1-\theta}{\theta}}$ tends to 0 when θ goes to 1 and tends to infinity when θ goes to 0.

2.3 A minoration depending on the radial behaviour of φ

We are using Theorem 2 to give, as in [11], Theorem 3.2, a lower bound for $a_n(C_\varphi)$ which depends on the behaviour of φ near $\partial\mathbb{D}$.

We recall first (see [11], Section 3) that an analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ is said to be *real* if it takes real values on $]-1,1[$. If $\omega\colon [0,1] \to [0,2]$ is a modulus of continuity (meaning that ω is continuous, increasing, sub-additive, vanishing at 0, and concave), φ is said to be an *ω-radial symbol* if it is real and:

$$1 - \varphi(r) \leq \omega(1-r)\,, \quad 0 \leq r < 1\,. \tag{2.22}$$

We have the following result.

Theorem 3. *Let φ be an ω-radial symbol. Then, for $1 \leq p < \infty$, the approximation numbers of the composition operator $C_\varphi\colon H^p \to H^p$ satisfy:*

$$a_n(C_\varphi) \geq c'_p \sup_{0<\sigma<1} \left[\left(\frac{\omega^{-1}(a\,\sigma^n)}{a\,\sigma^n}\right)^{1/p} (1-\sigma)^{1/\max(p^*,2)} \exp\left(-\frac{5}{1-\sigma}\right)\right], \tag{2.23}$$

where c'_p is a constant depending only on p, p^* is the conjugate exponent of p, and $a = 1 - \varphi(0) > 0$.

Proof. As in [11], p. 556, we fix $0 < \sigma < 1$ and define inductively $u_j \in [0,1)$ by $u_0 = 0$ and, using the intermediate value theorem:

$$1 - \varphi(u_{j+1}) = \sigma\,[1 - \varphi(u_j)]\,, \quad \text{with } 1 > u_{j+1} > u_j\,.$$

We set $v_j = \varphi(u_j)$. We have $-1 < v_j < 1$ and $1 - v_n = a\,\sigma^n$. We proved in [11], p. 556, that:

$$\frac{1-|u_j|^2}{1-|v_j|^2} \geq \frac{1}{2}\,\frac{\omega^{-1}(a\,\sigma^n)}{a\,\sigma^n}\,. \tag{2.24}$$

Moreover, we proved in [11], p. 557, that the uniform separation constant of $v = (v_1, \ldots, v_n)$ is such that:

$$\delta_v \geq \exp\left(-\frac{5}{1-\sigma}\right). \tag{2.25}$$

Since $\delta_u \geq \delta_v$, we get, from (2.17), that:

$$\kappa_u \leq 12\left(\frac{6-\sigma}{1-\sigma}\right) \exp\left(\frac{5}{1-\sigma}\right) \leq 60\left(\frac{1}{1-\sigma}\right) \exp\left(\frac{5}{1-\sigma}\right). \tag{2.26}$$

Using now (2.4) of Theorem 2 and combining (2.24), (2.25) and (2.26), we get Theorem 3. □

Example 1: lens maps. Let us come back to the lens maps λ_θ for testing Theorem 3. We have $\omega^{-1}(h) \approx h^{1/\theta}$ (see [8], Lemma 2.5) and $a = 1 - \lambda_\theta(0) = 1$. Setting $K = \frac{1}{10\sqrt{p}}\sqrt{\frac{1-\theta}{\theta}}$ and taking, for n large enough, $\sigma = 1 - \frac{1}{K\sqrt{n}}$, we have, using that $\mathrm{e}^{-s} \leq 1 - \frac{4}{5}s$ for $s > 0$ small enough, $\sigma^n \geq \exp\big(-\frac{5}{4K}\sqrt{n}\big)$ and hence:

$$a_n(C_{\lambda_\theta}) \geq c_{\theta,p}\, n^{-\frac{1}{2\max(p^*,2)}} \exp\left[-\frac{5}{\sqrt{p}}\sqrt{\frac{1-\theta}{\theta}}\sqrt{n}\right].$$

Note that the coefficient of $\sqrt{n}$ in the exponential is slightly different of that in (2.21), but of the same order.

Example 2: cusp map. We refer to [11], Section 4, for its definition and properties. It is the conformal mapping χ from $\mathbb{D}$ onto the domain represented on Fig. 1 such that $\chi(1) = 1$, $\chi(-1) = 0$, $\chi(i) = (1+i)/2$ and $\chi(-i) = (1-i)/2$. We proved in [11], Lemma 4.2, that, for $0 \leq r < 1$, one has:

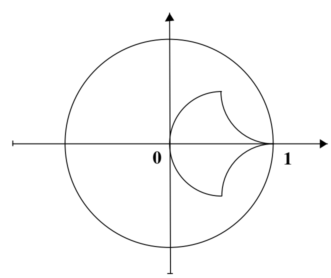

Figure 1: *Cusp map domain*

$$1 - \chi(r) = \frac{1}{1 + \frac{2}{\pi}\log\left[1/2\,\arctan\left(\frac{1-r}{1+r}\right)\right]}.$$

Since $1 - \frac{2}{\pi}\log 2 > 0$ and $\arctan x \leq x$ for $x \geq 0$, we get that:

$$1 - \chi(r) \leq \frac{\pi}{2}\,\frac{1}{\log\left(\frac{1+r}{1-r}\right)} \leq \frac{\pi}{2}\,\frac{1}{\log\left(\frac{1}{1-r}\right)} \leq 2\,\frac{1}{\log\left(\frac{1}{1-r}\right)}.$$

Hence χ is an ω-radial symbol with $\omega(x) = 2/\log(1/x)$. Then $\omega^{-1}(h) = \mathrm{e}^{-2/h}$. By choosing $\sigma = 1 - \frac{\log n}{4n}$ in (2.23), we get, using that $\log(1-x) \geq -2x$ for $x > 0$ small enough, that, for n large enough, $\sigma^n \geq 1/\sqrt{n}$; hence:

$$a_n(C_\chi) \geq c''_p\left(\sqrt{n}\exp\left[-(2\,a)\sqrt{n}\right]\right)^{1/p}\left(\frac{\log n}{n}\right)^{1/\max(p^*,2)} \exp\left(-\frac{20n}{\log n}\right).$$

It follows that, for some constant $C_p > 0$ depending only on p, we have:

$$a_n(C_\chi) \geq C_p \exp\left(-\frac{25n}{\log n}\right). \tag{2.27}$$

It has to be stressed that the term in the exponential does not depend on p.

Example 3: Shapiro-Taylor's maps. These maps ς_θ, for $\theta > 0$, were defined in [20]. Let us recall their definition. For $\varepsilon > 0$, we set $V_\varepsilon = \{z \in \mathbb{C}\,;\ \Re z > 0 \text{ and } |z| < \varepsilon\}$. For $\varepsilon = \varepsilon_\theta > 0$ small enough, one can define

$$f_\theta(z) = z(-\log z)^\theta, \tag{2.28}$$

for $z \in V_\varepsilon$, where $\log z$ will be the principal determination of the logarithm. Let now g_θ be the conformal mapping from $\mathbb{D}$ onto V_ε, which maps $\mathbb{T} = \partial\mathbb{D}$ onto ∂V_ε, defined by $g_\theta(z) = \varepsilon\,\varphi_0(z)$, where φ_0 is the conformal map from $\mathbb{D}$ onto V_1, given by:

$$\varphi_0(z) = \frac{\left(\frac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\frac{z-i}{iz-1}\right)^{1/2} + 1}\,. \tag{2.29}$$

Then, we define:

$$\varsigma_\theta = \exp(-f_\theta \circ g_\theta). \tag{2.30}$$

We saw in [11], p. 560, that $\omega^{-1}(h) = K_\theta\, h\big(\log(1/h)\big)^{-\theta}$. Hence, choosing $\sigma = 1/(\mathrm{e}\,\alpha_\theta^{1/n})$, where $\alpha_\theta = 1 - \varsigma_\theta(0)$, we get that:

$$a_n(C_{\varsigma_\theta}) \geq c_{p,\theta}.\frac{1}{n^{\theta/2p}}\,. \tag{2.31}$$

However, we already remarked in [11], Section 4.2, that, even for $p = 2$, this result is not optimal.

3 Upper bound

For upper bounds, there is essentially no change with regard to the case $p = 2$. Hence we essentially only state some results.

We have the following upper bound, which can be obtained with the same proof as in [8].

Theorem 4. *Let $C_\varphi \colon H^p \to H^p$, $1 \leq p < \infty$, a composition operator, and $n \geq 1$. Then, for every Blaschke product B with (strictly) less than n zeros, each counted with its multiplicity, one has, for some universal constant C:*

$$a_n(C_\varphi) \leq C\sqrt{n}\left(\sup_{\substack{0<h<1\\ \xi\in\mathbb{T}}} \frac{1}{h}\int_{S(\xi,h)} |B|^p\, dm_\varphi\right)^{1/p},$$

where m_φ is the pullback measure of m, the normalized Lebesgue measure on $\mathbb{T}$, under φ and $S(\xi, h) = \mathbb{D} \cap D(\xi, h)$ is the Carleson window of size h centered at $\xi \in \mathbb{T}$.

Proof. We first estimate the Gelfand number $c_n(C_\varphi)$ by restricting to the subspace BH^p which is of codimension $< n$. As in [8], Lemma 2.4:

$$c_n(C_\varphi) \lesssim \left(\sup_{\substack{0<h<1\\ \xi\in\mathbb{T}}} \frac{1}{h}\int_{S(\xi,h)} |B|^p\, dm_\varphi\right)^{1/p}.$$

Now (see [2], Proposition 2.4.3), one has $a_n(C_\varphi) \leq \sqrt{2n}\, c_n(C_\varphi)$, hence the result. □

Recall ([11], Definition 2.2) that a symbol $\varphi \in A(\mathbb{D})$ (i.e. $\varphi \colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is continuous and analytic in $\mathbb{D}$) is said to be *globally regular* if $\varphi(\overline{\mathbb{D}}) \cap \partial\mathbb{D} = \{\xi_1, \ldots, \xi_l\}$ and there exists a modulus of continuity ω (i.e. a continuous, increasing and sub-additive function $\omega \colon [0, A] \to \mathbb{R}^+$, which vanishes at zero, and that we may assume to be concave), such that, writing $E_{\xi_j} = \{t\,;\, \gamma(t) = \xi_j\}$, one has $\mathbb{T} = \bigcup_{j=1}^{l} \big(E_{\xi_j} + [-r_j, r_j]\big)$ for some $r_1, \ldots, r_l > 0$, and for some positive constants $C, c > 0$:

$$|\gamma(t) - \gamma(t_j)| \leq C\big(1 - |\gamma(t)|\big) \tag{3.1}$$

$$c\ \omega(|t - t_j|) \leq |\gamma(t) - \gamma(t_j)| \tag{3.2}$$

for $j = 1, \ldots, l$, all $t_j \in E_{\xi_j}$ with $|t - t_j| \leq r_j$.

We can then deduce of Theorem 4 the following version of [11], Theorem 2.3, with the same proof.

Theorem 5. *Let φ be a symbol in $A(\mathbb{D})$ whose image touches $\partial\mathbb{D}$ exactly at the points $\xi_1, \ldots, \xi_l$ and which is globally-regular. Then there are constants $\kappa, K, L > 0$, depending only on φ, such that, for every $k \geq 1$:*

$$a_k(C_\varphi) \leq K \left[\frac{\omega^{-1}(\kappa\, 2^{-N_k})}{\kappa\, 2^{-N_k}}\right]^{1/p}, \tag{3.3}$$

where N_k is the largest integer such that $lNd_N < k$ and d_N is the integer part of $\left[\log \frac{\kappa\, 2^{-N}}{\omega^{-1}(\kappa\, 2^{-N})} \Big/ \log(\chi^{-p})\right] + 1$, with $0 < \chi < 1$ an absolute constant.

As a corollary, we get for lens maps λ_θ (as well as for polygonal maps), in the same way as Theorem 2.4 in [11], p. 550 (recall that then $\omega(h) \approx h^\theta$), the following upper bound.

Theorem 6. *Let $\varphi = \lambda_\theta$ be the lens map of parameter θ acting on H^p, $1 < p < \infty$. Then, for positive constants b and c depending only on θ and p:*

$$a_n(C_{\lambda_\theta}) \leq c\, \mathrm{e}^{-b\sqrt{n}}.$$

For the cusp map, we also have as in [11], Theorem 4.3 (here, $\omega(h) \approx 1/\log(1/h)$).

Theorem 7. *Let $\varphi = \chi$ be the cusp map. For some positive constants b and c depending only on p, one has:*

$$a_n(C_\chi) \leq c\, \mathrm{e}^{-b\, n/\log n}.$$

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