Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org
pp. **X–XX**

NULL-EXACT CONTROLLABILITY OF A SEMILINEAR CASCADE SYSTEM OF PARABOLIC-HYPERBOLIC EQUATIONS

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(Communicated by Martino Bardi)

Abstract. This paper is concerned with the null-exact controllability of a cascade system formed by a semilinear heat and a semilinear wave equation in a cylinder $\Omega\times(0,T)$. More precisely, we intend to drive the solution of the heat equation (resp. the wave equation) exactly to zero (resp. exactly to a prescribed but arbitrary final state). The control acts only on the heat equation and is supported by a set of the form $\omega\times(0,T)$, where $\omega\subset\Omega$. In the wave equation, the restriction of the solution to the heat equation to another set $\mathcal{O}\times(0,T)$ appears. The nonlinear terms are assumed to be globally Lipschitz-continuous. In the main result in this paper, we show that, under appropriate assumptions on T, ω and $\mathcal{O}$, the equations are simultaneously controllable.

1. **Introduction. The main result.** Let $\Omega\subset\mathbb{R}^N$ be a bounded domain of class C^2 $(N\geq 1)$. Let ω and $\mathcal{O}$ be two nonempty open subsets of Ω. Let $T>0$ and set $Q=\Omega\times(0,T)$ and $\Sigma=\partial\Omega\times(0,T)$. We will consider the following cascade system:

$$\left\{\begin{array}{ll} y_t-\Delta y+f_1(x,t;y,q)=h_\omega & \text{in } Q,\\ y=0 & \text{on } \Sigma,\\ y(x,0)=y^0(x) & \text{in } \Omega, \end{array}\right. \tag{1}$$

$$\left\{\begin{array}{ll} q_{tt}-\Delta q+f_2(x,t;q)=y\,\mathbb{1}_{\mathcal{O}} & \text{in } Q,\\ q=0 & \text{on } \Sigma,\\ q(x,0)=q^0(x),\quad q_t(x,0)=q^1(x) & \text{in } \Omega. \end{array}\right. \tag{2}$$

Here, y^0 and (q^0,q^1) are given, h_ω is a control with support in $\overline{\omega}\times[0,T]$, $\mathbb{1}_{\mathcal{O}}$ is the characteristic function of the set $\mathcal{O}$ and f_1 and f_2 are appropriate Carathéodory functions (measurable in (x,t) and continuous in the other variables).

We address the following question:

2000 *Mathematics Subject Classification.* 35M20, 93B05, 93B07.

Key words and phrases. Semilinear systems, parabolic-hyperbolic equations, controllability, observability inequalities.

Supported by grant BFM2003-06446 of the D.G.E.S. (Spain) and by project IN102799 of D.G.A.P.A. (Mexico).

Under which assumptions on T, ω, $\mathcal{O}$, f_1, f_2, y^0, (q^0, q^1) and (r^0, r^1) does there exist a control h_ω supported by $\overline{\omega} \times [0, T]$ such that the corresponding solution of (1)–(2) satisfies simultaneously

$$y(x, T) = 0, \quad q(x, T) = r^0(x) \quad \text{and} \quad q_t(x, T) = r^1(x) \quad \text{in} \quad \Omega? \tag{3}$$

The physical situation described by (1)–(2) is the following. We are assuming that Ω is a N-dimensional medium whose particles are heat-conducting and reacting and, at the same time, can propagate waves. The initial temperature distribution y^0 and the initial and final vibrations (q^0, q^1) and (r^0, r^1) are given. We also assume that a heat source h_ω at our disposal can be applied on $\omega \times (0, T)$. Finally, it is accepted that the temperature on $\mathcal{O}$ behaves as a source of vibrations for all $t \in (0, T)$. Hence, the question is whether we can choose the heat source h_ω so as to vanish the temperature and get desired vibrations (r^0, r^1) at time $t = T$.

This system and this control question can be viewed as a first step in the analysis of other more complex and realistic situations. Thus, in forthcoming papers, we will be concerned with

• Cascade Navier-Stokes-Lamé systems of the form

$$\begin{cases} y_t + (y \cdot \nabla) y - \nu \Delta y + \nabla p = h_\omega, \quad \nabla \cdot y = 0, \\ q_{tt} - \mu \Delta q - \lambda \nabla (\nabla \cdot q) = y \, 1\!\!1_{\mathcal{O}}, \end{cases}$$

again completed with initial and boundary conditions for y and q.

• Cascade heat-wave (or Navier-Stokes-Lamé) systems in different domains. For instance, if $\Omega = G \times (0, L)$ where $G \subset \mathbb{R}^2$ is a bounded regular domain, we may consider the system

$$\begin{cases} y_t - y_{x_1 x_1} - y_{x_2 x_2} - y_{x_3 x_3} = h_\omega & \text{in } \Omega \times (0, T), \\ q_{tt} - q_{x_1 x_1} - q_{x_2 x_2} = y|_{x_3 = 0} \, 1\!\!1_{\mathcal{O}} & \text{in } G \times (0, T), \end{cases}$$

where $\omega \subset \Omega$ and $\mathcal{O} \subset G$.

Our aim is to understand and explain the control mechanisms for (1)–(2). We believe that this will be useful to deal with similar controllability questions for the previous systems.

Observe that, in (3), we are concerned with a *null-exact controllability* problem. However, the control acts in the equation satisfied by q indirectly through the variable y and, accordingly, the question under consideration is more intricate than in the standard situation of the exact controllability problem of the classical wave equation. In order to deal with the controllability properties of system (1)–(2), an additional assumption must be imposed on $\omega \cap \mathcal{O}$; see (9). In particular, this assumption implies that $\omega \cap \mathcal{O} \neq \emptyset$.

It will be convenient to introduce several functions, sets and spaces. Let $x^0 \in \mathbb{R}^N$. We set $m(x) = x - x^0$,

$$\Gamma(x^0) = \{ \, x \in \partial\Omega : m(x) \cdot \nu(x) > 0 \, \},$$

where $\nu(x)$ denotes the unit outwards normal vector to $\partial\Omega$ at x,

$$\Sigma(x^0) = \Gamma(x^0) \times (0, T) \quad \text{and} \quad R(x^0) = \max_{x \in \bar{\Omega}} |m(x)|. \tag{4}$$

Let $\delta > 0$ be given. We will consider the sets

$$\mathcal{B}_\delta(x^0) = \bigcup_{x \in \Gamma(x^0)} B(x; \delta) \quad \text{and} \quad \mathcal{G}_\delta(x^0) = \mathcal{B}_\delta(x^0) \cap \Omega, \tag{5}$$

where $B(x;\delta)$ is the ball centered at x of radius δ. Finally, we will use the Hilbert space $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, for instance endowed with the norm

$$\|v\|_{D(-\Delta)} = \left(\int_\Omega (|\Delta v|^2 + |v|^2)\, dx \right)^{1/2}.$$

For each $f \in H^{-1}(\Omega)$, we will denote by u_f the solution to the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, we will consider the following scalar product in $H^{-1}(\Omega)$:

$$(f,g)_{H^{-1}} = (u_f, u_g)_{H_0^1} = \int_\Omega \nabla u_f \cdot \nabla u_g \, dx \quad \forall f, g \in H^{-1}(\Omega).$$

Notice that the norm $\| \cdot \|_{H^{-1}}$ induced by $(\cdot\,,\cdot)_{H^{-1}}$ is also the norm associated to $\| \cdot \|_{H_0^1}$ by duality.

We will assume that the function $f_1 = f_1(x,t;s,r)$ is globally Lipschitz in the variable (s,r) and satisfies

$$|f_1(x,t;s,r)| \le C|s| \quad \forall (x,t;s,r) \in Q \times \mathbb{R}^2 \tag{6}$$

for some $C > 0$. We will also assume that the function $f_2 = f_2(x,t;r)$ satisfies

$$f_2(\cdot\,,\cdot\,;0) \in L^2(Q) \tag{7}$$

and is globally Lipschitz in the variable r:

$$|f_2(x,t;r') - f_2(x,t;r)| \le C|r'-r| \quad \forall (x,t;r), (x,t;r') \in Q \times \mathbb{R}. \tag{8}$$

Our main result is the following:

Theorem 1. *Assume that, for some $x^0 \in \mathbb{R}^N$ and some $\delta > 0$, there exists a set of the form $\mathcal{G}_\delta(x^0)$ satisfying*

$$\mathcal{G}_\delta(x^0) \subset \omega \cap \mathcal{O}. \tag{9}$$

Assume that $T > 2R(x^0)$ and f_1 and f_2 are globally Lipschitz-continuous and satisfy (6)*–*(8)*. Then, for any $y^0 \in H^{-1}(\Omega)$, $(q^0, q^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $(r^0, r^1) \in H_0^1(\Omega) \times L^2(\Omega)$ there exist controls h_ω in the space $L^2(0,T;D(-\Delta)')$ with* Supp $h_\omega \subset$ *$\overline{\omega} \times [0,T]$ and associated solutions $(y,q) \in L^2(Q) \times C^0([0,T];H_0^1(\Omega))$ to* (1)*–*(2) *that satisfy* (3)*.*

Remark 1. In order to prove theorem 1, a fixed point argument will be performed. In particular, we will see that the couple (y,q) satisfies $y \in C^0([0,T];H^{-1}(\Omega))$, and $q \in C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ and solves the system (11)–(12) for some appropriate h_ω and $a, b \in L^\infty(Q)$ (which depend on y and q). We will see that y is a *solution by transposition* of (1) (for the definition of solution by transposition, see subsection 2.1) and the equalities in (3) are satisfied in $H^{-1}(\Omega)$, $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively. ■

Remark 2. It may seem that the regularity of h_ω is not satisfactory. However, it is clear that, in order to get the exact controllability in $H_0^1(\Omega) \times L^2(\Omega)$ of (2), $y\mathbb{1}_\mathcal{O}$ must not be better than $L^2(Q)$ and consequently h_ω must not be better than $L^2(0,T;D(-\Delta)')$. Accordingly, the previous assertion is reasonable. ■

Remark 3. In the particular case $f_1 \equiv f_2 \equiv 0$, the controllability properties of the cascade system (1)–(2) were analyzed in [4]. There, a result very similar to theorem 1 was proved. ■

Remark 4. It is well known that, under the assumptions $\mathcal{G}_\delta(x^0) \subset \omega$ and $T > 2R(x^0)$, the classical wave equation is exactly controllable with L^2 controls supported by $\overline{\omega} \times [0,T]$. In other words, for any $(v^0, v^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exist controls $f \in L^2(\omega \times (0,T))$ such that the solution of the associated system

$$\begin{cases} v_{tt} - \Delta v = f\,\mathbb{1}_\omega & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(x,0) = v^0(x), \quad v_t(x,0) = v^1(x) & \text{in } \Omega, \end{cases} \tag{10}$$

satisfies

$$v(x,T) = 0, \quad v_t(x,T) = 0.$$

This is a consequence of an observability estimate that will be recalled below, see [10]. On the other hand, (10) is not exactly controllable in general. The precise necessary and sufficient conditions on ω and T that guarantee exact controllability are given in [2] (more details will be recalled in Section 4). Therefore, the hypotheses on ω, Ω and T in theorem 1 are, at first sight, appropriate. ■

The proof of theorem 1 is divided in two parts. We will first prove the null-exact controllability of similar cascade linear systems with potentials $a, b \in L^\infty(Q)$ and source $g \in L^2(Q)$:

$$\begin{cases} y_t - \Delta y + a(x,t)y = h_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{11}$$

$$\begin{cases} q_{tt} - \Delta q + b(x,t)q = y\,\mathbb{1}_{\mathcal{O}} + g(x,t) & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(x,0) = q^0(x), \quad q_t(x,0) = q^1(x) & \text{in } \Omega. \end{cases} \tag{12}$$

More precisely, the following result will be established:

Theorem 2. *Let $a, b \in L^\infty(Q)$ and $g \in L^2(Q)$. Under the assumptions of theorem 1 there exist a positive constant $C = C(\|a\|_\infty, \|b\|_\infty, \|g\|_{L^2(Q)}, \omega, \mathcal{O}, \Omega, T)$ and controls $h_\omega \in L^2(0,T; D(-\Delta)')$, with* Supp $h_\omega \subset \overline{\omega} \times [0,T]$ *and*

$$\|h_\omega\|_{L^2(0,T;D(-\Delta)')} \le C\, \|(y^0, q^0 - \tilde{q}^0, q^1 - \tilde{q}^1)\|_{H^{-1} \times H_0^1 \times L^2}, \tag{13}$$

such that the corresponding solutions (y, q) to (11)–(12) satisfy (3). In (13),

$$(\tilde{q}^0, \tilde{q}^1) = (\tilde{q}, \tilde{q}_t)(\cdot, 0),$$

where $\tilde{q}$ is the solution of the uncontrolled system

$$\begin{cases} \tilde{q}_{tt} - \Delta \tilde{q} + b(x,t)\tilde{q} = g(x,t) & \text{in } Q, \\ \tilde{q} = 0 & \text{on } \Sigma, \\ \tilde{q}(x,T) = r^0(x), \quad \tilde{q}_t(x,T) = r^1(x) & \text{in } \Omega. \end{cases} \tag{14}$$

In a second step, using a fixed point argument we will obtain the desired controllability result for the nonlinear system.

Remark 5. The lack of regularity of the control provided by theorem 2 introduces some technical difficulties in our analysis. To be precise, the fixed point argument will be formulated in $L^2(0,T; H^{-1}(\Omega) \times L^2(\Omega))$ and consequently, in order to define a set-valued mapping we need to apply a regularization process. The fixed point argument does not lead directly to the solution. To obtain our result, we still have to absorb the non regular part of the limit in the control (see section 3). ■

The proof of theorem 2 is based on the existence of a positive constant $C = C(\|a\|_\infty, \|b\|_\infty, \omega, \mathcal{O}, \Omega, T)$ such that the observability inequality

$$\|(z,p,p_t)(\cdot,0)\|^2_{H^1_0\times L^2\times H^{-1}} \leq C\int_0^T\int_\Omega \rho_\omega\left(|\Delta z|^2+|z|^2\right)\,dx\,dt \tag{15}$$

holds true for *any* solution of the adjoint system

$$\begin{cases} p_{tt}-\Delta p+b(x,t)p=0 & \text{in } Q,\\ p=0 & \text{on } \Sigma,\\ p(x,T)=p^0(x), \quad p_t(x,T)=p^1(x) & \text{in } \Omega, \end{cases} \tag{16}$$

$$\begin{cases} -z_t-\Delta z+a(x,t)z=p\,\mathbb{1}_{\mathcal{O}} & \text{in } Q,\\ z=0 & \text{on } \Sigma,\\ z(x,T)=z^0(x) & \text{in } \Omega \end{cases} \tag{17}$$

associated to final data $z^0 \in H^1_0(\Omega)$ and $(p^0,p^1)\in L^2(\Omega)\times H^{-1}(\Omega)$. In (15), $\rho_\omega=\rho_\omega(x)$ is an appropriate regular approximation of the characteristic function $\mathbb{1}_\omega$. Among other things, we will assume that $\rho_\omega\in C^1(\overline{\Omega})$, $\rho_\omega(x)=1$ for all $x\in\omega'\subset\omega$ and $\rho_\omega(x)=0$ for all $x\notin\omega$. As we shall see in Section 4, we must use a smooth approximation of the characteristic function of the set ω in order to guarantee the regularity of the control.

The rest of the paper is organized as follows. In Section 2, we will recall some existence and regularity results for the solutions of the wave and the heat equations and then we will prove theorem 2, i.e. the null-exact controllability of the linear system (11)–(12), assuming that the observability inequality (15) holds true. Section 3 is devoted to prove theorem 1. Finally, Section 4 is devoted to prove (15). This relies mainly on an observability estimate for the solutions of (16), i.e. the exact controllability of (10) with controls in $L^2(\omega\times(0,T))$ and a (global) Carleman estimate for the heat equation taken from [6].

2. **Preliminaries and the linear case.**

2.1. **Preliminaries.** We begin this Section by recalling some existence and regularity results for wave and heat equations. For more complete treatises, see for instance [1] and [8].

In the sequel, C, C_1, C_2,... stand for generic positive constants, depending on Ω, T, ω, $\mathcal{O}$ and maybe the coefficients of the considered equations. We will sometimes (but not always) indicate this dependence explicitly.

For any Banach space X considered below, the usual norm in X will be denoted by $\|\cdot\|_X$. In the particular cases of $L^2(\Omega)$, $H^1_0(\Omega)$, etc., the corresponding norms and scalar products will be respectively denoted by $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^1_0}$, $(\cdot\,,\cdot)_{L^2}$, $(\cdot\,,\cdot)_{H^1_0}$, etc.

Let $c\in L^\infty(Q)$, $k\in L^2(Q)$ and $(v^0,v^1)\in H^1_0(\Omega)\times L^2(\Omega)$ be given. It is well known that the solution v of the linear problem

$$\begin{cases} v_{tt}-\Delta v+c(x,t)v=k(x,t) & \text{in } Q,\\ v=0 & \text{on } \Sigma,\\ v(x,0)=v^0(x), \quad v_t(x,0)=v^1(x) & \text{in } \Omega \end{cases} \tag{18}$$

satisfies $(v,v_t)\in C^0([0,T];H^1_0(\Omega)\times L^2(\Omega))$ and

$$\|(v,v_t)\|^2_{C^0([0,T];H^1_0\times L^2)} \leq e^{CT(1+\|c\|_\infty)}\left(\|(v^0,v^1)\|^2_{H^1_0\times L^2}+\|k\|^2_{L^2(Q)}\right), \tag{19}$$

where C only depends on Ω. We can also solve (18) when the data $(v^0, v^1) \in L^2(\Omega)\times H^{-1}(\Omega)$. For instance, when $k \equiv 0$, we have $(v, v_t) \in C^0([0,T]; L^2(\Omega) \times H^{-1}(\Omega))$ and

$$\|(v,v_t)\|^2_{C^0([0,T];L^2\times H^{-1})} \le e^{CT(1+\|c\|_\infty)} \|(v^0,v^1)\|^2_{L^2\times H^{-1}},$$

with C again depending on Ω.

In the sequel, for any couple of Banach spaces X and Y satisfying $X \hookrightarrow Y$ with a continuous embedding, we will use the following notation:

$$W(0,T;X,Y) = \{\, v \in L^2(0,T;X) : v_t \in L^2(0,T;Y)\,\},$$
$$\|v\|_{W(0,T;X,Y)} = \left(\|v\|^2_{L^2(0,T;X)} + \|v_t\|^2_{L^2(0,T;Y)}\right)^{1/2}.$$

It is then well known that, for any $c \in L^\infty(Q)$, $k \in L^2(0,T;H^{-1}(\Omega))$, and $w^0 \in L^2(\Omega)$, the solution w of the parabolic problem

$$\begin{cases} w_t - \Delta w + c(x,t)w = k & \text{in } Q,\\ w = 0 & \text{on } \Sigma,\\ w(x,0) = w^0(x) & \text{in } \Omega, \end{cases} \tag{20}$$

satisfies $w \in W(0,T;H^1_0(\Omega),H^{-1}(\Omega))$ and consequently $w \in C^0([0,T];L^2(\Omega))$ and the estimate

$$\begin{cases} \|w\|_{W(0,T;H^1_0,H^{-1})} + \|w\|_{C^0([0,T];L^2)} \\ \quad \le e^{CT(1+\|c\|_\infty)} \left(\|w^0\|_{L^2} + \|k\|_{L^2(0,T;H^{-1})}\right). \end{cases} \tag{21}$$

Let us assume that $k \in L^2(Q)$ and $w^0 \in H^1_0(\Omega)$. Then the solution satisfies $w \in L^2(0,T;D(-\Delta)) \cap C^0([0,T];H^1_0(\Omega))$ and $w_t \in L^2(Q)$ and we have

$$\begin{cases} \|w\|_{L^2(0,T;D(-\Delta))} + \|w\|_{C^0([0,T];H^1_0)} + \|w_t\|_{L^2(Q)} \\ \quad \le e^{CT(1+\|c\|_\infty)} \left(\|w^0\|_{H^1_0} + \|k\|_{L^2(Q)}\right). \end{cases} \tag{22}$$

Of course, in (21) and (22) the constants C depend on Ω.

In this paper, we will also have to solve systems of the form (11) with $h \in L^2(0,T;D(-\Delta)')$ and $y^0 \in H^{-1}(\Omega)$. The appropriate concept is the *solution by transposition*.

Thus, assume that h is given in $L^2(0,T;D(-\Delta)')$, $a \in L^\infty(Q)$ and $y^0 \in H^{-1}(\Omega)$. By definition, the solution by transposition of

$$\begin{cases} y_t - \Delta y + a(x,t)y = h & \text{in } Q,\\ y = 0 & \text{on } \Sigma,\\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{23}$$

is the unique function $y \in L^2(Q)$ satisfying

$$\begin{cases} \displaystyle\int_0^T\!\!\int_\Omega y\, g\, dx\, dt = \int_0^T \langle\!\langle h(t), \varphi_g(\cdot,t)\rangle\!\rangle\, dt + \langle y^0, \varphi_g(\cdot,0)\rangle \\ \forall g \in L^2(Q). \end{cases} \tag{24}$$

Here, for each $g \in L^2(Q)$, we have denoted by φ_g the solution to the corresponding adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi + a(x,t)\varphi = g & \text{in } Q,\\ \varphi = 0 & \text{on } \Sigma,\\ \varphi(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{25}$$

In (24) and also in the sequel, $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ and $\langle \cdot\,,\cdot\rangle$ stand for the usual duality pairings associated to $D(-\Delta)'$ and $D(-\Delta)$ and $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, respectively. Notice that the solution of (25) satisfies

$$\varphi_g \in L^2(0,T;D(-\Delta)) \cap C^0([0,T];H_0^1(\Omega))$$

and

$$\|\varphi_g\|_{L^2(0,T;D(-\Delta))} + \|\varphi_g(\cdot,0)\|_{H_0^1(\Omega)} \le e^{CT(1+\|a\|_\infty)}\|g\|_{L^2(Q)}.$$

Hence (24) makes sense, y is well defined and one has

$$\|y\|_{L^2(Q)} \le e^{CT(1+\|a\|_\infty)}(\|h\|_{L^2(0,T;D(-\Delta)')} + \|y^0\|_{H^{-1}(\Omega)}). \tag{26}$$

On the other hand, it is clear that y solves the partial differential equation in (23) in the distributional sense, i.e.

$$y_t - \Delta y + a(x,t)y = h^* \quad \text{in} \quad \mathcal{D}'(Q),$$

where h^* is the distribution in $L^2(0,T;H^{-2}(\Omega))$ given by

$$\langle h^*, \varphi\rangle_{\mathcal{D}'(Q),\mathcal{D}(Q)} = \int_0^T \langle\!\langle h(t), \varphi(\cdot,t)\rangle\!\rangle\, dt \quad \forall \varphi \in \mathcal{D}(Q).$$

Therefore, we also have $y_t \in L^2(0,T;H^{-2}(\Omega))$ and $y \in C^0([0,T];H^{-1}(\Omega))$, with

$$\|y\|_{C^0([0,T];H^{-1})} \le e^{CT(1+\|a\|_\infty)}(\|h\|_{L^2(0,T;D(-\Delta)')} + \|y^0\|_{H^{-1}(\Omega)}).$$

Observe that, for any $g \in L^2(Q)$ and any $\psi^0 \in H_0^1(\Omega)$, the solution by transposition of (23) satisfies

$$\int_0^T\int_\Omega y\, g\, dx\, dt + \langle y(\cdot,T), \psi^0\rangle = \int_0^T \langle\!\langle h(t), \psi_g(\cdot,t)\rangle\!\rangle\, dt + \langle y^0, \psi_g(\cdot,0)\rangle, \tag{27}$$

where ψ_g is the solution to the linear problem

$$\begin{cases} -\psi_t - \Delta\psi + a(x,t)\psi = g & \text{in } Q,\\ \psi = 0 & \text{on } \Sigma,\\ \psi(x,T) = \psi^0 & \text{in } \Omega. \end{cases}$$

Let $\mathcal{G}_\delta(x^0)$ and $R(x^0)$ be as in the previous Section (see (4) and (5)). Let us introduce two positive parameters $\kappa, \kappa_1 \in (0,\delta)$ with $\kappa < \kappa_1$ and let ω_0, ω_1 be the following open sets:

$$\omega_0 = \mathcal{G}_\kappa(x^0) = \mathcal{B}_\kappa(x^0)\cap\Omega \quad \text{and} \quad \omega_1 = \mathcal{G}_{\kappa_1}(x^0) = \mathcal{B}_{\kappa_1}(x^0)\cap\Omega. \tag{28}$$

Finally, let ρ_ω be a function satisfying

$$\begin{cases} \rho_\omega \in C^2(\overline{\Omega}), \quad 0 \le \rho_\omega \le 1,\\ \rho_\omega(x) = 1 \ \text{ in } \omega_1,\\ \rho_\omega(x) = 0 \ \text{ for all } \ x \in \Omega\setminus\omega. \end{cases} \tag{29}$$

As mentioned above, the proof of theorem 2 relies on an observability inequality for the adjoint of the linear cascade system (11)–(12). This is given in the following result:

Proposition 1. *Assume that $a, b \in L^\infty(Q)$ and $T > 2R(x^0)$. Then there exists a positive constant $C = C(\|a\|_\infty, \|b\|_\infty, \omega, \mathcal{O}, \Omega, T)$ such that any solution of* (16)–(17) *associated to a final data $(z^0, p^0, p^1) \in H_0^1(\Omega)\times L^2(\Omega)\times H^{-1}(\Omega)$ satisfies:*

$$\|(z,p,p_t)(\cdot,0)\|^2_{H_0^1\times L^2\times H^{-1}} \le C\int_0^T\int_\Omega \rho_\omega\left(|\Delta z|^2 + |z|^2\right)\, dx\, dt. \tag{30}$$

The proof of this result is given in Section 4.

Remark 6. Observe that *any* solution of (16)–(17) with $(z^0, p^0, p^1) \in H_0^1(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$ satisfies

$$(p, p_t) \in C^0([0,T]; L^2(\Omega) \times H^{-1}(\Omega)).$$

We also get

$$z \in L^2(0,T; D(-\Delta)) \cap C^0([0,T]; H_0^1(\Omega))$$

and, in particular, (30) makes sense. ∎

2.2. **Proof of theorem 2.** First of all, let us observe that it suffices to prove theorem 2 when $g \equiv 0$ and $(r^0, r^1) = (0,0)$.

Indeed, let us assume that the result is true in this case and let $(y^0, q^0, q^1) \in H^{-1}(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$, $g \in L^2(Q)$ and $(r^0, r^1) \in H_0^1(\Omega) \times L^2(Q)$ be given. Let us introduce the solution $\tilde{q}$ of (14) and let us set $(\tilde{q}^0, \tilde{q}^1) = (\tilde{q}, \tilde{q}_t)(\cdot, 0)$. Then, by hypothesis, there exists $h_\omega \in L^2(0,T; D(-\Delta)')$ with Supp $h_\omega \subset \overline{\omega} \times [0,T]$ such that (13) holds and the solution $(\hat{y}, \hat{q})$ of (11)–(12) associated to $g \equiv 0$ and initial data $(y^0, q^0 - \tilde{q}^0, q^1 - \tilde{q}^1)$ satisfies $\hat{y}(\cdot, T) = 0$, $\hat{q}(\cdot, T) = 0$ and $\hat{q}_t(\cdot, T) = 0$ in Ω.

Since $(y, q) = (\hat{y}, \hat{q} + \tilde{q})$ solves (11)–(12) for this h_ω and satisfies

$$y(x,T) = 0, \quad q(x,T) = r^0(x) \quad \text{and} \quad q_t(x,T) = r^1(x) \quad \text{in} \quad \Omega,$$

we deduce that theorem 2 also holds for general g and (r^0, r^1).

Thus, let us assume that $g \equiv 0$ and $(r^0, r^1) = (0,0)$ and let us consider the null controllability problem for

$$\begin{cases} y_t - \Delta y + a(x,t) y = h_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{31}$$

$$\begin{cases} q_{tt} - \Delta q + b(x,t) q = y \, \mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(x,0) = q^0(x), \quad q_t(x,0) = q^1(x) & \text{in } \Omega, \end{cases} \tag{32}$$

where $y_0 \in H^{-1}(\Omega)$ and $(q^0, q^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

There are several ways to deduce the null controllability of (31)–(32) from the observability inequality in proposition 1. We will use here a well known argument which relies on the construction of a sequence of minimal norm controls h^n that provide states that converge to the desired target $(0,0,0)$ as $n \to +\infty$.

Let $(y^0, q^0, q^1) \in H^{-1}(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ and $\varepsilon > 0$ be given. Let us introduce the functional J_ε with

$$\begin{cases} J_\varepsilon(z^0, p^0, p^1) = \dfrac{1}{2} \displaystyle\int_0^T \int_\Omega \rho_\omega \left(|\Delta z|^2 + |z|^2 \right) dx\, dt \\ \qquad + \varepsilon \|(z^0, p^0, p^1)\|_{H_0^1 \times L^2 \times H^{-1}} \\ \qquad + \langle y^0, z(\cdot, 0) \rangle - \langle p_t(\cdot, 0), q^0 \rangle + (p(\cdot, 0), q^1)_{L^2} \\ \forall (z^0, p^0, p^1) \in H_0^1(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega), \end{cases} \tag{33}$$

where (p, z) denotes the solution to the adjoint system (16)–(17).

We then have the following result:

Proposition 2. *The functional J_ε is continuous and strictly convex and satisfies*

$$\liminf_{\|(z^0,p^0,p^1)\|_{H_0^1 \times L^2 \times H^{-1}} \to \infty} \frac{J_\varepsilon(z^0, p^0, p^1)}{\|(z^0, p^0, p^1)\|_{H_0^1 \times L^2 \times H^{-1}}} \geq \varepsilon. \tag{34}$$

Therefore, J_ε reaches its minimum at a unique point $(z^0_\varepsilon, p^0_\varepsilon, p^1_\varepsilon) \in H^1_0(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$. One has $(z^0_\varepsilon, p^0_\varepsilon, p^1_\varepsilon) = (0,0,0)$ if and only if the solution (y,q) to (31)–(32) *associated to $h_\omega \equiv 0$ verifies*

$$\|(y,q,q_t)(\cdot,T)\|_{H^{-1}\times H^1_0\times L^2} \leq \varepsilon.$$

When $(z^0_\varepsilon, p^0_\varepsilon, p^1_\varepsilon) \neq (0,0,0)$, the following optimality condition is satisfied:

$$\begin{cases} \displaystyle\int_0^T\int_\Omega \rho_\omega\left((\Delta z_\varepsilon)(\Delta z) + z_\varepsilon z\right)\,dx\,dt \\ \displaystyle +\frac{\varepsilon}{\|(z^0_\varepsilon, p^0_\varepsilon, p^1_\varepsilon)\|_{H^1_0\times L^2\times H^{-1}}}\left(\int_\Omega (\nabla z^0_\varepsilon\cdot\nabla z^0 + p^0_\varepsilon\, p^0)\,dx + (p^1_\varepsilon, p^1)_{H^{-1}}\right) \\ +\langle y^0, z(\cdot,0)\rangle - \langle p_t(\cdot,0), q^0\rangle + (p(\cdot,0), q^1)_{L^2} = 0 \\ \forall (z^0,p^0,p^1) \in H^1_0(\Omega)\times L^2(\Omega)\times H^{-1}(\Omega), \end{cases} \tag{35}$$

where (z,p) and $(z_\varepsilon, p_\varepsilon)$ are, respectively, the solutions to (16)–(17) *corresponding to (z^0,p^0,p^1) and $(z^0_\varepsilon, p^0_\varepsilon, p^1_\varepsilon)$. Furthermore, one has*

$$\int_0^T\int_\Omega \rho_\omega\left(|\Delta z_\varepsilon|^2 + |z_\varepsilon|^2\right)\,dx\,dt \leq C\|(y^0,q^0,q^1)\|^2_{H^{-1}\times H^1_0\times L^2} \tag{36}$$

where the positive constant $C = C(\|a\|_\infty, \|b\|_\infty, \omega, \mathcal{O}, \Omega, T)$ is given in proposition 1.

Proof: The continuity and strict convexity of J_ε are straightforward, in view of the regularity properties recalled in the previous paragraph.

Indeed, for any $(p^0,p^1) \in L^2(\Omega)\times H^{-1}(\Omega)$, the corresponding solution to (16) satisfies $(p,p_t) \in C^0([0,T]; L^2(\Omega)\times H^{-1}(\Omega))$ and consequently we have $p\mathbb{1}_{\mathcal{O}} \in C^0([0,T];L^2(\Omega))$. Therefore, for any $z^0 \in H^1_0(\Omega)$, the solution to (17) satisfies

$$z \in L^2(0,T;D(-\Delta)) \cap C^0([0,T];H^1_0(\Omega)).$$

From (30) and (33), we have:

$$\begin{aligned} J_\varepsilon(z^0,p^0,p^1) &\geq \frac{1}{2}\int_0^T\int_\Omega \rho_\omega\left(|\Delta z|^2 + |z|^2\right)\,dx\,dt \\ &\quad + \varepsilon\|(z^0,p^0,p^1)\|_{H^1_0\times L^2\times H^{-1}} \\ &\quad - \|(y^0,q^0,q^1)\|_{H^{-1}\times H^1_0\times L^2}\|(z,p,p_t)(\cdot,0)\|_{H^1_0\times L^2\times H^{-1}} \\ &\geq \frac{1}{4}\int_0^T\int_\Omega \rho_\omega\left(|\Delta z|^2 + |z|^2\right)\,dx\,dt + \varepsilon\|(z^0,p^0,p^1)\|_{H^1_0\times L^2\times H^{-1}} \\ &\quad - C\|(y^0,q^0,q^1)\|^2_{H^{-1}\times H^1_0\times L^2}, \end{aligned}$$

whence we immediately obtain (34).

The proof of (35) is standard. Finally, in order to prove (36), let us observe that, as a consequence of the optimality condition, one has

$$\begin{aligned} &\int_0^T\int_\Omega \rho_\omega\left(|\Delta z_\varepsilon|^2 + |z_\varepsilon|^2\right)\,dx\,dt + \varepsilon\|(z^0_\varepsilon, p^0_\varepsilon, p^1_\varepsilon)\|_{H^1_0\times L^2\times H^{-1}} \\ &\qquad = -\langle y^0, z_\varepsilon(\cdot,0)\rangle + \langle p_{\varepsilon,t}(\cdot,0), q^0\rangle - \langle p_\varepsilon(\cdot,0), q^1\rangle. \end{aligned}$$

This, together with the observability inequality (30), gives

$$\begin{aligned}&\int_0^T\int_\Omega \rho_\omega\left(|\Delta z_\varepsilon|^2+|z_\varepsilon|^2\right)\,dx\,dt+\varepsilon\|(z^0,p^0,p^1)\|_{H_0^1\times L^2\times H^{-1}}\\ &\le\left(C\int_0^T\int_\Omega \rho_\omega\left(|\Delta z_\varepsilon|^2+|z_\varepsilon|^2\right)\,dx\,dt\right)^{1/2}\|(y^0,q^0,q^1)\|_{H^{-1}\times H_0^1\times L^2}\end{aligned}$$

which implies (36). □

We can now finish the proof of theorem 2. For each $n\ge 1$, let us introduce h^n with

$$\begin{cases}\displaystyle\int_0^T \langle\langle h^n,w\rangle\rangle\,dt=\int_0^T\int_\Omega \rho_\omega\left((\Delta z_{1/n})(\Delta w)+z_{1/n}\,w\right)\,dx\,dt\\ \forall w\in L^2(0,T;D(-\Delta)),\quad h^n\in L^2(0,T;D(-\Delta)').\end{cases}$$

Here, $z_{1/n}$ is, together with $p_{1/n}$, the solution of (16)–(17) corresponding to the unique initial data $(z^0_{1/n},p^0_{1/n},p^1_{1/n})$ which minimizes $J_{1/n}$ in $H_0^1(\Omega)\times L^2(\Omega)\times H^{-1}(\Omega)$.

Then, in view of (36),

$$\begin{aligned}\|h^n\|_{L^2(0,T;D(-\Delta)')}&\le C\left(\int_0^T\int_\Omega \rho_\omega\left(|\Delta z_{1/n}|^2+|z_{1/n}|^2\right)\,dx\,dt\right)^{1/2}\\ &\le C\|(y^0,q^0,q^1)\|_{H^{-1}\times H_0^1\times L^2}\end{aligned}\tag{37}$$

for all $n\ge 1$ ($C=C(\|a\|_\infty,\|b\|_\infty,\|g\|_{L^2(Q)},\omega,\mathcal{O},\Omega,T)$ is a new positive constant).

Let us introduce the solution (y^n,q^n) to (31)–(32) associated to the control h^n. Of course, y^n is the solution to (31) in the sense of (24)–(25) and, in view of (26) and (37), we also have

$$\|y^n\|_{L^2(Q)}\le C\|(y^0,q^0,q^1)\|_{H^{-1}\times H_0^1\times L^2}$$

and, consequently, $\|(q^n,q^n_t)\|_{C^0([0,T];H_0^1\times L^2)}$ is bounded independently of n.

From (35) written for $\varepsilon=1/n$ and the definition of h^n, we have

$$\begin{cases}\displaystyle\int_0^T\langle\langle h^n,z\rangle\rangle\,dt\\ \displaystyle+\frac{1}{n}\Big(\frac{(z^0_{1/n},p^0_{1/n},p^1_{1/n})}{\|(z^0_{1/n},p^0_{1/n},p^1_{1/n})\|_{H_0^1\times L^2\times H^{-1}}},(z^0,p^0,p^1)\Big)_{H_0^1\times L^2\times H^{-1}}\\ +\langle y^0,z(\cdot,0)\rangle-\langle p_t(\cdot,0),q^0\rangle+(p(\cdot,0),q^1)_{L^2}=0\\ \forall(z^0,p^0,p^1)\in H_0^1(\Omega)\times L^2(\Omega)\times H^{-1}(\Omega),\end{cases}\tag{38}$$

where (p,z) is the solution to (16)–(17) associated to (z^0,p^0,p^1). On the other hand, from (27) written for $y=y^n$ and $\psi_g=z$ (the solution to (17)), we also find that

$$\int_0^T\langle\langle h^n,z\rangle\rangle\,dt+\langle y^0,z(\cdot,0)\rangle=\int_0^T\int_\Omega y^n p\,\mathbb{1}_{\mathcal{O}}\,dx\,dt+\langle y^n(\cdot,T),z^0\rangle.\tag{39}$$

Taking into account that q^n solves (32) with $y = y^n$, we see that

$$\left\{\begin{aligned} &\int_0^T\int_\Omega y^n p\,\mathbb{1}_{\mathcal{O}}\,dx\,dt \\ &\quad = (p(\cdot,t), q_t^n(\cdot,t))_{L^2}\big|_{t=0}^{t=T} - \langle p_t(\cdot,t), q^n(\cdot,t)\rangle\big|_{t=0}^{t=T} \\ &\quad = (p^0, q_t^n(\cdot,T))_{L^2} - \langle p^1, q^n(\cdot,T)\rangle \\ &\quad -(p(\cdot,0), q^1)_{L^2} + \langle p_t(\cdot,0), q^0\rangle. \end{aligned}\right. \tag{40}$$

Thus, combining (38), (39) and (40), the following is found:

$$\left\{\begin{aligned} &\langle y^n(\cdot,T), z^0\rangle + (p^0, q_t^n(\cdot,T))_{L^2} - \langle p^1, q^n(\cdot,T)\rangle \\ &= -\frac{1}{n}\Big(\frac{(z^0_{1/n}, p^0_{1/n}, p^1_{1/n})}{\|(z^0_{1/n}, p^0_{1/n}, p^1_{1/n})\|_{H_0^1\times L^2\times H^{-1}}}, (z^0, p^0, p^1)\Big)_{H_0^1\times L^2\times H^{-1}} \\ &\forall (z^0, p^0, p^1) \in H_0^1(\Omega)\times L^2(\Omega)\times H^{-1}(\Omega). \end{aligned}\right.$$

Obviously, this indicates that

$$\|(y^n, q^n, q_t^n)(\cdot,T))\|_{H^{-1}\times H_0^1\times L^2} \le \frac{1}{n} \qquad \forall n \ge 1. \tag{41}$$

From (37), at least for a subsequence again denoted by n, we have

$$\begin{aligned} &h^n \to \hat{h} \text{ weakly in } L^2(0,T;D(-\Delta)'), \\ &y^n \to \hat{y} \text{ weakly in } L^2(Q) \text{ and weakly-}* \text{ in } L^\infty(0,T;H^{-1}(\Omega)), \\ &(q^n, q_t^n) \to (\hat{q}, \hat{q}_t) \text{ weakly-}* \text{ in } L^\infty(0,T;H_0^1(\Omega)\times L^2(\Omega)), \end{aligned}$$

where $(\hat{y}, \hat{q})$ solves (31)–(32) for $h_\omega = \hat{h}$. Furthermore, from (37) we see that

$$\|\hat{h}\|_{L^2(0,T,D(-\Delta)')} \le C\|(y^0, q^0, q^1)\|,$$

where C is given in proposition 1. Since Supp $h^n \subset \overline{\omega}\times[0,T]$ for all n, the support of $\hat{h}$ is also contained in $\overline{\omega}\times[0,T]$. From (41), we also see that

$$(\hat{y}(\cdot,T), \hat{q}(\cdot,T), \hat{q}_t(\cdot,T)) = (0,0,0).$$

Hence, we have found a control $\hat{h} \in L^2(0,T;D(-\Delta)')$ with support in $\overline{\omega}\times[0,T]$ that drives the state exactly to $(0,0,0)$.

This proves the null controllability of system (31)–(32) and ends the proof of theorem 2.

3. **Proof of theorem 1: The fixed point argument.** As mentioned above, for the proof of theorem 1 we will use the controllability result in theorem 2 and a fixed point argument. This strategy was introduced in [12] in the framework of the exact controllability of the semilinear wave equation. Since then, it has been used in several different contexts; for instance, see [13], [3] and [6] for results concerning the approximate and null controllability of semilinear wave and heat equations with Dirichlet or Neumann boundary conditions. Let us also mention the paper [9], where the authors analyzed the null controllability of semilinear abstract systems (and in particular semilinear wave equations) using a global inverse function theorem.

3.1. **The case in which f_1 and f_2 are C^1.** Let us introduce the functions g_i with

$$g_1(x,t;s,r) = \begin{cases} \dfrac{f_1(x,t;s,r)}{s} & \text{if } s \neq 0, \\ D_s f_1(x,t;0,r) & \text{if } s = 0 \end{cases}$$

and

$$g_2(x,t;r) = \begin{cases} \dfrac{f_2(x,t;r) - f_2(x,t;0)}{r} & \text{if } r \neq 0, \\ D_r f_2(x,t;0) & \text{if } r = 0. \end{cases}$$

Under the assumptions imposed in theorem 1 on the functions f_1 and f_2, one has

$$|g_1(x,t;s,r)| \leq L_1 \quad \text{and} \quad |g_2(x,t;r)| \leq L_2 \quad \text{a.e. in} \quad Q,$$

where L_1 and L_2 are Lipschitz constants for f_1 and f_2 respectively.

Let us introduce the space $Z = L^2(0,T;H^{-1}(\Omega) \times L^2(\Omega))$. Let us fix $\varepsilon > 0$. To each $(v,\xi) \in Z$, we can associate the solution v^ε of the linear problem

$$v^\varepsilon - \varepsilon \Delta v^\varepsilon = v \text{ in } \Omega, \quad v^\varepsilon = 0 \text{ on } \partial\Omega \tag{42}$$

and the functions $G_1 = g_1(x,t;v^\varepsilon,\xi)$ and $G_2 = g_2(x,t;\xi)$. Observe that G_1 and G_2 belong to $L^\infty(Q)$.

Recall that ω satisfies (9) for some $\delta > 0$. Let us choose δ_1 and δ_2 such that $0 < \delta_1 < \delta_2 < \delta$ and let us set $\omega_i = \mathcal{G}_{\delta_i}(x^0)$ for $i = 1$ and $i = 2$. In view of theorem 2, there exist controls $h^\varepsilon_{\omega_1} \in L^2(0,T;D(-\Delta)')$ supported by $\overline{\omega}_1 \times [0,T]$ such that

$$\|h^\varepsilon_{\omega_1}\|_{L^2(0,T;D(-\Delta)')} \leq C \|(y^0, \tilde{q}^0 - r^0, \tilde{q}^1 - r^1)\|_{H^{-1} \times H^1_0 \times L^2} \tag{43}$$

for some C only depending on L_1, L_2, ω, $\mathcal{O}$, Ω and T and the associated solutions to

$$\begin{cases} y^\varepsilon_t - \Delta y^\varepsilon + g_1(x,t;v^\varepsilon,\xi) y^\varepsilon = h^\varepsilon_{\omega_1} & \text{in } Q, \\ y^\varepsilon = 0 & \text{on } \Sigma, \\ y^\varepsilon(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{44}$$

$$\begin{cases} q^\varepsilon_{tt} - \Delta q^\varepsilon + g_2(x,t;\xi) q^\varepsilon = y^\varepsilon \, 1\!\!1_{\mathcal{O}} + f_2(x,t;0) & \text{in } Q, \\ q^\varepsilon = 0 & \text{on } \Sigma, \\ q^\varepsilon(x,0) = q^0(x), \quad q^\varepsilon_t(x,0) = q^1(x) & \text{in } \Omega. \end{cases} \tag{45}$$

satisfy (3). In (43), we have denoted by $(\tilde{q}^0, \tilde{q}^1)$ the couple $(\tilde{q}, \tilde{q}_t)(\cdot, T)$, where $\tilde{q}$ is the solution of (14) with $b(x,t) \equiv g_2(x,t;\xi)$ and $g(x,t) \equiv f_2(x,t;0)$. Consequently, we also have

$$\|h^\varepsilon_{\omega_1}\|_{L^2(0,T;D(-\Delta)')} \leq C \left(\|y^0\|_{H^{-1}}, \|(q^0,q^1)\|_{H^1_0 \times L^2}, \|(r^0,r^1)\|_{H^1_0 \times L^2} \right)$$

for some $C = C(f_1, f_2, \omega, \mathcal{O}, \Omega, T)$.

We will denote by $U_\varepsilon(v,\xi)$ the set of these controls.

Let us introduce the set-valued mapping $\Lambda^\varepsilon : Z \mapsto 2^Z$, with

$$\Lambda^\varepsilon(v,\xi) = \{ \, (y^\varepsilon, q^\varepsilon) : (y^\varepsilon, q^\varepsilon) \text{ solves (44)–(45) for some } h^\varepsilon \in U_\varepsilon(v,\xi) \, \}.$$

We have the following result:

Proposition 3. *Under the assumptions of theorem 1, there exists a compact set $K \subset Z$ such that, for every $(v,\xi) \in Z$, one has $\Lambda_\varepsilon(v,\xi) \subset K$. Furthermore, for every (v,ξ), $\Lambda_\varepsilon(v,\xi)$ is a non-empty convex compact subset of Z and the mapping*

Λ_ε is upper hemicontinuous, that is to say, for each linear continuous form $\mu \in Z'$, the real-valued function

$$(v,\xi) \in Z \mapsto \sup_{(y^\varepsilon,q^\varepsilon)\in\Lambda_\varepsilon(v,\xi)} \langle \mu, (y^\varepsilon, q^\varepsilon)\rangle_{Z',Z}$$

is upper semicontinuous.

Proof: Observe that, for each $(v,\xi) \in Z$, the solution to (44)–(45) associated to a control $h \in U_\varepsilon(v,\xi)$ is such that

$$\|(y^\varepsilon, q^\varepsilon)\|_{\widetilde{W}} \le C\,(\|y^0\|_{H^{-1}}, \|(q^0,q^1)\|_{H_0^1\times L^2}, \|(r^0,r^1)\|_{H_0^1\times L^2}),$$

where C only depends on f_1, f_2, ω, $\mathcal{O}$, Ω and T and $\widetilde{W}$ is the space

$$\widetilde{W} = W(0,T;L^2(\Omega),D(-\Delta)') \times W(0,T;H_0^1(\Omega),L^2(\Omega)).$$

Since $\widetilde{W} \hookrightarrow Z$ with a compact embedding, there exists a compact set K such that

$$\Lambda_\varepsilon(v,\xi) \subset K \quad \forall (v,\xi) \in Z.$$

On the other hand, from theorem 2 and the properties satisfied by ω_1 and T, we know that $\Lambda_\varepsilon(v,\xi)$ is non-empty. Since $U_\varepsilon(v,\xi)$ is convex, the fact that the system (44)–(45) is linear implies that $\Lambda_\varepsilon(v,\xi)$ is also a convex set.

Since $\Lambda_\varepsilon(v,\xi) \subset K$ for some compact set K of Z, in order to prove that $\Lambda_\varepsilon(v,\xi)$ is compact, we only need to check that it is closed.

Thus, let $\{(y_n^\varepsilon, q_n^\varepsilon)\}$ be a sequence in $\Lambda_\varepsilon(v,\xi)$ that converges in Z:

$$(y_n^\varepsilon, q_n^\varepsilon) \to (y^\varepsilon, q^\varepsilon) \text{ strongly in } Z.$$

We have to prove that $(y^\varepsilon, q^\varepsilon) \in \Lambda_\varepsilon(v,\xi)$. Observe that, associated to each $(y_n^\varepsilon, q_n^\varepsilon)$, there is a control $h_n^\varepsilon \in U_\varepsilon(v,\xi)$ and consequently

$$\|h_n^\varepsilon\| \le C\,(\|y^0\|_{H^{-1}}, \|(q^0,q^1)\|_{H_0^1\times L^2}, \|(r^0,r^1)\|_{H_0^1\times L^2})$$

for every n. This means that, at least for a subsequence, one has:

$$h_n^\varepsilon \to h^\varepsilon \text{ weakly in } L^2(0,T;D(-\Delta)').$$

Let us denote by $(\tilde{y}^\varepsilon, \tilde{q}^\varepsilon)$ the solution to (44)–(45) associated to the control h^ε. Then it is easy to see that

$$y_n^\varepsilon \to \tilde{y}^\varepsilon \text{ weakly in } L^2(Q),$$
$$q_n^\varepsilon \to \tilde{q}^\varepsilon \text{ weakly in } L^2(Q)$$

and

$$q_{n,t}^\varepsilon \to \tilde{q}_t^\varepsilon \text{ weakly in } L^2(Q).$$

From the uniqueness of the weak limit, we thus have

$$(y^\varepsilon, q^\varepsilon, q_t^\varepsilon) = (\tilde{y}^\varepsilon, \tilde{q}^\varepsilon, \tilde{q}_t^\varepsilon).$$

Moreover, it is not difficult to see that $y_n^\varepsilon(\cdot,T)$, $q_n^\varepsilon(\cdot,T)$ and $q_{n,t}^\varepsilon(\cdot,T)$ converge, at least weakly in $H^{-1}(\Omega)$, respectively to $y^\varepsilon(\cdot,T)$, $q^\varepsilon(\cdot,T)$ and $q_t^\varepsilon(\cdot,T)$. Consequently, $y^\varepsilon(\cdot,T) = 0$, $q^\varepsilon(\cdot,T) = r^0$ and $q_t^\varepsilon(\cdot,T) = r^1$. Therefore, $h^\varepsilon \in U_\varepsilon(v,\xi)$ and $(y^\varepsilon, q^\varepsilon) \in \Lambda_\varepsilon(v,\xi)$.

Finally, let us prove that Λ_ε is upper hemicontinuous. We have to check that the set

$$B_{\alpha,\mu} = \{\, (v,\xi) : \sup_{(y^\varepsilon,q^\varepsilon)\in\Lambda_\varepsilon(v,\xi)} \langle \mu, (y^\varepsilon, q^\varepsilon)\rangle_{Z',Z} \ge \alpha \,\}$$

is closed for every $\alpha \in \mathbb{R}$ and every $\mu \in Z'$.

Thus, let $\{(v_n, \xi_n)\}$ be a sequence in $B_{\alpha,\mu}$ such that $(v_n, \xi_n) \to (v, \xi)$ in Z. It is clear that $v_n^\varepsilon \to v^\varepsilon$ strongly in $L^2(0, T, H_0^1(\Omega))$. Furthermore, from the continuity of g_1 and g_2, we have

$$g_1(x, t; v_n^\varepsilon, \xi_n) \to g_1(x, t; v^\varepsilon, \xi) \text{ weakly-}* \text{ in } L^\infty(Q) \text{ and strongly in } L^p(\Omega)$$

and

$$g_2(x, t; \xi_n) \to g_2(x, t; \xi) \text{ weakly-}* \text{ in } L^\infty(Q) \text{ and strongly in } L^p(\Omega)$$

for all finite $p \geq 1$.

Since all the sets $\Lambda_\varepsilon(v_n^\varepsilon, \xi_n)$ are compact and contained in the same compact set K, for each $n \geq 1$ we have

$$\sup_{(y^\varepsilon, q^\varepsilon) \in \Lambda_\varepsilon(v_n, \xi_n)} \langle \mu, (y^\varepsilon, q^\varepsilon) \rangle_{Z', Z} = \langle \mu, (y_n^\varepsilon, q_n^\varepsilon) \rangle_{Z', Z} \geq \alpha$$

for some $(y_n^\varepsilon, q_n^\varepsilon) \in \Lambda_\varepsilon(v_n, \xi_n) \subset K$.

From the definitions of Λ_ε and U_ε, for each $n \geq 1$ there exists $h_{n,\omega_1}^\varepsilon$,

$$h_{n,\omega_1}^\varepsilon \in L^2(0, T; D(-\Delta)'),$$

with support in $\overline{\omega}_1 \times [0, T]$ such that

$$\begin{cases} y_{n,t}^\varepsilon - \Delta y_n^\varepsilon + g_1(x, t; v_n^\varepsilon, \xi_n) y_n^\varepsilon = h_{n,\omega_1}^\varepsilon & \text{in } Q, \\ y_n^\varepsilon = 0 & \text{on } \Sigma, \\ y_n^\varepsilon(x, 0) = y^0(x) & \text{in } \Omega, \end{cases}$$

$$\begin{cases} q_{n,tt}^\varepsilon - \Delta q_n^\varepsilon + g_2(x, t; \xi_n) q_n^\varepsilon = y_n^\varepsilon \mathbb{1}_{\mathcal{O}} + f_2(x, t; 0) & \text{in } Q, \\ q_n^\varepsilon = 0 & \text{on } \Sigma, \\ q_n^\varepsilon(x, 0) = q^0(x), \quad q_{n,t}^\varepsilon(x, 0) = q^1(x) & \text{in } \Omega. \end{cases}$$

Furthermore, $(y_n^\varepsilon, q_n^\varepsilon)$ verifies (3) and

$$\|h_{n,\omega_1}^\varepsilon\|_{L^2(0,T;D(-\Delta)')} \leq C\,(\|y^0\|_{H^{-1}}, \|(q^0, q^1)\|_{H_0^1 \times L^2}, \|(r^0, r^1)\|_{H_0^1 \times L^2}),$$

where C is independent of n. Therefore, at least for a subsequence, one has

$$h_{n,\omega_1}^\varepsilon \to \hat{h}_{\omega_1}^\varepsilon \text{ weakly in } L^2(0, T; D(-\Delta)')$$

and

$$(y_n^\varepsilon, q_n^\varepsilon) \to (\hat{y}^\varepsilon, \hat{q}^\varepsilon) \text{ weakly in } \widetilde{W} \text{ and strongly in } Z,$$

where $(\hat{y}^\varepsilon, \hat{q}^\varepsilon)$ solves the system

$$\begin{cases} \hat{y}_t^\varepsilon - \Delta \hat{y}^\varepsilon + g_1(x, t; v^\varepsilon, \xi) \hat{y}^\varepsilon = \hat{h}_{\omega_1}^\varepsilon & \text{in } Q, \\ \hat{y}^\varepsilon = 0 & \text{on } \Sigma, \\ \hat{y}^\varepsilon(x, 0) = y^0(x) & \text{in } \Omega, \\ \hat{y}^\varepsilon(x, T) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \hat{q}_{tt}^\varepsilon - \Delta \hat{q}^\varepsilon + g_2(x, t; \xi) \hat{q}^\varepsilon = \hat{y}^\varepsilon \mathbb{1}_{\mathcal{O}} + f_2(x, t; 0) & \text{in } Q, \\ \hat{q}^\varepsilon = 0 & \text{on } \Sigma, \\ \hat{q}^\varepsilon(x, 0) = q^0(x), \quad \hat{q}_t^\varepsilon(x, 0) = q^1(x) & \text{in } \Omega, \\ \hat{q}^\varepsilon(x, T) = r^0(x), \quad \hat{q}_t^\varepsilon(x, T) = r^1(x) & \text{in } \Omega. \end{cases}$$

That is, $\hat{h}_{\omega_1}^\varepsilon \in U_\varepsilon(v, \xi)$ and $(\hat{y}^\varepsilon, \hat{q}^\varepsilon) \in \Lambda_\varepsilon(v, \xi)$. Passing to the limit, we get

$$\sup_{(y^\varepsilon, q^\varepsilon) \in \Lambda_\varepsilon(v, \varepsilon)} \langle \mu, (y^\varepsilon, q^\varepsilon) \rangle_{Z', Z} \geq \langle \mu, (\hat{y}^\varepsilon, \hat{q}^\varepsilon) \rangle_{Z', Z} \geq \alpha,$$

i.e. (v, ξ) belongs to the set $B_{\alpha,\mu}$.

This ends the proof of proposition 3. $\square$

As a consequence of proposition 3, Kakutani's theorem can be applied for every $\varepsilon > 0$ and there exists a fixed point $(y^\varepsilon, q^\varepsilon)$ of the mapping Λ_ε. If we denote by $y^\varepsilon_\varepsilon$ the solution to the linear problem (42) with $v = y^\varepsilon$, then $(y^\varepsilon, q^\varepsilon)$ verifies

$$\begin{cases} y^\varepsilon_t - \Delta y^\varepsilon + g_1(x,t;y^\varepsilon_\varepsilon, q^\varepsilon) y^\varepsilon = h^\varepsilon_{\omega_1} & \text{in } Q, \\ y^\varepsilon = 0 & \text{on } \Sigma, \\ y^\varepsilon(x,0) = y^0(x) & \text{in } \Omega, \end{cases}$$

$$\begin{cases} q^\varepsilon_{tt} - \Delta q^\varepsilon + f_2(x,t;q^\varepsilon) = y^\varepsilon\, 1\!\!1_{\mathcal{O}} & \text{in } Q, \\ q^\varepsilon = 0 & \text{on } \Sigma, \\ q^\varepsilon(x,0) = q^0(x), \quad q^\varepsilon_t(x,0) = q^1(x) & \text{in } \Omega. \end{cases}$$

Observe that, for a positive constant C independent of ε and which only depends on f_1, f_2, ω, $\mathcal{O}$, Ω and T, one has

$$\|h^\varepsilon_{\omega_1}\|_{L^2(0,T;D(-\Delta)')} \leq C\,(\|y^0\|_{H^{-1}}, \|(q^0,q^1)\|_{H^1_0\times L^2}, \|(r^0,r^1)\|_{H^1_0\times L^2})$$

and

$$\|y^\varepsilon\|_{L^2(Q)} \leq C\,(\|y^0\|_{H^{-1}}, \|(q^0,q^1)\|_{H^1_0\times L^2}, \|(r^0,r^1)\|_{H^1_0\times L^2})$$

for all $\varepsilon > 0$, whence we can assume that

$$h^\varepsilon_{\omega_1} \to h_{\omega_1} \text{ weakly in } L^2(0,T;D(-\Delta)'), \quad y^\varepsilon \to y \text{ weakly in } L^2(Q)$$

and therefore

$$q^\varepsilon \to q \text{ strongly in } L^2(Q)$$

as $\varepsilon \to 0$.

Let us now see that, at least for a subsequence, the sequence $\{y^\varepsilon\}$ converges strongly in $L^2(0,T;L^2(\Omega\setminus\overline{\omega}_2))$.

For every $\varepsilon > 0$, let us put $y^\varepsilon = Y + w^\varepsilon$, where Y is the solution to

$$\begin{cases} Y_t - \Delta Y = 0 & \text{in } Q, \\ Y = 0 & \text{on } \Sigma, \\ Y(x,0) = y^0(x) & \text{in } \Omega, \end{cases}$$

and w^ε is the solution to

$$\begin{cases} w^\varepsilon_t - \Delta w^\varepsilon + g_1(x,t;y^\varepsilon_\varepsilon, q^\varepsilon) w^\varepsilon = -g_1(x,t;y^\varepsilon_\varepsilon, q^\varepsilon) Y + h^\varepsilon_{\omega_1} & \text{in } Q, \\ w^\varepsilon = 0 & \text{on } \Sigma, \\ w^\varepsilon(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Then we have the following:

- Y is a fixed function in $L^2(Q)$.
- On the other hand, the unique reason for the lack of regularity of w^ε is the lack of regularity of $h^\varepsilon_{\omega_1}$. For every $p \in [1,\infty)$, let us introduce the spaces

  $$X^p = \{\, u \in L^p(0,T;W^{2,p}(\Omega\setminus\overline{\omega}_2)) : u_t \in L^p(0,T;L^p(\Omega\setminus\overline{\omega}_2)) \,\}$$

  and the associated norms

  $$\|u\|_{X^p} = \Big(\|u\|^p_{L^p(0,T;W^{2,p}(\Omega\setminus\overline{\omega}_2))} + \|u_t\|^p_{L^p(0,T;L^p(\Omega\setminus\overline{\omega}_2))}\Big)^{1/p}.$$

  Since the support of $h^\varepsilon_{\omega_1}$ is contained in $\overline{\omega}_1 \times [0,T]$, as a consequence of the regularizing effect of the heat equation and the choice we have made of ω_1 and ω_2, we have $w^\varepsilon \in X^2$ and

  $$\|w^\varepsilon\|_{X^2} \leq C\,(\|y^0\|_{H^{-1}}, \|(q^0,q^1)\|_{H^1_0\times L^2}, \|(r^0,r^1)\|_{H^1_0\times L^2})$$

  for some $C > 0$ independent of ε (see for instance [7]).

Hence, it can be assumed that $y^\varepsilon \to y$ strongly in $L^2\left((\Omega \setminus \overline{\omega}_2) \times (0,T)\right)$ and a.e. in $(\Omega \setminus \overline{\omega}_2) \times (0,T)$, the functions $g_1^\varepsilon = g_1(x,t;y_\varepsilon^\varepsilon, q^\varepsilon)y^\varepsilon$ satisfy

$$g_1^\varepsilon \mathbb{1}_{\Omega\setminus\overline{\omega}_2} \to g_1(x,t,y,q)y\mathbb{1}_{\Omega\setminus\overline{\omega}_2} \text{ strongly in } L^2(Q)$$

and

$$g_1^\varepsilon \mathbb{1}_{\omega_2} \to \tilde{g}\mathbb{1}_{\omega_2} \text{ weakly in } L^2(\omega \times (0,T)).$$

By introducing the new control h with

$$h = h_{\omega_1} - \tilde{g}\mathbb{1}_{\omega_2} + g_1(x,t;y,q)y\mathbb{1}_{\omega_2},$$

we see that the couple (y,q) satisfies

$$\begin{cases} y_t - \Delta y + f_1(x,t;y,q) = h & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega, \\ y(x,T) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} q_{tt} - \Delta q + f_2(x,t;q) = y\,\mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(x,0) = q^0(x), \quad q_t(x,0) = q^1(x) & \text{in } \Omega \\ q(x,T) = r^1(x), \quad q_t(x,T) = r^1(x) & \text{in } \Omega. \end{cases}$$

Furthermore, $h \in L^2(0,T,D(-\Delta)')$, Supp $h \subset \overline{\omega}_2 \times [0,T]$ and

$$\|h\|_{L^2(0,T;D(-\Delta)')} \le C\,(\|y^0\|_{H^{-1}}, \|(q^0,q^1)\|_{H_0^1\times L^2}, \|(r^0,r^1)\|_{H_0^1\times L^2}),$$

where C only depends on f_1, f_2, ω, Ω and T.

This ends the proof of theorem 1 when f_1 and f_2 are C^1 functions.

3.2. **The general case.** Let us now suppose that f_1 and f_2 are globally Lipschitz-continuous functions and satisfy (6)–(8). Let us introduce the functions $\rho_1 \in \mathcal{D}(\mathbb{R}^2)$ and $\rho_2 \in \mathcal{D}(\mathbb{R})$, with $\rho_i \ge 0$, Supp $\rho_1 \subset \overline{B}(0,1)$, Supp $\rho_2 \subset [-1,1]$ and

$$\iint_{\mathbb{R}^2} \rho_1(s,r)\,ds\,dr = \int_{\mathbb{R}} \rho_2(r)\,dr = 1.$$

We will consider the functions $\rho_{1,n}$, $\rho_{2,n}$, $g_{1,n}$ and $g_{2,n}$, with

$$\rho_{1,n}(s,r) = n^2\rho_1(ns,nr) \quad \forall (s,r) \in \mathbb{R}^2, \quad \rho_{2,n}(r) = n\rho_2(nr) \quad \forall r \in \mathbb{R},$$

$$g_{1,n}(x,t;\cdot) = \rho_{1,n} * g_1, \quad g_{2,n}(x,t;\cdot) = \rho_{2,n} * g_2,$$

$g_1(x,t;s,r) = f_1(x,t;s,r)/s$ for $s \neq 0$ and $g_2(x,t;s,r) = (f_2(x,t;r) - f_2(x,t;0))/r$ for $r \neq 0$. Then it is not difficult to check that the following properties of g_1 and g_2 are satisfied:

1. For every $n \ge 1$, $g_{1,n}(x,t;\cdot) \in C^0(\mathbb{R}^2)$ and $g_{2,n}(x,t;\cdot) \in C^0(\mathbb{R})$ a.e. $(x,t) \in Q$.
2. If we put $f_{1,n}(x,t;s,r) = g_{1,n}(x,t;s,r)s$ for $(s,r) \in \mathbb{R}^2$ and $f_{2,n}(x,t;r) = g_{2,n}(x,t;r)r + f_2(x,t;0)$ for $r \in \mathbb{R}$, then

$$f_{1,n}(x,t;\cdot) \to f_1(x,t;\cdot) \quad (\text{resp. } f_{2,n}(x,t;\cdot) \to f_2(x,t;\cdot))$$

   uniformly in the compact sets of $\mathbb{R}^2$ (resp. in the compact sets of $\mathbb{R}$).
3. There exists a positive constant L such that

$$\sup_{(s,r)\in R^2} |g_{1,n}(x,t;s,r)| + \sup_{r\in R} |g_{2,n}(x,t;r)| \le L \quad \forall n \ge 1.$$

For every n we can argue as in the previous subsection and find a control $h_n \in L^2(0,T;D(-\Delta)')$ with Supp $h_n \subset \overline{\omega}_2 \times [0,T]$ such that the system

$$\begin{cases} y_{t,n} - \Delta y_n + f_{1,n}(x,t;y_n,q_n) = h_n & \text{in } Q, \\ y_n = 0 & \text{on } \Sigma, \\ y_n(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{46}$$

$$\begin{cases} q_{tt,n} - \Delta q_n + f_{2,n}(x,t;q_n) = y_n \mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ q_n = 0 & \text{on } \Sigma, \\ q_n(x,0) = q^0(x), \quad q_{t,n}(x,0) = q^1(x) & \text{in } \Omega. \end{cases} \tag{47}$$

possesses at least one solution $(y_n, q_n) \in \widetilde{W}$, with $\widetilde{W} = W(0,T;L^2(\Omega),D(-\Delta)') \times W(0,T;H_0^1(\Omega),L^2(\Omega))$, satisfying

$$y_n(x,T) = 0, \quad q_n(x,T) = r^0(x) \ \text{ and } \ q_{n,t}(x,T) = r^1(x) \ \text{ in } \ \Omega.$$

From the properties satisfied by $g_{1,n}$ and $g_{2,n}$ and thanks to the estimates obtained in Section 3.1, it can be assumed that, for some positive C independent of n, one has

$$\|h_n\|_{L^2(0,T;D(-\Delta)')} + \|(y_n,q_n)\|_{\widetilde{W}} \le C,$$

for all $n \ge 1$. In view of the arguments in Section 3.1, it can also be assumed that

$$\begin{gathered} h_n \to h_{\omega_2} \quad \text{weakly in } L^2(0,T;D(-\Delta)'), \\ y_n \to y \quad \text{weakly in } L^2(Q), \quad y_n \to y \quad \text{strongly in } L^2((\Omega \setminus \overline{\omega}_2) \times (0,T)), \\ q_n \to q \quad \text{strongly in } L^2(Q), \\ f_{1,n}(\cdot;y_n,q_n)\mathbb{1}_{\omega_2} \to \hat{g}\mathbb{1}_{\omega_2} \quad \text{weakly in } L^2(\omega \times (0,T)) \ \text{ and} \\ f_{1,n}(\cdot;y_n,q_n)\mathbb{1}_{\Omega\setminus\overline{\omega}_2} \to f_1(\cdot;y,q)\mathbb{1}_{\Omega\setminus\overline{\omega}_2} \quad \text{strongly in } L^2((\Omega \setminus \overline{\omega}) \times (0,T)). \end{gathered}$$

Thus, passing to the limit in (46)–(47), we deduce that (y,q) solves (1)–(2) and (3) with the control h_ω given by

$$h_\omega = h_{\omega_2} - \hat{g}\mathbb{1}_{\omega_2} + g_1(x,t;y,q)\mathbb{1}_{\omega_2}.$$

This ends the proof of theorem 1.

4. **The observability inequality.** This Section is devoted to prove the observability inequality (30) for the adjoint system (16)–(17).

Thus, let $\mathcal{G}_\delta(x^0)$ and $R(x^0)$ be as in (4) and (5), let ω_0 and ω_1 be given by (28) for some $\kappa, \kappa_1 \in (0,\delta)$ ($\kappa < \kappa_1$) and let ρ_ω satisfy (29). We will need an appropriate (global) Carleman inequality for the heat equation. This is given in the following result:

Proposition 4. *Assume that $c \in L^\infty(Q)$. There exist a positive function $\zeta \in C^2(\overline{\Omega})$ and a positive constant $C_1 > 0$ depending on $\|c\|_\infty$, x^0, κ and T such that, for any $w^0 \in L^2(\Omega)$, the solution of (20) satisfies:*

$$\begin{cases} \displaystyle\int_0^T\!\!\int_\Omega e^{-\frac{2\zeta(x)}{t(T-t)}}\phi(t)|\nabla w|^2\,dx\,dt \\ \displaystyle\le C_1\left(\int_0^T\!\!\int_\Omega e^{-\frac{2\zeta(x)}{t(T-t)}}|k|^2\,dx\,dt + \int_0^T\!\!\int_{\mathcal{G}_\kappa(x^0)} e^{-\frac{2\zeta(x)}{t(T-t)}}\phi(t)^3|w|^2\,dx\,dt\right). \end{cases} \tag{48}$$

Here, we have used the notation $\phi(t) = t^{-1}(T-t)^{-1}$.

This result is proved in [6] (see Ch. I, lemma 1.2 for the proof in a more general context; see also the Appendix of [5] for a simplified proof). In fact, a similar inequality holds for any $T > 0$ (with other appropriate ζ and C_1) if $\mathcal{G}_\kappa(x^0)$ is replaced in (48) by an arbitrary nonempty open set $D \subset \Omega$. Furthermore, as noticed in [5], the way the function ζ and the constant C_1 depend on $\|c\|_\infty$ can be found explicitly.

We will also need an observability inequality for the wave equation (here, the quantity $R(x^0)$ is as in Section 1):

Proposition 5. *Assume that $c \in L^\infty(Q)$, $\alpha \geq 0$, $\beta > 0$ and $T - 2\alpha > 2R(x^0)$. There exists a positive constant C_2 depending on $\|c\|_\infty$, x^0, β, Ω, α and T such that, for any solution v of* (18) *with $k \equiv 0$ and $(v^0, v^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, the following holds:*

$$\|(v^0, v^1)\|^2_{L^2 \times H^{-1}} \leq C_2 \int_\alpha^{T-\alpha} \int_{\mathcal{G}_\beta(x^0)} |v|^2 \, dx \, dt.$$

The proof of proposition 5 can be found in [11]. There, the way the constant C_2 depends on $\|c\|_\infty$ is explicitly indicated.

In this Section, we will assume that the positive parameters α and β have been fixed in such a way that $T - 2\alpha > 2R(x^0)$ and $0 < \beta < \kappa$, whence $\mathcal{G}_\beta(x^0) \subset \mathcal{G}_\kappa(x^0) \subset \omega \cap \mathcal{O}$.

Let $(z^0, p^0, p^1) \in H^1_0(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$ be given and let (z, p) be the associated solution to (16)–(17). We first notice that

$$\|z(\cdot, 0)\|^2_{H^1_0} \leq C \left(\int_0^T \int_\Omega |p \mathbb{1}_{\mathcal{O}}|^2 \, dx \, dt + \int_0^T \int_{\mathcal{G}_\kappa(x^0)} |z|^2 \, dx \, dt \right) \tag{49}$$

for some constant C independent of (z^0, p^0, p^1). Indeed, we have from proposition 4 that

$$\begin{aligned}
&\int_{T/4}^{3T/4} \int_\Omega |\nabla z|^2 \, dx \, dt \leq C \int_0^T \int_\Omega e^{-\frac{2\zeta(x)}{t(T-t)}} \phi(t) |\nabla z|^2 \, dx \, dt \\
&\leq C \left(\int_0^T \int_\Omega e^{-\frac{2\zeta(x)}{t(T-t)}} |p \mathbb{1}_{\mathcal{O}}|^2 \, dx \, dt + \int_0^T \int_{\mathcal{G}_\kappa(x^0)} e^{-\frac{2\zeta(x)}{t(T-t)}} \phi(t)^3 |z|^2 \, dx \, dt \right) \\
&\leq C \left(\int_0^T \int_\Omega |p \mathbb{1}_{\mathcal{O}}|^2 \, dx \, dt + \int_0^T \int_{\mathcal{G}_\kappa(x^0)} |z|^2 \, dx \, dt \right)
\end{aligned} \tag{50}$$

On the other hand, multiplying the equation in (17) by $-\Delta z$ and integrating with respect to space and time in $\Omega \times (0, t)$ for each $t \in (T/4, 3T/4)$, we find:

$$\begin{aligned}
\frac{1}{2} \|z(\cdot, 0)\|^2_{H^1_0} &+ \int_0^t \int_\Omega |\Delta z|^2 \, dx \, dt + \int_0^t \int_\Omega a z (-\Delta z) \, dx \, dt \\
&= \frac{1}{2} \|z(\cdot, t)\|^2_{H^1_0} + \int_0^t \int_\Omega p \mathbb{1}_{\mathcal{O}} \cdot (-\Delta z) \, dx \, dt,
\end{aligned}$$

whence

$$\|z(\cdot, 0)\|^2_{H^1_0} \leq C \left(\|z(\cdot, t)\|^2_{H^1_0} + \int_0^T \int_\Omega |p \mathbb{1}_{\mathcal{O}}|^2 \, dx \, dt \right)$$

and

$$\|z(\cdot,0)\|^2_{H^1_0} \le C \left(\int_{T/4}^{3T/4} \int_\Omega |\nabla z|^2 \, dx \, dt + \int_0^T \int_\Omega |p \mathbb{1}_{\mathcal{O}}|^2 \, dx \, dt \right). \tag{51}$$

Combining (50) and (51), we obtain (49).

Also, we can apply proposition 5 to p, which gives

$$\|(p^0, p^1)\|^2_{L^2 \times H^{-1}} \le C_2 \int_\alpha^{T-\alpha} \int_{\mathcal{G}_\beta(x^0)} |p|^2 \, dx \, dt \tag{52}$$

for some positive C_2 only depending on $\|b\|_\infty$, x^0, β, Ω, α and T. Let us introduce the C^2 auxiliary functions $\eta_1 = \eta_1(x)$ and $\eta_2 = \eta_2(t)$, with

$$\begin{array}{l} 0 \le \eta_1 \le 1, \quad \eta_1(x) = 1 \quad \forall x \in \mathcal{G}_\beta(x^0), \quad \eta_1(x) = 0 \quad \forall x \notin \mathcal{G}_\kappa(x^0); \\ 0 \le \eta_2 \le 1, \quad \eta_2(t) = 1 \quad \forall t \in (\alpha, T-\alpha), \quad \mathrm{Supp}\, \eta_2 \subset [\alpha/2, T-\alpha/2]. \end{array}$$

Then

$$\left\{ \begin{array}{l} \displaystyle \int_\alpha^{T-\alpha} \int_{\mathcal{G}_\beta(x^0)} |p|^2 \, dx \, dt \le \int_{\alpha/2}^{T-\alpha/2} \int_{\mathcal{G}_\kappa(x^0)} \eta_1 \, \eta_2 \, \mathbb{1}_{\mathcal{O}} |p|^2 \, dx \, dt \\ \displaystyle \qquad = - \int_{\alpha/2}^{T-\alpha/2} \int_{\mathcal{G}_\kappa(x^0)} \eta_1 \, \eta_2 \, p \, (z_t + \Delta z - az) \, dx \, dt. \end{array} \right.$$

Integrating by parts and using that η_2 is supported by $[\alpha/2, T-\alpha/2]$, we find:

$$\left\{ \begin{array}{l} \displaystyle -\int_{\alpha/2}^{T-\alpha/2} \int_{\mathcal{G}_\kappa(x^0)} \eta_1(x) \, \eta_2(t) \, p \, (z_t + \Delta z - az) \, dx \, dt \\ \displaystyle \quad = \int_{\alpha/2}^{T-\alpha/2} \int_{\mathcal{G}_\kappa(x^0)} \eta_1(x) \eta_2'(t) \, p \, z \, dx \, dt + \int_{\alpha/2}^{T-\alpha/2} \eta_2(t) \, \langle p_t, \eta_1 z \rangle \, dt \\ \displaystyle \quad - \int_{\alpha/2}^{T-\alpha/2} \int_{\mathcal{G}_\kappa(x^0)} \eta_1(x) \, \eta_2(t) \, p \, \Delta z \, dx \, dt \\ \displaystyle \quad + \int_{\alpha/2}^{T-\alpha/2} \int_{\mathcal{G}_\kappa(x^0)} \eta_1(x) \, \eta_2(t) \, a \, p \, z \, dx \, dt. \end{array} \right.$$

Therefore, we have the following for any small $\varepsilon > 0$:

$$\left\{ \begin{array}{l} \displaystyle \int_\alpha^{T-\alpha} \int_{\mathcal{G}_\beta(x^0)} |p|^2 \, dx \, dt \\ \displaystyle \qquad \le - \int_{\alpha/2}^{T-\alpha/2} \int_{\mathcal{G}_\kappa(x^0)} \eta_1 \, \eta_2 \, p \, (z_t + \Delta z - az) \, dx \, dt \\ \displaystyle \qquad \le \varepsilon \int_0^T \left(\|p(\cdot,t)\|^2_{L^2} + \|p_t(\cdot,t)\|^2_{H^{-1}} \right) \, dt \\ \displaystyle \qquad \quad + \frac{C}{\varepsilon} \int_{\alpha/2}^{T-\alpha/2} \int_{\mathcal{G}_\kappa(x^0)} \left(|\Delta z|^2 + |z|^2 + |\nabla(\eta_1 z)|^2 \right) \, dx \, dt. \end{array} \right.$$

In view of the energy estimate (19) written for p, we find that

$$\left\{\begin{aligned} &\int_\alpha^{T-\alpha}\int_{\mathcal{G}_\beta(x^0)} |p|^2\,dx\,dt \\ &\qquad \le C_3\varepsilon \|(p^0,p^1)\|^2_{L^2\times H^{-1}} \\ &\qquad +\frac{C}{\varepsilon}\int_{\alpha/2}^{T-\alpha/2}\int_{\mathcal{G}_\kappa(x^0)} \left(|\Delta z|^2+|\nabla z|^2+|z|^2\right)\,dx\,dt. \end{aligned}\right. \tag{53}$$

Let us choose ε such that $C_2C_3\,\varepsilon \le 1/2$. Then, from (52) and (53), we see that

$$\left\{\begin{aligned} &\|(p^0,p^1)\|^2_{L^2\times H^{-1}} \\ &\qquad \le 2C_2\frac{C}{\varepsilon}\int_{\alpha/2}^{T-\alpha/2}\int_{\mathcal{G}_\kappa(x^0)} \left(|\Delta z|^2+|\nabla z|^2+|z|^2\right)\,dx\,dt \end{aligned}\right. \tag{54}$$

and

$$\left\{\begin{aligned} &\int_\alpha^{T-\alpha}\int_{\mathcal{G}_\beta(x^0)} |p|^2\,dx\,dt \\ &\qquad \le \left(2C_2C_3C+\frac{C}{\varepsilon}\right)\int_{\alpha/2}^{T-\alpha/2}\int_{\mathcal{G}_\kappa(x^0)} \left(|\Delta z|^2+|\nabla z|^2+|z|^2\right)\,dx\,dt. \end{aligned}\right. \tag{55}$$

In view of (49), (54) and (55), the following holds:

$$\left\{\begin{aligned} &\|(z,p,p_t)(\cdot,0)\|^2_{H_0^1\times L^2\times H^{-1}} \\ &\quad \le C\left(\int_{\alpha/2}^{T-\alpha/2}\int_{\mathcal{G}_\kappa(x^0)} (|\Delta z|^2+|\nabla z|^2)\,dx\,dt+\int_0^T\int_{\mathcal{G}_\kappa(x^0)}|z|^2\,dx\,dt\right). \end{aligned}\right. \tag{56}$$

Finally, let us consider $\xi\in C^2(\overline{\Omega})$ such that $0\le\xi\le 1$ in Ω,

$$\xi\equiv 1 \text{ in a neighborhood of } \omega_0 \text{ and } \xi\equiv 0 \text{ in } \Omega\setminus\omega_1$$

Taking into account the properties of ξ and the fact that $z=0$ on Σ, we can easily deduce the following:

$$\left\{\begin{aligned} &\int_0^T\int_{\mathcal{G}_\kappa(x^0)}|\nabla z|^2\,dx\,dt \le \int_0^T\int_\Omega \xi\,|\nabla z|^2\,dx\,dt = -\int_0^T\int_\Omega \nabla\cdot(\xi\,\nabla z)\,z\,dx\,dt \\ &\qquad = -\frac{1}{2}\int_0^T\int_\Omega \nabla\xi\cdot\nabla|z|^2\,dx\,dt-\int_0^T\int_\Omega \xi\,\Delta z\,z\,dx\,dt \\ &\qquad \le C\left(\int_0^T\int_{\omega_1}\left(|\Delta z|^2+|z|^2\right)\,dx\,dt\right). \end{aligned}\right.$$

Combining the properties of function ρ_ω (see (29)), the previous inequality and the estimate (56), we readily obtain (30). This ends the proof of proposition 1.

Remark 7. From the expressions of the function ζ and the constants C_1 and C_2 that can be found in [6], [5] and [11], it is not difficult to deduce an estimate of the constant C in (30) in terms of $\|a\|_\infty$ and $\|b\|_\infty$. More precisely, C can be taken of the form

$$C=e^{M(1+\|a\|_\infty+\|b\|^2_\infty)}$$

for some positive $M=M(\omega,\Omega,T)$. ∎

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