New phenomena for the null controllability of parabolic systems: Minimal time and geometrical dependence

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Abstract

We consider the null controllability problem for two coupled parabolic equations with a space-depending coupling term. We analyze both boundary and distributed null controllability. In each case, we exhibit a minimal time of control, that is to say, a time $T_0 \in [0, \infty]$ such that the corresponding system is null controllable at any time $T > T_0$ and is not if $T < T_0$. In the distributed case, this minimal time depends on the relative position of the control interval and the support of the coupling term. We also prove that, for a fixed control interval and a time $\tau_0 \in [0, \infty]$, there exist coupling terms such that the associated minimal time is τ_0.

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1 Introduction and main results

This paper deals with the controllability of non-scalar parabolic equations with a reduced number of controls. The control of parabolic systems is a challenging issue, which has attracted the interest of the control community in the last decade. These parabolic systems arise, for example, in the study of chemical reactions and in a wide variety of mathematical biology and physical situations (see e.g. [23], [33], [16], ...). More precisely, the aim of this paper is to investigate the relationship between the location of the controls and the action of the coupling terms. We will see that in this framework new phenomena arise.

To this end, let us fix $T > 0$ and $\omega = (a, b) \subset (0, \pi)$ and consider the following control problems:

$$\begin{cases} y_t - y_{xx} + q(x)A_0 y = 0 & \text{in } Q_T := (0,\pi) \times (0,T), \\ y(0,\cdot) = Bu, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \tag{1.1}$$

and

$$\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bv1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \tag{1.2}$$

where $A_0 \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ are respectively given by:

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.3}$$

In systems (1.1) and (1.2), $q \in L^\infty(0,\pi)$ is a given function, y_0 is the initial datum and $u \in L^2(0,T)$ and $v \in L^2(Q_T)$ are the control functions.

Let us remark that for every $u \in L^2(0,T)$ (resp., $v \in L^2(Q_T)$) and $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$ (resp., $y_0 \in L^2(0,\pi;\mathbb{R}^2)$), system (1.1) (resp., system (1.2)) possesses a unique solution defined by transposition (resp., a unique weak solution) which satisfies

$$y \in L^2(Q_T;\mathbb{R}^2) \cap C^0([0,T];H^{-1}(0,\pi;\mathbb{R}^2))$$
$$(\text{resp., } y \in L^2(0,T;H_0^1(0,\pi;\mathbb{R}^2)) \cap C^0([0,T];L^2(0,\pi;\mathbb{R}^2)))$$

and depends continuously on the data u and y_0, i.e., there exists a constant $C = C(T) > 0$ such that

$$\|y\|_{L^2(Q_T;\mathbb{R}^2)} + \|y\|_{C^0([0,T];H^{-1}(0,\pi;\mathbb{R}^2))} \le C\left(\|y_0\|_{H^{-1}(0,\pi;\mathbb{R}^2)} + \|u\|_{L^2(0,T)}\right)$$
$$(\text{resp., } \|y\|_{L^2(0,T;H_0^1(0,\pi;\mathbb{R}^2))} + \|y\|_{C^0([0,T];L^2(0,\pi;\mathbb{R}^2))} \le C\left(\|y_0\|_{L^2(0,\pi;\mathbb{R}^2)} + \|v\|_{L^2(Q_T)}\right)).$$

Let us recall that the function $y^* \in L^2(Q_T;\mathbb{R}^2) \cap C^0([0,T];H^{-1}(0,\pi;\mathbb{R}^2))$ (resp., the function $y^* \in L^2(0,T;H_0^1(0,\pi;\mathbb{R}^2)) \cap C^0([0,T];L^2(0,\pi;\mathbb{R}^2)))$ is a trajectory of system (1.1) (resp., of system (1.2)) if y^* is the solution of (1.1) (resp., of (1.2)) corresponding to the data $u^* \in L^2(0,T)$ and $y_0^* \in H^{-1}(0,\pi;\mathbb{R}^2)$ (resp., $v^* \in L^2(Q_T)$ and $y_0^* \in L^2(0,\pi;\mathbb{R}^2)$). With the previous notations, we define:

Definition 1.1. 1. It will be said that system (1.1) (resp., system (1.2)) is approximately controllable in $H^{-1}(0,\pi;\mathbb{R}^2)$ (resp., in $L^2(0,\pi;\mathbb{R}^2)$) at time T if for every $y_0, y_d \in H^{-1}(0,\pi;\mathbb{R}^2)$ (resp., $y_0, y_d \in L^2(0,\pi;\mathbb{R}^2)$) and for every $\varepsilon > 0$, there exists a control $u \in L^2(0,T)$ (resp., $v \in L^2(Q_T)$) such that the solution y to (1.1) (resp., to (1.2)) satisfies

$$\|y(\cdot,T) - y_d\|_{H^{-1}(0,\pi;\mathbb{R}^2)} \le \varepsilon \quad (\text{resp., } \|y(\cdot,T) - y_d\|_{L^2(0,\pi;\mathbb{R}^2)} \le \varepsilon).$$

2. It will be said that system (1.1) (resp., system (1.2)) is null controllable at time T if for every $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$ (resp., $y_0 \in L^2(0,\pi;\mathbb{R}^2)$), there exists a control $u \in L^2(0,T)$ (resp., $v \in L^2(Q_T)$) such that the solution y to (1.1) (resp., to (1.2)) satisfies

$$y(\cdot,T) = 0 \text{ in } H^{-1}(0,\pi;\mathbb{R}^2) \quad (\text{resp., in } L^2(0,\pi;\mathbb{R}^2)).$$

3. Finally, it will be said that system (1.1) (resp., system (1.2)) is exactly controllable to trajectories at time $T > 0$ if for every $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$ and every trajectory y^* of system (1.1) (resp., for every $y_0 \in L^2(0,\pi;\mathbb{R}^2)$ and every trajectory y^* of system (1.2)), there exists a control $u \in L^2(0,T)$ (resp., $v \in L^2(Q_T)$) such that the solution y to (1.1) (resp., to (1.2)) satisfies

$$y(\cdot,T) = y^*(\cdot,T) \text{ in } H^{-1}(0,\pi;\mathbb{R}^2) \quad (\text{resp., in } L^2(0,\pi;\mathbb{R}^2)).$$

In this work we are interested in studying the controllability properties of systems (1.1) and (1.2). Let us observe that we are exerting only one control force on the systems (a boundary or distributed control) but we want to control the corresponding state y which has two components. In fact, the first equation in (1.1) and (1.2) is indirectly controlled by means of the term $q(x)y_2$. Of course, this coupling term must be different from zero, i.e., $q \not\equiv 0$. On the other hand, using the linearity of systems (1.1) and (1.2), it is easy to see that the null controllability property at time T of the previous systems is equivalent to the exact controllability to trajectories at time T for these systems.

Systems (1.1) and (1.2) are particular classes of more general $n \times n$ parabolic control systems of the form:

$$\begin{cases} y_t - D\Delta y + A\,(x,t)\,y = Bv1_\omega & \text{in } Q_T := \Omega \times (0,T), \\ y = Cu1_{\Gamma_0}, & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

where ω and Γ_0 are, respectively, open subsets of the smooth bounded domain $\Omega \subset \mathbb{R}^N$ and of its boundary $\partial\Omega$, $D = \operatorname{diag}(d_1, \cdots, d_n) \in \mathcal{L}(\mathbb{R}^n)$, with $n \geq 1$, is a positive matrix, $B, C \in \mathcal{L}(\mathbb{R}^m,\ \mathbb{R}^n)$, with $m \leq n$, are given matrices, and $A = (a_{ij})_{1 \leq i,j \leq 2} \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^n))$ is a matrix-valued function. When $m < n$, the issue for this system is to control the whole components of the system with a control function acting, locally in space or on a part of the boundary, only on some of them. We refer to [5] for a review of results for the controllability problem of system (1.4).

The first results on controllability of the scalar case, $n = 1$, concerns the one-dimensional case $N = 1$. They have been established by H.O. Fattorini and D.L. Russell (see [19, 20]) through the moment method. The controllability of the N-dimensional case, still for the scalar equation ($n = 1$), has been established later by G. Lebeau and L. Robbiano in [31] and by A. Fursikov and O. Yu. Imanuvilov in [22] using Carleman estimates. It is interesting to point out that the boundary and distributed null controllability of scalar parabolic problems is valid for any positive time T, for any $\Gamma_0 \subset \partial\Omega$ and for any $\omega \subset \Omega$.

Let us also underline the reference [18], where the author proves the existence of a minimal control time for the one-dimensional heat equation with controls on the form $f(x)u(t)$, with $f \in H^{-1}(0,\pi)$, a given fixed function, and $u \in L^2(0,T)$.

The first results on controllability of coupled parabolic equations ($n > 1$) have been established in [35, 13, 3, 25]. They concern mainly system (1.4) with $n = 2$, $C = 0$ (distributed control) and

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In all the previous works the authors use Carleman inequalities for the corresponding adjoint system to (1.4). The main assumption on the matrix-valued function A is that there exist an open subset $\omega_0 \subset \omega$ and a positive constant σ such that

$$a_{12} \geq \sigma > 0 \quad \text{or} \quad a_{12} \leq -\sigma < 0 \quad \text{in} \quad \omega_0 \times (0,T). \tag{1.5}$$

It is interesting to point out that in [25], under the weaker assumption

$$|a_{12}| \geq \sigma > 0 \quad \text{in} \quad \omega_0 \times (0, T), \tag{1.6}$$

the authors prove a null controllability result at time $T > 0$ for some generalizations of system (1.4).

The previous controllability results have been extended in [26] to $n \geq 2$ when system (1.4) has a particular structure: *cascade systems*. To this end, the authors assume a generalization of assumption (1.5) on the coupling matrix $A(\cdot, \cdot)$ and, again, use Carleman inequalities for the adjoint problem for proving the null controllability result.

In [4], a necessary and sufficient condition for the approximate and null controllability at time $T > 0$ is established when A is a constant matrix. This condition does not depend on T and generalizes the algebraic Kalman condition (see [28]), well-known for the controllability of finite dimensional systems. In the case $n = 2$, this necessary and sufficient condition reduces to $a_{12} \neq 0$.

Let us now describe the existing results on boundary controllability of system (1.4) ($B = 0$). There are few results on this framework and most of them concern the one-dimensional case ($N = 1$), $D = Id$ and A a constant matrix. When $D = Id$ and A is a constant matrix, a necessary and sufficient condition is exhibited in [21] and [6]. This condition is different from the one that characterizes the distributed null controllability of system (1.4) in the constant case (see [4]). As a consequence and unlike the scalar case, we deduce that the distributed and boundary null controllability properties of non-scalar parabolic systems are in general not equivalent.

The boundary null controllability problem for system (1.4) in the constant case is more intricate if $D \neq Id$. When $n = 2$, the boundary null controllability property holds if T is greater than a minimal time $T_0 \in [0, \infty]$ which depends on the coefficients of the constant matrices D and A (see [10]). For instance, if the diffusion matrix D, the coupling matrix A and the control vector C are given by

$$D = \operatorname{diag}(1, d^2), \quad d \neq 0, 1, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1.7}$$

then system (1.4) is approximately controllable at time $T > 0$ if and only if d is an irrational number and the minimal time T_0 of null controllability depends on the diophantine approximation of d. Let us also underline that this phenomenon (minimal time of controllability for parabolic equations) has been observed for the first time in the scalar case in [18], but concerning pointwise controls. We would like to comment that, for system (1.4) with the previous data (1.7), it is possible to select positive numbers $d > 0$ for which system (1.4) is approximately controllable at any positive time T and never null controllable (see [10]). Unlike the scalar case, from the results in [10] we infer that the approximate and null boundary controllability properties for non-scalar parabolic systems are, in general, not equivalent.

In [21], [6] and [10], the authors use the moment method (see [19, 20]) to prove the positive null controllability result at time T. They carry out a study on bounds of biorthogonal families to exponentials associated to complex sequences. In fact, the previous minimal time T_0 is related to the index of condensation of the sequence of eigenvalues of the operator associated to the system (see [10]).

Finally, in [12] the authors extend the one-dimensional boundary null controllability results from [21] and [6] to the N-dimensional case when the domain Ω is a cylindrical domain.

Unlike the distributed controllability problem for system (1.4), Carleman estimates for the corresponding adjoint system seem not to be suitable when dealing with the boundary null controllability problem of system (1.4).

Let us come back to systems (1.1) and (1.2). First, observe that from the null controllability result stated in [25], if the function q satisfies (1.6), with $\sigma > 0$ and $\omega_0 \subset \omega$ an open interval, then system (1.2) is null controllable at any positive time T. Therefore, a natural question arises: what happens if $\operatorname{Supp} q \cap \omega = \emptyset$? The first and partial answer concerns the approximate controllability of this system. More precisely, in [29], the approximate controllability of system (1.2) at every time $T > 0$ is proved when $q = 1_{\mathcal{O}}$ with $\mathcal{O}$ a nonempty open subset of Ω. Later, other partial answers are given for the null controllability of systems (1.1) and (1.2) under sign conditions on the function q (see [2], [34], [1] and [17]):

$$q \not\equiv 0 \quad \text{and} \quad q \geq 0 \quad \text{or} \quad q \leq 0 \quad \text{ in } (0, \pi). \tag{1.8}$$

These results have been obtained as a consequence of the corresponding hyperbolic results by using the transmutation strategy (see [32]). Of course in the N-dimensional case ($N \geq 2$), they assume the Geometric Control Condition (CGC) defined in [11] on both sets ω and $\operatorname{Supp} q$. Clearly these assumptions are not necessary in the parabolic setting.

The first satisfying answer without sign conditions on q concerns the null controllability of systems (1.1) and (1.2) when q satisfies the condition

$$\int_0^\pi q(x)\, dx \neq 0. \tag{1.9}$$

Under this condition, in [7] the authors give a necessary and sufficient condition for the approximate and null controllability at time $T > 0$ of system (1.1). As a consequence, they also obtain the null controllability property at any positive time T for (1.2) under the same conditions.

The first general result for the distributed controllability of system (1.2) concerns the approximate controllability and is proved in [14]. For general open sets ω, the authors provide a necessary and sufficient condition for the approximate controllability of system (1.2) in terms of $\operatorname{Supp} q$ and the connected components of $\overline{\Omega} \setminus \omega$.

Some results presented here have been announced in [9]. In fact, in [9] the controllability of system (1.1) and the controllability of system (1.2) when the function q and the control interval $\omega = (a, b)$ satisfy the geometrical condition

$$\operatorname{Supp} q \subset [0, a] \quad \text{or} \quad \operatorname{Supp} q \subset [b, \pi] \tag{1.10}$$

are analyzed. Under the previous condition (1.10), a minimal time of boundary and distributed null controllability, $T_0(q) \in [0, +\infty]$, arises in such a way that these systems are null controllable at time T if $T \in (T_0(q), \infty)$ and are not when $T \in (0, T_0(q))$.

In this paper, we are going to provide a complete answer to the controllability problem of system (1.1) and system (1.2) without imposing condition (1.10) and when the control domain is an interval, $\omega = (a, b)$. More precisely, we will analyze the controllability properties of system (1.1), for a general function $q \in L^\infty(0, \pi)$, and of system (1.2), when q satisfies

$$\operatorname{Supp} q \cap \omega = \emptyset, \tag{1.11}$$

i.e., when $\operatorname{Supp} q \subset [0, a] \cup [b, \pi]$.

In the sequel, we set φ_k the normalized eigenvectors of the Dirichlet laplacian in $(0, \pi)$, i.e.,

$$\varphi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx), \quad \forall x \in (0, \pi), \quad k \geq 1.$$

On the other hand, the corresponding eigenvalues are given by k^2, $k \geq 1$.

For any $k \geq 1$, we associate with the function $q \in L^\infty(0, \pi)$ satisfying (1.11) the sequences $\{I_k(q)\}_{k\geq 1}$ and $\{I_{i,k}(q)\}_{k \geq 1}$, $i = 1, 2$, given by

$$\begin{cases} I_{1,k}(q) := \displaystyle\int_0^a q(x)|\varphi_k(x)|^2\, dx, \quad I_{2,k}(q) := \int_b^\pi q(x)|\varphi_k(x)|^2\, dx, \\ I_k(q) := I_{1,k}(q) + I_{2,k}(q) = \displaystyle\int_0^\pi q(x)|\varphi_k(x)|^2\, dx. \end{cases} \tag{1.12}$$

Let us present our boundary control results, that is, our main result related to system (1.1).

Theorem 1.1. *Let us consider A_0 and B given by (1.3) and $q \in L^\infty(0, \pi)$, a given function. Then, one has:*

1. *System (1.1) is approximately controllable at time $T > 0$ if and only if*

$$I_k(q) \neq 0 \quad \forall k \geq 1. \tag{1.13}$$

2. *Assume that condition (1.13) holds and define*

$$\widetilde{T}_0(q) := \limsup \frac{-\log |I_k(q)|}{k^2} \in [0, \infty] \,. \tag{1.14}$$

Then, if $T > \widetilde{T}_0(q)$ system (1.1) is null controllable at time T. On the other hand, if $T < \widetilde{T}_0(q)$ system (1.1) is not null controllable at time T.

This result has been announced in [9].

Remark 1.2. The approximate controllability result stated in Theorem 1.1 does not depend on the final time T: approximate controllability of system (1.1) at a time $T_0 > 0$ is equivalent to the approximate controllability of system (1.1) at *any* time $T > 0$. On the other hand, condition (1.13) characterizes the approximate controllability property of system (1.1). Thus, (1.13) is a necessary condition for the null controllability at time $T > 0$ of this system.

Remark 1.3. Note that the sequences $\{I_{i,k}(q)\}_{k\geq 1}$, $i = 1, 2$, and $\{I_k(q)\}_{k\geq 1}$ are convergent and from a simple computation one has:

$$\lim I_k(q) = \frac{1}{\pi} \int_0^\pi q(x)\, dx, \quad \lim I_{1,k}(q) = \frac{1}{\pi} \int_0^a q(x)\, dx, \quad \lim I_{2,k}(q) = \frac{1}{\pi} \int_b^\pi q(x)\, dx.$$

From this, it readily follows that the sequence $\{I_k(q)^{-1}\}_{k\in\Lambda}$ is bounded and $\widetilde{T}_0(q) = 0$ whenever condition (1.9) holds (for the expression of the set Λ, see (1.16)). Observe that, under condition (1.8) on the function q, (1.13) holds and $\widetilde{T}_0(q) = 0$. In particular, Theorem 1.1 generalizes the one-dimensional parabolic boundary controllability results obtained in [1] and [34].

Remark 1.4. We will see in Section 7 that there are functions $q \in L^\infty(0, \pi)$ such that $\widetilde{T}_0(q) > 0$ (in fact, $\widetilde{T}_0(q)$ may take any value in $[0, \infty]$). In particular, Theorem 1.1 implies that, even in a parabolic setting, a positive time of control may appear and that, unlike the scalar case, boundary approximate and null controllability are not equivalent properties in the non-scalar case (see also [10] for a similar result). On the other hand, Theorem 1.1 also infers negative boundary controllability results for hyperbolic versions of system (1.1). Indeed, if $q \in L^\infty(0, \pi)$ is such that $\widetilde{T}_0(q) > 0$, the transmutation strategy (see [32]) implies that the corresponding hyperbolic version of (1.1) is not controllable in the natural space associated to the system (see Theorem 3.6 in [1]) at any time $T > 0$.

For the distributed control problem (system (1.2)), let us first recall a recent result on approximate controllability:

Theorem 1.2 ([14])**.** *Let us consider A_0 and B given by (1.3) and $q \in L^\infty(0, \pi)$, a function satisfying (1.11). Then, system (1.2) is approximately controllable at time $T > 0$ if and only if*

$$|I_k(q)| + |I_{1,k}(q)| \neq 0 \quad \forall k \geq 1. \tag{1.15}$$

For the sake of completeness this result will be proved in Section 4.

Remark 1.5. As in the boundary case, the approximate controllability result for system (1.2) does not depend on the final time T: system (1.2) is approximately controllable at a time $T_0 > 0$ if and only if it is approximately controllable at any time $T > 0$.

To state our null controllability result for system (1.2) when $q \in L^\infty(0,\pi)$ satisfies (1.11), we need some definitions and notations. First, let us define the sets

$$\begin{cases} \Lambda := \{k \geq 1 : I_k(q) \neq 0\} = \Lambda_1 \cup \Lambda_2, \\ \Lambda_1 := \{k \in \Lambda : I_{1,k}(q) \neq 0\}, \quad \Lambda_2 := \{k \in \Lambda : I_{1,k}(q) = 0\} \quad \text{and} \\ \Lambda_3 := \{k \geq 1 : I_k(q) = 0\}, \end{cases} \tag{1.16}$$

where $I_k(q)$ and $I_{1,k}(q)$ are given in (1.12). Observe that Λ_1, Λ_2 and Λ_3 are disjoint sets and, of course, $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 = \Lambda \cup \Lambda_3 = \mathbb{N}^*$.

On the other hand, let us assume that the function $q \in L^\infty(0,\pi)$ is such that condition (1.15) holds. Thus, we can introduce the quantities $-\log|I_{1,k}(q)|$ and $-\log|I_k(q)|$ where we will use the notation $-\log|x| = \infty$ when $x = 0$. With this notation and under assumption (1.15), we deduce

$$\min\{-\log|I_{1,k}(q)|, -\log|I_k(q)|\} \in \mathbb{R}, \quad \forall k \geq 1.$$

One has:

Theorem 1.3. *Let us consider $A_0 \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$, given by (1.3), and $q \in L^\infty(0,\pi)$, a function satisfying (1.11). Let us also assume condition (1.15), and define*

$$T_0(q) := \limsup \frac{\min\{-\log|I_{1,k}(q)|, -\log|I_k(q)|\}}{k^2}. \tag{1.17}$$

Then, given $T > 0$, one has:

1. *Assume that $T > T_0(q)$. Then, system (1.2) is null controllable at time T.*
2. *If $T < T_0(q)$, then system (1.2) is not null controllable at time T.*

As in the boundary case, condition (1.15) characterizes the distributed approximate controllability of system (1.2). This implies that (1.15) is a necessary condition for the distributed null controllability at time $T > 0$ of (1.2).

We end the presentation of our main results with some remarks.

Remark 1.6. Under condition (1.15), the minimal time $T_0(q)$ is well-defined and, taking into account Remark 1.3, satisfies $T_0(q) \in [0,\infty]$. We will check in Section 7 that, given the control interval $\omega = (a,b)$, there are functions $q \in L^\infty(0,\pi)$, which fulfill condition (1.11), for which $T_0(q) > 0$ and even $T_0(q) = \infty$. For such functions, system (1.2) is approximately controllable at all positive time T but it is not null controllable at time T if $T \in (0, T_0(q))$. Again and unlike the scalar case, the distributed approximate property is not equivalent, in general, to the distributed null controllability property in the non-scalar case. Also, following the reasoning in Remark 1.4, from Theorem 1.3, we deduce that when $q \in L^\infty(0,\pi)$ satisfies (1.11) and $T_0(q) > 0$, the hyperbolic version of system (1.2) is not controllable in the natural space associated to the system (see Theorem 3.5 in [1] and Definition 1.1 in [17] for the definition of this space) at any time $T > 0$.

Remark 1.7. Let us fix a function $q \in L^\infty(0,\pi)$ and a control interval ω that satisfies (1.11). Taking into account that condition (1.13) implies (1.15) and the inequality $T_0(q) \leq \widetilde{T}_0(q)$, the boundary controllability at time $T > 0$ of system (1.1) implies the distributed controllability at time $T > 0$ of system (1.2) provided the control interval ω satisfies (1.11). But, it is interesting to note that there exist functions $q \in L^\infty(0,\pi)$ and control intervals ω fulfilling condition (1.11) for which $T_0(q) < \widetilde{T}_0(q)$ (see Example 7.3). This provides another difference with the scalar case: boundary and distributed controllability are not equivalent in the non-scalar parabolic setting. However, if $\omega = (a,b)$ and $q \in L^\infty(0,\pi)$ are such that (1.10) holds, then $T_0(q) = \widetilde{T}_0(q)$ and system (1.1) is null controllable at time $T > 0$ if and only if system (1.2) is also null controllable at time T.

Remark 1.8. The minimal time $T_0(q)$ (see (1.17)) depends on the function q but also on the position of the control interval ω (satisfying (1.11)). This fact provides a new phenomenon in the framework of the distributed controllability of non-scalar parabolic problems: the dependence of the controllability result on the position of the control set. Indeed, we will also see in Section 7 (see Example 7.3) that, given $\tau_0 \in (0, \infty]$ (which could be $\tau_0 = \infty$), there exist a function $q \in L^\infty(0, \pi)$ and control intervals $\omega_1, \omega_2 \subset (0, \pi)$, satisfying (1.11), such that (1.15) holds, for ω_1 and ω_2, and

$$T_0^{(1)}(q) = 0 \quad \text{and} \quad T_0^{(2)}(q) = \tau_0 > 0.$$

In the previous equalities, $T_0^{(i)}(q)$ is the minimal time associated to the function q and to the interval ω_i (see (1.17)). In conclusion, system (1.2) is null controllable at every positive time T, if the control is exerted on ω_1, but it is not null controllable at time T if $T \in (0, \tau_0)$ and the control is exerted on ω_2. This is another big difference with the scalar parabolic case. This dependence of the zone of control was highlighted in [14], in the case of the approximate controllability of system (1.2).

Remark 1.9. Under assumption (1.8) on the function q, conditions (1.13) and (1.15) hold for every interval $\omega \subset (0, \pi)$ satisfying (1.11). This means that systems (1.1) and (1.2) are approximately controllable at any positive time T. In fact, taking into account Remark 1.3, we get that $\widetilde{T}_0(q) = T_0(q) = 0$ and systems (1.1) and (1.2) are also null controllable at any positive time T. Thus, our results recover the one-dimensional parabolic version of the results in [29], [34], [1] and [17], with less restrictive assumptions on q.

The rest of the paper is organized as follows: In Section 2 we set and analyze some preliminary results related to the spectrum and the (generalized) eigenspaces of the operator associated with systems (1.1) and (1.2). Section 3 is devoted to studying the boundary controllability problem for system (1.1), namely to the proof of Theorem 1.1. For clarity, this section has been divided into two subsections; in the first one it can be found the proofs concerning the approximate controllability of system (1.1). In Subsection 3.2, the null-controllability property of this system is proved. The distributed approximate controllability problem is considered in Section 4. Theorem 1.3 is proved in Sections 5 (the positive null-controllability part) and 6 (negative null-controllability part). The last section contains some complementary results and some examples that illustrate the different situations.

2 Some preliminary results

In this section we will give some properties which will be used below. Let us consider the vectorial operator

$$L := -\frac{d^2}{dx^2} Id + q(x) A_0 : D(L) \subset L^2(0, \pi; \mathbb{R}^2) \longrightarrow L^2(0, \pi; \mathbb{R}^2) \tag{2.1}$$

with domain $D(L) = H^2(0, \pi; \mathbb{R}^2) \cap H_0^1(0, \pi; \mathbb{R}^2)$ and also its adjoint L^*. We will always denote by $\langle \cdot, \cdot \rangle$ the standard scalar product of either $L^2(0, \pi; \mathbb{R})$ or $L^2(0, \pi; \mathbb{R}^2)$, by $\langle \cdot, \cdot \rangle_{X', X}$ the duality pairing between the Hilert space X and its dual X'.

We are interested in studying the spectrum of the operators L and L^*. To this end, given a function $q \in L^\infty(0, \pi)$, we consider the quantity $I_k(q)$ given by (1.12), $k \geq 1$. With this notation, one has:

Proposition 2.1. *Let A_0 be given by (1.3) and consider the operator L given by (2.1) and its adjoint L^*. Then,*

1. *The spectra of L and L^* are given by $\sigma(L) = \sigma(L^*) = \{k^2 : k \geq 1\}$.*
2. *Given $k \geq 1$, if*

$$\Phi_{1,k} = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}, \quad \Phi_{2,k} = \begin{pmatrix} \psi_k \\ \varphi_k \end{pmatrix},$$

(resp., if

$$\Phi^*_{1,k} := \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix}, \quad \Phi^*_{2,k} := \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}),$$

where ψ_k is the unique solution of the non-homogeneous Sturm-Liouville problem:

$$\begin{cases} -\psi_{xx} - k^2\psi = [I_k(q) - q(x)]\, \varphi_k \ \text{in } (0,\pi), \\ \psi(0) = 0, \quad \psi(\pi) = 0, \\ \displaystyle\int_0^\pi \psi(x)\varphi_k(x)\, dx = 0, \end{cases} \tag{2.2}$$

then,

$$(L - k^2 I_d)\Phi_{1,k} = 0 \quad \text{and} \quad (L - k^2 I_d)\Phi_{2,k} = I_k(q)\Phi_{1,k} \tag{2.3}$$

(resp.,

$$\left(L^* - k^2 I_d\right) \Phi^*_{1,k} = I_k(q)\Phi^*_{2,k} \quad \text{and} \quad \left(L^* - k^2 I_d\right) \Phi^*_{2,k} = 0). \tag{2.4}$$

In particular, if $k \in \Lambda$ then k^2 is a simple eigenvalue and $\Phi_{1,k}$ and $\Phi_{2,k}$ (resp., $\Phi^*_{2,k}$ and $\Phi^*_{1,k}$) are, respectively, an eigenfunction and a generalized eigenfunction of the operator L (resp., L^*) associated to k^2, while if $k \in \Lambda_3$ then $\Phi_{1,k}$ and $\Phi_{2,k}$ are both eigenfunctions of L (resp., L^*) associated to k^2.

Proof. First, L can be written

$$L = \begin{pmatrix} -\Delta & q \\ 0 & -\Delta \end{pmatrix}$$

where $\Delta = \frac{d^2}{dx^2} : L^2(0,\pi) \longrightarrow L^2(0,\pi)$ with domain $D(\Delta) = H^2(0,\pi) \cap H^1_0(0,\pi)$ is, as is well-known, boundedly invertible with compact inverse. We can check that:

$$L^{-1} = \begin{pmatrix} (-\Delta)^{-1} & -(-\Delta)^{-1} \circ q \circ (-\Delta)^{-1} \\ 0 & (-\Delta)^{-1} \end{pmatrix}$$

which readily implies that L^{-1} is a compact operator on $L^2(0,\pi;\mathbb{R}^2)$. Thus, the spectrum of L reduces to its point spectrum.

We have now to solve the eigenvalue problem:

$$\begin{cases} -y_1'' + qy_2 = \lambda y_1 \text{ in } (0,\pi), \\ -y_2'' = \lambda y_2 \text{ in } (0,\pi), \\ y_1(0) = y_2(0) = 0, \quad y_1(\pi) = y_2(\pi) = 0. \end{cases}$$

If $y_2 \equiv 0$, then, $\lambda = k^2$ is an eigenvalue of L and taking $y_1 = \varphi_k$ we obtain $\Phi_{1,k}$ as associated eigenfunction of L. If we now assume that $y_2 \not\equiv 0$, then, again $\lambda = k^2$ and $y_2 = \varphi_k$ is a (normalized) solution to the second o.d.e. Observe that the first equation admits a solution if and only if $k \in \Lambda_3$, i.e., $I_k(q) = 0$. In this case, $\Phi_{2,k}$ is a second associated eigenfunction of L. In conclusion, if $k \in \Lambda_3$, then k^2 is a double eigenvalue of L.

From the above considerations, it is clear that if $I_k(q) \neq 0$, then the eigenvalue k^2 of L is simple and $\Phi_{1,k}$ is an associated eigenfunction. Observe that, taking $\Phi_{2,k} = (y_1, y_2)$, the equation $(L - k^2 I_d)\Phi_{2,k} = c\Phi_{1,k}$ writes:

$$\begin{cases} -y_1'' - k^2 y_1 = c\varphi_k - qy_2 \text{ in } (0,\pi), \\ -y_2'' - k^2 y_2 = 0 \text{ in } (0,\pi), \\ y_1(0) = y_2(0) = 0, \quad y_1(\pi) = y_2(\pi) = 0. \end{cases}$$

Thus, again, choosing $y_2 = \varphi_k$ and inserting this expression in the first equation, we get for y_1:

$$\begin{cases} -y_1'' - k^2 y_1 = [c - q]\varphi_k, \\ y_1(0) = y_2(0) = 0 \end{cases}$$

A necessary and sufficient condition for the previous nonhomogeneous Sturm-Liouville problem to have a solution is that

$$\int_0^\pi [c - q(x)] |\varphi_k(x)|^2 \, dx = 0, \quad \text{i.e.,} \quad c = I_k(q).$$

With this value of c, the Sturm-Liouville problem has a continuum of solutions given by $y_1 = \gamma\varphi_k + \psi_k$ where $\gamma \in \mathbb{R}$ is arbitrary and ψ_k is the unique solution of (2.2). This proves that, for $k \in \Lambda$, $\Phi_{2,k}$ is a generalized eigenfunction of L associated to k^2.

Note that since the eigenvalues of L are real, then $\sigma(L^*) = \sigma(L)$ and the corresponding eigenspaces have the same dimension. Finally, reasoning as before, it is not difficult to prove the assertions concerning L^*. This ends the proof. □

In the next result we are going to give an explicit expression and some properties of the function ψ_k. This expression and properties will be used later and will be crucial in the proof of Theorems 1.1, 1.2 and 1.3. One has:

Proposition 2.2. *Let us fix $q \in L^\infty(0,\pi)$ and take $k \geq 1$. Then, one has:*

$$\begin{cases} \psi_k(x) = \alpha_k \varphi_k(x) - \dfrac{1}{k} \displaystyle\int_0^x \sin(k(x-\xi)) \, [I_k(q) - q(\xi)] \, \varphi_k(\xi) \, d\xi, \\ \alpha_k = \dfrac{1}{k} \displaystyle\int_0^\pi \int_0^x \sin(k(x-\xi)) \, [I_k(q) - q(\xi)] \, \varphi_k(\xi) \varphi_k(x) \, d\xi \, dx. \end{cases} \tag{2.5}$$

In addition, there exists a constant $C > 0$ such that

$$|\alpha_k| \leq \frac{C}{k}, \quad \|\psi_k\|_{L^\infty(0,\pi)} \leq \frac{C}{k}, \quad \|\psi_k'\|_{L^\infty(0,\pi)} \leq C, \quad \forall k \geq 1. \tag{2.6}$$

Proof. Let us fix $k \geq 1$. Starting from formulae (2.5), it is straightforward that ψ_k satisfies (2.2).

Finally, the properties (2.6) can be easily deduced from the formulae (2.5) . This finalizes the proof. □

Using the eigenfunctions and the generalized eigenfunctions of the operators L and L^* (see Proposition 2.1), we are going to construct two bases of the space $L^2(0,\pi;\mathbb{R}^2)$. To this end, let us consider the sets

$$\begin{cases} \mathcal{B} = \{\Phi_{1,k}, \Phi_{2,k} : k \in \mathbb{N}^*\}, \\ \mathcal{B}^* = \left\{\Phi^*_{1,k}, \Phi^*_{2,k} : k \in \mathbb{N}^*\right\}, \end{cases} \tag{2.7}$$

where $\Phi_{i,k}$ and $\Phi^*_{i,k}$, $i = 1, 2$, are given in the statement of Proposition 2.1. The next result states that $\mathcal{B}$ and $\mathcal{B}^*$ are bases for the space $L^2(0,\pi;\mathbb{R}^2)$. One has:

Lemma 2.3. *Given a function $q \in L^\infty(0,\pi)$, the sequences $\mathcal{B}$ and $\mathcal{B}^*$ are biorthogonal Riesz bases in $L^2(0,\pi;\mathbb{R}^2)$.*

Proof. Let us first prove that $\mathcal{B}$ and $\mathcal{B}^*$ are biorthogonal families. Indeed, using (2.3) and (2.4), it readily follows from the equality:

$$\left\langle (L - k^2 I_d)\Phi_{\mu,k}, \Phi^*_{\nu,j} \right\rangle = \left\langle \Phi_{\mu,k}, (L^* - k^2 I_d)\Phi^*_{\nu,j} \right\rangle$$

that

$$\delta_{2\mu} I_k(q) \left\langle \Phi_{1,k}, \Phi^*_{\nu,j} \right\rangle = (j^2 - k^2) \left\langle \Phi_{\mu,k}, \Phi^*_{\nu,j} \right\rangle + \delta_{1\nu} I_j(q) \left\langle \Phi_{\mu,k}, \Phi^*_{2,j} \right\rangle \tag{2.8}$$

where $k, j \geq 1, \nu, \mu \in \{1, 2\}$ and $\delta_{\mu\nu}$ is the Kronecker symbol (equal to 1 if $\mu = \nu$, and to 0 otherwise). We claim that

$$j \neq k \Rightarrow \left\langle \Phi_{\mu,k}, \Phi^*_{\nu,j} \right\rangle = 0, \text{ for } \nu, \mu \in \{1, 2\}.$$

Actually, if $j \neq k$, setting $(\mu, \nu) = (1, 2)$ in (2.8) leads to $\left\langle \Phi_{1,k}, \Phi^*_{2,j} \right\rangle = 0$ and setting $(\mu, \nu) = (1, 1)$ gives $\left\langle \Phi_{1,k}, \Phi^*_{1,j} \right\rangle = 0$. Using this and considering the cases $(\mu, \nu) = (2, 2)$ and then $(\mu, \nu) = (2, 1)$ give the other orthogonality relations. For $j = k$, direct computations show that $\left\langle \Phi_{\nu,k}, \Phi^*_{\mu,k} \right\rangle = \delta_{\nu\mu}$ for $\nu, \mu = 1, 2$.

Let us show that $\mathcal{B}$ is complete in $L^2(0, \pi; \mathbb{R}^2)$. Indeed, if $f = (f_1, f_2)$ is such that

$$\langle f, \Phi_{\nu,k} \rangle = 0, \ \forall k \geq 1, \quad \forall \nu = 1, 2,$$

then in particular

$$\forall k \geq 1, \quad \begin{cases} \langle f_1, \varphi_k \rangle = 0, \\ \langle f_1, \psi_k \rangle + \langle f_2, \varphi_k \rangle = 0. \end{cases}$$

This implies that $f_1 = f_2 = 0$ (since $\{\varphi_k\}_{k \geq 1}$ is an orthonormal basis in $L^2(0, \pi)$) and proves the completeness of $\mathcal{B}$. We prove in the same way that $\mathcal{B}^*$ is complete in $L^2(0, \pi; \mathbb{R}^2)$.

We are now ready to show that $\mathcal{B}$ is a Riesz basis for the space $L^2(0, \pi; \mathbb{R}^2)$. To this end, we use the following result which can be found in [24] or [27] for instance:

Lemma 2.4. *Let $\{x_k\}_{k \geq 1}$ be a sequence in a Hilbert space X. Then the following statements are equivalent.*

1. *$\{x_k\}_{k \geq 1}$ is a Riesz basis in X.*
2. *$\{x_k\}_{k \geq 1}$ is a complete Bessel sequence in X and possesses a biorthogonal system $\{y_k\}_{k \geq 1}$ that is also a complete Bessel sequence in X.* □

We recall that the sequence $\{x_k\}_{k \geq 1}$ in the Hilbert space X is a Bessel sequence if it satisfies

$$\sum_{k \geq 1} | \langle x, x_k \rangle_X |^2 < \infty, \quad \forall x \in X.$$

As a consequence of the previous lemma, the task consists in showing that the series

$$S_1 = \sum_{k \geq 1} \left[\langle f, \Phi_{1,k} \rangle^2 + \langle f, \Phi_{2,k} \rangle^2 \right] \quad \text{and} \quad S_2 = \sum_{k \geq 1} \left[\left\langle f, \Phi^*_{1,k} \right\rangle^2 + \left\langle f, \Phi^*_{2,k} \right\rangle^2 \right]$$

converge for any $f \in L^2(0, \pi; \mathbb{R}^2)$.

It is easy to see that:

$$S_1 = \sum_{k \geq 1} \left[|f_{1,k}|^2 + |\langle f_1, \psi_k \rangle + f_{2,k}|^2 \right], \ S_2 = \sum_{k \geq 1} \left[|f_{1,k} + \langle f_2, \psi_k \rangle|^2 + |f_{2,k}|^2 \right]$$

where $f_{i,k}$ is the Fourier coefficient of the function f_i with respect to φ_k. Relation (2.6) allows to bound S_i $(i = 1, 2)$ as follows:

$$S_i \leq C \sum_{k \geq 1} \left(|f_{1,k}|^2 + |f_{2,k}|^2 + \frac{1}{k^2} \right) < \infty.$$

This proves the convergence of the series S_1 and S_2 and finishes the proof. □

Remark 2.1. It is interesting to point out that, indeed, $\mathcal{B}$ is quadratically close to the canonical orthonormal basis of $L^2(0, \pi; \mathbb{R}^2)$ (see [24] for a definition):

$$\widetilde{\mathcal{B}} = \left\{ \Theta_{1,k} = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}, \Theta_{2,k} = \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} \right\}_{k \geq 1}$$

since, thanks to (2.6)

$$\sum_{k \geq 1} \left(\|\Phi_{1,k} - \Theta_{1,k}\|^2_{L^2(0,\pi;\mathbb{R}^2)} + \|\Phi_{2,k} - \Theta_{2,k}\|^2_{L^2(0,\pi;\mathbb{R}^2)} \right) = \sum_{k \geq 1} \|\psi_k\|^2_{L^2(0,\pi)} < \infty.$$

Corollary 2.5. *Given a function $q \in L^\infty(0,\pi)$, then $\mathcal{B}^*$ is a basis in $H_0^1(0,\pi;\mathbb{R}^2)$, biorthogonal to $\mathcal{B} \subset H^{-1}(0,\pi;\mathbb{R}^2)$, where $\mathcal{B}^*$ and $\mathcal{B}$ are given in (2.7).*

Proof. We take $L^2(0,\pi;\mathbb{R}^2)$ as a pivot space and then

$$H_0^1(0,\pi;\mathbb{R}^2) \hookrightarrow L^2(0,\pi;\mathbb{R}^2) \hookrightarrow H^{-1}(0,\pi;\mathbb{R}^2) = \left(H_0^1(0,\pi;\mathbb{R}^2)\right)'.$$

First, it is clear that $\mathcal{B}^* \subset H_0^1(0,\pi;\mathbb{R}^2)$ and is complete in this space since it is in $L^2(0,\pi;\mathbb{R}^2)$. On the other hand, by definition of the duality pairing, $\left\langle \Phi_{\nu,k}, \Phi^*_{\mu,j}\right\rangle_{H^{-1},H_0^1} = \left\langle \Phi_{\nu,k}, \Phi^*_{\mu,j}\right\rangle = \delta_{\nu\mu}\delta_{kj}$ for $k,j \geq 1$ and $\nu,\mu \in \{1,2\}$. Thus $\mathcal{B} \subset H^{-1}(0,\pi;\mathbb{R}^2)$ is biorthogonal to $\mathcal{B}^*$ and $\mathcal{B}^*$ is minimal in $H_0^1(0,\pi;\mathbb{R}^2)$. It remains to prove that for any $f = (f_1,f_2) \in H_0^1(0,\pi;\mathbb{R}^2)$, the series

$$\sum_{k\geq 1}\left\{\langle f,\Phi_{1,k}\rangle \Phi^*_{1,k} + \langle f,\Phi_{2,k}\rangle \Phi^*_{2,k}\right\} = \begin{pmatrix} \displaystyle\sum_{k\geq 1} f_{1,k}\varphi_k \\ \displaystyle\sum_{k\geq 1} f_{1,k}\psi_k + \sum_{k\geq 1}\left(\int_0^\pi f_1(x)\psi_k(x)\,dx + f_{2,k}\right)\varphi_k \end{pmatrix} \tag{2.9}$$

converges in $H_0^1(0,\pi;\mathbb{R}^2)$. But $\sum_{k\geq 1} f_{i,k}\varphi_k$, $i=1,2$, converges in $H_0^1(0,\pi)$ ($f_{i,k}$ is the Fourier coefficient of the function $f_i \in H_0^1(0,\pi)$ with respect to φ_k). On the other hand, the series $\sum_{k\geq 1} f_{1,k}\psi_k$ also converges in $H_0^1(0,\pi)$. Indeed, since $f_1 \in H_0^1(0,\pi)$, one has

$$\sum_{k\geq 1} |f_{1,k}|\, \|\psi_k'\|_{L^2(0,\pi)} \leq C\sum_{k\geq 1}|f_{1,k}| \leq C\left(\sum_{k\geq 1} k^2\,|f_{1,k}|^2 + \sum_{k\geq 1}\frac{1}{k^2}\right) < \infty,$$

where we have used the properties in (2.6).

Let us assume that there exists a constant $C>0$ such that the inequality

$$\left|\int_0^\pi f_1(x)\psi_k(x)\,dx\right| \leq \frac{C}{k^2}, \quad \forall k \geq 1, \tag{2.10}$$

holds for any function $f_1 \in H_0^1(0,\pi)$. This implies that the series

$$\sum_{k\geq 1} k^2 \left|\int_0^\pi f_1(x)\psi_k(x)\,dx\right|^2$$

converges and assures the convergence in $H_0^1(0,\pi)$ of the series

$$\sum_{k\geq 1}\left(\int_0^\pi f_1(x)\psi_k(x)\,dx\right)\varphi_k.$$

This completes the proof of the convergence in $H_0^1(0,\pi;\mathbb{R}^2)$ of the series in (2.9) and finalizes the proof.

Let us see (2.10) for a function $f \in H_0^1(0,\pi)$:

$$\int_0^\pi f(x)\psi_k(x)\,dx = -\int_0^\pi f'(x)\int_0^x \psi_k(s)\,ds\,dx.$$

From the expression of ψ_k (see (2.5) and (2.6)), we get $\psi_k(x) = \alpha_k\varphi_k(x) - \frac{1}{k}H_k(x)$ with $|\alpha_k| \leq C/k$ and

$$H_k(x) = \sin(kx)\int_0^x \cos(k\xi)h_k(\xi)\varphi_k(\xi)\,d\xi - \cos(kx)\int_0^x \sin(k\xi)h_k(\xi)\varphi_k(\xi)\,d\xi,$$

with $h_k = I_k(q) - q$. Therefore, the function h_k is uniformly bounded in $(0,\pi)$. Also, we have:

$$\int_0^x \psi_k(s)\,ds = \sqrt{\frac{2}{\pi}}\frac{\alpha_k}{k}\left(1-\cos(kx)\right) - \frac{1}{k}\int_0^x H_k(s)\,ds,$$

and

$$\int_0^x H_k(s)\,ds = -\frac{1}{k}\cos(kx)\int_0^x \cos(k\xi)h_k(\xi)\varphi_k(\xi)\,d\xi + \frac{1}{k}\int_0^x \cos^2(k\xi)h_k(\xi)\varphi_k(\xi)\,d\xi$$
$$- \frac{1}{k}\sin(kx)\int_0^x \sin(k\xi)h_k(\xi)\varphi_k(\xi)\,d\xi + \frac{1}{k}\int_0^x \sin^2(k\xi)h_k(\xi)\varphi_k(\xi)\,d\xi.$$

Therefore, for some positive constant C,

$$\left|\int_0^x \psi_k(s)\,ds\right| \leq \frac{C}{k^2}, \quad \forall x \in (0,\pi), \quad k \geq 1.$$

Combining the previous formulas we get (2.10). □

Let us finish this section by giving an expression in ω of the function ψ_k, the solution of (2.2). This expression will be crucial in the proof of Theorem 1.3.

Proposition 2.6. *Let $q \in L^\infty(0,\pi)$ be a function satisfying (1.11) with $\omega = (a,b) \subset (0,\pi)$. Let us also consider the function ψ_k constructed in Proposition 2.1. Then, for any $k \geq 1$:*

$$\psi_k(x) = \tau_k\varphi_k(x) + g_k(x), \quad \forall x \in \omega,$$

with

$$\tau_k = \alpha_k + \sqrt{\frac{\pi}{2}}\frac{1}{k}\int_0^a q(\xi)\varphi_k(\xi)\cos(k\xi)\,d\xi,$$

and

$$g_k(x) = -\frac{I_k(q)}{k}\int_0^x \sin(k(x-\xi))\varphi_k(\xi)\,d\xi - \sqrt{\frac{\pi}{2}}\frac{I_{1,k}(q)}{k}\cos(kx), \quad \forall x \in \omega,$$

where the quantities α_k, $I_k(q)$ and $I_{1,k}(q)$ are respectively given in (2.5) and (1.12).

Proof. Fix $k \geq 1$. The function ψ_k is given by (2.5). Taking into account (1.11), if $x \in \omega$, one gets from formulae (2.5)

$$\begin{aligned}\psi_k(x) &= \alpha_k\varphi_k(x) - \frac{I_k(q)}{k}\int_0^x \sin(k(x-\xi))\varphi_k(\xi)\,d\xi + \frac{1}{k}\int_0^a \sin(k(x-\xi))q(\xi)\varphi_k(\xi)\,d\xi \\ &= \tau_k\varphi_k(x) + g_k(x),\end{aligned}$$

where τ_k and g_k are given above. □

3 Boundary controllability problem

In this section we will prove Theorem 1.1. To this end, let $q \in L^\infty(0,\pi)$ and A_0 given by (1.3). We introduce the backward adjoint problem associated with systems (1.1) and (1.2):

$$\begin{cases} -\theta_t - \theta_{xx} + q(x)A_0^*\theta = 0 & \text{in } Q_T, \\ \theta(0,\cdot) = 0, \quad \theta(\pi,\cdot) = 0 & \text{on } (0,T), \\ \theta(\cdot,T) = \theta_0 & \text{in } (0,\pi), \end{cases} \tag{3.1}$$

where $\theta_0 \in L^2(0,\pi;\mathbb{R}^2)$ is a given initial datum. Let us first see that this problem is well posed. One has:

Proposition 3.1. *Assume that $\theta_0 \in L^2(0,\pi;\mathbb{R}^2)$ is given. Then, system (3.1) admits a unique solution $\theta \in L^2(0,T;H_0^1(0,\pi;\mathbb{R}^2)) \cap C^0([0,T];L^2(0,\pi;\mathbb{R}^2))$ which writes*

$$\theta(\cdot,t) = \sum_{k\geq 1} e^{-k^2(T-t)}\left(\langle\theta_0,\Phi_{1,k}\rangle\left(\Phi_{1,k}^* - (T-t)\,I_k(q)\,\Phi_{2,k}^*\right) + \langle\theta_0,\Phi_{2,k}\rangle\,\Phi_{2,k}^*\right) \tag{3.2}$$

and in addition satisfies

$$\|\theta\|_{L^2(0,T;H_0^1(0,\pi;\mathbb{R}^2))} + \|\theta\|_{C^0([0,T];L^2(0,\pi;\mathbb{R}^2))} \leq C\|\theta_0\|_{L^2(0,\pi;\mathbb{R}^2)},$$

for a positive constant C independent of θ_0. Furthermore, if $\theta_0 \in H_0^1(0,\pi;\mathbb{R}^2)$, then the solution θ of the adjoint problem (3.1) satisfies

$$\theta \in L^2(0,T;H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)) \cap C^0([0,T];H_0^1(0,\pi;\mathbb{R}^2)),$$

and

$$\|\theta\|_{L^2(0,T;H^2\cap H_0^1(0,\pi;\mathbb{R}^2))} + \|\theta\|_{C^0([0,T];H_0^1(0,\pi;\mathbb{R}^2))} \leq C\|\theta_0\|_{H_0^1(0,\pi;\mathbb{R}^2)},$$

for a new constant $C > 0$ independent of θ_0.

The next proposition provides a relation between systems (1.1) and (3.1):

Proposition 3.2. *Let us consider A_0 and B given by (1.3) and $q \in L^\infty(0,\pi)$. Then, for any $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, $u \in L^2(0,T;\mathbb{R}^2)$ and $\theta_0 \in H_0^1(0,\pi;\mathbb{R}^2)$, one has*

$$\int_0^T u(t)B^*\theta_x(0,t)\,dt = \langle y(\cdot,T),\theta_0\rangle_{H^{-1},H_0^1} - \langle y_0,\theta(\cdot,0)\rangle_{H^{-1},H_0^1},$$

where $y \in L^2(Q_T;\mathbb{R}^2) \cap C^0([0,T];H^{-1}(0,\pi;\mathbb{R}^2))$ and $\theta \in L^2(0,T;H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)) \cap C^0([0,T];H_0^1(0,\pi;\mathbb{R}^2))$ are, resp., the solutions to (1.1) and (3.1) associated with (u,y_0) and θ_0.

For a proof of the previous results see for instance [36] or [21].

The controllability of system (1.1) can be characterized in terms of appropriate properties of the solutions to the adjoint problem (3.1). More precisely, we have:

Proposition 3.3. *Under the previous assumptions, one has:*

1. *System (1.1) is approximately controllable at time $T > 0$ if and only if the following unique continuation property holds:*

   "Let $\theta_0 \in H_0^1(0,\pi;R^2)$ be given and let θ be the corresponding solution of the adjoint problem (3.1). Then, if $B^*\theta_x(0,t) = 0$ on $(0,T)$, one has $\theta_0 \equiv 0$ in $(0,\pi)$."

2. *System (1.1) is null controllable at time $T > 0$ if and only if there exists a positive constant C such that the observability inequality*

$$\|\theta(\cdot,0)\|^2_{H_0^1(0,\pi;\mathbb{R}^2)} \leq C\int_0^T |B^*\theta_x(0,t)|^2\,dt \tag{3.3}$$

   holds for every $\theta_0 \in H_0^1(0,\pi;\mathbb{R}^2)$. In (3.3), θ is the adjoint state associated with θ_0.

Again, this result is well known. For a proof see, for instance, [37], [15], [36] or [21].

3.1 Boundary approximate controllability

This subsection is devoted to proving the approximate controllability of system (1.1), that is to say, the first point of Theorem 1.1. To this end, we are going to apply item 1 of Proposition 3.3. Recall that $A_0 \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ are given in (1.3) and $q \in L^\infty(0,\pi)$ is a given function.

Necessary condition: By contradiction, let us assume that condition (1.13) does not hold, i.e., that there is $k_0 \geq 1$ for which $I_{k_0}(q) = 0$. Let us see that the unique continuation property for the adjoint system (3.1) is no longer valid. Indeed, let us take $\theta_0 = a\Phi^*_{1,k_0} + b\Phi^*_{2,k_0} \in H_0^1(0,\pi;\mathbb{R}^2)$, with $a,b \in \mathbb{R}$ to be determined. In this case, $k_0 \in \Lambda_3$ (see (1.16)) and the functions Φ^*_{1,k_0} and Φ^*_{2,k_0} are eigenfunctions of the operator L^* (see Proposition 2.1) associated with the eigenvalue

k_0^2. Thus, it is not difficult to see that the corresponding solution to the adjoint problem (3.1) is given by

$$\theta(\cdot,t) = e^{-k_0^2(T-t)}\left[a\Phi^*_{1,k_0} + b\Phi^*_{2,k_0}\right] = e^{-k_0^2(T-t)}\begin{pmatrix} a\varphi_{k_0} \\ a\psi_{k_0} + b\varphi_{k_0}\end{pmatrix} \quad \text{in } Q_T. \tag{3.4}$$

Therefore,

$$B^*\theta_x(0,t) = e^{-k_0^2(T-t)}\left(a\psi'_{k_0}(0) + bk_0\sqrt{\frac{2}{\pi}}\right), \quad \forall t \in (0,T).$$

Just taking $a = k_0\sqrt{2/\pi}$ and $b = -\psi'_{k_0}(0)$ we get $B^*\theta_x(0,\cdot) = 0$ on $(0,T)$ but $\theta \not\equiv 0$. So, system (1.1) is not approximately controllable at time $T>0$. This proves the necessary part of the first point of Theorem 1.1.

Sufficient condition: Let us now assume that condition (1.13) holds. The task now is to prove that the unique continuation property for the solutions of the adjoint problem (3.1) holds. To this end, let us take $\theta_0 \in H_0^1(0,\pi;\mathbb{R}^2)$ and assume that the corresponding solution θ of (3.1) satisfies

$$B^*\theta_x(0,t) = 0 \quad \forall t \in (0,T).$$

From Corollary 2.5, we know that $\mathcal{B}^*$ is a basis for $H_0^1(0,\pi;\mathbb{R}^2)$ (for the expression of $\mathcal{B}^*$, see (2.7)). Therefore,

$$\theta_0(x) = \sum_{k\ge 1}\left(a_k\Phi^*_{1,k}(x) + b_k\Phi^*_{2,k}(x)\right),$$

where the previous series converges in $H_0^1(0,\pi;\mathbb{R}^2)$. In this series the coefficients are given by $a_k = \langle\theta_0, \Phi_{1,k}\rangle$ and $b_k = \langle\theta_0,\Phi_{2,k}\rangle$ for any $k\ge 1$. In view of (3.2) in Proposition 3.1, we have:

$$\theta(x,t) = \sum_{k\ge 1} e^{-k^2(T-t)}\left[a_k\left(\Phi^*_{1,k} - (T-t)\,I_k\,(q)\,\Phi^*_{2,k}\right) + b_k\Phi^*_{2,k}\right].$$

This series converges in $C^0([0,T];H_0^1(0,\pi;\mathbb{R}^2))$ and this property allows us to write:

$$\begin{cases} B^*\theta_x(0,t) = \displaystyle\sum_{k\ge 1} e^{-k^2(T-t)}\left\{\left[a_kB^*\Phi^*_{1,k,x}(0) + b_kB^*\Phi^*_{2,k,x}(0)\right] - (T-t)I_k\,(q)\,a_kB^*\Phi^*_{2,k,x}(0)\right\} \\ \qquad = \displaystyle\sum_{k\ge1} e^{-k^2(T-t)}\left[a_k\psi'_k(0) + b_kk\sqrt{\frac{2}{\pi}}\right] - \sum_{k\ge 1}(T-t)e^{-k^2(T-t)}I_k\,(q)\,a_kk\sqrt{\frac{2}{\pi}} \end{cases}$$

for any $t\in(0,T)$. On the other hand, from the results proved in [21] and [6] about biorthogonal families to exponentials, we infer that the family $\{e^{-k^2t}, te^{-k^2t}\}_{k\ge1} \subset L^2(0,T)$ is minimal[1] in $L^2(0,T)$. Recall that we have assumed $B^*\theta_x(0,\cdot)\equiv 0$ on the interval $(0,T)$. Then, the expression of $B^*\theta_x(0,\cdot)$ together with the property of the exponentials imply $a_k = b_k = 0$ for any $k\ge 1$. This proves the continuation property for the solutions to the adjoint problem (3.1) and, thanks to Proposition 3.3, the approximate controllability of system (1.1) at any positive time T.

3.2 Boundary null controllability

In this subsection we will study the null controllability properties of system (1.1), i.e., we will prove the second item in Theorem 1.1. Let us first observe that condition (1.13) is a necessary condition for having the null controllability property of system (1.1) at time $T>0$. This, in particular, implies that $\Lambda = \mathbb{N}^*$ (the set Λ is given in (1.16)) and $\widetilde{T}_0(q)$, given by (1.14), is well defined and $\widetilde{T}_0(q)\in[0,\infty]$.

[1] We say that a sequence $\{x_k\}_{k\ge1}$ in a Banach space X is minimal if for any $l\ge1$ one has $x_l \notin \overline{\mathrm{span}}\,\{x_k : k\ne l\}$.

3.2.1 Positive boundary controllability result

Let us assume that $T > \widetilde{T}_0(q) \in [0,\infty)$ (see (1.14)). Our objective is to prove that system (1.1) is exactly controllable to zero at time T. To this end, for $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, we will reformulate the null controllability problem as a moment problem.

Using Propositions 3.1 and 3.2, we deduce that the control $u \in L^2(0,T)$ drives the solution of (1.1) to zero at time T if and only if $u \in L^2(0,T)$ satisfies

$$\int_0^T u(t)B^*\theta_x(0,t)\,dt = -\langle y_0, \theta(\cdot,0)\rangle_{H^{-1},H_0^1}, \quad \forall \theta_0 \in H_0^1(0,\pi;\mathbb{R}^2),$$

where $\theta \in C^0([0,T];H_0^1(0,\pi;\mathbb{R}^2))$ is the solution to the adjoint problem (3.1) associated with θ_0. Since $\mathcal{B}^*$ is a basis of $H_0^1(0,\pi;\mathbb{R}^2)$ (see Corollary 2.5), the null controllability property at time T for system (1.1) is equivalent to find $u \in L^2(0,T)$ such that

$$\int_0^T u(t)B^*\theta_x^{i,k}(0,t)\,dt = -\langle y_0, \theta^{i,k}(\cdot,0)\rangle_{H^{-1},H_0^1}, \quad \forall k\geq 1, \quad \forall i = 1,2, \tag{3.5}$$

where $\theta^{i,k}$ is the solution of system (3.1) associated with $\theta_0 = \Phi^*_{i,k}$ (for the expression of the function $\Phi^*_{i,k}$, see Proposition 2.1). Let us take

$$u(t) = v(T-t), \quad t\in(0,T).$$

Developing the equality (3.5), one has:

1. If we take $\theta_0 = \Phi^*_{2,k}$, the solution of the adjoint problem is $\theta^{2,k}(\cdot,t) = e^{-k^2(T-t)}\Phi^*_{2,k}$ and (3.5) becomes,
$$\int_0^T e^{-k^2t}v(t)\,dt = -\frac{1}{k}\sqrt{\frac{\pi}{2}}e^{-k^2T}\langle y_0,\Phi^*_{2,k}\rangle_{H^{-1},H_0^1} := e^{-k^2T}\widetilde{M}_1^{(k)}(y_0), \quad \forall k\geq 1.$$
It is easy to see that
$$\left|\widetilde{M}_1^{(k)}(y_0)\right| \leq C\|y_0\|_{H^{-1}(0,\pi;\mathbb{R}^2)}, \quad \forall k\geq 1, \tag{3.6}$$
for a positive constant C independent of k and y_0.

2. Let us now take $\theta_0 = \Phi^*_{1,k}$. In this case the solution of the adjoint system (3.1) is
$$\theta^{1,k}(\cdot,t) = e^{-k^2(T-t)}\Phi^*_{1,k} - (T-t)\,I_k\,(q)\,e^{-k^2(T-t)}\Phi^*_{2,k}$$
and, then, the equality (3.5) transforms into ($u(t) = v(T-t)$, $t\in(0,T)$)
$$\left\{\begin{array}{l} \psi_k'(0)\displaystyle\int_0^T e^{-k^2t}v(t)\,dt - I_k(q)\varphi_k'(0)\int_0^T te^{-k^2t}v(t)\,dt \\ \qquad\qquad\qquad = -e^{-k^2T}\left[\langle y_0,\Phi^*_{1,k}\rangle_{H^{-1},H_0^1} - TI_k\,(q)\,\langle y_0,\Phi^*_{2,k}\rangle_{H^{-1},H_0^1}\right]. \end{array}\right.$$
Thus, the control $u = v(T-\cdot)$ must also satisfy
$$\int_0^T te^{-k^2t}v(t)\,dt = \frac{e^{-k^2T}}{I_k(q)}\widetilde{M}_2^{(k)}(y_0), \quad \forall k\geq 1,$$
where
$$\widetilde{M}_2^{(k)}(y_0) := \frac{1}{k}\sqrt{\frac{\pi}{2}}\left\{\psi_k'(0)\widetilde{M}_1^{(k)}(y_0) + \left[\langle y_0,\Phi^*_{1,k}\rangle_{H^{-1},H_0^1} - TI_k\,(q)\,\langle y_0,\Phi^*_{2,k}\rangle_{H^{-1},H_0^1}\right]\right\}.$$
Using the properties of the function ψ_k stated in Proposition 2.2 (see (2.6)), one has
$$\left|\widetilde{M}_2^{(k)}(y_0)\right| \leq C\|y_0\|_{H^{-1}(0,\pi;\mathbb{R}^2)}, \quad \forall k\geq 1. \tag{3.7}$$
for a new positive constant C independent of k and y_0.

Summarizing, we have proved that $u \in L^2(0,T)$ is such that the solution y of system (1.1) satisfies $y(\cdot,T)=0$ in $(0,\pi)$ if and only if $v=u(T-\cdot)\in L^2(0,T)$ satisfies

$$\left\{ \begin{array}{l} \displaystyle\int_0^T e^{-k^2 t} v(t)\,dt = e^{-k^2 T}\widetilde{M}_1^{(k)}(y_0), \\ \displaystyle\int_0^T te^{-k^2 t} v(t)\,dt = \frac{e^{-k^2 T}}{I_k(q)}\widetilde{M}_2^{(k)}(y_0), \quad \forall k \geq 1, \end{array} \right. \tag{3.8}$$

with $\widetilde{M}_1^{(k)}(y_0)$ and $\widetilde{M}_2^{(k)}(y_0)$ satisfying (3.6) and (3.7).

From the results in [21] (see also [6]), we can conclude that the sequence

$$\left\{ e_{1,k} := e^{-k^2 t}, e_{2,k} := te^{-k^2 t} \right\}_{k\geq 1}$$

admits a biorthogonal family $\{q_{1,k}, q_{2,k}\}_{k\geq1}$ in $L^2(0,T)$, i.e., a family $\{q_{1,k}, q_{2,k}\}_{k\geq1}$ in $L^2(0,T)$ satisfying

$$\int_0^T e_{r,k} q_{s,j}(t)\,dt = \delta_{kj}\delta_{rs}, \quad \forall k,j \geq 1, \quad 1 \leq r,s \leq 2, \tag{3.9}$$

which moreover satisfies that for every $\varepsilon > 0$ there exists a constant $C_{\varepsilon,T} > 0$ such that

$$\|q_{i,k}\|_{L^2(0,T)} \leq C_{\varepsilon,T}\, e^{\varepsilon k^2}, \quad \forall k \geq 1, \quad i = 1,2. \tag{3.10}$$

Using the formulas in (3.8) and the property (3.9), we infer that an explicit formal solution of the moment problem (3.5) is given by

$$u(T-t) = v(t) = \sum_{k\geq1} e^{-k^2 T}\left(\widetilde{M}_1^{(k)}(y_0) q_{1,k}(t) + \frac{1}{I_k(q)}\widetilde{M}_2^{(k)}(y_0) q_{2,k}(t) \right).$$

Let us see that this series defines an element of $L^2(0,T)$ when $T > \widetilde{T}_0(q)$, i.e., the previous series converges in $L^2(0,T)$ if $T > \widetilde{T}_0(q)$. Indeed, from the definition of the minimal time $\widetilde{T}_0(q)$ (see (1.14)) and for any fixed $\varepsilon > 0$, we can infer that there exists a positive constant C_ε such that

$$\frac{1}{|I_k(q)|} \leq C_\varepsilon e^{k^2\left(\widetilde{T}_0(q)+\varepsilon\right)}, \quad \forall k \geq 1.$$

On the other hand, we can use the bound (3.10) and get a new positive constant $C_{\varepsilon,T}$ for which

$$\left\{ \begin{array}{l} \displaystyle\left\| e^{-k^2 T}\left(\widetilde{M}_1^{(k)}(y_0) q_{1,k} + \frac{1}{I_k(q)}\widetilde{M}_2^{(k)}(y_0) q_{2,k} \right) \right\|_{L^2(0,T)} \leq C_{\varepsilon,T}\, \frac{e^{-k^2 T} e^{\varepsilon k^2}}{|I_k(q)|} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq C_{\varepsilon,T}\, e^{-k^2\left(T-\widetilde{T}_0(q)-2\varepsilon\right)}. \end{array} \right.$$

This last inequality proves the absolute convergence of the series which defines the control u since ε may be chosen arbitrarily small. This proves the null controllability of system (1.1) at time T when $T > \widetilde{T}_0(q)$.

3.2.2 Negative boundary controllability result

In order to finish the proof of Theorem 1.1, let us prove that if $0 < T < \widetilde{T}_0(q)$, then system (1.1) is not null controllable at time T. Recall that condition (1.13) holds. We argue by contradiction.

Assume that system (1.1) is null controllable at time $T < \widetilde{T}_0(q)$. By means of Proposition 3.3, this last fact is equivalent to the existence of a positive constant C such that the observability inequality (3.3) holds for every solution θ of the adjoint problem (3.1). Let us work with the particular solutions associated with initial data $\theta_0 = a_k\Phi^*_{1,k} + b_k\Phi^*_{2,k}$, with $a_k, b_k \in \mathbb{R}$, to be determined, and $\Phi^*_{1,k}$ and $\Phi^*_{2,k}$ given in Proposition 2.1. With this choice, the solution θ^k of (3.1) is given by

$$\theta^k(\cdot,t) = a_k e^{-k^2(T-t)}\left(\Phi^*_{1,k} - (T-t) I_k(q)\,\Phi^*_{2,k}\right) + b_k e^{-k^2(T-t)}\Phi^*_{2,k}, \quad \forall k \geq 1.$$

Thus, the observability inequality (3.3) becomes

$$A_{1,k} \leq C A_{2,k}, \quad \forall k \geq 1,$$

with

$$A_{1,k} := e^{-2k^2 T}\left\{k^2|a_k|^2 + \left[|a_k|^2\|\psi_k'\|^2_{L^2(0,\pi)} + k^2\left(b_k - T I_k(q) a_k\right)^2\right]\right\} \geq e^{-2k^2T}k^2|a_k|^2, \quad \forall k \geq 1,$$

and

$$A_{2,k} := \int_0^T e^{-2k^2 t}\left|a_k\psi_k'(0) + (b_k - tI_k(q)a_k)\varphi_k'(0)\right|^2\,dt, \quad \forall k \geq 1.$$

Taking $a_k = 1$ and $b_k = -\psi_k'(0)/\varphi_k'(0) = -\frac{1}{k}\sqrt{\frac{\pi}{2}}\psi_k'(0)$, the inequality observability transforms into

$$e^{-2k^2T}k^2 \leq A_{1,k} \leq CA_{2,k} = C\frac{2}{\pi}\left|I_k(q)\right|^2 k^2\int_0^T t^2 e^{-2k^2 t}\,dt, \quad \forall k \geq 1,$$

that is to say, for a new constant $C > 0$ not depending on k, one has,

$$1 \leq Ce^{2k^2T}\left|I_k(q)\right|^2, \quad \forall k \geq 1. \tag{3.11}$$

From the definition of $\widetilde{T}_0(q)$, we obtain the existence of an increasing unbounded subsequence $\{k_n\}_{n\geq 1}$ such that

$$\widetilde{T}_0(q) = \lim_{n\to\infty}\frac{-\log|I_{k_n}(q)|}{k_n^2} \in (0,\infty].$$

Assume that $0 < \widetilde{T}_0(q) < \infty$ (the case $\widetilde{T}_0(q) = \infty$ is much simpler and the details are left to the reader). In this case, for every $\varepsilon > 0$, there exits a positive integer n_ε such that

$$\widetilde{T}_0(q) - \varepsilon \leq \frac{-\log|I_{k_n}(q)|}{k_n^2}, \quad \forall n \geq n_\varepsilon.$$

This last inequality together with (3.11) provide the new inequality

$$1 \leq Ce^{-2k_n^2\left(\widetilde{T}_0(q) - T - \varepsilon\right)}, \quad \forall n \geq n_\varepsilon.$$

The previous inequality gives a contradiction if we take $0 < \varepsilon < \left(\widetilde{T}_0(q) - T\right)/2$. This ends the proof.

4 Distributed approximate controllability

In this section we will address the problem of the approximate controllability at time $T > 0$ of system (1.2), i.e, we will prove Theorem 1.2. As said above, Theorem 1.2 is a direct consequence of the results on approximate controllability stated in [14]. For the sake of completeness we will provide a direct proof of the result.

As in Section 3, we will first establish the relation between system (1.2) and (3.1). On the other hand, we will also give a general characterization of the controllability properties of system (1.2). One has:

Proposition 4.1. *Let us consider A_0 and B given by (1.3) and $q \in L^\infty(0,\pi)$, a given function. Then, for any $y_0 \in L^2(0,\pi;\mathbb{R}^2)$, $v \in L^2(Q_T)$ and $\theta_0 \in L^2(0,\pi;\mathbb{R}^2)$, one has*

$$\iint_{Q_T} v(x,t) 1_\omega B^*\theta(x,t)\, dx\, dt = \langle y(\cdot,T), \theta_0\rangle - \langle y_0, \theta(\cdot,0)\rangle ,$$

where $y, \theta \in L^2(0,T;H_0^1(0,\pi;\mathbb{R}^2)) \cap C^0([0,T];L^2(0,\pi;\mathbb{R}^2))$ are, resp., the solutions to (1.2) and (3.1) associated to (y_0,v) and θ_0.

For a proof of the previous result see for instance [15], [36] or [21].

Proposition 4.2. *Under assumptions of Proposition 4.1, one has:*

1. *System (1.2) is approximately controllable at time $T>0$ if and only if the following unique continuation property holds:*

   "Let $\theta_0 \in L^2(0,\pi;R^2)$ be given and let θ be the corresponding solution of the adjoint problem (3.1). Then, if $B^*\theta = 0$ in $\omega \times (0,T)$, one has $\theta_0 \equiv 0$ in $(0,\pi)$."

2. *System (1.2) is null controllable at time $T>0$ if and only if there exists a positive constant C such that the observability inequality*

$$\|\theta(\cdot,0)\|^2_{L^2(0,\pi;\mathbb{R}^2)} \le C \iint_{\omega\times(0,T)} |B^*\theta(x,t)|^2\, dx\, dt \tag{4.1}$$

   holds for every $\theta_0 \in L^2(0,\pi;\mathbb{R}^2)$. In (4.1), θ is the adjoint state associated to θ_0, i.e., the solution of (3.1) associated to θ_0.

Again, this result is very well known. For a proof see, for instance, [37], [15] or [36].

We can already prove Theorem 1.2. The arguments will be similar to those used in Section 3. We recall that $q \in L^\infty(0,\pi)$ is a function satisfying (1.11), where $\omega = (a,b)$.

Necessary condition: Again, we argue by contradiction. Let us suppose that condition (1.15) does not hold, i.e., that there exists $k_0 \ge 1$ such that $I_{k_0}(q) = I_{1,k_0}(q) = 0$. We will see that the distributed unique continuation property for the adjoint system (3.1) fails to be true.

First, from Proposition 2.6, the function ψ_{k_0} is given by:

$$\psi_{k_0}(x) = \tau_{k_0}\varphi_{k_0}(x), \quad \forall x \in \omega, \tag{4.2}$$

(since $I_{k_0}(q) = I_{1,k_0}(q) = 0$) where τ_{k_0} is given in Proposition 2.6.

On the other hand, let us take $\theta_0 = a\Phi^*_{1,k_0} + b\Phi^*_{2,k_0} \in L^2(0,\pi;\mathbb{R}^2)$, with $a,b \in \mathbb{R}$ to be determined. Again, the functions Φ^*_{1,k_0} and Φ^*_{2,k_0} are eigenfunctions of the operator L^* (see Proposition 2.1). Thus, the solution of the adjoint problem (3.1) is given by (3.4), so that:

$$B^*\theta(x,t) = e^{-k_0^2(T-t)}\left(a\psi_{k_0}(x) + b\varphi_{k_0}(x)\right) = e^{-k_0^2(T-t)}(a\tau_{k_0} + b)\varphi_{k_0}(x), \quad \forall (x,t) \in \omega \times (0,T),$$

thanks to (4.2). Just taking $a=1$ and $b = -\tau_{k_0}$ we obtain $B^*\theta \equiv 0$ in $\omega \times (0,T)$ and $\theta \not\equiv 0$. This contradicts the distributed unique continuation property for system (3.1). So, system (1.2) is not approximately controllable at time $T>0$. This proves the necessary part of Theorem 1.2.

Sufficient condition: Let us assume that condition (1.15) holds. The objective is to show that system (1.2) is approximately controllable at time T, when $q \in L^\infty(0,\pi)$ satisfies (1.11). This amounts to prove the distributed unique continuation property for system (3.1) stated in Proposition 4.2.

Let us fix $\theta_0 \in L^2(0,\pi;\mathbb{R}^2)$ and assume that the corresponding solution θ of (3.1) satisfies

$$B^*\theta \equiv 0 \quad \text{in} \quad \omega \times (0,T).$$

Since $\mathcal{B}^*$ is a basis for $L^2(0,\pi;\mathbb{R}^2)$ (for the expression of $\mathcal{B}^*$, see (2.7)), we can write

$$\theta_0 = \sum_{k\ge1}\left(a_k\Phi^*_{1,k} + b_k\Phi^*_{2,k}\right),$$

where the coefficients are given by $a_k = \langle \theta_0, \Phi_{1,k} \rangle$ and $b_k = \langle \theta_0, \Phi_{2,k} \rangle$ for any $k \geq 1$. As it has been already observed, we have:

$$\theta(\cdot, t) = \sum_{k \geq 1} e^{-k^2(T-t)} \left\{ a_k \left[\Phi^*_{1,k} - (T-t) I_k(q) \Phi^*_{2,k} \right] + b_k \Phi^*_{2,k} \right\} \quad \text{in} \quad Q_T.$$

In fact, following the ideas in Lemma 2.3, it is not difficult to prove the convergence of this series in $C^0([0,T]; L^2(0,\pi;\mathbb{R}^2))$. Thus,

$$\begin{cases} B^*\theta(\cdot,t)|_\omega = \displaystyle\sum_{k\geq 1} e^{-k^2(T-t)} \left[a_k B^*\Phi^*_{1,k}|_\omega + b_k B^*\Phi^*_{2,k}|_\omega \right] - \sum_{k \geq 1} (T-t) e^{-k^2(T-t)} a_k I_k(q) B^*\Phi^*_{2,k}|_\omega \\ \qquad = \displaystyle\sum_{k\geq 1} e^{-k^2(T-t)} \left[a_k \psi_k|_\omega + b_k \varphi_k|_\omega \right] - \sum_{k\geq 1} (T-t) e^{-k^2(T-t)} a_k I_k(q) \varphi_k|_\omega \end{cases}$$

for any $t \in (0,T)$. Using again that the family $\{e^{-k^2 t}, te^{-k^2 t}\}_{k\geq 1} \subset L^2(0,T)$ is minimal in $L^2(0,T)$ and the assumption $B^*\theta \equiv 0$ in $\omega \times (0,T)$ we get

$$a_k \psi_k|_\omega + b_k \varphi_k|_\omega \equiv 0 \quad \text{and} \quad a_k I_k(q) \varphi_k|_\omega \equiv 0 \quad \forall k \geq 1$$

It is clear that from the previous identities that $a_k = b_k = 0$ for all $k \in \Lambda$. On the other hand, taking into account the expression of the ψ_k in ω (see Proposition 2.6), the last equality becomes

$$(a_k \tau_k + b_k)\varphi_k(x) - \sqrt{\frac{\pi}{2}} \frac{I_{1,k}(q)}{k} a_k \cos(kx) = 0 \quad \forall x \in \omega, \quad \forall k \in \Lambda_3.$$

Using the independence of φ_k and the function $\cos(k\cdot)$ in ω, we conclude that $a_k = b_k = 0$ for every $k \in \Lambda_3$. This proves that $\theta_0 \equiv 0$. Therefore, we have proved the distributed continuation property for the solutions to the adjoint problem (3.1) and the approximate controllability of system (1.2) at any positive time T.

5 Proof of Theorem 1.3: The positive null controllability result

This section will be devoted to proving the null controllability of system (1.2) at time $T > 0$, when this time satisfies $T > T_0(q)$ ($T_0(q)$, given by (1.17), is assumed to be finite in this section). In order to make the proof clearer, we will divide it into several steps.

5.1 The moment problem

We start the proof of the first point of Theorem 1.3 by reformulating the null controllability property for system (1.2) as a moment problem. To this end, let us consider $T > T_0(q)$ ($T_0(q)$ is given by (1.17)). The aim is to prove that for any $y_0 \in L^2(0,\pi;\mathbb{R}^2)$ there exists a control $v \in L^2(Q_T)$ such that the corresponding solution y of system (1.2) satisfies $y(\cdot,T) = 0$ in $(0,\pi)$.

Let us fix an initial datum $y_0 \in L^2(0,\pi;\mathbb{R}^2)$. Thanks to Proposition 3.1 and 4.1, it is easy to see that the solution $y \in C^0([0,T];L^2(0,\pi;\mathbb{R}^2))$ of system (1.2) associated with y_0 and a control $v \in L^2(Q_T)$ satisfies $y(\cdot,T) = 0$ in $(0,\pi)$ if and only if the control $v \in L^2(Q_T)$ satisfies

$$\iint_{Q_T} v(x,t) 1_\omega B^*\theta(x,t)\, dx\, dt = -\langle y_0, \theta(\cdot,0) \rangle, \quad \forall \theta_0 \in L^2(0,\pi;\mathbb{R}^2),$$

where θ is the solution of the adjoint problem (3.1) corresponding to θ_0. Using that $\mathcal{B}^*$ is a basis of $L^2(0,\pi;\mathbb{R}^2)$ (see Lemma 2.3) this last property is equivalent to $v \in L^2(Q_T)$ and satisfies

$$\iint_{Q_T} v(x,t) 1_\omega B^*\theta_{i,k}(x,t)\, dx\, dt = -\langle y_0, \theta_{i,k}(\cdot,0) \rangle, \quad \forall k \geq 1, \quad \forall i = 1,2, \tag{5.1}$$

where $\theta_{i,k}$ denotes the solution of system (3.1) associated with $\theta_0 = \Phi^*_{i,k}$. By means of the previous problem we have reformulated the null controllability property for system (1.2) as a moment problem.

In order to solve the moment problem (5.1), the first main idea is to search controls under the particular form

$$v(x,t) = f_1(x)v_1(T-t) + f_2(x)v_2(T-t), \quad (x,t) \in Q_T, \tag{5.2}$$

where $v_1, v_2 \in L^2(0,T)$ are new controls, only depending on t, and $f_1, f_2 \in L^2(0,\pi)$ are appropriate functions satisfying the condition $\operatorname{Supp} f_1, \operatorname{Supp} f_2 \subseteq \omega = (a,b)$. This choice will be made clearer a little further in the text.

For $k \geq 1$ and $\theta_0 = \Phi^*_{2,k}$, the solution to (3.1) is given by $\theta_{2,k}(\cdot,t) = e^{-k^2(T-t)}\Phi^*_{2,k}$. Thus, after a change of variables, the moment problem (5.1) with controls v given by (5.2) reads as follows:

$$f_{1,k}\int_0^T v_1(t)e^{-k^2t}\,dt + f_{2,k}\int_0^T v_2(t)e^{-k^2t}\,dt = -e^{-k^2T}\left\langle y_0, \Phi^*_{2,k}\right\rangle,$$

where $f_{1,k}$, $f_{2,k}$ are, respectively, the Fourier coefficients with respect to φ_k corresponding to f_1, f_2:

$$f_{i,k} := \int_0^\pi f_i(x)\varphi_k(x)\,dx, \quad i = 1,2, \quad \forall k \geq 1. \tag{5.3}$$

For $\theta_0 = \Phi^*_{1,k}$, the corresponding solution of (3.1) is given by

$$\theta(\cdot,t) = e^{-k^2(T-t)}\left(\Phi^*_{1,k} - (T-t)I_k(q)\,\Phi^*_{2,k}\right).$$

From the expression of functions $\Phi^*_{i,k}$ (see the statement of Proposition 2.1), for $k \geq 1$ and $i = 1$, the equality (5.1) with controls v given by (5.2) changes into

$$\left\{\begin{aligned} &\widetilde{f}_{1,k}\int_0^T v_1(t)e^{-k^2t}\,dt + \widetilde{f}_{2,k}\int_0^T v_2(t)e^{-k^2t}\,dt \\ &\qquad - I_k(q)f_{1,k}\int_0^T v_1(t)\,te^{-k^2t}\,dt - I_k(q)f_{2,k}\int_0^T v_2(t)\,te^{-k^2t}\,dt \\ &\qquad = -e^{-k^2T}\left(\left\langle y_0, \Phi^*_{1,k}\right\rangle - TI_k(q)\left\langle y_0, \Phi^*_{2,k}\right\rangle\right), \end{aligned}\right.$$

where, for $k \geq 1$, $\widetilde{f}_{1,k}$, $\widetilde{f}_{2,k}$ are given by

$$\widetilde{f}_{i,k} := \int_0^\pi f_i(x)\psi_k(x)\,dx, \quad i = 1,2. \tag{5.4}$$

Let us point out that, thanks to the properties of the function ψ_k (see (2.6)), one has

$$\left|\widetilde{f}_{i,k}\right| \leq \frac{C}{k}, \quad i = 1,2, \qquad \text{if } k \geq 1, \tag{5.5}$$

for some positive constant C.

Summarizing, we have transformed the null-controllability problem at time $T > 0$ for system (1.2) into the following moment problem: Find $v \in L^2(Q_T)$ under the form (5.2) such that $v_1, v_2 \in L^2(0,T)$ satisfy

$$\left\{\begin{aligned} &f_{1,k}\int_0^T v_1(t)e^{-k^2t}\,dt + f_{2,k}\int_0^T v_2(t)e^{-k^2t}\,dt = -e^{-k^2T}\left\langle y_0, \Phi^*_{2,k}\right\rangle \\ &\widetilde{f}_{1,k}\int_0^T v_1(t)e^{-k^2t}\,dt + \widetilde{f}_{2,k}\int_0^T v_2(t)e^{-k^2t}\,dt \\ &\qquad - I_k(q)f_{1,k}\int_0^T v_1(t)\,te^{-k^2t}\,dt - I_k(q)f_{2,k}\int_0^T v_2(t)\,te^{-k^2t}\,dt \\ &\qquad = -e^{-k^2T}\left(\left\langle y_0, \Phi^*_{1,k}\right\rangle - TI_k(q)\left\langle y_0, \Phi^*_{2,k}\right\rangle\right), \end{aligned}\right. \quad k \geq 1, \tag{5.6}$$

with the notations in (5.3) and (5.4).

Our objective is to solve the previous moment problem under the assumption (1.15) and when $T > T_0(q)$ (see (1.17)). To this end, we will construct appropriate functions $f_1, f_2 \in L^2(0,\pi)$ satisfying $\operatorname{Supp} f_1, \operatorname{Supp} f_2 \subseteq \omega = (a,b)$. Let us remark that, if we fix $k \geq 1$, (5.6) is a linear system of two equations and four unknown quantities:

$$\int_0^T v_1(t)e^{-k^2 t}\,dt, \quad \int_0^T v_2(t)e^{-k^2 t}\,dt, \quad \int_0^T v_1(t)\,te^{-k^2 t}\,dt \quad \text{and} \quad \int_0^T v_2(t)\,te^{-k^2 t}\,dt.$$

The moment problem (5.6) can be written as

$$A_k V_k + \widetilde{A}_k \widetilde{V}_k = F_k \quad \forall k \geq 1, \tag{5.7}$$

whith for $k \geq 1$:

$$A_k = \begin{pmatrix} f_{1,k} & f_{2,k} \\ \widetilde{f}_{1,k} & \widetilde{f}_{2,k} \end{pmatrix}, \ \widetilde{A}_k = \begin{pmatrix} 0 & 0 \\ -I_k(q) f_{1,k} & -I_k(q) f_{2,k} \end{pmatrix} \tag{5.8}$$

$$V_k := \begin{pmatrix} \displaystyle\int_0^T v_1(t)e^{-k^2 t}\,dt \\ \displaystyle\int_0^T v_2(t)e^{-k^2 t}\,dt \end{pmatrix}, \quad \widetilde{V}_k := \begin{pmatrix} \displaystyle\int_0^T v_1(t)te^{-k^2 t}\,dt \\ \displaystyle\int_0^T v_2(t)te^{-k^2 k^2 t}\,dt \end{pmatrix}, \tag{5.9}$$

and

$$F_k = \begin{pmatrix} -e^{-k^2 T}\left\langle y_0, \Phi^*_{2,k}\right\rangle \\ -e^{-k^2 T}\left(\left\langle y_0, \Phi^*_{1,k}\right\rangle - T I_k(q)\left\langle y_0, \Phi^*_{2,k}\right\rangle\right) \end{pmatrix}. \tag{5.10}$$

Remind that $f_{i,k}$ is the Fourier coefficient of f_i with respect to φ_k and $\widetilde{f}_{i,k}$ is given by (5.4).

5.2 Construction of the functions f_1 and f_2

In this subsection we will construct appropriate functions $f_1, f_2 \in L^2(0,T)$ satisfying

$$\operatorname{Supp} f_1, \operatorname{Supp} f_2 \subseteq \omega,$$

which will allow us to solve the moment problem (5.7) . One has:

Lemma 5.1. *There exist functions $f_1, f_2 \in L^2(0,\pi)$ satisfying $\operatorname{Supp} f_1, \operatorname{Supp} f_2 \subseteq \omega$ and such that*

$$\begin{cases} \min\{|f_{1,k}|, |f_{2,k}|\} \geq \dfrac{C}{k^3}, \quad \forall k \geq 1, \\ |B_k| := \left| f_{1,k}\widehat{f}_{2,k} - f_{2,k}\widehat{f}_{1,k}\right| \geq \dfrac{C}{k^5}, \quad \forall k \geq 1. \end{cases} \tag{5.11}$$

In (5.11) C is a positive constant only depending on f_1 and f_2, $f_{i,k}$ ($i = 1,2$) is the Fourier coefficient of the function f_i with respect to φ_k and $\widehat{f}_{i,k}$ is given by

$$\widehat{f}_{i,k} = \int_0^\pi f_i(x)\cos(kx)\,dx, \quad k \geq 1, \quad i = 1,2. \tag{5.12}$$

Proof. Let us consider the functions $f_1 := 1_{(a_1,b_1)}$ and $f_2 := 1_{(a_2,b_2)}$ with $a_1, b_1, a_2, b_2 \in \omega$ and $a_i < b_i$, $i = 1,2$. Then,

$$\begin{cases} f_{i,k} = \displaystyle\int_0^\pi f_i(x)\varphi_k(x)\,dx = \frac{2}{k}\sqrt{\frac{2}{\pi}}\sin\left(k\frac{a_i+b_i}{2}\right)\sin\left(k\frac{b_i-a_i}{2}\right), \\ \widehat{f}_{i,k} = \displaystyle\int_0^\pi f_i(x)\cos(kx)\,dx = \frac{2}{k}\cos\left(k\frac{a_i+b_i}{2}\right)\sin\left(k\frac{b_i-a_i}{2}\right). \end{cases}$$

Direct computations show that

$$|B_k| = \frac{4}{k^2}\sqrt{\frac{2}{\pi}}\left|\sin\left(k\frac{b_1-a_1}{2}\right)\sin\left(k\frac{b_2-a_2}{2}\right)\sin\left(k\frac{a_1+b_1-a_2-b_2}{2}\right)\right|.$$

Let us now take $b_1 = a_1 + 2\ell$, $a_2 = a_1 + \ell$ and $b_2 = a_1 + 3\ell$, with $a_1 \in (a, (3a+b)/4)$ and $\ell \in (0, (b-a)/4)$ such that a_1/π is a rational number and ℓ/π is an irrational algebraic number of order 2. In this case, we have that $(a_1+\ell)/\pi$ and $(a_1+2\ell)/\pi$ are also irrational algebraic numbers of order 2. Thus, $a_1, b_1, a_2, b_2 \in \omega$ and $a_i < b_i$, $i = 1, 2$. On the other hand, let us admit the following property which will be proved below: if $\xi/\pi \in (0, \infty)$ is an irrational algebraic number of order 2, then

$$\inf_{k\geq 1}\left(k\left|\sin(k\xi)\right|\right) \geq C, \tag{5.13}$$

for a positive constant C only depending on ξ.

Coming back to the expressions of $f_{1,k}$, $f_{2,k}$ and $|B_k|$ and taking into account the previous property, one obtains

$$\begin{cases} |f_{1,k}| = \dfrac{2}{k}\sqrt{\dfrac{2}{\pi}}\left|\sin\left(k\left(a_1+\ell\right)\right)\right|\left|\sin\left(k\ell\right)\right| \geq \dfrac{C_1}{k^3}, \\ |f_{2,k}| = \dfrac{2}{k}\sqrt{\dfrac{2}{\pi}}\left|\sin\left(k\left(a_1+2\ell\right)\right)\right|\left|\sin\left(k\ell\right)\right| \geq \dfrac{C_2}{k^3}, \\ |B_k| = \dfrac{4}{k^2}\sqrt{\dfrac{2}{\pi}}\left|\sin\left(k\ell\right)\right|^3 \geq \dfrac{C_3}{k^5}, \quad \forall k \geq 1, \end{cases}$$

with C_1, C_2 and C_3 positive constants only depending on a_1 and ℓ. This proves (5.11).

Let us finalize the proof showing inequality (5.13). This inequality is a consequence of Liouville's theorem on diophantine approximation:

Lemma 5.2 ([30])**.** *Let ν be an irrational algebraic number of degree $n \geq 2$, i.e., ν is an irrational number which is the root of a polynomial of degree n with integer coefficients. Then, there exists a positive number C, depending on ν, such that*

$$\left|\nu - \frac{p}{q}\right| > \frac{C}{q^n}, \quad \forall p, q \in \mathbb{N}^*, \quad q > 0.$$

Let us consider $\xi > 0$ such that ξ/π is an irrational algebraic number of degree 2 and let us see inequality (5.13). First, for any $k \geq 1$ there exists $h_k \in \mathbb{N}^*$ such that

$$\left|k\frac{\xi}{\pi} - h_k\right| \leq \frac{1}{2}, \quad \forall k \geq 1.$$

Indeed, we can take $h_k = \lfloor k\xi/\pi \rfloor$ if $k\xi/\pi - \lfloor k\xi/\pi \rfloor \leq 1/2$ or $h_k = \lfloor k\xi/\pi \rfloor + 1$ otherwise ($\lfloor\cdot\rfloor$ is the floor function, i.e., for $x \in \mathbb{R}$, $\lfloor x \rfloor$ gives the largest integer less than or equal to x).

If we now apply Lemma 5.2 with $\nu = \xi/\pi$, $n = 2$, $q = k$ and $p = h_k$ we get

$$\frac{C\pi}{k} \leq |k\xi - h_k\pi| \leq \frac{\pi}{2}, \quad \forall k \geq 1,$$

and

$$k\left|\sin\left(k\xi\right)\right| = k\left|\sin\left(k\xi - h_k\pi\right)\right| = k\sin|k\xi - h_k\pi| \geq k\sin\left(\frac{C\pi}{k}\right) \geq 2C, \quad \forall k \geq 1.$$

In the last inequality we have used

$$\frac{\sin x}{x} \geq \frac{2}{\pi}, \quad \forall x \in (0, \pi/2].$$

This proves inequality (5.13). $\square$

As a consequence of the previous result, we also have:

Corollary 5.3. *Let us consider the functions f_1 and f_2 provided by Lemma 5.1 and the associated matrix A_k given in (5.8). Then, there exists positive constants C_1 and C_2 (only depending on f_1 and f_2) such that*

$$|\det A_k| \geq C_1 \frac{|I_{1,k}(q)|}{k^6} - C_2 \frac{|I_k(q)|}{k}, \quad \forall k \geq 1. \tag{5.14}$$

Proof. Let $k \geq 1$. We have (see (5.8))

$$\det A_k = \left(f_{1k} \widetilde{f}_{2,k} - f_{2,k} \widetilde{f}_{1,k} \right),$$

where f_{1k} and f_{2k} are the Fourier coefficients of f_1 and f_2 and where $\widetilde{f}_{1,k}$ and $\widetilde{f}_{2,k}$ are given by (5.4). Using Proposition 2.6 and taking into account that Supp $f_i \subset \omega$, one gets

$$\widetilde{f}_{i,k} = \tau_k f_{i,k} + \int_0^\pi f_i(x)\, g_k(x)\, dx.$$

So

$$\det A_k = \left(f_{1,k} \int_0^\pi f_2(x)\, g_k(x)\, dx - f_{2,k} \int_0^\pi f_1(x)\, g_k(x)\, dx \right).$$

Using again Proposition 2.6, g_k can be written as

$$g_k(x) = -\frac{I_k(q)}{k} \int_0^x \sin(k(x-\xi))\varphi_k(\xi)\, d\xi - \sqrt{\frac{\pi}{2}} \frac{I_{1,k}(q)}{k} \cos(kx), \quad \forall x \in \omega, \quad \forall k \geq 1.$$

We deduce then that

$$\det A_k = -\sqrt{\frac{\pi}{2}} \frac{I_{1,k}(q)}{k} \left(f_{1,k} \widehat{f}_{2,k} - f_{2,k} \widehat{f}_{1,k} \right) - \frac{I_k(q)}{k} \left(f_{1,k} G_{2,k} - f_{2,k} G_{1,k} \right), \quad k \geq 1,$$

where $\widehat{f}_{i,k}$ is given in (5.12) and

$$G_{i,k} = \int_0^\pi \int_0^x f_i(x) \sin(k(x-\xi))\varphi_k(\xi)\, d\xi\, dx,$$

for $i = 1, 2$ and $k \geq 1$. Finally, from (5.11) and using that the sequence $\{G_{i,k}\}_{k \geq 1}$ $(i = 1, 2)$ is bounded, we deduce (5.14) for $k \geq 1$. This ends the proof. $\square$

5.3 Solving the moment problem

We will devote this subsection to solving the moment problem (5.7) when $T > T_0(q)$ ($T_0(q)$, given by (1.17), is assumed to be finite in this section). To this end, we will work with the functions f_1 and f_2 provided by Lemma 5.1 and Corollary 5.3.

Theorem 5.4. *Let $y_0 \in L^2(0, \pi; \mathbb{R}^2)$ be given and let us consider the moment problem (5.7). Then, we can find a solution of this problem under the form*

$$\begin{cases} \displaystyle\int_0^T v_i(t) e^{-k^2 t}\, dt = e^{-k^2 T} M_{1,i}^{(k)}(y_0), \\ \displaystyle\int_0^T v_i(t)\, t e^{-k^2 t}\, dt = e^{-k^2 T} M_{2,i}^{(k)}(y_0), \end{cases} \tag{5.15}$$

where the quantities $M_{i,j}^{(k)}(y_0) \in \mathbb{R}$, with $k \geq 1$ and $1 \leq i, j \leq 2$, satisfy the following property: for any $\varepsilon > 0$ there exists a positive constant C_ε (only depending on ε) such that

$$\left| M_{i,j}^{(k)}(y_0) \right| \leq C_\varepsilon e^{k^2(T_0(q)+2\varepsilon)} \|y_0\|_{L^2(0,\pi;\mathbb{R}^2)}, \quad \forall k \geq 1, \quad 1 \leq i, j \leq 2. \tag{5.16}$$

In the sequel, let us fix $\varepsilon > 0$. From the definition of the minimal time $T_0(q)$, we can infer the existence of a positive integer k_ε for which

$$\frac{\min\left\{-\log\left|I_{1,k}(q)\right|, -\log\left|I_k(q)\right|\right\}}{k^2} < T_0(q) + \varepsilon, \quad \forall k > k_\varepsilon. \tag{5.17}$$

In order to find a solution of the moment problem (5.7) under the form (5.15), we are going to distinguish if k belongs to the set Λ_1, the set Λ_2 or the set Λ_3 (see (1.16)).

5.3.1 The case $k \in \Lambda_1$

Let us start solving the moment problem (5.7) when $k \in \Lambda_1$ (for the definition of Λ_1, see (1.16)).
1. Let us first consider $k \in \Lambda_1$ with $k \le k_\varepsilon$. Thanks to Lemma 5.1 (see (5.11)) we can deduce that $f_{1,k} f_{2,k} \neq 0$ for any $k \ge 1$. In this case, we solve the moment problem (5.7) as follows. Take

$$\int_0^T v_2(t) e^{-k^2 t}\, dt = \int_0^T v_2(t)\, t e^{-k^2 t}\, dt = 0, \quad \forall k \in \Lambda_1, \quad k \le k_\varepsilon.$$

With this choice, system (5.7) is equivalent to

$$\begin{cases} f_{1,k} \displaystyle\int_0^T v_1(t) e^{-k^2 t}\, dt = F_k^{(1)}, \\ \widetilde{f}_{1,k} \displaystyle\int_0^T v_1(t) e^{-k^2 t}\, dt - I_k(q) f_{1,k} \int_0^T v_1(t)\, t e^{-k^2 t}\, dt = F_k^{(2)}, \end{cases} \tag{5.18}$$

with $k \in \Lambda_1$, $k \le k_\varepsilon$ and where $F_k^{(i)}$, $i = 1, 2$, are the components of F_k (see (5.10)). Observe that in the set Λ_1 one has $I_k(q) \neq 0$. Therefore, the previous problem can be solved as in the boundary case (see Section 3.2) obtaining a solution under the form (5.15), for any $k \in \Lambda_1$ with $k \le k_\varepsilon$. In particular, $M_{1,2}^{(k)}(y_0) = M_{2,2}^{(k)}(y_0) = 0$.

Using the properties of $\widetilde{f}_{i,k}$ (see (5.5)) and taking into account that $k \in \Lambda_1$ and $k \le k_\varepsilon$, we deduce the existence of a positive constant C_ε such that

$$\left| M_{i,j}^{(k)}(y_0) \right| \le C_\varepsilon \|y_0\|_{L^2(0,\pi;\mathbb{R}^2)}, \quad \forall k \in \Lambda_1, \quad k \le k_\varepsilon, \quad 1 \le i, j \le 2. \tag{5.19}$$

As a consequence, we get inequality (5.16) for any $k \in \Lambda_1$, with $k \le k_\varepsilon$.
2. Let us now deal with the case $k \in \Lambda_1$ and $k > k_\varepsilon$. As before, our objective is to solve the moment problem (5.7). To this end, for $k > k_\varepsilon$, let us split the set Λ_1 into two subsets

$$\begin{cases} \Lambda_{1,\varepsilon}^\star := \left\{ k \in \Lambda_1 : k > k_\varepsilon \text{ and } -\dfrac{1}{k^2} \log\left|I_k(q)\right| \le T_0(q) + \dfrac{3}{2}\varepsilon \right\}, \\ \Lambda_{1,\varepsilon} := \left\{ k \in \Lambda_1 : k > k_\varepsilon \text{ and } -\dfrac{1}{k^2} \log\left|I_k(q)\right| > T_0(q) + \dfrac{3}{2}\varepsilon \right\}. \end{cases}$$

If $k \in \Lambda_{1,\varepsilon}^\star$, then we reason as in the previous case. We take

$$\int_0^T v_2(t) e^{-k^2 t}\, dt = \int_0^T v_2(t)\, t e^{-k^2 t}\, dt = 0, \quad \forall k \in \Lambda_{1,\varepsilon}^\star,$$

and the moment problem (5.7) is equivalent to (5.18), with $k \in \Lambda_{1,\varepsilon}^\star$. Again, we can compute the solution of this system, which is given by (5.15) ($k \in \Lambda_{1,\varepsilon}^\star$), where $M_{1,2}^{(k)}(y_0) = M_{2,2}^{(k)}(y_0) = 0$ and

$$\begin{cases} M_{1,1}^{(k)}(y_0) = \dfrac{-1}{f_{1,k}} \left\langle y_0, \Phi_{2,k}^* \right\rangle, \\ M_{2,1}^{(k)}(y_0) = \dfrac{-1}{f_{1,k} I_k(q)} \left(\left\langle y_0, \Phi_{1,k}^* \right\rangle - T I_k(q) \left\langle y_0, \Phi_{2,k}^* \right\rangle + \dfrac{\widetilde{f}_{1,k}}{f_{1,k}} \left\langle y_0, \Phi_{2,k}^* \right\rangle \right). \end{cases}$$

From the properties satisfied by k_ε, $f_{1,k}$, $\widetilde{f}_{1,k}$ and the definition of $\Lambda_{1,\varepsilon}^\star$ (see (5.17), (5.11) and (5.5)), we get

$$\begin{cases} \left| M_{1,i}^{(k)}(y_0) \right| \leq Ck^3 \|y_0\|_{L^2(0,\pi;\mathbb{R}^2)} \leq C_\varepsilon e^{\frac{1}{2}\varepsilon k^2} \\ \left| M_{2,i}^{(k)}(y_0) \right| \leq C_\varepsilon \dfrac{e^{\frac{1}{2}\varepsilon k^2}}{|I_k(q)|} \|y_0\|_{L^2(0,\pi;\mathbb{R}^2)} \leq C_\varepsilon e^{k^2(T_0(q)+2\varepsilon)} \|y_0\|_{L^2(0,\pi;\mathbb{R}^2)}, \quad \forall k \in \Lambda_{1,\varepsilon}^\star, \end{cases}$$

for a positive constant C_ε independent of k and y_0. We have then proved the bounds (5.16) in the case $k \in \Lambda_{1,\varepsilon}^\star$

Let us now consider $k \in \Lambda_{1,\varepsilon}$. From the definition of the set $\Lambda_{1,\varepsilon}$ and the inequality (5.17), it is easy to see that the minimun in (5.17) is reached in $-\log|I_{1,k}(q)|$ and therefore

$$\frac{-\log|I_{1,k}(q)|}{k^2} < T_0(q) + \varepsilon < T_0(q) + \frac{3}{2}\varepsilon < \frac{-\log|I_k(q)|}{k^2}, \quad \forall k \in \Lambda_{1,\varepsilon},$$

whence

$$|I_k(q)| < e^{-\frac{1}{2}\varepsilon k^2} |I_{1,k}(q)|, \quad \forall k \in \Lambda_{1,\varepsilon}.$$

This last inequality together with inequality (5.14), allow us to write:

$$|\det A_k| > C_\varepsilon e^{-\frac{1}{2}\varepsilon k^2} |I_{1,k}(q)|, \quad \forall k \in \Lambda_{1,\varepsilon} \tag{5.20}$$

(possibly for an integer k_ε larger than before). The matrix A_k is given by (5.8) and C_ε is a new positive constant depending on $\varepsilon > 0$.

We can now solve the moment problem (5.7) when $k \in \Lambda_{1,\varepsilon}$. To this end, we will take

$$\int_0^T v_1(t)\, te^{-k^2 t}\, dt = \int_0^T v_2(t)\, te^{-k^2 t}\, dt = 0, \quad \forall k \in \Lambda_{1,\varepsilon}.$$

Thus, the moment problem (5.7) is equivalent to $A_k V_k = F_k$, whith A_k, V_k and F_k respectively given by (5.8), (5.9) and (5.10). The solution of the system is explicitely given by $V_k = A_k^{-1} F_k$, i.e.,

$$\int_0^T v_i(t) e^{-k^2 t}\, dt = e^{-k^2 T} M_{1,i}^{(k)}(y_0), \quad k \in \Lambda_{1,\varepsilon}.$$

Taking into account inequality (5.20), the expression of F_k (see (5.10)) and the properties of $\widetilde{f}_{i,k}$, it is not difficult to prove that $M_{i,j}^{(k)}(y_0)$ satisfies (5.16) for any $i,j : 1 \leq i,j \leq 2$ and $k \in \Lambda_{1,\varepsilon}$.

This finalizes the proof of Theorem 5.4 in the case $k \in \Lambda_1$.

5.3.2 The case $k \in \Lambda_2$

Let us continue with the proof of Theorem 5.4 in the case $k \in \Lambda_2$. Observe that in this case $I_{1,k}(q) = 0$ and $I_k(q) \neq 0$ (see (1.16)) and therefore, inequality (5.17) changes into

$$\frac{-\log|I_k(q)|}{k^2} < T_0(q) + \varepsilon, \quad \forall k > k_\varepsilon, \quad k \in \Lambda_2. \tag{5.21}$$

When $k \in \Lambda_2$ and $k \leq k_\varepsilon$ we can repeat the arguments developed for $k \in \Lambda_1$ and $k \leq k_\varepsilon$ and obtain that we can solve the moment problem (5.7) with a solution under the form (5.15) where $M_{i,j}^{(k)}(y_0)$ satisfies (5.19) for every $i,j : 1 \leq i,j \leq 2$ ($k \in \Lambda_2$ and $k \leq k_\varepsilon$). In particular, we deduce (5.15) and (5.16) for $k \in \Lambda_2$ and $k \leq k_\varepsilon$.

Let us now consider an integer $k \in \Lambda_2$ such that $k > k_\varepsilon$. In this case we can reason as in the case $k \in \Lambda_{1,\varepsilon}^\star$. Indeed, we set

$$\int_0^T v_2(t) e^{-k^2 t}\, dt = \int_0^T v_2(t)\, te^{-k^2 t}\, dt = 0, \quad \forall k \in \Lambda_{1,\varepsilon}^\star,$$

and (5.7) becomes (5.15) ($k \in \Lambda_2$), where $M^{(k)}_{1,2}(y_0) = M^{(k)}_{2,2}(y_0) = 0$ and

$$\begin{cases} M^{(k)}_{1,1}(y_0) = \dfrac{-1}{f_{1,k}} \left\langle y_0, \Phi^*_{2,k} \right\rangle, \\ M^{(k)}_{2,1}(y_0) = \dfrac{-1}{f_{1,k} I_k(q)} \left(\left\langle y_0, \Phi^*_{1,k} \right\rangle - T I_k\left(q\right) \left\langle y_0, \Phi^*_{2,k} \right\rangle + \dfrac{\widetilde{f}_{1,k}}{f_{1,k}} \left\langle y_0, \Phi^*_{2,k} \right\rangle \right). \end{cases}$$

Combining the previous expressions, the inequality (5.21) and the properties of $f_{1,k}$ (see (5.11)) and $\widetilde{f}_{1,k}$ (see (5.5)), we infer that the coefficients $M^{(k)}_{i,j}(y_0)$ satisfy the bounds (5.16) for any $k \in \Lambda_2$, with $k > k_\varepsilon$, and $1 \le i, j \le 2$. This completes the proof of Theorem 5.4 in the case $k \in \Lambda_2$.

5.3.3 The case $k \in \Lambda_3$

In order to finish the proof of Theorem 5.4, let us deal with the case $k \in \Lambda_3$, with Λ_3 given in (1.16). In this case $I_k(q) = 0$ and the inequality (5.17) reads as follows

$$\frac{-\log |I_{1,k}(q)|}{k^2} < T_0(q) + \varepsilon, \quad \forall k > k_\varepsilon, \quad k \in \Lambda_3. \tag{5.22}$$

When $k \in \Lambda_3$ the moment problem (5.7) is simpler. It can be written as $A_k V_k = F_k$ (A_k, V_k and F_k are given in (5.8), (5.9) and (5.10)). Using inequality (5.14) we deduce $\det A_k \neq 0$ for any $k \in \Lambda_3$ and the solution of (5.7) is given by $V_k = A_k^{-1} F_k$. Combining inequalities (5.5), again (5.14) and (5.22), we get the formulas (5.15) ($M^{(k)}_{2,i}(y_0) = 0$, $i = 1, 2$, in this case) and (5.16) for $k \in \Lambda_3$.

5.4 Conclusion

In this subsection we will finish the proof of the null controllability result for system (1.2). To this end, we will show that if $T > T_0(q)$, $T_0(q)$ given by (1.17), there exist controls $v_1, v_2 \in L^2(0,T)$ such that the control $v \in L^2(Q)$, given by (5.2) (f_1 and f_2 are the functions provided by Lemma 5.1), satisfies the moment problem (5.1) or, equivalently, (5.7). Thanks to Theorem 5.4, this amounts to the existence of controls $v_1, v_2 \in L^2(0,T)$ which satisfies (5.15) for coefficients $M^{(k)}_{i,j}(y_0)$, $1 \le i, j \le 2$, $k \ge 1$, that fulfils the bounds (5.16).

Let us find controls v_1, v_2 in $L^2(0,T)$ satisfying (5.15). To this effect, we are going to reason as in Subsection 3.2.1. Indeed, using the property (3.9), we can obtain an explicit formula for these controls:

$$v_i(t) = \sum_{k \ge 1} e^{-k^2 T} \left(M^{(k)}_{1,i}(y_0) q_{1,k}(t) + M^{(k)}_{2,i}(y_0) q_{2,k}(t) \right), \quad i = 1, 2.$$

Then, the control v given by (5.2) is a solution of the moment problem (5.1) as soon as the previous two series converge in $L^2(0,T)$. But taking into account the bounds (5.16) and the property of the biorthogonal sequence $\{q_{1,k}, q_{2,k}\}_{k \ge 1}$, we can conclude that the series are absolutely convergent in $L^2(0,T)$ if $T > T_0(q)$. Indeed, we can write

$$\left\| e^{-k^2 T} M^{(k)}_{\ell,i}(y_0) q_{\ell,k}(t) \right\|_{L^2(0,T)} \le C_{\varepsilon,T}\, e^{-k^2 T} e^{k^2 (T_0(q) + 2\varepsilon)} e^{\varepsilon k^2} \equiv C_{\varepsilon,T}\, e^{-k^2 (T - T_0(q) - 3\varepsilon)},$$

for any $k \ge 1$ and $\ell, i : 1 \le \ell, i \le 2$. Observe that if we take

$$\varepsilon \in \left(0, \frac{T - T_0(q)}{3} \right)$$

we can conclude the absolute convergence in $L^2(0,T)$ of the series defining v_1 and v_2. This finalizes the proof of the positive null controllability result stated in Theorem 1.3.

Remark 5.1. An inspection of the previous proof shows that the null controllability result for system (1.2) at time T holds if the function $q \in L^\infty(0,\pi)$ satisfies

$$I_k(q) = \int_0^\pi q(x)|\varphi_k(x)|^2\,dx \neq 0, \quad \forall k \geq 1,$$

and $T > \widetilde{T}_0(q)$ (see (1.14)). In particular, this result occurs if the open interval $\omega = (a,b)$ satisfies

$$\operatorname{Supp} q \cap \omega \neq \emptyset.$$

This result is not optimal because if $q \in C^0(0,\pi) \cap L^\infty(0,\pi)$ and $\operatorname{Supp} q \cap \omega \neq \emptyset$, then there exist an open interval $\omega_0 \subset \omega$ and $\sigma > 0$ such that one has condition (1.6). From [35, 13, 3, 25], we know that the null controllability result for system (1.2) is valid for any positiver time T.

6 Proof of Theorem 1.3: The negative null controllability result

In order to prove the negative null controllability result stated in Theorem 1.3, let us assume that $T \in (0, T_0(q))$, where $T_0(q)$ is given in (1.17). In particular, we assume that $T_0(q) > 0$, otherwise there is nothing to prove. We are going to follow the same argument developed for the boundary controllability problem for system (1.1) (see Subsection 3.2.2). Indeed, we will prove that system (1.2) is not null-controllable at time T by contradiction. As in the boundary case, system (1.2) is null-controllable at time T if and only if there exists a constant $C > 0$ such that any solution θ of the adjoint problem (3.1) satisfies the observability inequality (4.1) (see Proposition 4.2).

Let us fix an arbitrary $k \geq 1$. For $\theta_0 = a_k\Phi^*_{1,k} + b_k\Phi^*_{2,k}$, with $(a_k, b_k) \in \mathbb{R}^2$ and $\Phi^*_{i,k}$ given in Proposition 2.1, the previous inequality reads as

$$A_{1,k} \leq C A_{2,k}, \tag{6.1}$$

with

$$A_{1,k} := e^{-2k^2T}\left\{|a_k|^2 + \left[|a_k|^2|\,\|\psi_k\|^2_{L^2(0,\pi)} + (b_k - Ta_kI_k(q))^2\right]\right\}$$

and

$$A_{2,k} := \int_0^T\!\!\int_\omega e^{-2k^2t}\left|a_k\psi_k(x) + (b_k - ta_kI_k(q))\varphi_k(x)\right|^2\,dx.$$

Using the expression of $\psi_k(x)$ given in Proposition 2.6 and

$$g_k(x) = -\frac{I_k(q)}{k}\int_0^x \sin(k(x-\xi))\varphi_k(\xi)\,d\xi - \sqrt{\frac{\pi}{2}}\frac{1}{k}I_{1,k}(q)\cos(kx), \quad \forall x \in \omega,$$

then by choosing $a_k = 1$ and $b_k = -\tau_k$, we get:

$$A_{2,k} = \int_0^T\!\!\int_\omega e^{-2k^2t}\left|-\sqrt{\frac{\pi}{2}}\frac{I_{1,k}(q)}{k}\cos(kx) + I_k(q)\left(\widetilde{g}_k(x) - t\varphi_k(x)\right)\right|^2\,dx\,dt$$

with $\widetilde{g}_k$ defined by

$$\widetilde{g}_k(x) = -\frac{1}{k}\int_0^x \sin\left(k(x-\xi)\right)\varphi_k(\xi)\,d\xi, \quad k \geq 1.$$

From the expression of $A_{1,k}$ we directly obtain the inequality $A_{1,k} \geq e^{-2k^2T}$. Therefore, inequality (6.1) can be rewritten as

$$\left\{\begin{array}{l} 1 \leq Ce^{2k^2T}A_{2,k} \leq Ce^{2k^2T}\left(|I_{1,k}(q)|^2 + |I_k(q)|^2\right) = Ce^{2k^2T}\left(e^{2\log|I_{1,k}(q)|} + e^{2\log|I_k(q)|}\right) \\ \quad \leq Ce^{-2k^2\left[\frac{1}{k^2}\min(-\log|I_{1,k}(q)|,-\log|I_k(q)|)-T\right]}. \end{array}\right. \tag{6.2}$$

From the definition of $T_0(q)$ (see (1.17)) there exists a subsequence of indices $\{k_n\}_{n\geq 1} \subseteq \mathbb{N}^*$ satisfying:

$$T_0(q) = \lim_{n\to\infty} \frac{\min\left(-\log\left|I_{1,k_n}(q)\right|, -\log\left|I_{k_n}(q)\right|\right)}{k_n^2}.$$

If $T_0(q) < \infty$, as a consequence, we deduce that for any $\varepsilon > 0$ there is $n_\varepsilon \geq 1$ such that

$$\frac{\min\left(-\log\left|I_{1,k_n}(q)\right|, -\log\left|I_{k_n}(q)\right|\right)}{k_n^2} \geq T_0(q) - \varepsilon, \quad \forall n \geq n_\varepsilon.$$

Coming back to inequality (6.2), we obtain

$$1 \leq Ce^{-2k_n^2[T_0(q)-\varepsilon-T]}, \quad \forall n \geq n_\varepsilon,$$

which gives a contradiction if we take $\varepsilon \in (0, T_0(q) - T)$. In the case in which $T_0(q) = \infty$, the reasoning is easier and we also get a contradiction. This proves that the observability inequality (4.1) does not hold and finishes the proof of the negative null controllability result of Theorem 1.3.

Remark 6.1. For proving the second item in Theorem 1.3, the assumption (1.11) on the support of q has been strongly used. To be more precise, observe that (see Proposition 2.2)

$$\psi_k(x) = \tau_k(x)\varphi_k(x) + g_k(x), \quad \forall x \in (0, \pi).$$

But thanks to assumption (1.11), τ_k is a constant function on ω. This is the key point in the contradiction argument.

7 Complementary results. Some examples

We will devote this section to giving some complementary results on the minimal times $\widetilde{T}_0(q)$ and $T_0(q)$ (see (1.14) and (1.17)) associated to the null controllability of systems (1.1) and (1.2). In the distributed case (1.2), we will also provide some examples which clarify the dependence of this minimal time $T_0(q)$ on the coefficient $q \in L^\infty(0,\pi)$ and on the position of the control interval ω with respect to $\operatorname{Supp} q$ when condition (1.11) holds.

Before giving these complementary results and examples, let us state a technical result which will be used later:

Lemma 7.1. *Let us fixed $\tau_0 \in [0, \infty]$, $x_0 \in [0, \infty)$ and $\varepsilon > 0$. Then, there exist an irrational number $\nu > 0$ such that $|\nu - x_0| \leq \varepsilon$ and*

$$\limsup \frac{-\log\left|\sin\left(k\nu\pi\right)\right|}{k^2} = \tau_0. \tag{7.1}$$

This result has been essentially proved in [18] and [10]. For the sake of completeness we will include its proof in Appendix A.

The first result reads as follows:

Theorem 7.2. *For any $\tau_0 \in [0, \infty]$, there exists $q \in L^\infty(0, \pi)$ satisfying (1.13) such that the minimal time $\widetilde{T}_0(q)$ associated to the system (1.1) (see (1.14)) is given by $\widetilde{T}_0(q) = \tau_0$. Moreover, the function q can be chosen such that $q > 0$ in an open interval $[0, c)$ $(c > 0)$ and*

$$\operatorname{Supp} q \equiv [0, \pi].$$

Proof. The proof is a direct consequence of Lemma 7.1. Indeed, let us fix $\tau_0 \in [0, \infty]$. Applying Lemma 7.1 with $x_0 = 1$ and $\varepsilon = 1/2$, we deduce the existence of an irrational number $\nu \in [1/2, 3/2]$ satisfying (7.1). Let us take $\alpha = \nu/2 \in [1/4, 3/4]$ and consider the function $q \in L^\infty(0, \pi)$ given by

$$q(x) = \begin{cases} 1 & \text{if } x \in [0, \alpha\pi), \\ -\dfrac{\alpha}{1-\alpha} & \text{if } x \in [\alpha\pi, \pi]. \end{cases}$$

Clearly, $q > 0$ in $[0, \alpha\pi)$ and $\operatorname{Supp} q \equiv [0, \pi]$. On the other hand,

$$I_k(q) := \int_0^\pi q(x)|\varphi_k(x)|^2 \, dx = \frac{-1}{\pi k(1-\alpha)} \sin(2k\alpha\pi).$$

Therefore $I_k(q) \neq 0$, for any $k \geq 1$, and

$$\widetilde{T}_0(q) = \limsup \frac{-\log|I_k(q)|}{k^2} = \limsup \frac{-\log|\sin(2k\alpha\pi)|}{k^2} = \limsup \frac{-\log|\sin(k\nu\pi)|}{k^2} = \tau_0.$$

This ends the proof. □

Remark 7.1. It is interesting to observe that there is not a clear relation between the minimal time of null controllability of system (1.1) (see (1.14)) and the length of the set

$$\{x \in [0, \pi] : q(x) > 0\}.$$

(resp., the set

$$\{x \in [0, \pi] : q(x) < 0\}).$$

With the previous ideas, we can prove:

"For any $\tau_0 \in [0, \infty]$ and $\varepsilon \in (0, \pi)$, there exists $q \in L^\infty(0, \pi)$ satisfying $q > 0$ in $[0, \pi - \varepsilon]$ (resp., $q < 0$ in $[0, \pi - \varepsilon]$), $\operatorname{Supp} q = [0, \pi]$ and $\widetilde{T}_0(q) = \tau_0$*."*

As said before, if $q \not\equiv 0$ and $q \geq 0$ in $[0, \pi]$ (resp., $q \leq 0$ in $[0, \pi]$), then $\widetilde{T}_0(q) = 0$, i.e., system (1.1) is null controllable at time T, for any $T > 0$.

The next result is related to the minimal time of distributed null controllability $T_0(q)$ (see (1.17)) of system (1.2). One has:

Theorem 7.3. *Let us fix $\omega = (a, b) \subset (0, \pi)$. Then, for any $\tau_0 \in [0, \infty]$, there exists a function $q \in L^\infty(0, \pi)$ satisfying (1.11) and (1.15) such that the minimal time of null controllability, $T_0(q)$, associated to the system (1.2) (see (1.17)) is given by*

$$T_0(q) = \tau_0.$$

Proof. The proof is very similar to the one done in Theorem 7.2. Indeed, let us assume that $a > 0$. The proof is similar if $b < \pi$. We are going to work with the function $q(x)$ given by:

$$q(x) := \begin{cases} 1 & \text{if } x \in [a_1\pi, (a_1 + \alpha)\pi), \\ -1 & \text{if } x \in [(a_1 + \alpha)\pi, (a_1 + 2\alpha)\pi], \end{cases}$$

with $a_1, \alpha \in (0, 1)$ to be determined. Suppose that $(a_1 + 2\alpha)\pi < a$. Then, it is not difficult to show (see (1.12)) that

$$I_{1,k}(q) = I_k(q) = \int_0^a q(x)|\varphi_k(x)|^2 \, dx = -\frac{2}{k\pi} \sin^2(k\alpha\pi) \sin(2k\pi(a_1 + \alpha)).$$

Given $\tau_0 \in [0, \infty]$, from Lemma 7.1, let us take $\alpha \in (0, a/(3\pi))$, an irrational number satisfying (7.1) for $\tau_0/2$, i.e.,

$$\limsup \frac{-\log|\sin(k\alpha\pi)|}{k^2} = \frac{\tau_0}{2}. \tag{7.2}$$

On the other hand, let us also take ℓ, an irrational algebraic number of degree 2, such that $\ell \in (2a/(3\pi), 4a/(3\pi))$. With these two quantities, let us set

$$a_1 = \frac{\ell}{2} - \alpha.$$

Whit this choice, it is easy to check

$$0 < a_1\pi < (a_1 + 2\alpha)\pi < a$$

and therefore, $\operatorname{Supp} q \subset (0,a)$ and (1.11) holds. The previous choice also gives

$$I_{1,k}(q) = I_k(q) = -\frac{2}{k\pi}\sin^2(k\alpha\pi)\sin(k\pi\ell),$$

and (1.15) holds (α, ℓ are irrational numbers). Using inequality (5.13) ($\xi = \pi\ell$), we deduce

$$-\log\left(\frac{2}{k\pi}\right) - 2\log|\sin(k\alpha\pi)| \leq -\log|I_k(q)| \leq -\log\left(\frac{2}{k\pi}\right) - 2\log|\sin(k\alpha\pi)| - \log\left(\frac{C}{k}\right),$$

for a positive constant C. These last inequalities and (7.2) provide

$$T_0(q) = \limsup \frac{\min\{-\log|I_{1,k}(q)|, -\log|I_k(q)|\}}{k^2} = \tau_0.$$

This ends the proof of Theorem 7.3. $\square$

Remark 7.2. Following the ideas in the proof of Theorem 7.3, it is possible to give an example of function q and control interval ω, with a positive minimal time of controllability, such that condition (1.11) holds and

$$\operatorname{Supp} q \cup \overline{\omega} = [0,\pi].$$

Indeed, let us fix $\tau_0 \in (0,\infty]$ and consider $\alpha \in (0,1/2)$, an irrational number which will be chosen later, $\omega = (\pi/2,\pi)$ and

$$q(x) := \begin{cases} \dfrac{1}{2} - \alpha & \text{if } x \in [0,\alpha\pi), \\ -\alpha & \text{if } x \in \left[\alpha\pi, \dfrac{\pi}{2}\right]. \end{cases}$$

With these data, one has condition (1.11) and

$$I_{1,k}(q) = I_k(q) = -\frac{1}{4k\pi}\sin(2k\alpha\pi).$$

Since α is an irrational number, again, condition (1.15) holds. Finally, as a direct consequence of Lemma 7.1, it is posible to find $\nu = 2\alpha \in (0,1)$ for which condition (7.1) holds. Taking into account this last property and the expression of the distributed minimal time (see (1.17)) we can conclude $T_0(q) = \tau_0$.

Let us finalize giving an example which reveals the dependence of the minimal time $T_0(q)$ for the null controllability of system (1.2) on the position of the control interval ω.

Example 7.3. Let us consider $\alpha_1 \in (4/5,1)$, $\alpha_2 \in (1/5,2/5)$, two irrational algebraic numbers of degree 2. Let us also take $\ell \in (0,1/5)$, another irrational number which (will be selected later) satisfying appropriated properties. On the other hand, let us set

$$a_1 = \frac{1}{2}(\alpha_1 - \alpha_2 - \ell)\pi, \quad a_2 = \frac{1}{2}(\alpha_1 + \alpha_2 - \ell)\pi.$$

With the previous choice, one has

$$0 < a_1 < a_1 + \ell\pi < a_2 < a_2 + \ell\pi < \pi.$$

Indeed,

$$\begin{cases} a_1 = \dfrac{1}{2}(\alpha_1 - \alpha_2 - \ell)\pi > \dfrac{1}{2}\left(\dfrac{4}{5} - \dfrac{2}{5} - \dfrac{1}{5}\right)\pi = \dfrac{\pi}{10} > 0; \\ a_2 - (a_1 + \ell\pi) = (\alpha_2 - \ell)\pi > 0; \quad a_2 + \ell\pi = \dfrac{1}{2}(\alpha_1 + \alpha_2 + \ell)\pi < \dfrac{1}{2}\left(1 + \dfrac{2}{5} + \dfrac{1}{5}\right) = \dfrac{4}{5}\pi < \pi. \end{cases}$$

Let us also introduce the function q:

$$q(x) := \begin{cases} 1 & \text{if } x \in [a_1, a_1 + \ell\pi], \\ -1 & \text{if } x \in [a_2 + \ell\pi, a_2 + \ell\pi]. \end{cases}$$

With this function q, the objective is to analyze the dependence of the minimal time of null controllability for system (1.2) on the position of the control open set $\omega = (a, b) \subset (0, \pi)$. To this end, we will consider three different situations:

1. $\boxed{\operatorname{Supp} q \cap \omega \neq \emptyset}$: In this case, system (1.2) is a particular case of system (1.4) ($C \equiv 0$) where the coefficient $a_{12} = q$ satisfies condition (1.5) with $\sigma = 1$ and ω_0 could be a connected component of the interior of the set $\operatorname{Supp} q \cap \omega \neq \emptyset$. From very well-known results (see for instance [35], [25] or [26]), we deduce that system (1.2) is null controllable at time T for any $T > 0$, that is to say, the minimal time of distributed null controllability is zero: $T_0(q) = 0$.

2. $\boxed{a_1 + \ell \leq a < b \leq a_2}$: In this case, condition (1.11) holds and it is easy to show (see (1.12))

$$\begin{aligned} I_{1,k}(q) &= \frac{1}{\pi}\left[\ell - \frac{1}{k}\sin(k\ell\pi)\cos(k(2a_1 + \ell\pi))\right], \\ I_{2,k}(q) &= -\frac{1}{\pi}\left[\ell - \frac{1}{k}\sin(k\ell)\cos\left(k\left(2a_1 + \frac{3}{2}\ell\pi\right)\right)\right], \\ I_k(q) &= I_{1,k}(q) + I_{2,k}(q) = -\frac{2}{k\pi}\sin(k\ell\pi)\sin(k(a_1 + a_2 + \ell\pi))\sin(k(a_2 - a_1)) \\ &= -\frac{2}{k\pi}\sin(k\ell\pi)\sin(k\alpha_1\pi)\sin(k\alpha_2\pi). \end{aligned}$$

   Thanks to the assumption on α_1, α_2 and ℓ, we deduce that $I_k(q) \neq 0$ for any $k \geq 1$ and q fulfills condition (1.15). Since $\ell > 0$, we also obtain the existence of $k_0 \geq 1$ such that $|I_{1,k}(q)| > |I_k(q)|$ for all $k \geq k_0$. Therefore (see (1.17)),

$$T_0(q) = \limsup \frac{-\log|I_{1,k}(q)|}{k^2} = 0.$$

   In conclusion, under the previous geometrical situation, one obtains that system (1.2) is approximately and null controllable at any positive time T. Observe that the null controllability property of system (1.2) is independent of the diophantine approximation properties of the irrational number ℓ.

3. $\boxed{0 \leq a < b \leq a_1}$ or $\boxed{a_2 + \ell \leq a < b \leq \pi}$: In this case, condition (1.11) also holds. Let us analyze the case $0 \leq a < b \leq a_1$. An analogous result can be obtained in the case $a_2 + \ell \leq a < b \leq \pi$. With the previous choice, $I_{1,k}(q) = 0$,

$$I_k(q) = -\frac{2}{k\pi}\sin(k\ell\pi)\sin(k\alpha_1\pi)\sin(k\alpha_2\pi),$$

   and

$$T_0(q) = \limsup \frac{-\log|I_k(q)|}{k^2}\left(= \widetilde{T}_0(q)\right).$$

   Again, we will use the properties of irrational algebraic numbers proved before. To be precise, as a consequence of inequality (5.13) applied to $\alpha_1\pi$ and $\alpha_2\pi$, we deduce the existence of two positive constants C_1 and C_2 such that

$$-\log\left(\frac{2}{k\pi}\right) - \log|\sin(k\ell\pi)| \leq -\log|I_k(q)| \leq -\log\left(\frac{2C_1C_2}{k^3\pi}\right) - \log|\sin(k\ell\pi)|, \quad \forall k \geq 1.$$

   As a consequence,

$$T_0(q) = \limsup \frac{-\log|\sin(k\ell\pi)|}{k^2},$$

and the minimal time of null controllability for system (1.2) depends on the diophantine approximation properties of the irrational number ℓ. Thanks to Lemma 7.1, given $\tau_0 \in [0, \infty]$, there is $\ell \in (0, 1/5)$ satisfying (7.1), that is to say, there is $\ell \in (0, 1/5)$ such that $T_0(q) = \tau_0$. In contrast with the geometrical situation in item 2, in the current case, the null controllability property of system (1.2) strongly depends on the diophantine approximation property of the irrational number ℓ.

Summarizing, with this example we have shown that, given a function $q \in L^\infty(0, \pi)$, the null controllability property of system (1.2) is different when the function q and the control interval ω satisfy $\operatorname{Supp} q \cap \omega \neq \emptyset$ or $\operatorname{Supp} q \cap \omega = \emptyset$. But even in this last case, i.e., in the case in which condition (1.11) holds, the distributed null controllability result depends on the relative position of the set $\operatorname{Supp} q$ and the control interval ω. For the same function q and the same non-scalar parabolic problem, we can find control intervals satisfying (1.11) for which the minimal time of null controllability can be zero and if we move the control interval (still satisfying (1.11)) the minimal time is positive or even ∞. This phenomenon is very well-known in the framework of the controllability of hyperbolic problems but, to our knowledge, is new in the parabolic framework.

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A Proof of Lemma 7.1

We will obtain the proof of Lemma 7.1 as a consequence of Lemma 5.2 and the result:

Lemma A.1. *1. Let us fixed $\tau_0 \in (0,\infty)$, $x_0 \in [0,\infty)$ and $\varepsilon > 0$. Then, there exist an irrational number $\nu > 0$ and a sequence of rational numbers $\{p_k/q_k\}_{k\geq 1}$ such that p_k and q_k are co-prime positive integers, the sequences $\{p_k\}_{k\geq 1}$ and $\{q_k\}_{k\geq 1}$ are strictly increasing,*

$$|\nu - x_0| \leq \varepsilon \quad \text{and} \quad \lim e^{\tau_0 q_k^2} \left| \nu - \frac{p_k}{q_k} \right| = 1. \tag{A.1}$$

Moreover, for any $k \geq 1$ one has

$$0 < |q_k \nu - p_k| \leq |q\nu - p|, \quad \forall p, q \in \mathbb{N}^*, \quad \text{with } q < q_{k+1}. \tag{A.2}$$

2. For any $\sigma \in (0,\infty)$, $x_0 \in [0,\infty)$ and $\varepsilon > 0$, there exists an irrational number $\nu > 0$ and a sequence of rational numbers $\{p_k/q_k\}_{k\geq 1}$ such that p_k and q_k are co-prime positive integers, the sequences $\{p_k\}_{k\geq 1}$ and $\{q_k\}_{k\geq 1}$ are strictly increasing and

$$|\nu - x_0| \leq \varepsilon \quad \text{and} \quad \lim e^{q_k^{2+\sigma}} \left| \nu - \frac{p_k}{q_k} \right| = 0. \tag{A.3}$$

The previous result has been proved in [10] (see Lemma 6.22, Corollary 6.25 and Appendix A).

Let us fix $x_0 \geq 0$ and $\varepsilon > 0$. In order to prove Lemma 7.1 we will use some ideas from [10]. We will divide the proof of Lemma 7.1 into three different cases:

Case $\tau_0 = 0$. Given $x_0 \geq 0$ and $\varepsilon > 0$, let us take $\nu \in [x_0 - \varepsilon, x_0 + \varepsilon]$ a positive irrational algebraic number of order 2. From (5.13) applied to $\xi = \nu\pi$, we deduce the existence of a positive constant C such that

$$|\sin(k\nu\pi)| \geq \frac{C}{k}, \quad \forall k \geq 1.$$

Thus,

$$\limsup \frac{-\log|\sin(k\nu\pi\pi)|}{k^2} \leq \limsup \frac{-\log(C/k)}{k^2} = 0.$$

Taking into account that the previous limit superior is always nonnegative, we deduce (7.1). This proves the result for $\tau_0 = 0$.

Case $\tau_0 \in (0, \infty)$. Given $x_0 \geq 0$, $\varepsilon > 0$ and τ_0, we can apply the first item in Lemma A.1 and conclude the existence of an irrational number $\nu \in [x_0 - \varepsilon, x_0 + \varepsilon]$ satisfying (A.1) and (A.2) for the sequences of positive integers $\{p_k\}_{k\geq 1}$ and $\{q_k\}_{k\geq 1}$. With this choice we deduce that ν satisfies $\lim p_k/q_k = \nu$,

$$\left\{ \begin{array}{l} \lim \left(\dfrac{1}{q_k} e^{\tau_0 q_k^2} |\nu q_k - p_k| \right) = 1 \quad \text{and} \\ 0 < |\nu q_k - p_k| \leq |\nu q - p|, \quad \forall p, q \in \mathbb{N}^*, \quad q < q_{k+1}. \end{array} \right. \tag{A.4}$$

Let us see that the previous number ν satisfies (7.1).

From the first equality in (A.4) we deduce $\lim |\nu q_k - p_k| = 0$ and

$$\begin{aligned} \widetilde{T}_0(q) &= \limsup \frac{-\log|\sin(\nu k\pi)|}{k^2} \geq \limsup \frac{-\log|\sin(\nu q_k\pi)|}{q_k^2} = \limsup \frac{-\log|\sin[\pi(\nu q_k - p_k)]|}{q_k^2} \\ &= \lim \frac{-\log[\pi|\nu q_k - p_k|]}{q_k^2} = \lim \frac{-\log\left(\pi q_k e^{-\tau_0 q_k^2}\right)}{q_k^2} = \tau_0. \end{aligned}$$

Then $\widetilde{T}_0(q) \geq \tau_0$. Observe that the previous reasoning also implies the existence of the following limit:

$$\lim \frac{-\log|\sin[\pi(\nu q_k - p_k)]|}{q_k^2} = \tau_0.$$

Let us now prove the inequality $\widetilde{T}_0(q) \leq \tau_0$. To this end, let us fix $\varepsilon > 0$. From the previous property, there exists $k_0(\varepsilon) \geq 1$ such that

$$\frac{-\log|\sin[\pi(\nu q_k - p_k)]|}{q_k^2} \leq \tau_0 + \varepsilon, \quad \forall k \geq k_0(\varepsilon). \tag{A.5}$$

As in the proof of Lemma 5.2, for every $n \geq 1$ there is $h_n \in \mathbb{N}^*$ for which

$$|\nu n - h_n| \leq \frac{1}{2}, \quad \forall n \geq 1.$$

Let us take $n_0(\varepsilon) = q_{k_0(\varepsilon)} \geq 1$. Thus, using that the sequence $\{q_k\}_{k\geq 1}$ is strictly increasing, if $n \geq n_0(\varepsilon)$, we infer the existence of $k \geq k_0(\varepsilon)$ such that $q_k \leq n < q_{k+1}$. These last properties together with the second formula in (A.4) allow us to write

$$0 < |\nu q_k - p_k| \leq |\nu n - h_n| \leq \frac{1}{2},$$

and

$$\left\{ \begin{array}{l} |\sin(\nu n\pi)| = |\sin(\nu n\pi - h_n\pi)| = \sin|\nu n\pi - h_n\pi| \\ \qquad\qquad \geq \sin|\nu q_k\pi - p_k\pi| = |\sin(\nu q_k\pi - p_k\pi)|, \quad \forall n \geq n_0(\varepsilon) : q_k \leq n < q_{k+1}. \end{array} \right.$$

Since $q_k \leq n < q_{k+1}$, the previous inequality and (A.5) give

$$\frac{-\log|\sin(\nu n\pi)|}{n^2} \leq \frac{-\log|\sin[\pi(\nu q_k - p_k)]|}{q_k^2} \leq \tau_0 + \varepsilon, \quad \forall n \geq n_0(\varepsilon) : q_k \leq n < q_{k+1},$$

and

$$\limsup \frac{-\log|\sin(\nu n\pi)|}{n^2} \leq \tau_0 + \varepsilon \quad \forall \varepsilon > 0.$$

In conclusion, we have obtained (7.1). This proves Lemma A.1 when $\tau_0 \in (0, \infty)$.

Case $\tau_0 = \infty$. For $\tau_0 = \infty$, $x_0 \geq 0$ and $\varepsilon > 0$, we apply the second item in Lemma A.1 with, for instance, $\sigma = 1/2$. We deduce the existence of a positive irrational number ν which fulfills property (A.2). Repeating the arguments of the previous point we deduce $\lim |\nu q_k - p_k| = 0$ and $(\sigma = 1/2)$

$$\widetilde{T}_0(q) = \limsup \frac{-\log|\sin(\nu k\pi)|}{k^2} \geq \limsup \frac{-\log|\sin(\nu q_k \pi)|}{q_k^2} = \limsup \frac{-\log|\sin[\pi(\nu q_k - p_k)]|}{q_k^2}$$
$$= \lim \frac{-\log[\pi|\nu q_k - p_k|]}{q_k^2} = \lim \frac{-\log\left(\pi q_k e^{-q_k^{2+\sigma}}\right)}{q_k^2} = \infty.$$

This shows that $\widetilde{T}_0(q) = \infty$ and finishes the third case and the proof of Lemma 7.1.