Large subspaces of compositionally universal functions with maximal cluster sets

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Abstract

Let (φ_n) be a sequence of holomorphic self-maps of a Jordan domain G in the complex plane. Under appropriate conditions on (φ_n), we construct an $H(G)$-dense linear manifold –as well as a closed infinite-dimensional linear manifold– all of whose non-zero functions have $H(G)$-dense orbits under the action of the sequence of composition operators associated to (φ_n). Simultaneously, these functions also present maximal cluster sets along each member of a large class of curves in G tending to the boundary.

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1. Introduction and notation

In recent years, there has been increasing interest in the study of the existence of strange mathematical objects enjoying, simultaneously, other

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different (often seemingly contradictory) properties. Moreover, the problem of determining large linear subspaces within nonlinear sets has recently attracted the attention of many mathematician across different subfields of infinite dimensional analysis. Its growing interest is evidenced in the recent terms of lineability and spaceability (see [1], [2], [3], [24]). We say that a subset of an infinite dimensional topological vector space X is *dense-lineable* or *algebraically generic* (*spaceable*, resp.) provided it contains, except possibly the origin, a dense (closed infinite-dimensional, resp.) linear subspace of X.

In this paper, we deal with the phenomenon of hypercyclicity and its compatibility with the maximality of cluster sets of holomorphic functions; and we are concerned with the *linear structure* of the family of functions exhibiting *doubly inner chaotic behavior* in a planar domain. To be more precise, our aim is to investigate both the dense-lineability and the spaceability of the family of holomorphic functions being compositionally universal with respect to a sequence of self-maps of a Jordan domain and, simultaneously, having maximal cluster sets along every admissible curve (see below for definitions). The exact statements will be provided in Section 3.

A domain in the complex plane $\mathbb{C}$ is a nonempty connected open subset $G \subset \mathbb{C}$. Recall that a domain G is said to be simply connected if $\mathbb{C}_\infty \setminus G$ is connected, where $\mathbb{C}_\infty$ denotes the extended complex plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. If $A \subset \mathbb{C}$, then $\overline{A}$ and ∂A will stand, respectively, for the closure and the boundary of A in $\mathbb{C}_\infty$. In a slightly more general way than usual, we define a *Jordan domain* as a domain $G \subset \mathbb{C}$ such that ∂G is a homeomorphic image of $\partial \mathbb{D}$ (so that, for instance, an open half-plane is a Jordan domain, but an open strip is not because its boundary in $\mathbb{C}_\infty$ attains twice the infinity point). Here $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk. Of course, every Jordan domain is simply connected.

If G is a domain in $\mathbb{C}$, then $H(G)$ denotes the vector space of holomorphic functions in G, endowed with the topology of uniform convergence on compacta. Under this topology, $H(G)$ becomes a complete metrizable separable topological vector space; in short, $H(G)$ is a separable F-space.

Assume that G is a domain in $\mathbb{C}$. If $f : G \to \mathbb{C}$ is a function and A is a subset of G, then the *cluster set of f along A* is defined as the set $C_A(f) = \{w \in \mathbb{C}_\infty :$ there exists a sequence $\{z_n\}_{n=1}^\infty$ in A tending to some point of ∂G such that $f(z_n) \to w\}$ (see [17] and [30] for surveys of results about cluster sets). It is clear that $C_A(f) \neq \emptyset$ if and only if A is not relatively compact in G.

An important special instance of such a set A is that of a *curve in G tending to the boundary of G*, that is, the trajectory of a continuous map $\gamma : [0,1) \to G$ such that for each compact set $K \subset G$ there is $u_0 = u_0(K) \in [0,1)$ with $\gamma(u) \in G \setminus K$ for all $u > u_0$. By abuse of language we sometimes identify $\gamma = \gamma([0,1))$. In this situation, we denote by $\Gamma(G)$ the family of all curves γ in G tending to the boundary and having non-total boundary oscillation, that is, $(\partial G) \setminus \overline{\gamma} \neq \emptyset$. The set $\Gamma(G)$ will be our family of "admissible" curves.

It is an interesting problem to obtain holomorphic functions with *maximal* cluster sets, that is, with cluster sets equal to $\mathbb{C}_\infty$ (see, for instance, the survey [31]). If $\mathcal{F}$ is a family of subsets of G then $MCS(\mathcal{F})$ will stand for the set of functions $f \in H(G)$ satisfying $C_A(f) = \mathbb{C}_\infty$ for all $A \in \mathcal{F}$. An answer to this problem is furnished by the next theorem.

Theorem 1.1. *Let G be a Jordan domain. We have:*
(a) *The set $MCS(\Gamma(G))$ is residual in $H(G)$.*
(b) *The set $MCS(\Gamma(G))$ is dense-lineable in $H(G)$.*

Part (a) of it tells us that –for a rather large family of curves– the set of such holomorphic functions is topologically large, while part (b) asserts that the same set is even algebraically large. Parts (a) and (b) can be found, respectively, in [27, Section 4] and [10, Theorem 2.1] (see also [11] for related results with operators). As a matter of fact, in [27] the original statement was slightly different, but its proof can be easily adapted.

Moving on to hypercyclicity, assume that X, Y are two (Hausdorff) topological vector spaces, and let $L(X,Y)$ denote the space of all continuous linear mappings from X to Y. As usual, we denote $L(X) := L(X,X) = \{\text{operators on } X\}$. Let $\mathbb{N}$ be the set of positive integers. A sequence $(T_n) \subset L(X,Y)$ is said to be *hypercyclic* or *universal* whenever there is a vector $x_0 \in X$, called *hypercyclic* or *universal* for (T_n), whose orbit $\{T_n x_0 : n \in \mathbb{N}\}$ under (T_n) is dense in Y. The hypercyclicity of (T_n) forces Y to be separable. The set of hypercyclic vectors for (T_n) is denoted by $HC((T_n))$. An operator T on X is said to be *hypercyclic* whenever there is a vector $x_0 \in X$, called *hypercyclic* for T, whose orbit $\{T^n x_0 : n \in \mathbb{N}\}$ under the sequence of iterates of T ($T^1 = T$, $T^2 = T \circ T$ and so on) is dense in X. We denote $HC(T) := HC((T^n))$. As for background on hypercyclicity, we refer to the surveys [4], [15], [20], [21] and [22].

It is well known that $HC(T)$ is dense as soon as T is hypercyclic. Nevertheless, for hypercyclic sequences of mappings T_n between topological vector spaces, the set $HC((T_n))$ need not be dense. For any sequence $(T_n) \subset$

$L(X,Y)$ we have that $HC((T_n))$ is a G_δ subset of X provided that Y is metrizable separable. The following theorem can be easily obtained.

Theorem 1.2. *Assume that X is Baire and that Y is metrizable and separable. Suppose that $(T_n) \subset L(X,Y)$. If $HC((T_n))$ is dense then $HC((T_n))$ is in fact residual in X. In particular, if X is a separable F-space and T is hypercyclic then $HC(T)$ is residual.*

Even a rich algebraic structure happens to be true. Namely, a result of Herrero-Bourdon-Bès-Wengenroth (see [26], [16], [14], [32]) asserts that if X is any topological vector space and $T \in L(X)$ is hypercyclic then there exists a dense T-invariant linear subspace M of X with $M \setminus \{0\} \subset HC(T)$ (if X is a Banach space, M can even be chosen such that $\dim(M)$ is the cardinality of the continuum, see [7]). For general sequences $(T_n) \subset L(X,Y)$ we have the assertion contained in the next Theorem 1.3, which is a slight improvement of [6, Theorem 2]. We say that a sequence $(T_n) \subset L(X,Y)$ is *hereditarily densely hypercyclic* if there is a strictly increasing sequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ such that $HC((T_{m_k}))$ is dense in X for each strictly increasing subsequence (m_k) of (n_k).

Theorem 1.3. *Let X,Y be metrizable and separable, and (T_n) be a sequence in $L(X,Y)$. If (T_n) is hereditarily densely hypercyclic, then $HC((T_n))$ is dense-lineable in X.*

A combination of Theorem 1.1(a) and Theorem 1.2 yields that, for a given sequence $(T_n) \subset L(H(G))$, with G Jordan and $HC((T_n))$ dense, the set of holomorphic functions in G having dense orbits under (T_n) and, simultaneously, having maximal cluster sets along every curve $\gamma \in \Gamma(G)$ is residual. Nevertheless, it is not clear whether from combining Theorem 1.1(b) and Theorem 1.3 (assuming (T_n) hereditarily densely hypercyclic) one can obtain the set $MCS(\Gamma(G)) \cap HC((T_n))$ to be dense-lineable. Indeed, the intersection of two dense-lineable sets may well be empty: for instance, the subsets $\mathcal{A} := \{\text{polynomials}\} \setminus \{0\}$ and $\mathcal{B} := \{p(z) \cdot e^z : p \in \mathcal{A}\}$ of $X := H(\mathbb{C})$ are evidently dense-lineable, but $\mathcal{A} \cap \mathcal{B} = \varnothing$.

Let us take a brief glance at the universality of sequences (C_{φ_n}) of *composition operators* on $H(G)$ generated by holomorphic self-maps $\varphi_n : G \to G$ ($n \in \mathbb{N}$). Recall that if $\varphi : G \to G$ is holomorphic, then the composition operator $C_\varphi : H(G) \to H(G)$ is defined as $C_\varphi f = f \circ \varphi$. This topic has been investigated by several mathematicians (see [23] and references therein). In some sense, a hypercyclic function with respect to composition with self-maps presents *inner* chaotic behavior. According to [12], a sequence $\varphi_n : G \to G$ ($n \in \mathbb{N}$) is called *runaway* if, for every compact set $K \subset G$ there exists $N \in \mathbb{N}$ such that $\varphi_N(K) \cap K = \emptyset$. In the case that the symbols φ_n are automorphisms (i.e., bijective holomorphic self-maps) of the domain G and G is not isomorphic to the punctured plane $\mathbb{C} \setminus \{0\}$, it is proved in [12] that the runaway property characterizes the hypercyclicity of (C_{φ_n}). If the φ_n's are not necessarily automorphic, Grosse-Erdmann and Mortini [23, Theorem 3.2] (see also [28]) have demonstrated the next theorem. We say that a sequence $\varphi_n : G \to G$ ($n \in \mathbb{N}$) is *injectively runaway* if, for every compact subset K of G, there is some $N = N(K) \in \mathbb{N}$ such that $\varphi_N(K) \cap K = \emptyset$ and the restriction $\varphi_N|_K$ is injective.

Theorem 1.4. *Let (φ_n) be a sequence of holomorphic self-maps on a simply connected domain $G \subset \mathbb{C}$. The following are equivalent:*

(a) *The sequence (C_{φ_n}) is universal on $H(G)$.*

(b) *The sequence (φ_n) is injectively runaway.*

(c) *The sequence (φ_n) has a subsequence (φ_{n_j}) for which every subsequence is injectively runaway.*

(d) *The set $HC((C_{\varphi_n}))$ is dense in $H(G)$.*

(e) *The sequence (C_{φ_n}) is hereditarily densely hypercyclic.*

(f) *The set $HC((C_{\varphi_n}))$ is dense-lineable in $H(G)$.*

We note that [23, Theorem 3.2] states the equivalence (a)–(c), but its proof gives that each of (d) and (e) are also equivalent statements. The equivalence of statement (f) follows immediately from Theorem 1.3.

Finally, turning our attention to *closed* subspaces, Montes and the first author [13] were able to prove in 1995 the following assertion.

Theorem 1.5. *If $G \subset \mathbb{C}$ is a domain that is not isomorphic to $\mathbb{C} \setminus \{0\}$ and (φ_n) is a runaway sequence of automorphisms of G, then the set $HC((C_{\varphi_n}))$ is spaceable in $H(G)$.*

2. Preliminary results

In order to study the algebraic structure of our family of chaotic functions, we need a number of technical lemmas.

The following result concerns extensions of isomorphisms to the boundaries and is due to Osgood and Carathéodory. It can be found in [25]. Recall that a homeomorphism between two topological spaces A, B is a bijective bicontinuous mapping $A \to B$, whereas an isomorphism between two planar domains G, Ω is a bijective holomorphic mapping $G \to \Omega$.

Theorem 2.1. *If G, Ω are Jordan domains of $\mathbb{C}$, then there exists a homeomorphism $\psi : \overline{G} \to \overline{\Omega}$ such that the restriction $\psi|_G : G \to \Omega$ is an isomorphism.*

In fact, *any* isomorphism $G \to \Omega$ between Jordan domains (whose existence is guaranteed by the Riemann mapping theorem) extends to a homeomorphism $\overline{G} \to \overline{\Omega}$.

Next, we consider the following important approximation theorem that is due to Nersesjan (see [19] and [29]). By $G_\infty := G \cup \{\omega\}$ we denote the one-point compactification of the domain G. If $A \subset \mathbb{C}$, then A^0 will stand for the interior of A.

Theorem 2.2. *Suppose that $G \subset \mathbb{C}$ is a domain and that F is a closed subset in G. Assume that $G_\infty \setminus F$ is connected and locally connected at ω. Assume also that F "lacks long islands" (see Figure 1), that is, for every compact subset $K \subset G$ there exists a neighborhood V of ω in G_∞ such that no component of F^0 intersects both K and V. Let $\varepsilon : F \to (0, +\infty)$ be continuous and $g : F \to \mathbb{C}$ be a function that is continuous on F and holomorphic in F^0. Then there exists a function $f \in H(G)$ such that*

$$|g(z) - f(z)| < \varepsilon(z) \quad \textit{for all} \quad z \in F.$$

Finally, we turn our attention to the Hilbert space $L^2(\partial\mathbb{D})$ of all (Lebesgue classes of) measurable functions $f : \partial\mathbb{D} \to \mathbb{C}$ with finite quadratic norm $\|f\|_2 := (\int_0^{2\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi})^{1/2}$. Since $\{z^n\}_{n=-\infty}^{\infty}$ is an orthonormal basis of $L^2(\partial\mathbb{D})$, we have that $\{z^n\}_{n\geq 1}$ is a basic sequence of $L^2(\partial\mathbb{D})$. Recall that a sequence $\{x_n\}_{n\geq 1}$ in a Banach space $(E, \|\cdot\|)$ is said to be a *basic sequence* whenever every vector $x \in E$ can be written as $x = \sum_{n=1}^{\infty} a_n x_n$ for a unique scalar sequence $\{a_n\}_{n\geq 1}$. Moreover, two basic sequences $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$

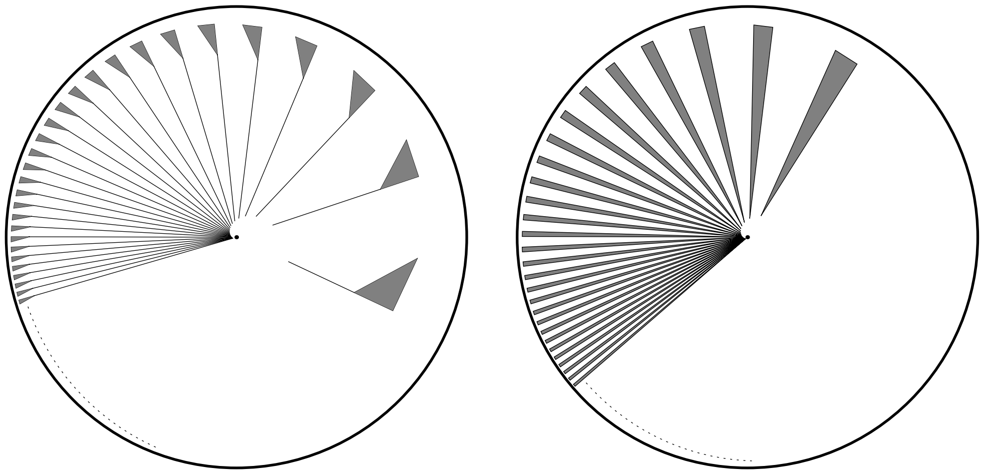

Figure 1: Example (left) and counterexample (right) of “lacks long island” property.

are said to be *equivalent* if, for every sequence $\{a_n\}_{n\geq 1}$ of scalars, the series $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if the series $\sum_{n=1}^{\infty} a_n y_n$ converges. This happens (see [5]) if and only if there exist two constants $m, M \in (0, +\infty)$ such that

$$m \left\| \sum_{j=1}^{J} a_j x_j \right\| \leq \left\| \sum_{j=1}^{J} a_j y_j \right\| \leq M \left\| \sum_{j=1}^{J} a_j x_j \right\|$$

for all scalars $a_1, \dots, a_J$ and all $J \in \mathbb{N}$. By using the first inequality, we are easily driven to the next result, whose proof can be found in [8, Lemma 2.1].

Lemma 2.3. *Assume that G is a domain with $\overline{\mathbb{D}} \subset G$ and that $\{f_j\}_{j\geq 1} \subset H(G)$ is a sequence such that it is a basic sequence in $L^2(\partial\mathbb{D})$ that is equivalent to $\{z^j\}_{j\geq 1}$. If $\{h_l := \sum_{j=1}^{J(l)} c_{j,l} f_j\}_{l\geq 1}$ is a sequence in* $\mathrm{span}\{f_j\}_{j\geq 1}$ *converging in $H(G)$, then* $\sup_{l\in\mathbb{N}} \sum_{j=1}^{J(l)} |c_{j,l}|^2 < +\infty$.

3. Algebraic genericity and spaceability

We are now ready to establish our theorems. Throughout this section we assume that G is a *Jordan domain* of $\mathbb{C}$ and that (φ_n) is an *injectively runaway* sequence of holomorphic self-maps of G.

Theorem 3.1. *The set $MCS(\Gamma(G)) \cap HC((C_{\varphi_n}))$ is dense-lineable in $H(G)$.*

Proof. By Theorem 2.1, there exists an isomorphism $\psi : G \to \mathbb{D}$ that extends to a homeomorphism $\psi : \overline{G} \to \overline{\mathbb{D}}$. Define $\psi_n := \psi \circ \varphi_n \circ \psi^{-1} \in H(\mathbb{D})$ $(n \in \mathbb{N})$. Since ψ and ψ^{-1} preserve compactness and interchange boundary points, we reach the following conclusions:

- (φ_n) is runaway in G if and only if (ψ_n) is runaway in $\mathbb{D}$.
- $f \in HC((C_{\varphi_n}))$ if and only if $f \circ \psi^{-1} \in HC((C_{\psi_n}))$.
- $\gamma \in \Gamma(G)$ if and only if $\psi \circ \gamma \in \Gamma(\mathbb{D})$.
- $f \in MCS((\Gamma(G)))$ if and only if $f \circ \psi^{-1} \in MCS(\Gamma(\mathbb{D}))$.

In view of these points, we obtain that if M is a dense linear subspace of $H(\mathbb{D})$ with $M \setminus \{0\} \subset MCS((\Gamma(\mathbb{D})) \cap HC((C_{\psi_n}))$, then $\widetilde{M} := \{h \circ \psi : h \in M\}$ is a dense linear subspace of $H(G)$ satisfying $\widetilde{M} \setminus \{0\} \subset MCS(\Gamma(G)) \cap HC((C_{\varphi_n}))$.

Therefore, we can assume without loss of generality that $G = \mathbb{D}$. By hypothesis, $\varphi_n : \mathbb{D} \to \mathbb{D}$ $(n \in \mathbb{N})$ is an injectively runaway sequence with $(\varphi_n) \subset H(\mathbb{D})$. By applying part (c) of Theorem 1.4, we can find a subsequence $\{n_1 < n_2 < n_3 < \cdots\} \subset \mathbb{N}$ such that for each compact subset K of $\mathbb{D}$ there is some $J \in \mathbb{N}$ for which $\varphi_{n_j}(K) \cap K = \emptyset$ and $\varphi_{n_j}|_K$ is injective for all $j \geq J$. We can consider only the subsequence (φ_{n_j}) and, after relabeling, assume that (φ_{n_j}) is the whole sequence (φ_n).

Let us prepare a number of tools. Let (q_j) be any fixed dense sequence in $\mathbb{C}$. Denote by (P_n) a countable dense subset of $H(\mathbb{D})$ (for instance, an enumeration of the holomorphic polynomials having coefficients with rational real and imaginary parts). If $0 < r < s < 1$, we denote by $S(r,s)$ the spiral compact set

$$S(r,s) = \left\{ \left(r + \frac{s-r}{4\pi}\theta \right) e^{i\theta} : \theta \in [0, 4\pi] \right\}.$$

Moreover, we divide $\mathbb{N}$ into infinitely many strictly increasing sequences $\{p(n,j) : j = 1, 2, \ldots\}$ $(n \in \mathbb{N})$.

The beginning of the following construction is sketched (non-scaled) in Figure 2. Fix a closed ball $B_1 \subset \mathbb{D}$ with center at the origin and radius $> 1/2$. Now, choose r_1, s_1 with $\text{radius}(B_1) < r_1 < s_1 < 1$. Set $S_1 := S(r_1, s_1)$ and let K_1 be a closed ball with center at the origin satisfying $K_1 \supset B_1 \cup S_1$. Next, select $m_1 \in \mathbb{N}$ such that $\varphi_n(K_1) \cap K_1 = \emptyset$ and $\varphi_n|_{K_1}$ is injective for all

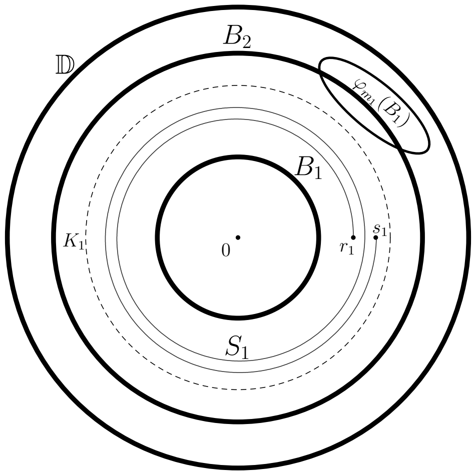

Figure 2: First step of the construction of sets B_n, S_n and numbers m_n.

$n \geq m_1$. Now, we begin the second step. Fix a closed ball B_2 with center at the origin and radius $> 2/3$. Then choose r_2, s_2 with

$$\max\{|z| : z \in B_2 \cup S_1 \cup \varphi_{m_1}(B_1)\} < r_2 < s_2 < 1.$$

Set $S_2 := S(r_2, s_2)$ and let K_2 be a closed ball with center at the origin containing $B_2 \cup S_2$. We can select $m_2 \in \mathbb{N}$ with $m_2 > m_1$ such that $\varphi_n(K_2) \cap K_2 = \emptyset$ and $\varphi_n|_{K_2}$ is injective for all $n \geq m_2$. By proceeding in this way, we obtain a sequence $\{m_1 < m_2 < \cdots < m_n < \cdots\}$ of natural numbers, a sequence (B_n) of balls with center at the origin, and a sequence (S_n) of spiral compact sets satisfying

$$\text{radius}\,(B_n) > \frac{n}{n+1} \quad \text{for all } n \in \mathbb{N},$$

$$B_n \cap S_k = \emptyset = B_n \cap \varphi_{m_k}(B_k) \quad \text{for all } n, k \in \mathbb{N} \text{ with } k \geq n,$$

$$\varphi_{m_n}(B_n) \cap \varphi_{m_k}(B_k) = \emptyset = S_n \cap S_k \quad \text{for all } n, k \in \mathbb{N} \text{ with } k \neq n,$$

$$S_n \cap \varphi_{m_k}(B_k) = \emptyset \quad \text{for all } n, k \in \mathbb{N}, \text{ and}$$

$$\varphi_n|_{B_k} \text{ is injective for all } n \geq m_k \text{ and all } k \in \mathbb{N}.$$

Observe that if $S_n = S(r_n, s_n)$ then $\lim_{n\to\infty} r_n = 1 = \lim_{n\to\infty} s_n$. Hence the sequence (S_n) "goes" to $\partial\mathbb{D}$. Moreover, (B_n) forms an exhaustive sequence of compact sets of $\mathbb{D}$.

Next, we consider the sets F_n $(n \in \mathbb{N})$ given by

$$F_n = B_n \cup \bigcup_{j=n}^{\infty} S_j \cup \bigcup_{j=n}^{\infty} \varphi_{m_j}(B_j).$$

Note that each F_n consists of infinitely many pairwise disjoint compact set without holes, say $F_n = \bigcup_{j=1}^{\infty} A_j$, (at this point, the fact that each $\varphi_{m_j}|_{B_j}$ is a homeomorphism from B_j onto $\varphi_{m_j}(B_j)$ is crucial; this follows from the fact that a bijective map $A \to B$ between topological spaces A, B –with A compact and B Hausdorff– is necessarily a homeomorphism) with $\mathrm{dist}(A_j, F_n \setminus A_j) > 0$. Therefore $\mathbb{D}_\infty \setminus F_n$ is connected as well as locally connected at ω, in fact, it is arc-connected. In addition, F_n clearly lacks long islands. Define the function $g_n : F_n \to \mathbb{C}$ by

$$g_n(z) = \begin{cases} P_n(z) & \text{if } z \in B_n \\ q_j & \text{if } z \in S_{p(n,j)} \text{ and } p(n,j) \geq n \\ 0 & \text{if } z \in S_{p(k,j)}\ (k \neq n) \text{ and } p(k,j) \geq n \\ P_j(\varphi^{-1}_{m_{p(n,j)}})(z) & \text{if } z \in \varphi_{m_{p(n,j)}}(B_{p(n,j)}) \text{ and } p(n,j) \geq n \\ 0 & \text{if } z \in \varphi_{m_{p(k,j)}}(B_{p(k,j)})\ (k \neq n) \text{ and } p(k,j) \geq n. \end{cases}$$

From the inverse mapping theorem and from each $\varphi_{m_j}|_{B_j} : B_j \to \varphi_{m_j}(B_j)$ being homeomorphism, one derives that g_n is continuous on F_n and holomorphic in F_n^0. With this in hand, we can apply Theorem 2.2 to $G = \mathbb{D}$, $F = F_n$, $g = g_n$ and $\varepsilon(z) := \frac{1-|z|}{n}$, so obtaining a function $f_n \in H(\mathbb{D})$ satisfying

$$|f_n(z) - g_n(z)| < \frac{1-|z|}{n} \quad (z \in F_n,\ n \in \mathbb{N}). \tag{1}$$

Define

$$M := \mathrm{span}\,\{f_n :\ n \in \mathbb{N}\},$$

the linear span generated by the functions f_n. Observe that (1) and the definition of g_n show that

$$|f_n(z) - P_n(z)| < \frac{1}{n} \quad \text{for all } z \in B_n \text{ and all } n \in \mathbb{N}.$$

Hence, $d(f_n, P_n) \longrightarrow 0$ $(n \to \infty)$ for any distance d inducing the topology of $H(\mathbb{D})$. The denseness of (P_n) in $H(G)$ together with this fact imply the denseness of (f_n). Consequently, M is a dense linear subspace of $H(\mathbb{D})$.

It remains to prove that $M \setminus \{0\} \subset MCS(\Gamma(\mathbb{D})) \cap HC((C_{\varphi_n}))$. For this, fix $f \in M \setminus \{0\}$. Since $MCS(\Gamma(\mathbb{D}))$ and $HC((C_{\varphi_n}))$ are invariant under scaling, we can assume that

$$f = \lambda_1 f_1 + \cdots + \lambda_{N-1} f_{N-1} + f_N. \tag{2}$$

for some $N \in \mathbb{N}$ and some complex scalars $\lambda_1, \ldots, \lambda_{N-1}$. Consider a curve $\gamma \in \Gamma(\mathbb{D})$. Then there is at least one point in $\partial\mathbb{D}$ that is not approximated by γ. Now, the shape of the sets S_j, the continuity of γ and the fact that γ escapes towards $\partial\mathbb{D}$ forces γ to intersect all spirals S_j from some j on. Therefore, there exists $j_0 \in \mathbb{N}$ such that $p(k, j_0) \geq N$ $(k = 1, \ldots, N)$ and $\gamma \cap S_{p(N,j)} \neq \emptyset$ $(j \geq j_0)$. Choose points $z_j \in \gamma \cap S_{p(N,j)}$ $(j \geq j_0)$. Note that $|z_j| \geq r_{p(N,j)} \geq r_j$. According to (1) we get, for every $j \geq j_0$,

$$|f_N(z_j) - q_j| = |f_N(z_j) - g_N(z_j)| < \frac{1 - |z_j|}{N} < 1 - |z_j| \leq 1 - r_j \quad \text{and}$$

$$|f_n(z_j)| = |f_n(z_j) - g_n(z_j)| < \frac{1 - |z_j|}{n} < 1 - |z_j| \leq 1 - r_j \quad (n = 1, \ldots, N-1).$$

Thus we obtain from (2) that

$$\begin{aligned} |f(z_j) - q_j| &\leq |f_N(z_j) - q_j| + \sum_{n=1}^{N-1} |\lambda_n f_n(z_j)| \\ &< (1 + \sum_{n=1}^{N-1} |\lambda_n|)(1 - r_j) \longrightarrow 0 \quad (j \to \infty). \end{aligned}$$

The denseness of (q_j) in $\mathbb{C}_\infty$ and the facts that $(z_j) \subset \gamma$ and (z_j) tends to $\partial\mathbb{D}$ show that $C_\gamma(f) = \mathbb{C}_\infty$.

Our next task is to demonstrate that such a function f is (C_{φ_n})-hypercyclic. For this, we again resort to (1) and the definition of the functions g_n. If f is as in (2), consider the sequence of balls $\{B_{p(N,j)}\}_{j \geq j_0}$, where j_0 is such that $p(N, j) \geq N$ for all $j \geq j_0$. Note that it is an exhaustive sequence of compact subsets of $\mathbb{D}$. If $j \geq j_0$ and $z \in \varphi_{m_{p(N,j)}}(B_{p(N,j)})$, we have that

$$|f_N(z) - P_j(\varphi^{-1}_{m_{p(N,j)}}(z))| < \frac{1 - |z|}{N} \leq 1 - |z| < 1 - \text{radius}\,(B_{p(N,j)}) \quad \text{and}$$

$$|f_n(z) - 0| < \frac{1 - |z|}{n} \leq 1 - |z| < 1 - \text{radius}\,(B_{p(N,j)}) \quad (n = 1, \ldots, N-1).$$

We have used that $B_{q(j)} \cap \varphi_{\nu(j)}(B_{q(j)}) = \emptyset$, where $q(j) := p(N,j)$ and $\nu(j) := m_{p(N,j)}$. By changing variables, we get

$$|f_N(\varphi_{\nu(j)}(z)) - P_j(z)| < \frac{1}{q(j)} \quad \text{and}$$

$$|f_n(\varphi_{\nu(j)}(z))| < \frac{1}{q(j)} \quad \text{for all } z \in B_{q(j)}, \text{ all } j \geq j_0 \text{ and all } n \in \{1, \ldots, N-1\}.$$

Putting everything together, we are driven to

$$\begin{aligned} |f(\varphi_{\nu(j)}(z)) - P_j(z)| &\leq |f_N(\varphi_{\nu(j)}(z)) - P_j(z)| + \sum_{n=1}^{N-1} |\lambda_n| \cdot |f_n(\varphi_{\nu(j)}(z))| \\ &< \left(1 + \sum_{n=1}^{N-1} |\lambda_n|\right) \frac{1}{q(j)} \end{aligned}$$

for all $z \in B_{q(j)}$ and all $j \geq j_0$. Therefore

$$\sup_{z \in B_{q(j)}} |f(\varphi_{\nu(j)}(z)) - P_j(z)| \longrightarrow 0 \quad \text{as } j \to \infty.$$

A reasoning similar to the one showing the denseness of (f_n) in $H(\mathbb{D})$ concludes the proof. $\square$

Theorem 3.2. *The set* $MCS(\Gamma(G)) \cap HC((C_{\varphi_n}))$ *is spaceable in* $H(G)$.

Proof. We maintain the notation and all constructions of the proof of Theorem 3.1. Again, if $\psi : G \to \mathbb{D}$ is an isomorphism and M were an infinite dimensional closed vector subspace of $H(\mathbb{D})$ with $M \setminus \{0\} \subset MCS(\Gamma(\mathbb{D})) \cap HC((C_{\psi_n}))$, then $\widetilde{M} := \{h \circ \psi : h \in M\}$ would be an infinite dimensional closed vector subspace of $H(G)$ satisfying $\widetilde{M} \setminus \{0\} \subset MCS(\Gamma(G)) \cap HC((C_{\varphi_n}))$. Consequently, we can assume without loss of generality that $G = \mathbb{D}$.

In this setting, we consider the circle $C = \{z : |z| = 1/2\} \subset \mathbb{D}$ and the space $L^2(C)$ of all (classes of) Lebesgue measurable functions $f : C \to \mathbb{C}$ with square-integrable modulus, endowed with the norm

$$\|f\|_2 := \left(\int_0^{2\pi} |f(e^{i\theta}/2)|^2 \, \frac{d\theta}{2\pi}\right)^{1/2}.$$

Then the sequence $\{\sigma_k(z) := (2z)^k\}_{k\geq 1}$ is an orthonormal basis of the subspace of $L^2(C)$ generated by it. In particular, $\{\sigma_k\}_{k\geq 1}$ is a basic sequence in $L^2(C)$. Note that convergence in $H(\mathbb{D})$ implies quadratic convergence in the space $L^2(C)$.

Denote $K_0 := \{z : |z| \leq 1/2\}$, so that $C = \partial K_0$. We define the new set

$$F := K_0 \cup \bigcup_{j=1}^{\infty} S_j \cup \bigcup_{j=1}^{\infty} \varphi_{m_j}(B_j).$$

Note that $K_0 \cap S_j = S_j \cap S_k = \varphi_{m_j}(B_j) \cap \varphi_{m_k}(B_k) = K_0 \cap \varphi_{m_j}(B_j) = S_j \cap \varphi_{m_j}(B_j) = S_j \cap \varphi_{m_k}(B_k) = \emptyset$ $(j, k \in \mathbb{N}; j \neq k)$. Observe that in contrast with the proof of Theorem 3.1, this time we are considering a unique set F and not a sequence $\{F_n\}_{n\geq 1}$ of sets. As in the proof of Theorem 3.1, $\mathbb{D}_\infty \setminus F$ is connected and locally connected at ω. It is plain that F satisfies the "long islands" property. Consider the function $\widetilde{g}_n : F \to \mathbb{C}$ given by

$$\widetilde{g}_n(z) = \begin{cases} (2z)^n & \text{if } z \in K_0 \\ q_j & \text{if } z \in S_{p(n,j)} \\ 0 & \text{if } z \in S_{p(k,j)} \text{ and } k \neq n \\ P_j(\varphi^{-1}_{m_{p(n,j)}})(z) & \text{if } z \in \varphi_{m_{p(n,j)}}(B_{p(n,j)}) \\ 0 & \text{if } z \in \varphi_{m_{p(k,j)}}(B_{p(k,j)}) \text{ and } k \neq n. \end{cases} \tag{3}$$

As in Theorem 3.1, $\widetilde{g}_n$ is continuous on F and holomorphic in F^0. An application of Theorem 2.2 yields the existence of a function $\widetilde{f}_n \in H(\mathbb{D})$ satisfying

$$|\widetilde{f}_n(z) - \widetilde{g}_n(z)| < \frac{1-|z|}{3^n} \quad (z \in F,\ n \in \mathbb{N}). \tag{4}$$

Then we define $\widetilde{M}$ as the closure in $H(\mathbb{D})$ of the linear manifold generated by the $\widetilde{f}_n$'s, that is,

$$\widetilde{M} := \overline{\text{span}}\,\{\widetilde{f}_n : n \in \mathbb{N}\}.$$

It is plain that $\widetilde{M}$ is a closed vector subspace of $H(\mathbb{D})$.

Let us prove that $\widetilde{M}$ makes $MCS(\Gamma(\mathbb{D})) \cap HC((C_{\varphi_n}))$ spaceable. First of all, observe that due to (4) and the definition of $\widetilde{g}_n$ we have

$$|\widetilde{f}_n(z) - (2z)^n| < \frac{1}{3^n} \quad (z \in C,\ n \in \mathbb{N}),$$

from which we derive that $\|\widetilde{f}_n - \sigma_n\|_2 < 1/3^n$ $(n \in \mathbb{N})$. Let $\{e_n^*\}_{n\geq 1}$ be the sequence of coefficient functionals corresponding to the basic sequence $\{\sigma_n\}_{n\geq 1}$. Since $\|e_n^*\|_2 = 1$ $(n \geq 1)$, one obtains

$$\sum_{n=1}^{\infty} \|e_n^*\|_2 \|\widetilde{f}_n - \sigma_n\|_2 < \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2} < 1. \tag{5}$$

From (5) and the basis perturbation theorem [18, p. 50] it follows that $\{\widetilde{f}_n\}_{n\geq 1}$ is also a basic sequence in $L^2(C)$ that is equivalent to $\{\sigma_n\}_{n\geq 1}$. In particular, the functions $\widetilde{f}_n$ $(n \in \mathbb{N})$ are linearly independent. Hence $\widetilde{M}$ has infinite dimension.

Fix $f \in \widetilde{M} \setminus \{0\}$. We show that $f \in MCS(\Gamma(\mathbb{D}))$ and $f \in HC((C_{\varphi_n}))$. Since the convergence in $H(\mathbb{D})$ is stronger than the convergence in $L^2(C)$, we have that (the restriction to C of) f is in $M_0 := \text{closure}_{L^2(C)}(\text{span}\,\{\widetilde{f}_n : n \in \mathbb{N}\})$. Therefore f has a (unique) representation $f = \sum_{n=1}^{\infty} c_n \widetilde{f}_n$ in $L^2(C)$, because $\{\widetilde{f}_n\}_{n\geq 1}$ is a basic sequence in this space. Since $f \not\equiv 0$, there exists $N \in \mathbb{N}$ with $c_N \neq 0$. Due to the invariance under scaling, we can assume $c_N = 1$. On the other hand, there is a sequence $\{h_l := \sum_{j=1}^{J(l)} c_{j,l} \widetilde{f}_j\}_{l\geq 1} \subset \text{span}\,\{\widetilde{f}_n : n \in \mathbb{N}\}$ (without loss of generality, we can assume that $J(l) \geq N$ for all l) converging to f compactly in $\mathbb{D}$. By Lemma 2.3 (to be more accurate, by a slightly modified version of such lemma where $\mathbb{D}$, $\partial\mathbb{D}$, $\{z^n\}_{n\geq 1}$ are respectively replaced by $\{|z| < 1/2\}$, C, $\{\sigma_n\}_{n\geq 1}$), one gets

$$\alpha := \sup_{l\in\mathbb{N}} \sum_{j=1}^{J(l)} |c_{j,l}|^2 < +\infty. \tag{6}$$

But $\{h_l\}_{l\geq 1}$ also converges to f in $L^2(C)$, so the continuity of each projection

$$\sum_{j=1}^{\infty} d_j \widetilde{f}_j \in M_0 \mapsto d_m \in \mathbb{C} \quad (m \in \mathbb{N})$$

yields that

$$\lim_{l\to\infty} c_{N,l} = 1. \tag{7}$$

In particular, the sequence $\{c_{N,l}\}_{l\geq 1}$ is bounded, say

$$|c_{N,l}| \leq \beta \quad (l \in \mathbb{N}). \tag{8}$$

As in the proof of Theorem 3.1, we get that for a prescribed curve $\gamma \in \Gamma(\mathbb{D})$, there is $j_0 \in \mathbb{N}$ with $\gamma \cap S_{p(N,j)} \neq \emptyset$ for all $j \geq j_0$. Then we select points $z_j \in \gamma \cap S_{p(N,j)}$ $(j \geq j_0)$. Since $|z| \geq r_k$ for all $z \in S_k$, it follows that

$$1 - |z_j| \leq 1 - r_{p(N,j)} \leq 1 - r_j \longrightarrow 0 \quad (j \to \infty). \tag{9}$$

Since $h_l \to f$ compactly in $\mathbb{D}$ and the singleton $\{z_j\}$ is compact, we have that for each $j \geq j_0$ there is $l_0(j) \in \mathbb{N}$ satisfying

$$|f(z_j) - h_l(z_j)| < \frac{1}{j} \quad (l \geq l_0(j)). \tag{10}$$

Moreover, from (7), the existence of a number $l = l(j) \in \mathbb{N}$ follows, with $l \geq l_0(j)$, such that

$$|c_{N,l} - 1| < \frac{1}{j(1 + |q_j|)}. \tag{11}$$

By using (4), (6), (8), (9), (10), (11), the triangle inequality and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|f(z_j) - q_j| &\leq |f(z_j) - h_l(z_j)| + |h_l(z_j) - q_j| \\
&\leq |f(z_j) - h_l(z_j)| + |c_{N,l}\widetilde{f}_N(z_j) - q_j| + \sum_{\substack{k=1 \\ k \neq N}}^{J(l)} |c_{k,l}\widetilde{f}_k(z_j)| \\
&\leq |f(z_j) - h_l(z_j)| + |c_{N,l}(\widetilde{f}_N(z_j) - \widetilde{g}_N(z_j))| + |(c_{N,l} - 1)q_j| \\
&\quad + \left(\sum_{k=1}^{J(l)} |c_{k,l}|^2 \right)^{1/2} \left(\sum_{\substack{k=1 \\ k \neq N}}^{\infty} |\widetilde{f}_k(z_j)|^2 \right)^{1/2} \\
&\leq \frac{1}{j} + \frac{\beta(1 - r_j)}{3^N} + \frac{|q_j|}{j(1 + |q_j|)} + \alpha^{1/2} \left(\sum_{k=1}^{\infty} \left(\frac{1 - r_j}{3^k} \right)^2 \right)^{1/2} \\
&< \frac{1}{j} + \beta(1 - r_j) + \frac{1}{j} + \left(\frac{\alpha}{8} \right)^{1/2} (1 - r_j) \longrightarrow 0 \quad (j \to \infty).
\end{aligned}$$

Then $\lim_{j\to\infty}(f(z_j) - q_j) = 0$. Since $\{q_j\}_{j\geq 1}$ is dense in $\mathbb{C}_\infty$, the sequence $\{f(z_j)\}_{j\geq 1}$ is also dense in $\mathbb{C}_\infty$, so $f \in MCS(\Gamma(\mathbb{D}))$.

It remains to demonstrate that $f \in HC((C_{\varphi_n}))$. As in the "hypercyclicity" part of the proof of Theorem 3.1, we set $q(j) := p(N,j)$, $\nu(j) := m_{p(N,j)}$ ($j \in \mathbb{N}$). If $z \in \varphi_{\nu(j)}(B_{q(j)})$ then we obtain from (4) that

$$|\widetilde{f}_N(z) - P_j(\varphi_{\nu(j)}^{-1}(z))| < \frac{1-|z|}{3^N} < 1 - |z| < \frac{1}{q(j)}$$

$$\text{and} \quad |\widetilde{f}_k(z)| < \frac{1-|z|}{3^k} \quad \text{for all } k \neq N.$$

By changing variables, we get for all $j \in \mathbb{N}$ and all $z \in B_{q(j)}$ that

$$|\widetilde{f}_N(\varphi_{\nu(j)}(z)) - P_j(z)| < \frac{1}{q(j)} \tag{12}$$

and

$$|\widetilde{f}_k(\varphi_{\nu(j)}(z))| < \frac{1}{q(j)3^k} \quad (k \neq N). \tag{13}$$

Now recall that $h_l \to f$ compactly in $\mathbb{D}$. Therefore for each $j \in \mathbb{N}$ there exists $l_0(j) \in \mathbb{N}$ satisfying

$$|f(z) - h_l(z)| < \frac{1}{j} \quad (z \in \varphi_{\nu(j)}(B_{q(j)}),\, l \geq l_0(j)). \tag{14}$$

Moreover, it follows from (7) that there is $l = l(j) \in \mathbb{N}$ with $l \geq l_0(j)$ for which

$$|c_{N,l} - 1| < \frac{1}{j(1 + \max_{z \in B_{q(j)}} |P_j(z)|)}. \tag{15}$$

Putting together (12), (13), (14) and (15), and using again the triangle inequality as well as the Cauchy-Schwarz inequality, we obtain for every $z \in B_{q(j)}$ that

$$\begin{aligned} |(C_{\varphi_{\nu(j)}} f)(z) - P_j(z)| &\leq |f(\varphi_{\nu(j)}(z)) - h_l(\varphi_{\nu(j)}(z))| + |h_l(\varphi_{\nu(j)}(z)) - P_j(z)| \\ &\leq |f(\varphi_{\nu(j)}(z)) - h_l(\varphi_{\nu(j)}(z))| \\ &\quad + |c_{N,l}\widetilde{f}_N(\varphi_{\nu(j)}(z)) - P_j(z)| + \sum_{\substack{k=1 \\ k \neq N}}^{J(l)} |c_{k,l}\widetilde{f}_k(\varphi_{\nu(j)}(z))| \end{aligned}$$

$$
\begin{aligned}
&< |f(\varphi_{\nu(j)}(z)) - h_l(\varphi_{\nu(j)}(z))| \\
&\quad + |c_{N,l}(\widetilde{f}_N(\varphi_{\nu(j)}(z)) - P_j(z))| + |(c_{N,l}-1)P_j(z)| \\
&\quad + \left(\sum_{\substack{k=1 \\ k\neq N}}^{J(l)} |c_{k,l}|^2 \right)^{1/2} \left(\sum_{\substack{k=1 \\ k\neq N}}^{\infty} |\widetilde{f}_k(P_j(z))|^2 \right)^{1/2} \\
&< \frac{1}{j} + \frac{\beta}{q(j)} + \frac{1}{j} + \frac{\alpha^{1/2}}{q(j)} \left(\sum_{k=1}^{\infty} \frac{1}{9^k} \right)^{1/2} \\
&< \frac{2+\beta+(\alpha/8)^{1/2}}{j}.
\end{aligned}
$$

Thus $\lim_{j\to\infty} \sup_{z\in B_{q(j)}} |(C_{\varphi_{\nu(j)}} f)(z) - P_j(z)| = 0$. Since $\{P_j\}_{j\geq 1}$ is dense in $H(\mathbb{D})$ and $\{B_{q(j)}\}_{j\geq 1}$ is an exhausting sequence of compact sets in $\mathbb{D}$, we derive that $\{C_{\varphi_{\nu(j)}} f\}_{j\geq 1}$ is dense in $H(\mathbb{D})$. In turn, this trivially implies that $\{C_{\varphi_n} f\}_{n\geq 1}$ is dense in $H(\mathbb{D})$, that is, $f \in HC((C_{\varphi_n}))$. This completes the proof. $\square$

Remark 3.3.

1. Theorems 3.1-3.2 complement the results in [9], where simultaneous *inner* and *outer* behavior has been discovered. Specifically, in [9] the dense-lineability and the spaceability of $MCS(\Gamma(\mathbb{D})) \cap \mathcal{U}(\mathbb{D})$ are stated, where $\mathcal{U}(\mathbb{D})$ denotes the family of functions $f \in H(\mathbb{D})$ satisfying that, for any fixed compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement, the Taylor partial sums of f approximate uniformly any continuous function on K that is holomorphic on K^0.
2. A minor change in the proof of the last theorem shows that, for an injectively runaway sequence (φ_n) of holomorphic self-maps on a simply connected domain G, the set $HC((C_{\varphi_n}))$ is always spaceable. Namely, consider the same set F as in the last proof but without the spirals (i.e.

$F = K_0 \cup \bigcup_{j=1}^{\infty} \varphi_{m_j}(B_j))$, and consider as $\widetilde{g}_n : F \to \mathbb{C}$ the function

$$\widetilde{g}_n(z) = \begin{cases} (2z)^n & \text{if } z \in K_0 \\ P_j(\varphi_{m_{p(n,j)}}^{-1})(z) & \text{if } z \in \varphi_{m_{p(n,j)}}(B_{p(n,j)}) \\ 0 & \text{if } z \in \varphi_{m_{p(k,j)}}(B_{p(k,j)}) \text{ and } k \neq n. \end{cases}$$

Then take $\widetilde{M}$ as in the proof of Theorem 3.2 and conclude the demonstration in a similar (but shorter) way. This extends [23, Theorem 3.1] due to Grosse-Erdmann and Mortini and complements Theorem 1.5.

We want to complete this study by showing that, as a matter of fact, the algebraic genericity enjoyed by our family of functions is even stronger than that exhibited in Theorem 3.1. To be more precise, we will be able to state the existence of a *dense* vector subspace of $H(G)$ with *maximal algebraic dimension* (that is, its dimension equals $\mathfrak{c} :=$ the cardinality of the continuum) all of whose non-zero members are compositionally hypercyclic and have maximal cluster sets along any admissible curve tending to the boundary. Note that, since $H(G)$ is a separable complete metrizable space, we have $\dim(H(G)) = \mathfrak{c}$. Thus $\mathfrak{c}$ is the maximal dimension allowed for any subspace of $H(G)$.

Theorem 3.4. *The set $MCS(\Gamma(G)) \cap HC((C_{\varphi_n}))$ is maximal dense-lineable in $H(G)$, that is, there exists a dense vector subspace $M_{\max}$ in $H(G)$ satisfying*

$$\dim(M_{\max}) = \mathfrak{c} \quad \text{and} \quad M_{\max} \setminus \{0\} \subset MCS(\Gamma(G)) \cap HC((C_{\varphi_n})).$$

Proof. We only sketch the proof, because it is based upon the constructions given in the proofs of Theorems 3.1-3.2, whose notation we keep. The details are left to the interested reader. Once more, it is enough to consider the case $G = \mathbb{D}$.

We divide $\mathbb{N}$ into infinitely many pairwise disjoint strictly increasing sequences $\{p(n,j) : j = 1, 2, \ldots\}$ $(n \geq 0)$. Observe that the sequence $\{p(0,j)\}_{j\geq 1}$ occurs here for the first time. In turn, we divide $\{p(0,j)\}_{j\geq 1}$ into infinitely many pairwise disjoint strictly increasing sequences $\{\lambda(n,j)\}_{j\geq 1}$ $(n \geq 1)$. Now we define $M := \operatorname{span}\{f_n\}_{n\geq 1}$ as in the proof of Theorem 3.1, with the sole exception that, in the selection of the corresponding "close" functions g_n $(n \geq 1)$, each of these is defined as 0 on $\bigcup_{p(0,j)\geq n} S_{p(0,j)} \cup \varphi_{m_{p(0,j)}}(B_{p(0,j)})$. On the other hand, to select the functions $\widetilde{f}_n$ via Nersesjan's theorem, we define the functions $\widetilde{g}_n$ similarly to the proof of Theorem 3.2, with the unique change that the numbers $p(n, j)$ in equation (3) are respectively replaced by the numbers $\lambda(n, j)$. Now, we define $\widetilde{M} := \overline{\text{span}}\,\{\widetilde{f}_n\}_{n\geq 1}$ and

$$M_{\max} := \text{span}\,(M \cup \widetilde{M}).$$

Because of the density of M in $H(\mathbb{D})$, we plainly have that $M_{\max}$ is a dense vector subspace of $H(\mathbb{D})$. Moreover, since $\dim(\widetilde{M}) = \mathfrak{c}$, we also have $\dim(M_{\max}) = \mathfrak{c}$.

Finally, let $f \in M_{\max} \setminus \{0\}$. If $f \in M$, we have already proved that $f \in MCS((\Gamma(\mathbb{D})) \cap HC((C_{\varphi_n}))$. If $f \in M_{\max} \setminus M$, then one can write

$$f = \widetilde{f} + \sum_{k=1}^{m} \alpha_k f_k,$$

for certain $\widetilde{f} \in \widetilde{M} \setminus \{0\}$, $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ and $m \in \mathbb{N}$. At this point, we would proceed by combining the techniques of the proofs of Theorems 3.1–3.2. The function $\widetilde{f}$ should assume the role of the main function for approximations, while the f_k's are small in the sets to be considered (here the role played by the new sequence $\{p(0, j)\}_{j\geq 1}$ is relevant). $\square$

We conclude this paper by posing the following problem, which was the original germ of this work.

Problem. Let (T_n) be a hereditarily densely hypercyclic sequence of operators on $H(G)$. Is the set $MCS(\Gamma(G)) \cap HC((T_n))$ dense-lineable? Under what conditions is it spaceable? We do not know the answer even in the simpler case $(T_n) = (T^n)$.

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