ANALYTICAL BEHAVIOR OF 2-D INCOMPRESSIBLE FLOW IN POROUS MEDIA

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Abstract. In this paper we study the analytic structure of a two-dimensional mass balance equation of an incompressible fluid in a porous medium given by Darcy's law. We obtain local existence and uniqueness by the particle-trajectory method and we present different global existence criterions. These analytical results with numerical simulations are used to indicate nonformation of singularities. Moreover, blow-up results are shown in a class of solutions with infinite energy.

1. Introduction

The dynamics of a fluid through a porous medium is a complex and not thoroughly understood phenomenon [3, 16]. The purpose of the paper is to study a nonlinear two-dimensional mass balance equation in porous media and the conditions of formation of singularities using analytical results and numerical calculations.

In real applications, one might be interested in the transport of a dissolved contaminant in porous media where the contaminant is convected with the subsurface water [17]. For example, one is usually interested in the time taken by the pollutant to reach the water table below. Such flows also occur in artificial recharge wells where water and (or) chemicals from the surface are transported into aquifers. In this case, the chemicals may be the nutrients required for degradation of harmful polluting hydrocarbons resident in the aquifer after a spillage.

We use Darcy's law to model the flow velocities, yielding the following relationship between the liquid discharge (flux per unit area) $v = (v_1, v_2)$ and the pressure gradient

$$v = -k\left(\nabla p + g\gamma\rho\right),$$

where k is the matrix position-independent medium permeabilities in the different directions respectively divided by the viscosity, ρ is the liquid density, g is the acceleration due to gravity and the vector $\gamma = (0, 1)$. While the Navier-Stokes equation and the Stokes equation are both microscopic equations, Darcy's law gives the macroscopic description of a flow in a porous medium [3].

The free boundary problem given by an incompressible 2-D fluid through porous media with two different constant densities and viscosities at each side of the interface is studied in [18, 1] (see references therein). Here, we analyze the dynamics of the density function $\rho = \rho(x_1, x_2, t)$ with a regular initial data $\rho_0 = \rho(x_1, x_2, 0)$.

The mass balance equation is given by

$$\phi\frac{D\rho}{Dt} = \phi\left(\frac{\partial\rho}{\partial t} + v\cdot\nabla\rho\right) = 0,$$

where ϕ denotes the porosity of the medium. To simplify the notation, we consider $k = g = \phi = 1$. Thus, our system of a two-dimensional mass balance equation in porous media (2DPM) is written as

$$\frac{D\rho}{Dt} = 0, \tag{1.1}$$

$$v = -\left(\nabla p + \gamma\rho\right). \tag{1.2}$$

We close the system assuming incompressibility, i.e.,

$$\mathrm{div}\, v = 0, \tag{1.3}$$

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therefore there exists a stream function $\psi(x,t)$ such that

$$v = \nabla^{\perp}\psi \equiv \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right). \tag{1.4}$$

Computing the curl of equation (1.2), we get the Poisson equation for ψ

$$-\Delta\psi = \frac{\partial \rho}{\partial x_1}. \tag{1.5}$$

A solution of this equation is given by the convolution of the Newtonian potential with the function $\partial_{x_1}\rho$

$$\psi(x,t) \;=\; -\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|\frac{\partial \rho}{\partial y_1}(y,t)dy, \qquad x\in\mathbb{R}^2. \tag{1.6}$$

Thus, the velocity v can be recovered from ψ by the operator $\nabla^{\perp}$ by the two equivalent formulas

$$v(x,t) \;=\; \int_{\mathbb{R}^2} K(x-y)\,\nabla^{\perp}\rho(y,t)dy, \qquad x\in\mathbb{R}^2, \tag{1.7}$$

$$v(x,t) \;=\; \mathrm{PV}\int_{\mathbb{R}^2} H(x-y)\,\rho(y,t)dy - \frac{1}{2}\left(0,\rho(x)\right), \qquad x\in\mathbb{R}^2, \tag{1.8}$$

where the kernels $K(\cdot)$ and $H(\cdot)$ are defined by

$$K(x) = -\frac{1}{2\pi}\frac{x_1}{|x|^2} \qquad \text{and} \qquad H(x) = \frac{1}{2\pi}\left(-2\frac{x_1x_2}{|x|^4}, \frac{x_1^2-x_2^2}{|x|^4}\right). \tag{1.9}$$

Differentiating the equation (1.1), we obtain the evolution equation for

$$\nabla^{\perp}\rho \equiv \left(-\frac{\partial \rho}{\partial x_2}, \frac{\partial \rho}{\partial x_1}\right) \tag{1.10}$$

which is given by

$$\frac{D\nabla^{\perp}\rho}{Dt} = (\nabla v)\nabla^{\perp}\rho. \tag{1.11}$$

Taking the divergence of the equation (1.2), we get

$$-\Delta p = \frac{\partial \rho}{\partial x_2}$$

and the pressure can be obtained as in (1.6).

The objective of this work is to analyze the behavior of the solutions of the system 2DPM (1.1)-(1.3). First, we present the existence of singularities in a class of solutions with infinite energy in 2DPM (see Proposition 2.2 and Remark 2.3).

In the case of solutions with regular initial data and finite energy, we get local well-posedness using the classical particle trajectories method. We illustrate a criterion of global existence solutions via the norm of the bounded mean oscillation space of (1.10). A similar result is known in the three dimensional Euler equation (3D Euler) [2]. Also, using the geometric structure of the level sets of the density (where ρ is constant) and the nonlinear evolution equations of the gradient of the arc length of the level sets, we establish that no singularities are possible under not very restrictive conditions. This result is comparable to the 3D Euler equations [7] and to the two-dimensional quasi geostrophic equation (2DQG) [8]. Applying these criterions, we find no evidence of formation of singularities in our numerical simulations.

The paper is organized as follows. In Section 2 we study the analytical behavior of solutions with infinite energy. In Section 3 we prove the existence and uniqueness for the 2DPM, show a characterization of formation of singular solutions and we present geometric constraints on singular solutions. Finally, in Section 4 we illustrate two numerical examples in which the analytical results are applied to show nonsingular solutions.

2. Singularities with infinite energy

Let the stream function ψ be defined by

$$\psi(x_1, x_2, t) = x_2 f(x_1, t) + g(x_1, t). \tag{2.1}$$

Note that under this hypothesis the solution of (1.1)–(1.3) has infinite energy.

We reduce the equations (1.1)–(1.3) to other system with respect to the functions f and g. From (1.5) the density, apart from a constant, satisfies

$$\rho(t, x_1, x_2) = -x_2 \frac{\partial f}{\partial x_1}(x_1, t) - \frac{\partial g}{\partial x_1}(x_1, t) = -x_2 f_{x_1} - g_{x_1}, \tag{2.2}$$

and, by (1.4), v verifies

$$v(t, x_1, x_2) = \left(-f(x_1, t), x_2 \frac{\partial f}{\partial x_1}(x_1, t) + \frac{\partial g}{\partial x_1}(x_1, t) \right) = (-f, x_2 f_{x_1} + g_{x_1}). \tag{2.3}$$

Therefore, the system (1.1)–(1.3) under the hypothesis (2.1) is equivalent to

$$(f_x)_t = f f_{xx} - (f_x)^2, \tag{2.4}$$

$$(g_x)_t = f g_{xx} - f_x g_x. \tag{2.5}$$

(Here and in the sequel of the section, we denote with subscript the derivatives with respect to x.) We note the non-linear character of the first equation. Thus, our study of formation of singularities is concentrated in the solutions of (2.4). The function g depends implicitly on f in equation (2.5).

Now, we show that the system (2.4) and (2.5) is local well posed in the Sobolev spaces $H_0^k(0,1)$.

Lemma 2.1. *Let $f^0 = f(x, 0)$ and $g^0 = g(x, 0)$ satisfy $f_x^0, g_x^0 \in H_0^k(0,1)$ with $k \geq 1$. Then, there exists $T > 0$ such that $f_x, g_x \in C^1([0,T]; H_0^k(0,1))$ are the unique solution of (2.4)–(2.5).*

Proof. By (2.4) and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|f_x\|_{L^2}^2 = \int_0^1 f_x f f_{xx} - \int_0^1 f_x^3 = -\frac{3}{2}\int_0^1 f_x^3 \leq C\|f_x\|_{L^\infty}\|f_x\|_{L^2}^2 \leq C\|f_x\|_{H_0^1}^3.$$

Analogously,

$$\frac{1}{2}\frac{d}{dt}\|f_{xx}\|_{L^2}^2 = -\int_0^1 f_{xx}^2 f_x - \int_0^1 f_{xx} f f_{xxx} = -\frac{1}{2}\int_0^1 f_{xx}^2 f_x \leq C\|f_x\|_{L^\infty}\|f_{xx}\|_{L^2}^2 \leq C\|f_x\|_{H_0^1}^3.$$

We can repeat for all $k \geq 1$ and we obtain

$$\frac{1}{2}\frac{d}{dt}\|f_x\|_{H_0^k}^2 \leq C\|f_x\|_{H_0^k}^3.$$

Integrating in time, we get

$$\|f_x\|_{H_0^k} \leq \frac{\|f_x^0\|_{H_0^k}}{1 - Ct\|f_x^0\|_{H_0^k}}.$$

On the other hand, by (2.4) and integrating by parts, we have for g_x the following inequalities

$$\frac{1}{2}\frac{d}{dt}\|g_x\|_{L^2}^2 = \int_0^1 g_x g_{xx} f - \int_0^1 g_x^2 f_x = -\frac{3}{2}\int_0^1 g_x^2 f_x \leq \|f_x\|_{L^\infty}\|g_x\|_{H_0^1}^2$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|g_{xx}\|_{L^2}^2 &= \int_0^1 g_{xx} g_{xxx} f - \int_0^1 g_{xx} g_x f_{xx} = -\frac{1}{2}\int_0^1 g_{xx}^2 f_x - \int_0^1 g_{xx} g_x f_{xx} \\ &\leq \|f_x\|_{L^\infty}\|g_{xx}\|_{L^2}^2 + \|g_{xx}\|_{L^2}\|g_x\|_{L^\infty}\|f_{xx}\|_{L^2} \leq \|f_x\|_{H_0^1}\|g_x\|_{H_0^1}^2.\end{aligned}$$

Thus, we obtain using Gronwall's Lemma

$$\|g_x\|_{H_0^k}^2 \leq \|g_x^0\|_{H_0^k}^2 \exp\left(C \int_0^t \|f_x\|_{H_0^k} ds \right)$$

and we have existence up to a time $T = T(\|f_x^0\|_{H_0^k})$.

In order to prove the uniqueness, let $f_x(x,t) = h_x(x,t) - k_x(x,t)$, with h_x, k_x two solutions of (2.4) with the same initial data f_x^0. Since h_x, k_x satisfy (2.4) and integrating by parts, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|f_x\|_{L^2}^2 &= \int_0^1 f_x(hh_{xx} - kk_{xx}) - \int_0^1 f_x(h_x^2 - k_x^2) \\
&= \int_0^1 f_x h f_{xx} + \int_0^1 f_x f k_{xx} - \int_0^1 (f_x)^2(h_x + k_x) \\
&= -\frac{1}{2}\int_0^1 (f_x)^2 h_x + \int_0^1 f_x f k_{xx} - \int_0^1 (f_x)^2(h_x + k_x).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|f_x\|_{L^2}^2 &\leq \|k_{xx}\|_{L^2}\|f_x\|_{L^2}\|f\|_{L^\infty} + C(\|h_x\|_{L^\infty} + \|k_x\|_{L^\infty})\|f_x\|_{L^2}^2 \\
&\leq C(\|h_x\|_{H_0^1} + \|k_x\|_{H_0^1})\|f_x\|_{L^2}^2
\end{aligned}$$

and using Gronwall's Lemma it follows $h_x = k_x$. Finally, we conclude the uniqueness of g_x since (2.5) is a linear differential equation . ∎

The following result shows that the solution of (2.4) blows up in finite time under certain conditions on the initial data.

Proposition 2.2. *Let f_x be a solution of (2.4) with initial data satisfies $f_x^0 \in H_0^2(0,1)$ and $\min_x f_x^0 < 0$. Then, $\|f_x\|_{L^\infty}$ blows up in finite time $T = -1/\min_x f_x^0$.*

Proof. By the local existence result, we have $f_x \in C^1([0,T]; H^2) \subset C^1([0,T] \times [0,1])$. We consider the application $m : [0,T] \to \mathbb{R}$ defined by $m(t) = \min_x f_x(x,t) = f_x(x_t, t)$. By Rademacher Theorem, it follows that m is differentiable at almost every point.

First, we calculate the derivative of m as in [5, 9]. Let s be a point of differentiability of $m(t)$, then for $\tau > 0$

$$\begin{aligned}
m'(s) &= \lim_{\tau\to 0}\frac{m(s+\tau) - m(s)}{\tau} = \lim_{\tau\to 0}\frac{f_x(x_{s+\tau}, s+\tau) - f_x(x_s, s)}{\tau} \\
&= \lim_{\tau\to 0}\frac{f_x(x_{s+\tau}, s+\tau) - f_x(x_s, s+\tau)}{\tau} + \frac{f_x(x_s, s+\tau) - f_x(x_s, s)}{\tau}.
\end{aligned}$$

Since $f_x(x, s+\tau)$ reaches the minimum at the point $x_{s+\tau}$, we obtain

$$m'(s) \leq \lim_{\tau\to 0}\frac{f_x(x_s, s+\tau) - f_x(x_s, s)}{\tau} = f_{xt}(x_s, s).$$

We compute the derivative with a sequence of negative $\tau < 0$ and, by the sign of τ, we get the opposite inequality and we conclude that

$$m'(s) = f_{xt}(x_s, s) \qquad \text{almost everywhere.}$$

We replace x for x_s in (2.4) and yields

$$m'(s) = -f_x^2(x_s, s) = -(m(s))^2,$$

due to $f_{xx}(x_s, s) = 0$, and the proof follows. ∎

Remark 2.3. *There are other blow-up results with an initial data of lower regularity. In particular, we consider $f_x^0 \in H_0^1$ and assuming that*

$$\int_0^1 f_x^0 \leq 0.$$

Thus, by (2.4), we have

$$\frac{d}{dt}\int_0^1 f_x = \int_0^1 f f_{xx} - \int_0^1 (f_x)^2 = -2\int_0^1 (f_x)^2 \geq -2\left(\int_0^1 f_x\right)^2.$$

Defining

$$c(t) = \int f_x,$$

and integrating, we get

$$c(t) \leq \frac{c(0)}{1 + 2tc(0)}.$$

Then, $c(t)$ blows up for $c(0) < 0$.

In the case $c(0) = 0$, we have $c'(t) < 0$ for all $t > 0$, therefore, $c(t)$ also blows up. ▪

Remark 2.4. *Let $x_1 = x_t$ be the point such that*

$$f_x(x_t, t) = \min_x f_x(x, t),$$

and consider

$$x_2 = 1 - \frac{g_x(x_t, t)}{f_x(x_t, t)}.$$

Then, by (2.2), $\rho(x_1, x_2, t) = -f_x(x_t, t)$ blows up in finite time by Proposition 2.2. Analogously, v defined in (2.3) *blows up in finite time.* ▪

3. Analysis of 2DPM with finite energy

3.1. **Local existence of 2DPM.** We derive a reformulation of the system as an integro-differential equation for the particle trajectories. Given a smooth field $v(x, t)$, the particle trajectories $\Phi(\alpha, t)$ satisfy

$$\frac{d\Phi}{dt}(\alpha, t) = v(\Phi(\alpha, t), t), \qquad \Phi(\alpha, t)|_{t=0} = \alpha. \tag{3.1}$$

The time-dependent map $\Phi(\cdot, t)$ connects the Lagrangian reference frame (with the variable α) to the Eulerian reference frame (with the variable x).

It is well known (Section 2.5 in [15]) that the equation (1.11) implies the following formula

$$\nabla^{\perp}\rho(\Phi(\alpha, t), t) \;=\; \nabla_{\alpha}\Phi(\alpha, t)\, \nabla^{\perp}\rho_0(\alpha),$$

where $\nabla^{\perp}\rho_0$ is the orthogonal gradient of the initial density. This last equality shows us that the orthogonal gradient of the density is stretched by $\nabla_{\alpha}\Phi(\alpha, t)$ along particle trajectories. We rewrite (1.7) as

$$v(\Phi(\alpha, t), t) = \int_{\mathbb{R}^2} K(\Phi(\alpha, t) - \Phi(\beta, t))\, \nabla_{\alpha}\Phi(\beta, t)\, \nabla^{\perp}\rho_0(\beta) d\beta. \tag{3.2}$$

We consider (3.1) as an ODE on a Banach space and using Picard Theorem the local in time existence follows. This is proved analogously like the existence and uniqueness of solutions to the inviscid Euler equation (see Section 4.1 in [15]). In fact, we consider $\nabla^{\perp}\rho_0 \in C^{\delta}(\mathbb{R}^2)$, $\delta \in (0, 1)$. Let $\mathbf{B}$ be the Banach space defined by

$$\mathbf{B} = \left\{\Phi : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{such that } |\Phi(0)| + |\nabla_{\alpha}\Phi|_0 + |\nabla_{\alpha}\Phi|_{\delta} < \infty\right\},$$

where $|\cdot|_0$ is the L^{∞}-norm and $|\cdot|_{\delta}$ is the Hölder semi-norm. Define $\mathcal{O}_M$, the open set of $\mathbf{B}$, as

$$\mathcal{O}_M = \left\{\Phi \in \mathbf{B} | \inf_{\alpha \in \mathbb{R}^2} \det \nabla_{\alpha}\Phi(\alpha) > \frac{1}{2} \text{ and } |\Phi(0)| + |\nabla_{\alpha}\Phi|_0 + |\nabla_{\alpha}\Phi|_{\delta} < M\right\}.$$

The mapping $v(\Phi)$, defined by (3.2), satisfies the assumptions of the Picard theorem, i.e., v is bounded and locally Lipschitz continuous on O_M. As a consequence, for any $M > 0$ there exists $T(M) > 0$ and a unique solution

$$\Phi \in C^1((-T(M), T(M)); O_M)$$

to the particle trajectories (3.1, 3.2).

Remark 3.1. *The 2DPM (1.1)–(1.3) has quantities conserved in time, the L^p norm of ρ for $1 \leq p \leq \infty$, i.e.,*

$$\|\rho(t)\|_p \;=\; \|\rho_0\|_p, \qquad \forall t > 0, \quad 1 \leq p \leq \infty. \tag{3.3}$$

The velocity is obtained from ρ by (1.8)*. These operators are singular integrals with Calderón-Zygmund kernels (see* [19]*). Then for $1 < p < \infty$ the L^p norm of the velocity is bounded for any time $t > 0$.*

3.2. **Blow-up criterion.** In order to estimate the growth of the Sobolev norms we use the space of functions of bounded mean oscillation (*BMO*) (see Chapter IV in [19] for an introduction of this function space).

Theorem 3.2. *Let ρ be the solution of equation* (1.1)–(1.3) *with initial data $\rho_0 \in H^s(\mathbb{R}^2)$ with $s > 2$. Then, the following are equivalent:*

(A) *The interval $[0, \infty)$ is the maximal interval of H^s existence for ρ.*

(B) *The quantity*

$$\int_0^T \|\nabla\rho\|_{BMO}(t)\,dt < \infty \qquad \forall T > 0. \tag{3.4}$$

Proof. We denote the operator Λ^s by $\Lambda^s \equiv (-\Delta)^{s/2}$. Since the fluid is incompressible, we have for $s > 2$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^s\rho\|_{L^2}^2 &= -\int_{\mathbb{R}^2} \Lambda^s\rho\Lambda^s(v\nabla\rho)\,dx = -\int_{\mathbb{R}^2} \Lambda^s\rho(\Lambda^s(v\nabla\rho) - v\Lambda^s(\nabla\rho))\,dx,\\ &\le C\|\Lambda^s\rho\|_{L^2}\|\Lambda^s(v\nabla\rho) - v\Lambda^s(\nabla\rho)\|_{L^2}.\end{aligned}$$

Using the following estimate (see [13])

$$\|\Lambda^s(fg) - f\Lambda^s(g)\|_{L^p} \le c\left(\|\nabla f\|_{L^\infty}\|\Lambda^{s-1}g\|_{L^p} + \|\Lambda^s f\|_{L^p}\|g\|_{L^\infty}\right) \qquad 1 < p < \infty,$$

we obtain for $p = 2$

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s\rho\|_{L^2}^2 \le C(\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty})\|\Lambda^s\rho\|_{L^2}^2. \tag{3.5}$$

Integrating, we get for any $t \le T$

$$\|\Lambda^s\rho\|_{L^2} \le \|\Lambda^s\rho_0\|_{L^2}\exp\left(C\int_0^T(\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty})\right). \tag{3.6}$$

Now, we use the following inequality given in [14]: Let $f \in W^{s,p}$ with $1 < p < \infty$ and $s > 2/p$, then, there exists a constant C=C(p,s) such that

$$\|f\|_{L^\infty} \le C(1 + \|f\|_{BMO}(1 + \ln^+\|f\|_{W^{s,p}})), \tag{3.7}$$

where $\ln^+(x) = \max(0, \ln(x))$. Therefore, for $s > 2$ we have

$$\|\nabla\rho\|_{L^\infty} \le C(1 + \|\nabla\rho\|_{BMO}(1 + \ln^+\|\nabla\rho\|_{H^{s-1}})),$$

and from (3.6), we obtain that

$$\|\nabla\rho\|_{L^\infty} \le C(1 + \ln^+(\|\rho_0\|_{H^s})\|\nabla\rho\|_{BMO}\int_0^T(\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty})\,dt). \tag{3.8}$$

On the other hand, applying (3.7) for $v \in H^s(\mathbb{R}^2)$, we have

$$\|\nabla v\|_{L^\infty} \le C(1 + \|\nabla v\|_{BMO}(1 + \ln^+\|\nabla v\|_{H^{s-1}})).$$

Since v satisfies (1.8) and the singular integrals are bounded operators in *BMO* (see [19]), we get

$$\|\nabla v\|_{L^\infty} \le C(1 + \|\nabla\rho\|_{BMO}(1 + \ln^+\|\nabla\rho\|_{H^{s-1}})),$$

and, using (3.6), we obtain

$$\|\nabla v\|_{L^\infty} \le C(1 + \ln^+(\|\rho_0\|_{H^s})\|\nabla\rho\|_{BMO}\int_0^T(\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty})\,dt). \tag{3.9}$$

From (3.8) and (3.9), follows

$$\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty} \le C(1 + \ln^+(\|\rho_0\|_{H^s})\|\nabla\rho\|_{BMO}\int_0^T(\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty})\,dt).$$

Applying Gronwall's inequality, we have

$$\int_0^T(\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty})\,dt \le CT\exp\left(\ln^+(\|\rho_0\|_{H^s})\int_0^T\|\nabla\rho\|_{BMO}\,dt\right),$$

and so (A) is a consequence of (B).

Finally, due to the inequality
$$\|\nabla\rho\|_{BMO} \leq \|\nabla\rho\|_{H^1},$$
we conclude that (A) implies (B). ▪

Remark 3.3. *Using that*
$$\|\nabla\rho\|_{BMO} \leq C\|\nabla\rho\|_{L^\infty},$$
we get a blow-up characterization for numerical simulations.

3.3. **Geometric constraints on singular solutions.** From equation (1.1) it follows that the level sets, $\rho =$ constant, move with the fluid flow. Then $\nabla^\perp\rho$, defined in (1.10), is tangent to these level sets.

For the 2DPM, the infinitesimal length of a level set for ρ is given by $|\nabla^\perp\rho|$ and from (1.11), the evolution equation for the infinitesimal arc length is given by

$$\frac{D|\nabla^\perp\rho|}{Dt} = \mathcal{L}\,|\nabla^\perp\rho|. \tag{3.10}$$

The factor $\mathcal{L}(x,t)$ is defined through by

$$\mathcal{L}(x,t) = \begin{cases} \mathcal{D}\eta\cdot\eta, & \eta\neq 0,\\ 0, & \eta = 0.\end{cases} \tag{3.11}$$

where the direction of $\nabla^\perp\rho$ is denoted by

$$\eta = \frac{\nabla^\perp\rho}{|\nabla^\perp\rho|} \tag{3.12}$$

and $\mathcal{D}(x,t)$ is the symmetric part of the deformation matrix defined by

$$\mathcal{D} = (\mathcal{D}_{ij}) = \left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right]. \tag{3.13}$$

Now, we show a singularity criterion of 2DPM using the geometric structure of the level sets and mild hypotheses of the solutions. The theorem states below is analogous to 3D Euler [7] and to 2DQG [8].

Recall that η is the direction field tangent to the level sets of ρ defined (3.12). Analogously to [8], a set Ω is *smoothly directed* if there exists $\delta > 0$ such that

$$\sup_{x\in\overline{\Omega}}\int_0^T \|\nabla\eta(\cdot,t)\|^2_{L^\infty(B_\delta(\Phi(x,t)))}dt \,<\, \infty, \tag{3.14}$$

where
$$B_\delta(x) = \{y\in\mathbb{R}^2 : \quad |x-y|<\delta\}, \qquad \overline{\Omega} = \{x\in\Omega;\ |\nabla\rho_0(x)|\neq 0\},$$
and Φ is the particle trajectories map. We define $\Omega(t) = \Phi(\Omega,t)$ and $\mathcal{O}_T(\Omega)$ the semi-orbit, i.e.,
$$\mathcal{O}_T(\Omega) = \bigcup_{0\leq t\leq T}\{t\}\times\Omega(t).$$

Theorem 3.4. *If Ω is smoothly directed and*

$$\int_0^T \|R_j\rho\|_{L^\infty}(t)dt \,<\, \infty, \quad j=1,2, \quad \forall T>0, \tag{3.15}$$

where R_j denotes the Riesz transform in the direction x_j, then
$$\sup_{\mathcal{O}_T(\Omega)} |\nabla\rho(x,t)| \,<\, \infty.$$

Remark 3.5. *Using the Remark 3.3, the previous theorem illustrates that finite-time singularities are impossible in smoothly directed sets.*

Proof. We show a similar formula of the level–set stretching factor $\mathcal{L}$ defined in (3.11). We start by computing the full gradient of the velocity v. From formula (1.7)
$$v(x) = \int_{\mathbb{R}^2} K(y)\,\nabla^\perp\rho(x-y)dy,$$

we have

$$\nabla v(x) = \int_{\mathbb{R}^2} K(y)\,(\nabla_y \nabla_y^\perp \rho)(x-y)dy.$$

Take the integral as a limit as $\varepsilon \to 0$ of integrals on $|y| > \varepsilon$ and integrate by parts. In this fashion, we obtain the formula

$$\nabla v(x) = \frac{1}{2\pi}\mathrm{PV}\int_{\mathbb{R}^2} \big(\nabla_y \rho(x-y)\big) \otimes \widetilde{y}\,\frac{dy}{|y|^2} \;-\; \frac{1}{2}\begin{pmatrix} 0 & 0 \\ \dfrac{\partial \rho}{\partial x_1}(x) & \dfrac{\partial \rho}{\partial x_2}(x) \end{pmatrix}, \tag{3.16}$$

where $\widetilde{y}$ is the unit vector defined by

$$\widetilde{y} = \left(-\frac{2y_1y_2}{|y|^2}, \frac{y_1^2 - y_2^2}{|y|^2}\right).$$

By definition of η in (3.12), we have $\eta\cdot\nabla\rho = 0$. Thus, computing we get

$$\mathcal{L}(x) = \frac{1}{2\pi}\mathrm{PV}\int_{\mathbb{R}^2} \big(\widetilde{y}\cdot\eta(x)\big)\big(\eta(x-y)\cdot\eta^\perp(x)\big)\}|\nabla^\perp\rho(x-y)|\frac{dy}{|y|^2}. \tag{3.17}$$

Let be $\phi \in C_c^\infty(\mathbb{R})$, $\phi \geq 0$, supp(ϕ) include in $[-1,1]$ and $\phi(s) = 1$ in $s \in [-1/2, 1/2]$. Consider $r > 0$ and decompose

$$\mathcal{L}(x) = I_1 + I_2,$$

with

$$|I_1| \leq \frac{1}{2\pi}\left|\int_{\mathbb{R}^2} (1 - \phi(|y|^2/r^2))\big(\widetilde{y}\cdot\eta(x)\big)\big(\nabla^\perp\rho(x-y)\cdot\eta^\perp(x)\big)\frac{dy}{|y|^2}\right|.$$

Integrating by parts and using Cauchy-Schwartz inequality, we get

$$I_1 \leq \frac{C}{r^2}\|\rho\|_{L^2} \leq \frac{C}{r^2}\|\rho_0\|_{L^2}.$$

We have for any $|y| < r$

$$|\eta(x-y)\cdot\eta^\perp(x)| \leq |y|\|\nabla\eta\|_{L^\infty(B_r(x))}.$$

Applying this in the integral I_2, we get

$$|I_2| \leq \int |\nabla^\perp\rho(x-y)|\phi(|y|^2/r^2)\frac{dy}{|y|}\|\nabla\eta\|_{L^\infty(B_r(x))}.$$

We integrate by parts and decompose

$$\int |\nabla^\perp\rho(x-y)|\phi(|y|^2/r^2)\frac{dy}{|y|} = \int \rho(x-y)\nabla^\perp\left(\eta(x-y)\phi(|y|^2/r^2)\frac{1}{|y|}\right)dy = J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 &= \int \rho(x-y)\eta(x-y)\nabla^\perp(\phi(|y|^2/r^2))\frac{dy}{|y|}, \\ J_2 &= \int \rho(x-y)\nabla^\perp(\eta(x-y))\phi(|y|^2/r^2)\frac{dy}{|y|}, \\ J_3 &= \int \rho(x-y)\eta(x-y)\phi(|y|^2/r^2)\frac{(-y_2,y_1)}{|y|^3}dy, \end{aligned}$$

obtaining the following estimates

$$|J_1| \leq c\|\rho_0\|_{L^\infty} \qquad \text{and} \qquad |J_2| \leq cr\|\rho_0\|_{L^\infty}\|\nabla\eta\|_{L^\infty(B_r(x))}.$$

The J_3 term can be bounded using the identity

$$J_3 = \eta(x)(-R_2(\rho)(x), R_1(\rho)(x)) + J_4,$$

getting the following estimate for J_4 in a similar way

$$|J_4| \leq r\|\rho_0\|_{L^\infty}\|\nabla\eta\|_{L^\infty(B_r(x))} + r^{-1}\|\rho_0\|_{L^2}.$$

Thus, we conclude the following estimate for the factor $\mathcal{L}$:

$$|\mathcal{L}(x)| \;\leq\; c\left[\|\nabla\eta\|_{L^\infty(B_r(x))}\max_{j=1,2}|R_j(\rho)| + (r\|\nabla\eta\|_{L^\infty(B_r(x))} + 1)(\|\nabla\eta\|_{L^\infty(B_r(x))}\|\rho_0\|_\infty + r^{-2}\|\rho_0\|_2)\right].$$

Using (3.10), we obtain by Gronwall's lemma

$$\sup_{\mathcal{O}_T(\Omega)} |\nabla\rho(x,t)| \leq \sup_{\Omega} |\nabla\rho_0| \exp\big(\sup_{y\in\Omega} \int_0^T M(t)dt\big),$$

where $M(t)$ is defined by

$$M(t) = c\left[\|\nabla\eta\|_{L^\infty(B_r(x))} \max_{j=1,2} |R_j(\rho)| + (r\|\nabla\eta\|_{L^\infty(B_r(x))} + 1)(\|\nabla\eta\|_{L^\infty(B_r(x))}\|\rho_0\|_\infty + r^{-2}\|\rho_0\|_2)\right],$$

with $x = \Phi(y,t)$. This concludes the proof of Theorem 3.4. ▪

Remark 3.6. *The condition* (3.15) *depending on the Riesz transform is different that in 2DQG (see* [8]*). This appears because the integral kernels* (1.9) *in 2DPM are different to the Kernels in 2DQG.*

Now, we present a geometric conserved quantity that relates the curvature of the level sets and the magnitude $|\nabla^\perp\rho|$ in a similar way as in [6] (see references therein for more details). In particular, if we define the curvature of the level sets κ by

$$\kappa(x,t) = (\eta\cdot\nabla\eta)\cdot\eta^\perp(x,t), \tag{3.18}$$

where η is the direction of $\nabla^\perp\rho$ (see (3.12)), the following identity is satisfied

$$\frac{D(\kappa|\nabla^\perp\rho|)}{Dt} = \nabla^\perp\rho\cdot\nabla\beta \tag{3.19}$$

with

$$\beta(x,t) = (\eta\cdot\nabla v)\cdot\eta^\perp(x,t). \tag{3.20}$$

Indeed, we now prove the identity (3.19). Since $\nabla^\perp\rho$ and $|\nabla^\perp\rho|$ satisfies (1.11) and (3.10) respectively, we get

$$\frac{D\eta}{Dt} = (\nabla v)\eta - \mathcal{L}\,\eta.$$

Using (3.16), we obtain

$$\frac{D\eta}{Dt} = \beta\,\eta^\perp,$$

with β defined in (3.20). By the definition of κ (3.18) and the previous formula, we have

$$\frac{D\kappa}{Dt} = ((\nabla v\cdot\eta - \mathcal{L}\,\eta)\nabla\eta)\cdot\eta^\perp + (\eta\cdot(\beta\nabla\eta^\perp + \eta^\perp\otimes\nabla\beta - \nabla\eta\nabla v))\cdot\eta^\perp - \beta(\eta\cdot\nabla\eta)\cdot\eta$$

and, simplifying,

$$\frac{D\kappa}{Dt} = (\nabla\beta)\eta - \mathcal{L}\,\kappa.$$

Using this identity and (3.10), (3.19) is satisfied.

Remark 3.7. *The integral of the quantity $\kappa|\nabla^\perp\rho|$ over a region given by two different level sets is conserved along the time, i.e.*

$$\frac{d}{dt}\Big(\int_{\{x:\, C_1\leq\rho(x,t)\leq C_2\}} \kappa|\nabla^\perp\rho|\,dx\Big) = 0. \tag{3.21}$$

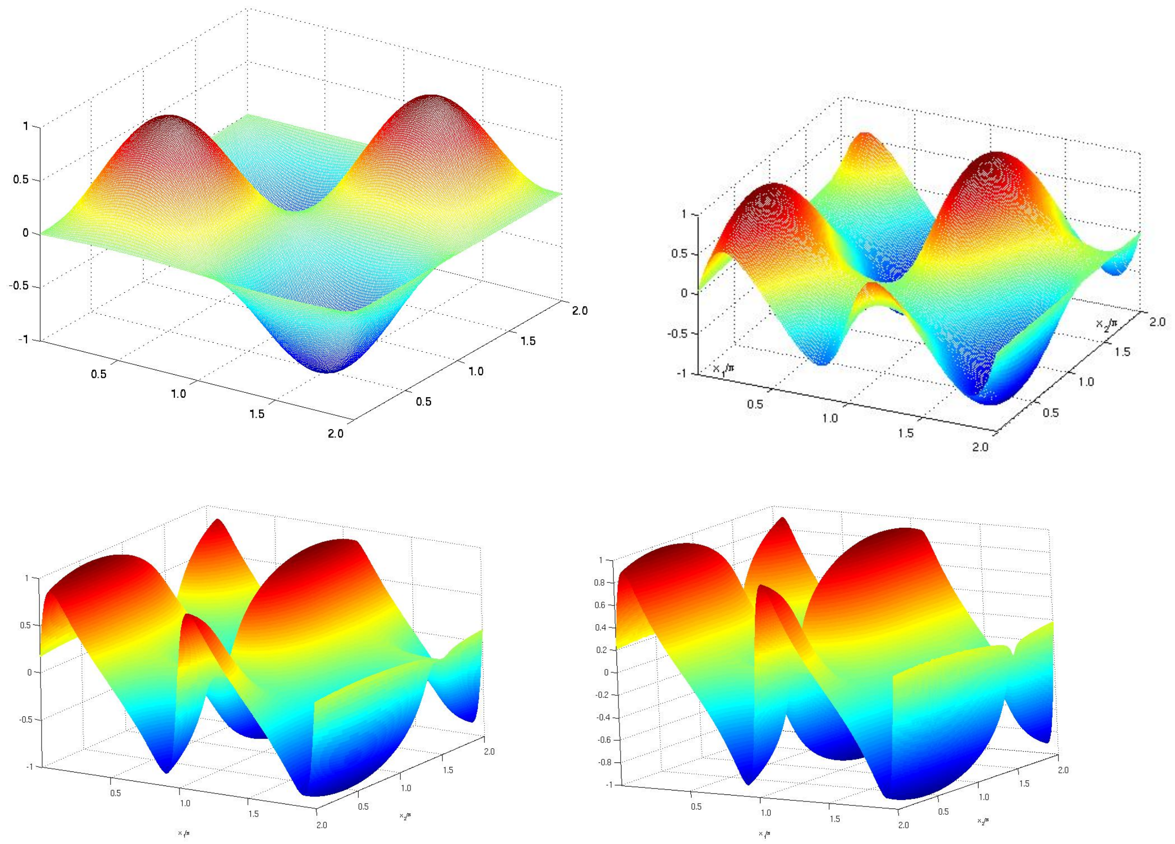
This can be showed using the equation (3.19) *and integrating by parts. Thus, in the case that $|\nabla^\perp\rho|$ is large by* (3.21) *the curvature κ is small if the level sets do not oscillate.*

In all of our numerical experiments, we find no evidence of level set oscillations. On the contrary, we observe that the level sets are flattering where the gradient of ρ is growing.

4. Numerical simulations

Here we present two examples of numerical simulations for solutions of the 2-D mass balance in porous media with initial data in a period-cell $[0,2\pi]^2$. Although periodic boundary conditions are rather unphysical, which does not matter because we are interested in the small-scale structures.

FIGURE 1. *Evolution of the density in the Case 1 for times* $t = 0, 3, 6, 8.5$.



The numerical method. We solve the equations (1.1)–(1.3) numerically on a 2π-periodic cell with a spectral method with smoothing. This numerical method is similar to the scheme developed by E and Shu [12] for incompressible flows and also used for the quasi-geostrophic active scalar in [8].

This algorithm is the standard Fourier-collocation method (see [4]) with smoothing. Roughly speaking, the differentiation operator is approximated in the Fourier space, while the nonlinear operations such as $v \cdot \nabla\rho$ are done in the physical space.

We smooth the gradients adding filters to the spectral method in order for the numerical solutions do not degrade catastrophically. A way of adding the filters is to replace the Fourier multiplier ik_j by $ik_j\varphi(|k_j|)$, where

$$\varphi(k) = e^{-a(k/N)^b}, \qquad \text{for } |k| \le N.$$

Here N is the numerical cutoff for the Fourier modes and a, b is chosen with the machine accuracy (see [20]).

For the temporal discretization, we use Runge-Kutta methods of various order. In our case, we have no explicit temporal dependence and we get a Runge-Kutta method of order 4 that requires at most three levels of storage, see in [4] p. 109.

We present the numerical approximation with an initial resolution of $(256)^2$ Fourier modes. We refine this resolution when the growth of $\|\nabla\rho\|_{L^\infty}$ is substantial to give additional insight preserving the relation space-time. We conclude our numerical simulations with a resolution of $(8192)^2$ Fourier modes.

The computational part of this work was performed on HPC320 (cluster of 8 SMP servers with 32 processors Alpha EV68 1 GHz) of the Centro de Supercomputación de Galicia (CESGA). We used the MATLAB routines to obtain the calculations.

FIGURE 2. *Evolution of the level sets -0.999, -0.99 (on the left of the figure) and 0.99, 0.999 (on the right) of the density in the Case 1 for times $t = 0, 3, 6$.*

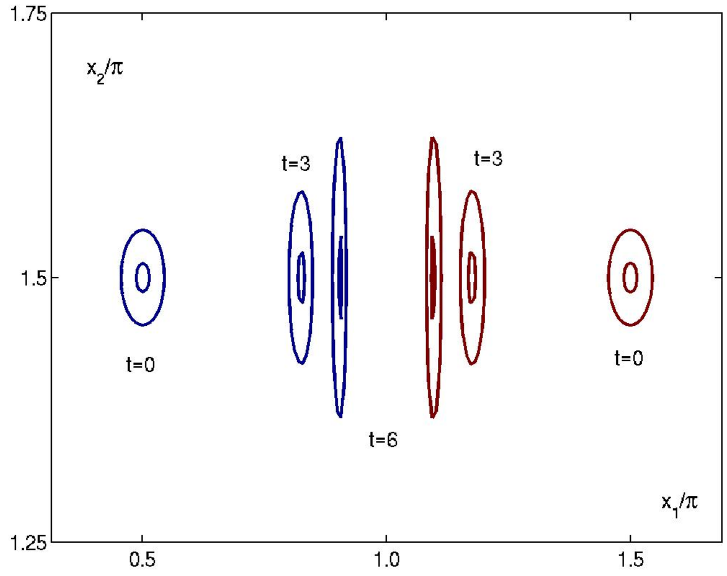
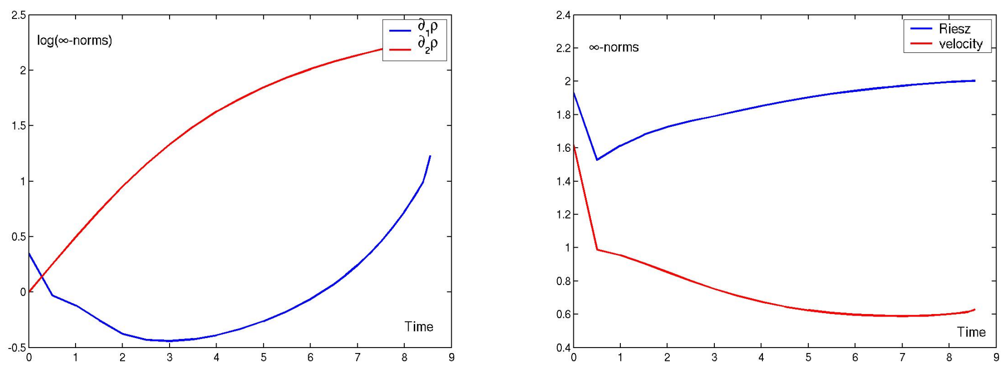


FIGURE 3. *Evolution of the L^∞-norms of $\nabla\rho$, the velocity v and the Riesz transforms $(R_1\rho, R_2\rho)$ for the Case 1.*



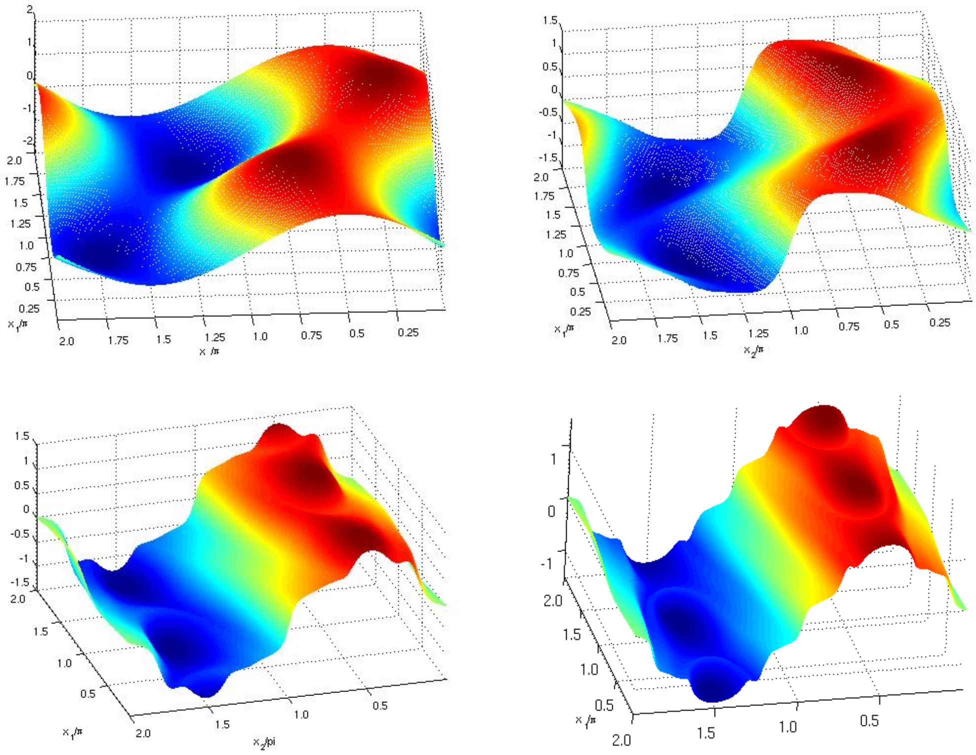
Case 1. We consider the initial datum

$$\rho(x_1, x_2) = \sin(x_1)\sin(x_2).$$

The time step is $\triangle t = 0.025$ from $t = 0$ to $t = 4.0$ with $a = 4.5$ and $b = 2.3$, stopping the experiment with $\triangle t = 0.00125$ and $a = 9.1$, $b = 7.1$. During the simulation the ratio $h/\triangle t$ is preserved getting a finer resolution as the gradients are growing. In this way the method conserves in time the L^∞-norm of the density.

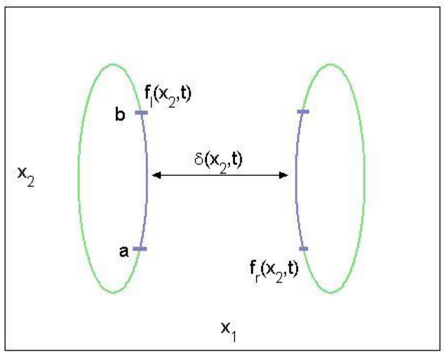
Figure 1 presents the density at times $t = 0, 3, 6, 8.5$ with a numerical resolution of $(256)^2$, $(256)^2$, $(1024)^2$, $(4096)^2$, respectively. The initial data has a hyperbolic saddle point that does not present a nonlinear behavior as time evolves. In all our numerical simulations we do not find any saddle point structures that present stronger front formation than in case 2 (see Figure 5). Nevertheless, the case 1

FIGURE 4. *Evolution of the density in Case 2 for times $t = 0, 3, 6, 9$.*



shows a front formation in the elliptic plots and approaching the maximum with the minimum (see Figure 2) as time evolves.

If we choose two level sets that are approaching each other we have the following scenario



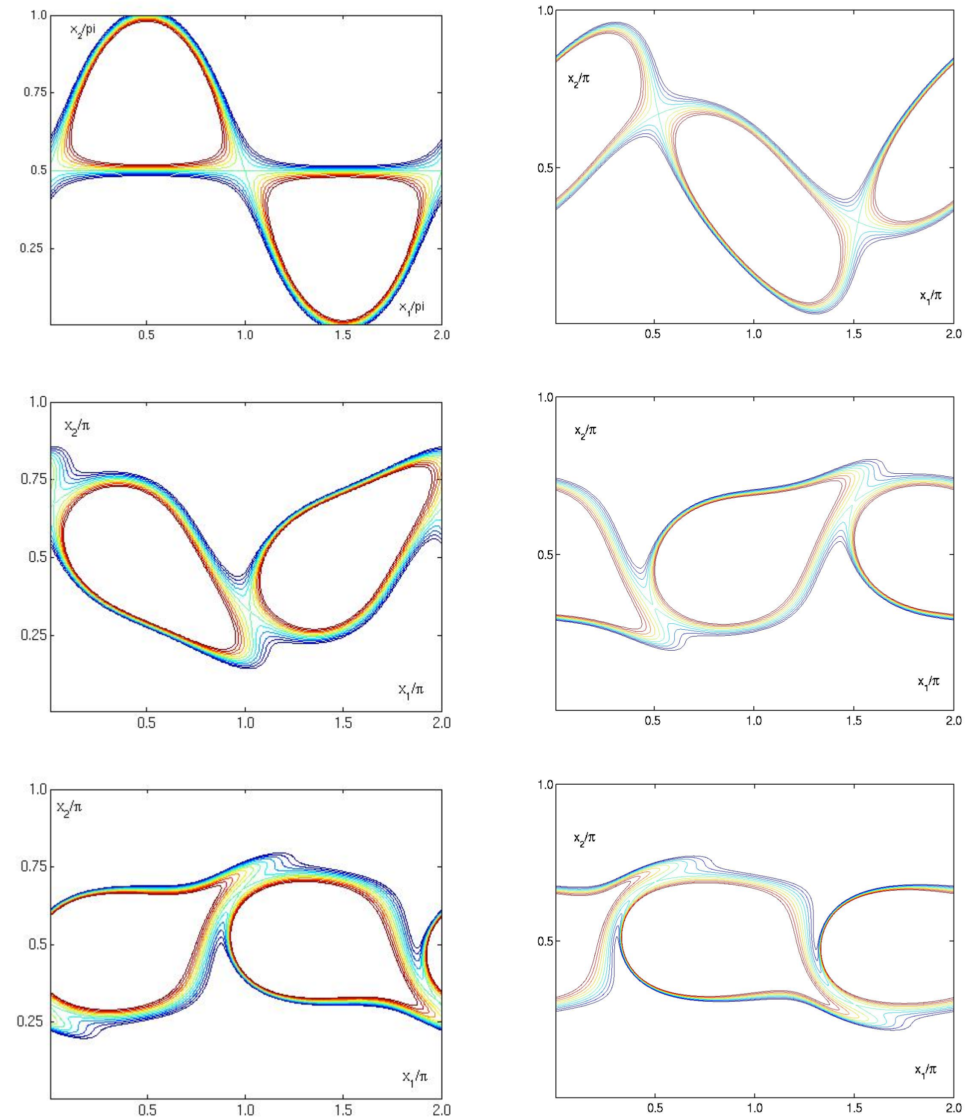
where $\delta = \delta(x_2, t)$ is the distance between the two counters. From previous work (see [10]) it is easy to check that in order for the two graphs f_l, f_r to collapse at time T in any interval $x_2 \in [a, b]$, i.e.,

$$\lim_{t \to T_-} [f_r(x_2, t) - f_l(x_2, t)] = 0 \qquad \forall x_2 \in [a, b]$$

it is necessary that

$$\int_0^T \|v\|_\infty(s) ds = \infty. \tag{4.1}$$

FIGURE 5. *Evolution of around contour $\rho(x_1, x_2) = 1$ in Case 2 for times $t = 0, 1.5, 3, 4.5, 6, 7.5$.*



In figure 3 we see no evidence for quantity $\int_0^T \|v\|_\infty(s)ds$ to blow up in finite time. Nevertheless, we can obtain an estimate of how close the two graphs approach each other without any assumption on the velocity. From figure 2 it seems reasonable to assume that the minimum and maximum of δ are comparable that means there exists and constant $c > 0$ such that

$$\min \delta(x_2, t) \leq c \max \delta(x_2, t) \qquad \forall x_2 \in [a, b].$$

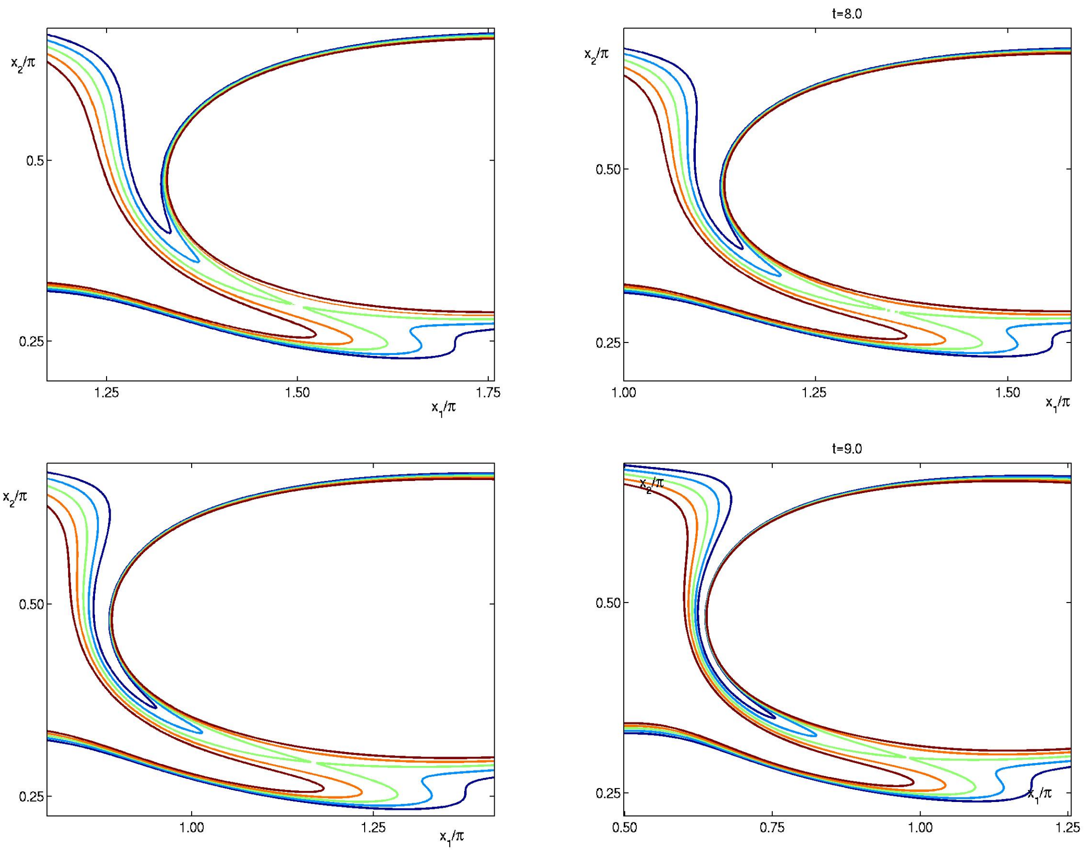
Then we can obtain an evolution equation for the area $A = A(t)$

$$A(t) = \frac{1}{b-a} \int_a^b [f_r(x_2, t) - f_l(x_2, t)]dx_2$$

in between the two graphs that satisfies (see [11] for more details)

$$\left|\frac{dA}{dt}(t)\right| \leq \frac{C}{b-a} \sup_{a \leq x_2 \leq b} |\psi(f_r(x_2, t), x_2, t) - \psi(f_l(x_2, t), x_2, , t)| \tag{4.2}$$

FIGURE 6. *Level contour (0.98,0.99,1.0,1.01,1.02) of ρ in Case 2 for times* $t = 7.5, 8, 8.5, 9$.



where ψ is the stream function (1.6)

$$\psi(x,t) \;=\; -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x_1-z_1}{|x-z|^2}\rho(z,t)dz, \qquad x\in\mathbb{R}^2.$$

Using this formula we obtain that for any $x, y \in \mathbb{R}^2$

$$|\psi(x,t)-\psi(y,t)| \le C(\|\rho_0\|_\infty, \|\rho_0\|_{L^2})|x-y|(1-ln|x-y|), \tag{4.3}$$

due to the following estimates

$$\begin{aligned}|\psi(x,t)-\psi(y,t)| &= |\frac{1}{2\pi}\int_{\mathbb{R}^2}(\frac{x_1-z_1}{|x-z|^2}-\frac{y_1-z_1}{|y-z|^2})\rho(z,t)dz| \\ &\le |\frac{1}{2\pi}\int_{B_{2r}(x)}| + |\frac{1}{2\pi}\int_{B_2(x)-B_{2r}(x)}| + |\frac{1}{2\pi}\int_{B_2(x)}| \\ &= I_1+I_2+I_3\end{aligned}$$

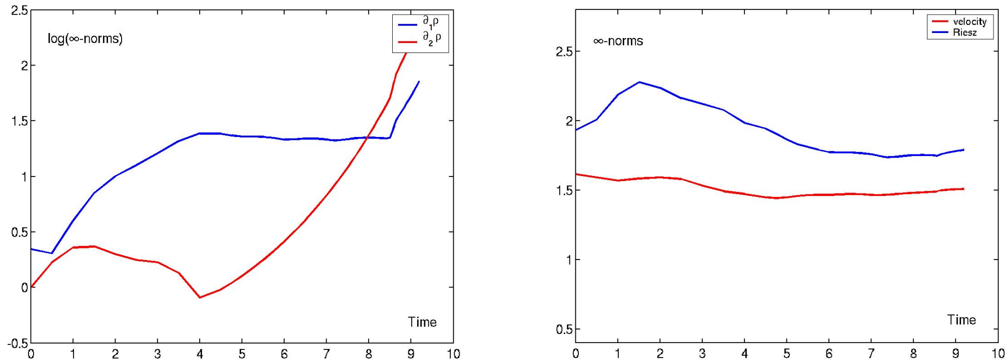
where $r = |x-y|$ and

$$\begin{aligned}&I_1 \le C\|\rho_0\|_\infty|x-y|,\\ &I_2 \le C\|\rho_0\|_\infty|x-y|\int_{2r}^{2}s^{-1}ds \le C\|\rho_0\|_\infty|x-y|(-ln|x-y|),\\ &I_3 \le C\|\rho_0\|_{L^2}|x-y|.\end{aligned}$$

Then, using (4.3) in (4.2) we get that the area $A(t)$ is bounded by

$$A(t) \ge A_0 e^{-Ce^t}.$$

FIGURE 7. *Evolution of L^∞-norms of $\nabla\rho$, the velocity v and the Riesz transforms $(R_1\rho, R_2\rho)$ for Case 2.*



In order to apply the global solution criterion (Theorem 3.2), in Figure 3 is plotted the logarithm of the L^∞-norms of $\partial_1\rho$ and $\partial_2\rho$ showing an exponential growth. The hypothesis (3.15) of the Theorem (3.4) is checked in this example (see Figure 3), so that using this result no singularity is possible due to the variation of the direction field tangent to the level sets is smooth.

Case 2. In this case, the initial datum is

$$\rho(x_1, x_2) = \sin(x_1)\cos(x_2) + \cos(x_1).$$

The time step is $\triangle t = 0.025$ from $t = 0$ to $t = 4.5$ with $a = 4.5$ and $b = 2.3$, stopping the experiment with $\triangle t = 0.001$ and $a = 11.4$, $b = 8$. The L^∞-norm is preserved during the simulation of this case.

Figure 4 presents the density at times $t = 0, 3, 6, 9$ with a numerical resolution of $(256)^2$, $(256)^2$, $(1024)^2$, $(8192)^2$, respectively. In Figure 7, we show log plots of $max|\partial_{x_1}\rho|$ and $max|\partial_{x_2}\rho|$, where we obtain an analogous exponential growth as in the case 1. This growth is not sufficient to guarantee a singularity.

The initial data for the density scalar clearly has a hyperbolic saddle and the numerical solution develops a front as time evolves (see Figures 5 and 6). We observe that the front does not develop nonlinear or potentially singular structure as time evolves. Where η is smoothly directed we observe the highest growth of $\|\nabla\rho\|_{L^\infty}$. Where η changes rapidly we obtain less growth of $\|\nabla\rho\|_{L^\infty}$. Using theorem 3.4 we show no evidence of singularities.

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