# Fixed point theorems for multivalued nonexpansive mappings satisfying inwardness conditions

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**Abstract**

Let $X$ be a Banach space whose characteristic of noncompact convexity is less than 1 and satisfies the non-strict Opial condition. Let $C$ be a bounded closed convex subset of $X$, $KC(X)$ the family of all compact convex subsets of $X$ and $T$ a nonexpansive mapping from $C$ into $KC(X)$ with bounded range. We prove that $T$ has a fixed point. The non-strict Opial condition can be removed if, in addition, $T$ is an 1-$\chi$-contractive mapping.

*Key words:* Fixed point, multivalued nonexpansive mapping, inwardness condition, characteristic of noncompact convexity of a Banach space, Opial condition.

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## 1 Introduction

Let $C$ be a bounded closed convex subset of a Banach space $X$ and $T : C \to C$ a nonexpansive mapping. The problem of finding suitable geometrical conditions on $X$ which assure the existence of a fixed point for $T$ has been widely studied in the last 40 years (see, for instance, [7]). In the case of multivalued nonexpansive mappings $T : C \to K(C)$ a very general problem is the following: Does $T$ have a fixed point under the suitable conditions on $X$ which assure the existence of fixed point for univalued mappings? The answer to this question is unknown, but some papers have appeared showing geometrical properties on $X$ which let state fixed point results for multivalued mappings.

One of the most general fixed point theorems for multivalued nonexpansive self-mappings was obtained by W. A. Kirk and S. Massa in 1990 [9], proving the existence of fixed points in Banach spaces for which the asymptotic center of a bounded sequence in a closed bounded convex subset is nonempty and compact. This occurs if $X$ is, for instance, a uniformly convex space but it is known (see [10]) that when $X$ is nearly uniformly convex (see definition in Section 2) the asymptotic center of a bounded sequence can be a noncompact set. Due to this fact, in [5] the authors establish a generalization of the Kirk-Massa theorem to a class of Banach spaces where the asymptotic center of a sequence is not necessary a compact set. Specifically, they give a fixed point theorem for a multivalued nonexpansive and 1-$\chi$-contractive compact convex valued mapping $T : C \to 2^C$ in the framework of a Banach space whose characteristic of noncompact convexity associated to the separation measure of noncompactness is less than 1. Also it is proved that the $\chi$-contractiveness assumption can be removed when, in addition, the space satisfies the non-strict

Opial condition.

In this paper we obtain similar results for non-self mappings $T : C \to 2^X$ satisfying a inwardness condition. In spite of the analogy between both problems, the arguments must be different. Indeed, in the case of a self-mapping, we can restrict to a separable setting. In this case a basic tool is the existence of a regular and asymptotically uniform subsequence of each bounded sequence. However, in the non-separable setting we need to use ultranets and to state (Theorem 3.1) a relationship between the Chebyshev radius of the asymptotic center of nets and the modulus of noncompact convexity of a Banach space associated to the Kuratowski measure of noncompactness.

## 2 Preliminaries

Let $X$ be a Banach space and $C$ a nonempty closed subset of $X$. We denote by $CB(C)$ the family of all nonempty closed bounded subsets of $C$ and by $K(C)$ (resp. $KC(C)$) the family of all nonempty compact (resp. compact convex) subsets of $C$.

On $CB(X)$ we have the Hausdorff metric $H$ given by

$$H(A,B) := \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}, \quad A,B \in CB(X)$$

where for $x \in X$ and $E \subset X$ $d(x,E) := \inf\{d(x,y) : y \in E\}$ is the distance from the point $x$ to the subset $E$.

A multivalued mapping $T : C \to CB(X)$ is said to be a contraction if there exists a constant $k \in [0,1)$ such that

$$H(Tx, Ty) \le k\|x - y\|, \quad x, y \in C,$$

and $T$ is said to be nonexpansive if

$$H(Tx, Ty) \le \|x - y\|, \quad x, y \in C.$$

Recall that the Kuratowski and Hausdorff measures of noncompactness of a nonempty bounded subset $B$ of $X$ are respectively defined as the numbers:

$$\alpha(B) = \inf\{d > 0 : B \text{ can be covered by finitely many sets of diameter} \le d\},$$

$$\chi(B) = \inf\{d > 0 : B \text{ can be covered by finitely many balls of radius} \le d\}.$$

Then a multivalued mapping $T : C \to CB(X)$ is called $\gamma$-condensing (resp. 1-$\gamma$-contractive) where $\gamma = \alpha(\cdot)$ or $\chi(\cdot)$ if, for each bounded subset $B$ of $C$ with $\gamma(B) > 0$, there holds the inequality

$$\gamma(T(B)) < \gamma(B) \quad (\text{resp.} \quad \gamma(T(B)) \le \gamma(B)).$$

Here $T(B) = \cup_{x\in B} Tx$.

Note that a multivalued mapping $T : C \to 2^X$ is said to be upper semicontinuous on $C$ if $\{x \in C : Tx \subset V\}$ is open in $C$ whenever $V \subset X$ is open; $T$ is said to be lower semicontinuous if $T^{-1}(V) := \{x \in C : Tx \cap V \neq \emptyset\}$ is open in $C$ whenever $V \subset X$ is open; and $T$ is said to be continuous if it is both upper and lower semicontinuous. There is another different kind of continuity for set-valued operators: $T : C \to CB(X)$ is said to be continuous on $C$ (with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \to 0$ whenever $x_n \to x$.

It is not hard to see (see [1] and [4]) that both definitions of continuity are equivalent if $Tx$ is compact for every $x \in C$. We say that $x \in C$ is a fixed point of $T$ if and only if $x$ is contained in $Tx$.

Recall that the inward set of $C$ at $x \in C$ is defined by

$$I_C(x) := \{x + \lambda(y - x) : \lambda \geq 0, y \in C\}.$$

Clearly $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set as $C$ does.

Next theorems will be very useful in order to prove our results on fixed points for multivalued mappings.

**Theorem 2.1** **([12],[13])** *Let $C$ be a closed convex subset of a Banach space $X$ and $F : C \to K(X)$ a contraction mapping. If $Fx \subset \overline{I_C(x)}$ for all $x \in C$, then $F$ has a fixed point.*

**Theorem 2.2** **([3],[13])** *Let $X$ be a Banach space and $\emptyset \neq D \subset X$ be closed bounded convex. Let $F : D \to 2^X$ be upper semicontinuous $\gamma$-condensing with closed convex values, where $\gamma(\cdot) = \alpha(\cdot)$ or $\chi(\cdot)$. If $Fx \cap \overline{I_D(x)} \neq \emptyset$ on $D$ then $F$ has a fixed point.*

Let us recall some geometric properties which are defined using the measures of noncompactness.

**Definition 2.3** *Let $X$ be a Banach space and $\phi = \alpha$ or $\chi$. The modulus of noncompact convexity associated to $\phi$ is defined in the following way*

$$\Delta_{X,\phi}(\epsilon) = \inf\{1 - d(0, A) : A \subset B_X \text{ is convex}, \ \phi(A) \geq \epsilon\}.$$

*($B_X$ is the unit ball of $X$).*

*The characteristic of noncompact convexity of $X$ associated with the measure of noncompactness $\phi$ is defined by*

$$\epsilon_\phi(X) = \sup\{\epsilon \geq 0 : \Delta_{X,\phi}(\epsilon) = 0\}.$$

The following relationships among the different moduli are easy to obtain

$$\Delta_{X,\alpha}(\epsilon) \leq \Delta_{X,\chi}(\epsilon),$$

and consequently

$$\epsilon_\alpha(X) \geq \epsilon_\chi(X).$$

The space $X$ is said to be nearly uniformly convex if $\epsilon_\phi(X) = 0$.

Let $C$ be a subset of a Banach space $X$, $\mathcal{D}$ be a directed set and $\{x_\alpha : \alpha \in \mathcal{D}\}$ a bounded net in $X$. For any $x \in C$, define

$$r(x, \{x_\alpha\}) = \inf\{\sup\{\|x_\beta - x\| : \beta \geq \alpha\} : \alpha \in \mathcal{D}\} := \limsup_\alpha \|x_\alpha - x\|;$$

$$r(C, \{x_\alpha\}) = \inf\{r(x, \{x_\alpha\}) : x \in C\};$$

$$A(C, \{x_\alpha\}) = \{x \in C : r(x, \{x_\alpha\}) = r(C, \{x_\alpha\})\}.$$

The number $r(C, \{x_\alpha\})$ and the (possibly empty) set $A(C, \{x_\alpha\})$ are called, respectively, the asymptotic radius and the asymptotic center of $\{x_\alpha : \alpha \in \mathcal{D}\}$ in $C$.

Obviously, the convexity of $C$ implies that $A(C, \{x_\alpha\})$ is convex. Notice that $A(C, \{x_\alpha\})$ is a nonempty weakly compact set if $C$ is weakly compact, or $C$ is a closed convex subset of a reflexive Banach space.

Let $S$ be a set and $H \subset S$. We shall say that a net $\{x_\alpha : \alpha \in \mathcal{D}\}$ in $S$ is eventually in $H$ if there exists $\alpha_o \in \mathcal{D}$ such that $x_\alpha \in H$ for all $\alpha \geq \alpha_o$.

**Definition 2.4** *A net $\{x_\alpha : \alpha \in \mathcal{D}\}$ in a set $S$ is called an ultranet if for each subset $G \subset S$, either $\{x_\alpha : \alpha \in \mathcal{D}\}$ is eventually in $G$ or $\{x_\alpha : \alpha \in \mathcal{D}\}$ is eventually in $S \setminus G$.*

The following facts concerning ultranets can be found in [8]:

(a) Every net in a set has a subnet which is an ultranet.

(b) Let $S_1$ and $S_2$ be two sets and $f : S_1 \to S_2$. If $\{x_\alpha : \alpha \in \mathcal{D}\}$ is an ultranet $S_1$, then $\{f(x_\alpha) : \alpha \in \mathcal{D}\}$ is an ultranet in $S_2$.

(c) If $S$ is a compact Hausdorff topological space and $\{x_\alpha : \alpha \in \mathcal{D}\}$ is an ultranet in $S$, then the limit $\lim_\alpha x_\alpha$ exists.

Finally recall that if $D$ is a bounded subset of $X$, the Chebyshev radius of $D$ relative to $C$ is defined by

$$r_C(D) := \inf\{\sup\{\|x - y\| : y \in D\} : x \in C\}.$$

## 3 Modulus of noncompact convexity. Fixed point theorems

Let us begin this Section by proving a connection between the asymptotic center of an ultranet and $\Delta_{X,\alpha}(\cdot)$. We shall use the following result which can be proved by standard arguments.

**Lemma 3.1** *Let $X$ be a Banach space and $\{x_\alpha : \alpha \in \mathcal{D}\}$ a net weakly convergent to $x \in X$. Let $A_\alpha = \overline{co}(\{x_\beta : \beta \geq \alpha\})$. Then*

$$\bigcap_{\alpha \in \mathcal{D}} A_\alpha = \{x\}.$$

**Theorem 3.2** *Let $C$ be a closed convex subset of a reflexive Banach space $X$ and let $\{x_\beta : \beta \in D\}$ be a bounded ultranet in $C$. Then*

$$r_C(A(C, \{x_\beta\})) \leq (1 - \Delta_{X,\alpha}(1^-))r(C, \{x_\beta\}).$$

**PROOF.** Denote $r = r(C, \{x_\beta\})$ and $A = A(C, \{x_\beta\})$ which is a nonempty set. Since $\overline{c}o(\{x_\beta : \beta \in D\}) \subset C$ is a weakly compact set, the ultranet $\{x_\beta : \beta \in D\}$ converges weakly to an element $z \in C$. Furthermore, for each $x \in C$, the limit $\lim_\beta \|x_\beta - x\|$ exists.

Let us first show that $\alpha(\{x_\beta : \beta \in D\}) \geq r$.

Indeed, let $d > \alpha(\{x_\beta : \beta \in D\})$. There exist $B_1, ..., B_n$ disjoint subsets of $C$ such that $\{x_\beta : \beta \in D\}$ is contained in $\displaystyle\bigcup_{i=1}^{n} B_i$ and $\text{diam}(B_i) \leq d$.

According to the definition of ultranet, $\{x_\beta : \beta \in D\}$ is either eventually in $B_1$ or eventually in $\cup_{i=2}^{n} B_i$. Suppose $\{x_\beta : \beta \in D\}$ is eventually in $B_1$, then

$\{x_\beta : \beta \geq \beta_o\} \subset B_1$, for some $\beta_o \in D$. In view of this, for every $x \in B_1$ we have

$$\|x_\beta - x\| \leq d, \quad \text{for all } \beta \geq \beta_o.$$

Hence

$$r \leq \lim_{\beta \geq \beta_o} \|x_\beta - x\| \leq d,$$

and thus $\alpha(\{x_\beta : \beta \in D\}) \geq r$.

In the second case, there exists $\beta_o \in D$ such that $\{x_\beta : \beta \geq \beta_o\} \subset \cup_{i=2}^n B_i$. Since $\{x_\beta : \beta \geq \beta_o\}$ is an ultranet, this net is either in $B_2$ or eventually in $\cup_{i=3}^n B_i$. In the first assumption, it is possible to repeat the above argument to obtain $\alpha(\{x_\beta : \beta \in D\}) \geq r$. Following this finite process we obtain the desired result.

It must be noted that this reasoning also allow us to prove that $\alpha(\{x_\gamma : \gamma \geq \beta\}) \geq r$, for every $\beta \in \mathcal{D}$.

Assume that $x$ lies in $A$. Since $\lim_\beta \|x_\beta - x\| = r$, given $\epsilon > 0$ we can find $\beta_0 \in \mathcal{D}$ such that $\|x_\beta - x\| < r + \epsilon$ for all $\beta \geq \beta_o$.

Thus, if we denote $A_\beta = \overline{co}(\{x_\gamma - x\}_{\gamma \geq \beta})$ we have that $A_\beta \subset B(0, r + \epsilon)$ for each $\beta \in \mathcal{D}$, $\beta \geq \beta_o$, and $\alpha(A_\beta) = \alpha(\{x_\gamma - x\}_{\gamma \geq \beta}) \geq r$.

From the definition of $\Delta_{X,\alpha}(\cdot)$ we deduce

$$\inf_{y \in A_\beta} \|y\| = d(0, A_\beta) \leq \left(1 - \Delta_{X,\alpha}\left(\frac{r}{r+\epsilon}\right)\right)(r + \epsilon),$$

for each $\beta \geq \beta_o$.

Since the set $A_\beta$ is a weakly compact set, it must have $\inf_{y \in A_\beta} \|y\| = \|y_\beta\|$ for some $y_\beta \in A_\beta$.

On the other hand, the net $\{y_\beta : \beta \geq \beta_o\} \subset A_{\beta_o}$ has a subnet weakly convergent to a point, say $y$, which clearly is a cluster point of $A_\beta$ for all $\beta \geq \beta_o$. Thus, it follows from Lemma 3.1 that $y = z - x = w - \lim_\beta y_\beta$.

Then the weakly lower semicontinuity of the norm implies

$$\|z - x\| \leq \left(1 - \Delta_{X,\alpha}\left(\frac{r}{r+\epsilon}\right)\right)(r + \epsilon).$$

Since the last inequality is true for every $\epsilon$, we have

$$\|z - x\| \leq \left(1 - \Delta_{X,\alpha}(1^-)\right)r.$$

This ends the proof because the last inequality holds for every $x \in A(C, \{x_\beta\})$.

**Remark 3.3**

In [5] the authors give a similar result to Theorem 3.2 for the asymptotic center of a regular sequence with respect to $C$ and the modulus $\Delta_{X,\beta}(\cdot)$, where $\beta$ is the separation measure of noncompactness ([2]). A sequence is called *regular with respect to* $C$ if each of its subsequences has the same asymptotic radius in $C$. Furthermore, they prove that the modulus $\Delta_{X,\chi}(\cdot)$ can be considered when $X$ satisfies the non-strict Opial condition (notice that $\Delta_{X,\beta}(\cdot) \leq \Delta_{X,\chi}(\cdot)$). A Banach space $X$ is said to satisfy the *non-strict Opial condition* if, whenever a sequence $\{x_n\}$ in $X$ converges weakly to $x$, then for $y \in X$

$$\limsup_n \|x_n - x\| \leq \limsup_n \|x_n - y\|.$$

Now we are ready to prove the main result of this paper.

**Theorem 3.4** *Let $X$ be a Banach space such that $\epsilon_\alpha(X) < 1$ and $C$ be a closed bounded convex subset of $X$. If $T : C \to KC(X)$ is a nonexpansive and 1-$\chi$-contractive mapping such that $T(C)$ is a bounded set, and which satisfies*

$$Tx \subset I_C(x) \quad \forall x \in C,$$

*then $T$ has a fixed point.*

**PROOF.** Let $x_0 \in C$ be fixed and consider for each $n \geq 1$ the contraction

$T_n : C \to KC(X)$ defined by

$$T_n x := \frac{1}{n}x_0 + (1 - \frac{1}{n})Tx, \quad x \in C.$$

Bearing in mind that for each $x \in C$ the set $I_C(x)$ is convex and contains $C$, it is easily seen that $T_n x \subset I_C(x)$ for all $x \in C$. We can apply Theorem 2.1 to obtain a fixed point $x_n \in C$ of $T_n$. Thus, we have a sequence $\{x_n\}$ in $C$ such that $\lim_n d(x_n, Tx_n) = 0$. Let $\{n_\alpha\}$ be an ultranet of the positive integers $\{n\}$.

Denote $A = A(C, \{x_{n_\alpha}\})$. We start by proving that

$$Tx \cap I_A(x) \neq \emptyset \quad \forall x \in A.$$

Indeed, the compactness of $Tx_{n_\alpha}$ implies that for each $n_\alpha$, we can take $y_{n_\alpha} \in Tx_{n_\alpha}$ such that

$$\|x_{n_\alpha} - y_{n_\alpha}\| = d(x_{n_\alpha}, Tx_{n_\alpha}).$$

Since $Tx$ is compact, for each $x \in A$, we can find $z_{n_\alpha} \in Tx$ such that

$$\|y_{n_\alpha} - z_{n_\alpha}\| = d(y_{n_\alpha}, Tx) \leq H(Tx_{n_\alpha}, Tx) \leq \|x_{n_\alpha} - x\|.$$

Let $z = \lim_\alpha z_{n_\alpha} \in Tx$. It should remain to prove $z \in I_A(x)$.

If $r = r(C, \{x_{n_\alpha}\})$, on the one hand we have

$$\lim_\alpha \|x_{n_\alpha} - z\| = \lim_\alpha \|y_{n_\alpha} - z_{n_\alpha}\| \leq \lim_\alpha \|x_{n_\alpha} - x\| = r,$$

and on the other hand, since $z \in Tx \subset I_C(x)$ there exists $\lambda \geq 0$ such that $z = x + \lambda(v - x)$ for some $v \in C$. If $\lambda \leq 1$ it is clear that $z \in C$ and hence, from the above inequality, $z \in A \subset I_A(x)$. So assume $\lambda > 1$ and write

$$v = \mu z + (1 - \mu)x, \quad \mu = \frac{1}{\lambda} \in (0, 1).$$

Therefore we have

$$\lim_\alpha \|x_{n_\alpha} - v\| \leq \mu \lim_\alpha \|x_{n_\alpha} - z\| + (1 - \mu) \lim_\alpha \|x_{n_\alpha} - x\| \leq r.$$

Hence $v \in A$ and thus $z \in I_A(x)$.

In this way, the mapping $T : A \to KC(X)$ is nonexpansive, 1-$\chi$-contractive and satisfies

$$Tx \cap I_A(x) \neq \emptyset \quad \forall x \in A.$$

Moreover, we can apply Theorem 3.2 to obtain

$$r_C(A) \leq \lambda r(C, \{x_{n_\alpha}\}),$$

where $\lambda := 1 - \Delta_{X,\alpha}(1^-) < 1$.

Now fix $x_1 \in A$ and for each number $\mu \in (0, 1]$ consider the contraction $T_\mu : A \to KC(X)$ defined by

$$T_\mu x = \mu x_1 + (1 - \mu)Tx \quad x \in A.$$

It is easily seen that $T_\mu$ is $\chi$-condensing (see [5]). Furthermore, since $I_A(x)$ is convex we also obtain

$$T_\mu x \cap I_A(x) \neq \emptyset, \quad \forall x \in A.$$

Hence by Theorem 2.2, $T_\mu$ has a fixed point. Consequently, we can get a sequence $\{x_n^1\}$ in $A$ satisfying $\lim_n d(x_n^1, Tx_n^1) = 0$. We proceed as before to obtain that

$$Tx \cap I_{A^1}(x) \neq \emptyset, \quad \forall x \in A^1 := A(C, \{x_{n_\alpha}^1\}),$$

and

$$r_C(A^1) \leq \lambda r(C, \{x_{n_\alpha}^1\}) \leq \lambda r_C(A).$$

By induction, for each integer $m \geq 1$ we take a sequence $\{x_n^m\}_n \subset A^{m-1}$ such that $\lim_n d(x_n^m, Tx_n^m) = 0$. By means of the ultranet $\{x_{n_\alpha}^m\}_\alpha$ we construct the set $A^m := A(C, \{x_{n_\alpha}^m\})$ such that

$$r_C(A^m) \leq \lambda^m r_C(A).$$

Choose $x_m \in A^m$. We shall prove that $\{x_m\}_m$ is a Cauchy sequence. For each $m \geq 1$ we have for any positive integer $n$

$$\|x_{m-1} - x_m\| \leq \|x_{m-1} - x_n^m\| + \|x_n^m - x_m\| \leq \operatorname{diam} A^{m-1} + \|x_n^m - x_m\|.$$

Taking upper limit as $n \to \infty$

$$\|x_{m-1} - x_m\| \leq \text{diam} A^{m-1} + \limsup_n \|x_n^m - x_m\| = \text{diam} A^{m-1} + r(C, \{x_n^m\})$$

$$\leq \text{diam} A^{m-1} + r_C(A^{m-1})$$

$$\leq 2r_C(A^{m-1}) + r_C(A^{m-1}) = 3r_C(A^{m-1}) \leq 3\lambda^{m-1} r_C(A).$$

Since $\lambda < 1$, we conclude that there exists $x \in C$ such that $x_m$ converges to $x$. Let us see that $x$ is a fixed point of $T$. For each $m \geq 1$,

$$d(x_m, Tx_m) \leq \|x_m - x_n^m\| + d(x_n^m, Tx_n^m) + H(Tx_n^m, Tx_m) \leq 2\|x_m - x_n^m\| + d(x_n^m, Tx_n^m).$$

Taking upper limit as $n \to \infty$

$$d(x_m, Tx_m) \leq 2 \limsup_n \|x_m - x_n^m\| \leq 2\lambda^{m-1} r_C(A).$$

Finally, taking limit in $m$ in both sides we obtain $\lim_m d(x_m, Tx_m) = 0$ and the continuity of $T$ implies that $d(x, Tx) = 0$ i.e. $x \in Tx$.

Simple examples show that we can not avoid nonexpansiveness assumption in the above theorem (see [5]). We do not know if $\chi$-contractiveness condition can be dropped in Theorem 3.4. In fact, it is an open problem if every nonexpansive mapping $T$ from $C$ to either $K(C)$ or $K(X)$ is 1-$\chi$-contractive even for single valued mappings. However, when $C$ is a weakly compact subset of a reflexive Banach space satisfying the non-strict Opial condition, we can follow the proof of Theorem 4.5 in [5] to deduce that a nonexpansive mapping $T : C \to K(X)$ with bounded range is 1-$\chi$-contractive. Then, in view of Theorem 3.4, we can state the following corollary.

**Corollary 3.5** *Let $X$ be a Banach space such that $\epsilon_\alpha(X) < 1$ satisfying the non-strict Opial condition and $C$ be closed bounded convex subset of $X$. If $T : C \to KC(X)$ is a nonexpansive mapping such that $T(C)$ is a bounded set, and which satisfies*

$$Tx \subset I_C(x) \quad \forall x \in C,$$

*then $T$ has a fixed point.*

Regarding the proof of Theorem 3.4 it is worthwhile to note that ultranets are needed due to the fact that the range of $T$ is not assumed to be contained in its domain and hence we cannot restrict to the case of a separable set $C$ (see [7] and [14]). However, if we assume that $C$ is separable and recall the first step of the induction method as applied in Theorem 3.4, then we can take a sequence of approximate fixed points of $T$ in $C$ such that it is regular and asymptotically uniform with respect to $C$ (see [6] and [11]). A sequence is said to be *asymptotically uniform with respect to $C$* if each of its subsequences has the same asymptotic center in $C$. Under this situation it is enough to consider a subsequence $\{x_n\}$ of the above-mentioned sequence such that

$$Tx \cap I_A(x) \neq \emptyset \quad \forall x \in A,$$

where $A = A(C, \{x_n\})$. The boundary condition imposed on $T$ allows us to rewrite the proof of Theorem 3.4 to the $\beta$ and $\chi$ moduli of noncompact convexity (see Remark 3.3). The following results are consequence of this fact.

**Theorem 3.6** *Let $X$ be a Banach space such that $\epsilon_\beta(X) < 1$ and $C$ be a closed bounded convex and separable subset of $X$. If $T : C \to KC(X)$ is a nonexpansive and 1-$\chi$-contractive mapping such that $T(C)$ is a bounded set, and which satisfies*

$$Tx \subset I_C(x) \quad \forall x \in C,$$

*then $T$ has a fixed point.*

**Theorem 3.7** *Let $X$ be a Banach space such that $\epsilon_\chi(X) < 1$ satisfying the non-strict Opial condition and $C$ be a closed bounded convex and separable subset of $X$. If $T : C \to KC(X)$ is a nonexpansive mapping such that $T(C)$ is a bounded set which satisfies*

$$Tx \subset I_C(x) \quad \forall x \in C,$$

*then $T$ has a fixed point.*

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