

ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN MODULAR FUNCTION SPACES

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ABSTRACT

In this paper, we prove that if ρ is a convex, σ -finite modular function satisfying a Δ_2 -type condition, C a convex, ρ -bounded, ρ -a.e. compact subset of L_ρ and $T : C \rightarrow C$ a ρ -asymptotically nonexpansive mapping, then T has a fixed point. In particular, any asymptotically nonexpansive self-map defined on a convex subset of $L^1(\Omega, \mu)$ which is compact for the topology of local convergence in measure has a fixed point.

1991 Mathematics subject classification : Primary 46E30; Secondary 47H09, 47H10.

Key Words: asymptotically nonexpansive mappings, fixed point, modular functions.

The first author is partially supported by PB-96-1338-C01-C02 and PAI-FMQ-0127.

INTRODUCTION

Let (M, d) be a metric space. A mapping, $T : M \rightarrow M$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y)$$

for any $x, y \in M$ and $n \in \mathbb{N}$. In 1970 Goebel and Kirk [5] proved that T has a fixed point whenever M is a convex bounded closed subset of a Banach space X . Further generalizations of this result were proved by Yu and Dai [14] when X is 2-uniformly rotund, by Martínez Yañez [10] and Xu [12] when X is k -uniformly rotund for some $k \geq 1$, by Xu [13] when X is nearly uniformly convex and by Kim and Xu [9] when X has uniform normal structure. Some special studies on the theory of the fixed point for asymptotically nonexpansive mappings were made by many other authors (see, for example, [2,11]).

The first fixed point results in modular function spaces were given by Khamsi, Kozłowski and Reich [7]. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated in modular spaces. For instance, fixed point theorems are proved in [6,7] for nonexpansive mappings, in [3] for asymptotically regular mappings and in [4] for uniformly Lipschitzian mappings. In this paper we will prove the existence of fixed points for asymptotically nonexpansive mappings in modular function spaces when the modular ρ satisfies some convexity and Δ_2 -type properties.

Our results can be, in particular, applied to $L^1(\Omega, \mu)$, showing that asymptotically nonexpansive mappings have a fixed point when they are defined on a convex subset of $L^1(\Omega, \mu)$ which is compact with respect to the topology of convergence local in measure.

1. PRELIMINARIES

We start by reviewing some basic facts about modular spaces as formulated by Kozłowski [8]. For more details the reader may consult [6,7].

Let Ω be a nonempty set and Σ be a nontrivial σ -algebra of subsets of Ω . Let \mathcal{P} be a δ -ring of subsets of Σ , such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that

$\Omega = \bigcup K_n$. By \mathcal{E} we denote the linear space of all simple functions with supports from \mathcal{P} . By \mathcal{M} we will denote the space of all measurable functions, i.e. all functions $f : \Omega \rightarrow \mathbb{R}$ such that there exists a sequence $\{g_n\} \in \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \rightarrow f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A .

Definition 1.1. A functional $\rho : \mathcal{E} \times \Sigma \rightarrow [0, \infty]$ is called a function modular if:

- (P₁) $\rho(0, E) = 0$ for any $E \in \Sigma$,
- (P₂) $\rho(f, E) \leq \rho(g, E)$ whenever $|f(\omega)| \leq |g(\omega)|$ for any $\omega \in \Omega$, $f, g \in \mathcal{E}$ and $E \in \Sigma$,
- (P₃) $\rho(f, \cdot) : \Sigma \rightarrow [0, \infty]$ is a σ -subadditive measure for every $f \in \mathcal{E}$,
- (P₄) $\rho(\alpha, A) \rightarrow 0$ as α decreases to 0 for every $A \in \mathcal{P}$, where $\rho(\alpha, A) = \rho(\alpha 1_A, A)$,
- (P₅) if there exists $\alpha > 0$ such that $\rho(\alpha, A) = 0$, then $\rho(\beta, A) = 0$ for every $\beta > 0$,
- (P₆) for any $\alpha > 0$ $\rho(\alpha, \cdot)$ is order continuous on \mathcal{P} , that is $\rho(\alpha, A_n) \rightarrow 0$ if $\{A_n\} \in \mathcal{P}$ and decreases to \emptyset .

The definition of ρ is then extended to $f \in \mathcal{M}$ by

$$\rho(f, E) = \sup\{\rho(g, E); g \in \mathcal{E}, |g(\omega)| \leq |f(\omega)| \text{ for every } \omega \in \Omega\}.$$

Definition 1.2. A set E is said to be ρ -null if and only if $\rho(\alpha, E) = 0$ for $\alpha > 0$. A property $p(\omega)$ is said to hold ρ -almost everywhere (ρ -a.e.) if the set $\{\omega \in \Omega; p(\omega) \text{ does not hold}\}$ is ρ -null. For example we will say frequently $f_n \rightarrow f$ ρ -a.e.

For the sake of simplicity we write $\rho(f)$ instead of $\rho(f, \Omega)$.

Definition 1.3. A modular function ρ is called σ -finite if there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $0 < \rho(K_n) < \infty$ and $\Omega = \bigcup K_n$.

It is easy to see that the functional $\rho : \mathcal{M} \rightarrow [0, \infty]$ is a modular and satisfies the following properties:

- (i) $\rho(f) = 0$ iff $f = 0$ ρ -a.e.

- (ii) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathcal{M}$.
- (iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha \geq 0, \beta \geq 0$ and $f, g \in \mathcal{M}$.

In addition, if the following property is satisfied

- (iii)' $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$; $\alpha \geq 0, \beta \geq 0$ and $f, g \in \mathcal{M}$,

we say that ρ is a convex modular.

The modular ρ defines a corresponding modular space, i.e the vector space L_ρ given by

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \rightarrow 0 \text{ as } \lambda \rightarrow 0\}.$$

When ρ is convex, the formula

$$\|f\|_\rho = \inf \left\{ \alpha > 0; \rho\left(\frac{f}{\alpha}\right) \leq 1 \right\}$$

defines a norm in the modular space L_ρ which is frequently called the Luxemburg norm. We can also consider the space

$$E_\rho = \{f \in \mathcal{M}; \rho(\alpha f, A_n) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for every } A_n \in \Sigma \text{ that decreases to } \emptyset \text{ and } \alpha > 0\}.$$

Definition 1.4. A function modular is said to satisfy the Δ_2 -condition if

$$\sup_{n \geq 1} \rho(2f_n, D_k) \rightarrow 0 \text{ as } k \rightarrow \infty \text{ whenever } \{f_n\}_{n \geq 1} \subset \mathcal{M}, D_k \in \Sigma \text{ decreases to } \emptyset \text{ and } \sup_{n \geq 1} \rho(f_n, D_k) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

We know from [8] that $E_\rho = L_\rho$ when ρ satisfies the Δ_2 -condition.

Definition 1.5. A function modular is said to satisfy the Δ_2 -type condition if there exists $K > 0$ such that for any $f \in L_\rho$ we have $\rho(2f) \leq K\rho(f)$.

In general, Δ_2 -type condition and Δ_2 -condition are not equivalent, even though it is obvious that Δ_2 -type condition implies Δ_2 -condition on the modular space L_ρ .

Definition 1.6. Let L_ρ be a modular space.

- (1) The sequence $\{f_n\}_n \subset L_\rho$ is said to be ρ -convergent to $f \in L_\rho$ if $\rho(f_n - f) \rightarrow 0$ as $n \rightarrow \infty$.
- (2) The sequence $\{f_n\}_n \subset L_\rho$ is said to be ρ -a.e. convergent to $f \in L_\rho$ if the set $\{\omega \in \Omega; f_n(\omega) \not\rightarrow f(\omega)\}$ is ρ -null.
- (3) The sequence $\{f_n\}_n \subset L_\rho$ is said to be ρ -Cauchy if $\rho(f_n - f_m) \rightarrow 0$ as n and m go to ∞ .
- (4) A subset C of L_ρ is called ρ -closed if the ρ -limit of a ρ -convergent sequence of C always belongs to C .
- (5) A subset C of L_ρ is called ρ -a.e. closed if the ρ -a.e. limit of a ρ -a.e. convergent sequence of C always belongs to C .
- (6) A subset C of L_ρ is called ρ -a.e. compact if every sequence in C has a ρ -a.e. convergent subsequence in C .
- (7) A subset C of L_ρ is called ρ -bounded if

$$\delta_\rho(C) = \sup\{\rho(f - g); f, g \in C\} < \infty.$$

We recall two basic results (see [7]) in the theory of modular spaces.

- (i) If there exists a number $\alpha > 0$ such that $\rho(\alpha(f_n - f)) \rightarrow 0$, then there exists a subsequence $\{g_n\}_n$ of $\{f_n\}_n$ such that $g_n \rightarrow f$ ρ -a.e.
- (ii) (Lebesgue's Theorem) If $f_n, f \in \mathcal{M}$, $f_n \rightarrow f$ ρ -a.e. and there exists a function $g \in E_\rho$ such that $|f_n| \leq |g|$ ρ -a.e. for all n , then $\|f_n - f\|_\rho \rightarrow 0$.

We know, by [6,7] that under Δ_2 -condition the norm convergence and modular convergence are equivalent, which implies that the norm and modular convergence are also the same when we deal with the Δ_2 -type condition.

In the sequel we will assume that the modular function ρ is convex and satisfies the Δ_2 -type condition.

Definition 1.7. Let ρ be as above. We define a growth function ω by:

$$\omega(t) = \sup \left\{ \frac{\rho(tf)}{\rho(f)}, f \in L_\rho \setminus \{0\} \right\} \quad \text{for all } 0 \leq t < \infty.$$

We have the following:

Lemma 1.1. [3] *Let ρ be as above. Then the growth function ω has the following properties:*

- (1) $\omega(t) < \infty, \forall t \in [0, \infty)$
- (2) $\omega : [0, \infty) \rightarrow [0, \infty)$ is a convex, strictly increasing function. So, it is continuous.
- (3) $\omega(\alpha\beta) \leq \omega(\alpha)\omega(\beta); \forall \alpha, \beta \in [0, \infty)$
- (4) $\omega^{-1}(\alpha)\omega^{-1}(\beta) \leq \omega^{-1}(\alpha\beta); \forall \alpha, \beta \in [0, \infty)$, where ω^{-1} is the function inverse of ω .

The following lemma shows that the growth function can be used to give an upper bound for the norm of a function.

Lemma 1.2. [3] *Let ρ be a convex function modular satisfying the Δ_2 -type condition. Then*

$$\|f\|_\rho \leq \frac{1}{\omega^{-1}\left(\frac{1}{\rho(f)}\right)} \quad \text{whenever } f \in L_\rho.$$

The next lemma will be of major interest throughout this work.

Lemma 1.3. [6] *Let ρ be a function modular satisfying the Δ_2 -condition and $\{f_n\}_n$ be a sequence in L_ρ such that $f_n \xrightarrow{\rho\text{-a.e.}} f \in L_\rho$ and there exists $k > 1$ such that $\sup_n \rho(k(f_n - f)) < \infty$. Then,*

$$\liminf_{n \rightarrow \infty} \rho(f_n - g) = \liminf_{n \rightarrow \infty} \rho(f_n - f) + \rho(f - g) \quad \text{for all } g \in L_\rho.$$

Moreover, we have

$$\rho(f) \leq \liminf_{n \rightarrow \infty} \rho(f_n).$$

2. AN EQUIVALENT TOPOLOGY

The concept of ρ -a.e. closed, compact sets have been studied extensively in the sequential case. One of the problem that many authors have found hard to circumvent is whether these notions are related to a topology. In this section we will discuss this problem. In particular, we will construct a topology τ for which the ρ -a.e. compactness is equivalent to the usual compactness for τ . This is crucial when we try to use Zorn's lemma.

From now on, we assume that the modular function ρ is, in addition, σ -finite. Set

$$d(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{1}{\rho(1_{K_k})} \rho \left(\frac{|f - g|}{1 + |f - g|} 1_{K_k} \right) \quad \text{for any } f, g \in L_\rho.$$

Some basic properties satisfied by d are discussed in the following proposition.

Proposition 2.1. The functional d satisfies the following:

- (1) $d(f, g) = 0$ if and only if $f = g$ ρ -a.e.;
- (2) $d(f, g) = d(g, f)$;
- (3) $d(f, g) \leq \frac{\omega(2)}{2} (d(f, h) + d(h, g))$;

for any f, g and h in L_ρ .

Proof. (1) and (2) are obvious. To prove (3) we only need to recall the inequality

$$\frac{|a + b|}{1 + |a + b|} \leq \frac{|a|}{1 + |a|} + \frac{|b|}{1 + |b|}$$

for all positive numbers a, b and use the definition of the growth function ω . □

Remark 2.1. The functional d is not a distance because of (3). But there are many mathematical objects which fail the triangle inequality but are very useful tools. That is the case with d .

In the next proposition, we discuss the relationship between ρ -a.e. convergence and the convergence for the functional d .

Proposition 2.2. Let ρ be a convex, σ -finite modular satisfying the Δ_2 -type condition and $\{f_n\}_n$ be a sequence of measurable functions. If $\{f_n\}_n$ is ρ -a.e. convergent to f , then

$$\lim_{n \rightarrow \infty} d(f_n, f) = 0.$$

Moreover, if

$$\lim_{n \rightarrow \infty} d(f_n, f) = 0,$$

then there exists a subsequence $\{f_{n_k}\}_k$ which converges ρ -a.e. to f .

Proof. Assume that $\{f_n\}_n$ ρ -a.e. converges to f . We will show that $\lim_{n \rightarrow \infty} d(f_n, f) =$

0. Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} \frac{1}{2^k} < \varepsilon$. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} d(f_n, f) &\leq \lim_{n \rightarrow \infty} \sum_{k=1}^N \frac{1}{2^k} \frac{1}{\rho(1_{K_k})} \rho\left(\frac{|f_n - f|}{1 + |f_n - f|} 1_{K_k}\right) + \varepsilon \\ &= \sum_{k=1}^N \lim_{n \rightarrow \infty} \frac{1}{2^k} \frac{1}{\rho(1_{K_k})} \rho\left(\frac{|f_n - f|}{1 + |f_n - f|} 1_{K_k}\right) + \varepsilon. \end{aligned}$$

Since

$$\frac{|f_n - f|}{1 + |f_n - f|} 1_{K_k} \xrightarrow{\rho\text{-a.e.}} 0 \quad \text{as } n \rightarrow \infty$$

for any $k \in \mathbb{N}$ and $\frac{|f_n - f|}{1 + |f_n - f|} 1_{K_k} \leq 1_{K_k}$, from Lebesgue's Theorem we obtain

$$\lim_{n \rightarrow \infty} \rho\left(\frac{|f_n - f|}{1 + |f_n - f|} 1_{K_k}\right) = 0 \text{ for every non null integer } k. \text{ Thus } \lim_{n \rightarrow \infty} d(f_n, f) \leq \varepsilon$$

for each $\varepsilon > 0$ which means that $\lim_{n \rightarrow \infty} d(f_n, f) = 0$.

Assume now that $\lim_{n \rightarrow \infty} d(f_n, f) = 0$. For every non null integer k we have

$$\lim_{n \rightarrow \infty} \rho\left(\frac{|f_n - f|}{1 + |f_n - f|} 1_{K_k}\right) = 0.$$

Thus, there exists a subsequence $\{f_n^1\}_n$ of $\{f_n\}_n$ such that $\frac{|f_n^1 - f|}{1 + |f_n^1 - f|} 1_{K_1} \xrightarrow{\rho\text{-a.e.}} 0$

and so $f_n^1 \xrightarrow{\rho\text{-a.e.}} f$ in K_1 i.e. $\lim_{n \rightarrow \infty} f_n^1(x) = f(x)$ whenever $x \in K_1 \setminus A_1$ where $A_1 \subset K_1$ and $\rho(1_{A_1}) = 0$.

By induction and using a diagonal argument we obtain a subsequence of $\{f_n\}_n$ which converges ρ -a.e. to f . \square

Definition 2.1. Let C be a subset of L_ρ .

- (a) C is said to be d -closed iff for any sequence $\{f_n\}_n$ in C which d -converges to f , then we have $f \in C$.
- (b) C is d -open iff $L_\rho \setminus C$ is d -closed.
- (c) C is said to be d -sequentially compact if for each sequence $\{f_n\}_n$ there exists a subsequence $\{f_{n_k}\}_k$ which d -converges to a point in C .

It is easily seen that the family of all d -open subsets of L_ρ form a topology on L_ρ . Furthermore, from proposition (2.2) d -sequentially compact sets and ρ -a.e. compact sets are identical. On the other hand, even though d satisfies (3) instead of the triangular inequality, the usual arguments which prove that sequential compactness and compactness are identical in metric spaces hold in this setting. We also have d -sequential compactness and d -compactness are identical.

3. TECHNICAL LEMMAS

In the sequel we assume that ρ is a convex, σ -finite modular function satisfying the Δ_2 -type condition, C is a convex, ρ -bounded and ρ -a.e. compact subset of the modular function space L_ρ and $T : C \rightarrow C$ is a ρ -asymptotically nonexpansive mapping, i.e. there exists a sequence of positive integers $\{k_n\}_n$ which converge to 1 such that for every $n \in \mathbb{N}$ and $f, g \in C$ we have $\rho(T^n f - T^n g) \leq k_n \rho(f - g)$.

Lemma 3.1. *Under the above assumptions, let $\{f_n\}_n$ be a sequence of elements of C . Consider the functional $\Phi : C \rightarrow R$ defined by $\Phi(g) = \limsup_{n \rightarrow \infty} \rho(f_n - g)$. Then, for any sequence $\{g_m\}_m$ in C which ρ -a.e. converges to $g \in C$ we have*

$$\Phi(g) \leq \liminf_{m \rightarrow \infty} \Phi(g_m).$$

Proof. Since C is ρ -a.e. compact, there exists a subsequence $\{f_{\phi(n)}\}_n$ of $\{f_n\}_n$ such that $f_{\phi(n)} \xrightarrow{\rho\text{-a.e.}} f \in C$ and $\lim_{n \rightarrow \infty} \rho(f_{\phi(n)} - g) = \limsup_{n \rightarrow \infty} \rho(f_n - g)$. Hence

$$\begin{aligned} \Phi(g_m) &= \limsup_{n \rightarrow \infty} \rho(f_n - g_m) \\ &\geq \limsup_{n \rightarrow \infty} \rho(f_{\phi(n)} - g_m) \\ &\geq \liminf_{n \rightarrow \infty} \rho(f_{\phi(n)} - g_m). \end{aligned}$$

Lemma (1.3) implies

$$\liminf_{n \rightarrow \infty} \rho(f_{\phi(n)} - g_m) = \liminf_{n \rightarrow \infty} \rho(f_{\phi(n)} - f) + \rho(f - g_m).$$

Thus, $\Phi(g_m) \geq \liminf_{n \rightarrow \infty} \rho(f_{\phi(n)} - f) + \rho(f - g_m)$, for any $m \geq 1$. Hence

$$\liminf_{m \rightarrow \infty} \Phi(g_m) \geq \liminf_{n \rightarrow \infty} \rho(f_{\phi(n)} - f) + \liminf_{m \rightarrow \infty} \rho(f - g_m).$$

Again using lemma (1.3), we have

$$\liminf_{m \rightarrow \infty} \rho(f - g_m) = \liminf_{m \rightarrow \infty} \rho(g_m - g) + \rho(g - f),$$

which implies

$$\liminf_{m \rightarrow \infty} \Phi(g_m) \geq \liminf_{n \rightarrow \infty} \rho(f_{\phi(n)} - f) + \liminf_{m \rightarrow \infty} \rho(g_m - g) + \rho(g - f) \quad (I).$$

On the other hand,

$$\Phi(g) = \limsup_{n \rightarrow \infty} \rho(f_n - g) = \lim_{n \rightarrow \infty} \rho(f_{\phi(n)} - g) = \liminf_{n \rightarrow \infty} \rho(f_{\phi(n)} - g)$$

which implies

$$\Phi(g) = \liminf_{n \rightarrow \infty} \rho(f_{\phi(n)} - f) + \rho(f - g) \quad (II).$$

From (I) and (II), it is clear that

$$\Phi(g) \leq \liminf_{m \rightarrow \infty} \Phi(g_m),$$

which completes the proof. □

Denote by \mathfrak{S} the family of all subsets K of C satisfying the following property: K is a nonempty, convex and ρ -a.e. closed subset of C such that

$$f \in K \quad \text{implies} \quad \Omega_{\rho\text{-a.e.}}(f) \subset K \quad (3.1)$$

where $\Omega_{\rho\text{-a.e.}}(f) = \{g \in L_\rho : g = \lim_{i \rightarrow \infty} T^{n_i}(f) \text{ } \rho\text{-a.e for some } n_i \uparrow \infty\}$. Ordering \mathfrak{S} by inclusion, there exists a nonempty minimal element H in \mathfrak{S} which satisfies (3.1) by using Zorn's lemma because C is compact for the topology generated by d .

The following lemma is the counterpart in modular function spaces of lemma (2.1) in [13] for Banach spaces.

Lemma 3.2. *Under the above assumptions, for each $f \in H$ define the functional*

$$r_f(g) = \limsup_{n \rightarrow \infty} \rho(T^n f - g)$$

for any $g \in L_\rho$. Then the functional $r_f(\cdot)$ is constant on H and this constant is independent of f in H .

Proof. Let $t > 0$ and $f \in H$. Set

$$H_t(f) = \{ g \in H, \quad r_f(g) \leq t \}.$$

It is easily seen that $H_t(f)$ is convex. We claim that $H_t(f)$ is ρ -a.e. closed. Indeed, assume that $\{g_m\}_m \in H_t(f)$ ρ -a.e. converges to $g \in H$. Using Lemma (3.1), we get

$$\limsup_{n \rightarrow \infty} \rho(T^n f - g) \leq \liminf_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \rho(T^n f - g_m) \leq t.$$

Hence $g \in H_t(f)$, which clearly implies that $H_t(f)$ is ρ -a.e. closed. Since H is ρ -a.e. compact we have that $H_t(f)$ is ρ -a.e. compact. Next, we claim that $H_t(f)$ satisfies property (3.1). Indeed, let $g \in H_t(f)$ and $h \in \Omega_{\rho\text{-a.e.}}(g)$. We need to check that $h \in H_t(f)$. By definition of $\Omega_{\rho\text{-a.e.}}(g)$, there exists an increasing sequence of integers $\{n_i\}_i$ such that $T^{n_i}(g) \xrightarrow{\rho\text{-a.e.}} h$. Lemma (3.1) implies

$$\begin{aligned} r_f(h) &= \limsup_{n \rightarrow \infty} \rho(T^n f - h) \leq \liminf_{i \rightarrow \infty} \limsup_{n \rightarrow \infty} \rho(T^n f - T^{n_i} g) \\ &\leq \liminf_{i \rightarrow \infty} r_f(T^{n_i}(g)) \leq \limsup_{i \rightarrow \infty} r_f(T^{n_i}(g)) \leq \limsup_{m \rightarrow \infty} r_f(T^m(g)) \\ &\leq \limsup_{m \rightarrow \infty} \left(\limsup_{n \rightarrow \infty} \rho(T^n f - T^m g) \right) \\ &\leq \limsup_{m \rightarrow \infty} \left(k_m \limsup_{n \rightarrow \infty} \rho(T^{n-m} f - g) \right) \\ &\leq \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \rho(T^n f - g) \leq t. \end{aligned}$$

Hence $h \in H_t(f)$ as claimed. The minimality of H implies that $H_t(f)$ is \emptyset or equal to H . From this, it is clear that $r_t(\cdot)$ is constant on H . In order to complete the proof of this lemma, we need to prove that r_f is independent of f . Let $f, g \in H$. Since C is ρ -a.e. compact, there exists a subsequence $\{T^{n_i}(g)\}_i$ of

$\{T^n(g)\}_n$ which ρ -a.e. converges to $h \in C$. Since H satisfies property (3.1), we have $h \in H$. Lemma (1.3) implies

$$\rho(T^n f - h) \leq \liminf_{i \rightarrow \infty} \rho(T^n f - T^{n_i} g).$$

Hence

$$\begin{aligned} r_f &= r_f(h) = \limsup_{n \rightarrow \infty} \rho(T^n f - h) \\ &\leq \limsup_{n \rightarrow \infty} \liminf_{i \rightarrow \infty} \rho(T^n f - T^{n_i} g) \\ &\leq \limsup_{n \rightarrow \infty} \limsup_{m \rightarrow \infty} \rho(T^n f - T^m g) \\ &\leq \limsup_{m \rightarrow \infty} \rho(f - T^m g) = r_g(f) = r_g, \end{aligned}$$

which obviously implies $r_g = r_f$. \square

Recall that if ρ satisfies the Δ_2 -type condition, then ρ -convergence and norm (i.e. Luxemburg norm) convergence coincide. We have the following result:

Lemma 3.3. *Let ρ be a convex modular function satisfying the Δ_2 -type condition. Let S be a nonempty, norm-compact subset of L_ρ with $\text{diam}_\rho(S) > 0$. Then there exists $f \in \overline{\text{conv}}(S)$ such that*

$$\sup\{\rho(g - f) : g \in S\} < \text{diam}_\rho(S).$$

Proof. The proof is similar to the classical one known in Banach spaces. Indeed, since S is compact and ρ is norm continuous, there exist $f_0, f_1 \in S$ such that $\rho(f_0 - f_1) = \text{diam}_\rho(S)$. Let S_0 be a maximal subset of S such that $f_0, f_1 \in S_0$ and for any $f, g \in S_0$, $f \neq g$, we have $\rho(f - g) = \text{diam}_\rho(S)$. Since S is compact, S_0 must be finite. Write $S_0 = \{f_0, f_1, f_2, \dots, f_n\}$ and define

$$h = \frac{f_0 + f_1 + \dots + f_n}{n + 1}.$$

Since S is compact, there exists $g_0 \in S$ such that

$$\rho(g_0 - h) = \sup\{\rho(g - h) : g \in S\}.$$

On the other hand, using the convexity of ρ , we get

$$\begin{aligned} \rho(g_0 - h) &= \rho\left(\sum_{k=0}^{k=n} \left(\frac{1}{n+1}\right) g_0 - \sum_{k=0}^{k=n} \left(\frac{1}{n+1}\right) f_k\right) \\ &\leq \sum_{k=0}^{k=n} \left(\frac{1}{n+1}\right) \rho(g_0 - f_k) \leq \text{diam}_\rho(S). \end{aligned}$$

If $\rho(g_0 - h) = \text{diam}_\rho(S)$, then we must have $\rho(g_0 - f_k) = \text{diam}_\rho(S)$, for $k = 0, 1, \dots, n$. This will contradict the maximality of S_0 . Hence

$$\sup\{\rho(g - h) : g \in S\} = \rho(g_0 - h) < \text{diam}_\rho(S).$$

□

4. MAIN RESULTS

Theorem 4.1. *Let ρ be a convex, σ -finite function modular satisfying the Δ_2 -type condition and C be a ρ -bounded, ρ -a.e. compact subset of L_ρ . Let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping. Let H be a nonempty convex subset of C such that:*

- (i) *if $f \in H$ then $\Omega_{\rho\text{-a.e.}}(f) \subset H$;*
- (ii) *for each $f \in H$, any subsequence $\{T^{n_i}(f)\}_i$ of $\{T^n(f)\}_n$, has a ρ -convergent subsequence.*

Then, T has a fixed point.

Proof. Consider the family \mathcal{F} of nonempty ρ -a.e. compact subsets of H which satisfy property (3.1). \mathcal{F} is not empty since $H \in \mathcal{F}$. By the previous results, \mathcal{F} has a minimal element. Let K be a minimal element of \mathcal{F} . Assume that K has more than one point, i.e. $\text{diam}_\rho(K) > 0$. Let $f \in K$. Set

$$S = \Omega_{\|\cdot\|}(f) = \{g \in H; T^{n_i}(f) \|\cdot\|\text{-converges to } g \text{ for some } n_i \uparrow \infty\}.$$

It is easy to see that $S \subset K$. We claim that $S = T(S)$. Indeed, let $g \in S$. Then there exists a sequence $\{T^{n_i}(f)\}_i$ which $\|\cdot\|$ -converges to g . Since T is continuous, we have $T^{n_i+1}(f) \xrightarrow{\|\cdot\|} T(g)$. By definition of S , we get $T(g) \in S$, i.e. $T(S) \subset S$. Let us show the other inclusion, i.e. $S \subset T(S)$. Let $g \in S$. Again by definition of S , there exists a sequence $\{T^{n_i}(f)\}_i$ which $\|\cdot\|$ -converges to g . The sequence

$\{T^{n_i-1}(f)\}_i$ has a norm convergent subsequence, say $\{T^{n_{\phi(i)}-1}(f)\}_i$. Let h be its $\|\cdot\|$ -limit. Since T is continuous, we get

$$T(h) = T\left(\lim_{i \rightarrow \infty} T^{n_{\phi(i)}-1}(f)\right) = \lim_{i \rightarrow \infty} T^{n_{\phi(i)}}(f) = g.$$

Hence $g \in T(S)$, i.e. $S \subset T(S)$. So our claim is proved, i.e. $T(S) = S$.

Next, notice that the assumption (ii) implies that S is norm compact. Lemma (3.3) implies the existence of $f_0 \in \overline{\text{conv}}(S) \subset K$ such that

$$\sup\{\rho(g - f_0) : g \in S\} < \text{diam}_\rho(S). \quad (A)$$

Let $r = \sup\{\rho(g - f_0) : g \in S\}$. Set

$$D = \{h \in K; \sup_{g \in S} \rho(g - h) \leq r\}.$$

Since $f_0 \in D$ and ρ is convex, D is a nonempty convex subset of K . We claim that $D = K$. Indeed, let us first show that D is ρ -a.e. compact. By the assumption (ii), it is enough to show that D is ρ -a.e. closed. Let $\{h_n\}_n$ be a sequence in D such that $h_n \xrightarrow{\rho\text{-a.e.}} h \in L_\rho$. Fix $g \in S$. Since $g - h_n \xrightarrow{\rho\text{-a.e.}} g - h$, Lemma (1.3) implies

$$\rho(g - h) \leq \liminf_{n \rightarrow \infty} \rho(g - h_n)$$

which yields

$$\rho(g - h) \leq \liminf_{n \rightarrow \infty} \left(\sup\{\rho(f - h_n) : f \in S\} \right) \leq r.$$

Hence $\sup\{\rho(h - g) : g \in S\} \leq r$, i.e. $h \in D$. Next we check that D satisfies property (3.1). Indeed, let $f \in D$ and $g \in \Omega_{\rho\text{-a.e.}}(f)$. Then there exists a sequence $\{T^{n_i}(f)\} \xrightarrow{\rho\text{-a.e.}} g$. Using Lemma (1.3) we obtain

$$\rho(g - h) \leq \liminf_{n \rightarrow \infty} \rho(T^{n_i}(f) - h) \leq \limsup_{n \rightarrow \infty} \rho(T^n f - h)$$

for any $h \in S$. Since $T(S) = S$, there exists a sequence $\{u_n\}_n$ in S such that $h = T^n(u_n)$, for any $n \geq 1$. Hence

$$\begin{aligned} \rho(g - h) &\leq \limsup_{n \rightarrow \infty} \rho(T^n f - T^n u_n) \leq \limsup_{n \rightarrow \infty} k_n \rho(f - u_n) \\ &\leq \limsup_{n \rightarrow \infty} \rho(f - u_n) \leq \sup\{\rho(f - u) : u \in S\} \leq r. \end{aligned}$$

So $\sup\{\rho(g-h) : h \in S\} \leq r$ which gives $g \in D$. Thus D satisfies property (3.1) and by minimality of K , we obtain $D = K$. But

$$\text{diam}_\rho(D) \leq r < \text{diam}_\rho(S) \leq \text{diam}_\rho(K),$$

which is a contradiction. Therefore, K is reduced to one point. Property (3.1) will force this point to be a fixed point of T . \square

Now we are ready to state and prove the main result of this work.

Theorem 4.2. *Let ρ be a convex, ρ is a convex, σ -finite function modular satisfying the Δ_2 -type condition and C be a convex ρ -bounded and ρ -a.e. compact subset of L_ρ . Let $T : C \rightarrow C$ be ρ -asymptotically nonexpansive. Then T has a fixed point.*

Proof. Let \mathcal{F} be the family of nonempty convex subsets of C which satisfy the property (3.1). \mathcal{F} is not empty since $C \in \mathcal{F}$. By Zorn's lemma, \mathcal{F} has a minimal element. Let H be a minimal element of \mathcal{F} . Let us show that H satisfies the hypothesis of Theorem (4.1). It suffices to check that H satisfies property (ii). Let r be defined on H as in Lemma (3.2). If $r = 0$ we have

$$\lim_{n \rightarrow \infty} T^n f = g$$

for any $f, g \in H$, which implies (ii). Otherwise, assume that $r > 0$. Let $f \in H$ such that there exists a sequence $\{T^{n_i} f\}_i$ which has no norm-convergent subsequence. Thus, there exists $\varepsilon > 0$ and a subsequence $\{T^{n(k)} f\}_k$ such that

$$\text{Sep}(\{T^{n(k)} f\}_k) = \inf\{\rho(T^{n(k)} f - T^{n(k')} f), k \neq k'\} \geq \varepsilon.$$

Since H is ρ -a.e. compact, there exists $f_\infty \in H$ such that $T^{n(k)} f \xrightarrow{\rho\text{-a.e.}} f_\infty \in H$ as $k \rightarrow \infty$. Without loss of generality, we may assume the existence of

$$\lim_{k \rightarrow \infty} \rho(T^{n(k)} f - f_\infty) = l.$$

Since $\limsup_{n \rightarrow \infty} \rho(T^n f - f) = r$, we choose $\eta > 0$ such that $\eta < \frac{\varepsilon}{2}$, and an integer $n_0 \geq 1$, such that for all $n \geq n_0$ we have

$$\rho(T^n f - f) < r + \eta.$$

Fix $n \geq n_0$. There exists $k_0 \geq 1$ such that for all $k \geq k_0$, we have $n(k) \geq n + n_0$ and

$$\begin{aligned} \rho(T^n f - T^{n(k)} f) &= \rho(T^n f - T^{n+(n(k)-n)} f) = \rho(T^n f - T^n(T^{n(k)-n} f)) \\ &\leq k_n \rho(f - T^{n(k)-n} f) < k_n(r + \eta). \end{aligned}$$

Note that if $f_n \xrightarrow{\rho\text{-a.e.}} f$ and $\text{Sep}\{f_n\}_n \geq \varepsilon$, then by Lemma (1.3), we have

$$\varepsilon \leq \liminf_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \rho(f_n - f_m) \leq 2 \liminf_{n \rightarrow \infty} \rho(f_n - f).$$

Combined with Lemma (1.3), we get

$$\liminf_{n \rightarrow \infty} \rho(f_n) = \liminf_{n \rightarrow \infty} \rho(f_n - f) + \rho(f) \geq \frac{\varepsilon}{2} + \rho(f).$$

In particular, since $\{T^{n(k)} f - T^n f\}_k$ is ρ -a.e. convergent to $f_\infty - T^n f$ as $k \rightarrow \infty$ and satisfies $\text{Sep}(\{T^{n(k)} f - T^n f\}_k) \geq \varepsilon$, we get

$$\rho(T^n f - f_\infty) \leq \liminf_{k \rightarrow \infty} \rho(T^{n(k)} f - T^n f) - \frac{\varepsilon}{2}.$$

Hence

$$\rho(f_\infty - T^n f) \leq r + \eta - \frac{\varepsilon}{2}$$

which implies

$$r = \limsup_{n \rightarrow \infty} \rho(f_\infty - T^n f) \leq r + \eta - \frac{\varepsilon}{2} < r.$$

This contradiction completes the proof of Theorem 4.2. □

Assume that $L_\rho = L_p(\Omega, \mu)$ for a σ -finite measure μ . If C is a convex, bounded and closed subset of L_p for $1 < p < \infty$ and $T : C \rightarrow C$ is asymptotically non-expansive, it is known that C has a fixed point because L_p is uniformly convex. However the result does not hold for $p = 1$ (even for nonexpansive mappings, see [1]). Since L_1 is a modular space, Theorem (4.1) implies the existence of fixed point if $p = 1$ when C is ρ -a.e. compact. Thus we can state.

Corollary 4.1. Let (Ω, μ) be as above, $C \subset L_1(\Omega, \mu)$ a convex bounded set which is compact for the topology of local convergence in measure and $T : C \rightarrow C$ asymptotically nonexpansive. Then, T has a fixed point.

Proof. Under the above hypothesis ρ -a.e. compact sets and compact sets in the topology of local convergence in measure are identical. □

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