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CONNECTIONS BETWEEN SOME MEASURES OF NON-COMPACTNESS AND ASSOCIATED OPERATORS

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Abstract

Some relationships between the Kuratowski's measure of noncompactness, the ball measure of noncompactness and the δ-separation of the points of a set are studied in special classes of Banach spaces. These relations are applied to compare operators which are contractive for these measures.

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1. INTRODUCTION

Let X be a metric space and $\mathcal{B}$ the collection of bounded subsets of X. A *measure of noncompactness* on X is a map $\mu : \mathcal{B} \to [0, +\infty)$ with the properties that (i) $\mu(B) = 0$ if and only if $Cl(B)$ is compact and (ii) $\mu(B) \le \mu(C)$ if $B \subset C$. Associated with the notion of measure of noncompactness is the concept of k-μ-contractive mapping, defined as follows: If $k \ge 0$ is a given real number, T a continuous mapping from X into another metric space Y, then T is said to be *k-μ-contractive* if, for any bounded subset B of X, $\mu(T(B)) \le k\mu(B)$. The most usual measures of noncompactness are the *set-measure* α defined by Kuratowski [6] and the *ball-measure* β defined by several authors (see for example [7, 10]). We denote in this case k-set-contraction (resp.: k-ball-contraction) instead k-α-contraction (resp.:k-β-contraction). The set-measure and ball-measure of noncompactness and the associated notions of k-set-contractions and k-ball-contractions have proved useful in several areas of functional analysis and differential equations (see for example [2, 9]).
Another measure of noncompactness is defined in [11, page 91] by $\delta(A) = \sup\{r \ge 0 :$ there exists an r-separated sequence in $A\}$ where a sequence $\{x_n\}$ is r-separated if $d(x_n, x_m) \ge r$ for every $n, m \in N; n \ne m$. These three measures of noncompactness are different. Indeed in [2, example 9.6] the difference between α and β is showed. On the other hand if U is the unit ball of ℓ^p, it is well known (see, for instance, [8, page 93]) that $\alpha(U) = \beta(U) = 2$, but the maximal separation for a sequence in U is $2^{1/p}$ (see [11, page 91]). However, in any metric space X the relation between these three measures of noncompactness is

$$\delta(A) \le \alpha(A) \le \beta(A) \le 2\delta(A)$$

for each bounded subset A of X, and therefore the standard relations between the associated operators are:
(a) k-ball-contraction imply 2k-set-contraction
(b) k-set-contraction imply 2k-ball-contraction
(c) k-set-contraction imply 2k-δ-contraction
(d) k-δ-contraction imply 2k-set-contraction
(e) k-ball-contraction imply 2k-δ-contraction
(f) k-δ-contraction imply 2k-ball-contraction

In this note we study some stronger relationships between these three measures of noncompactness and their associated operators in some classes of Banach spaces.

2. RESULTS

2.1 Definition. Let X be a metric space. We define the coefficients δ and δ' of X as the supremum and the infimum (respectively) of the set $\{\frac{\beta(A)}{\alpha(A)} : A$ is a bounded, α-minimal and nonprecompact subset of $X\}$. It is clear that $1 \leq \delta' \leq \delta \leq 2$.

2.2 Definition. Let X be a metric space. We define the packing rate of X by $\gamma(X) = \frac{\delta}{\delta'}$. Obviously $1 \leq \gamma(X) \leq 2$.

2.3 Remark This real number can be thought as a measure of the relationship between the maximal separation of the points in any subset A of X and the least radius of a ball containing A. We say that X is well"packed" when $\gamma(X)$ is near to 1. Actually $\gamma(X) = 1$ if X has the β-property (see [4, Definition 2.1]).

Following an argument similar as in theorem 2.5 of [4] and using the proposition 1.4 of [1], the lemmas 1.1 and 1.2 of [4] and the theorems 2.1 and 3.1 of [5] it is easy to prove the following result which states a relationship between set-contractions, ball-contractions and δ-contractions according to the packing rate of X.

2.4 Theorem. Let X be a separable normed space with packing rate $\gamma = \frac{\delta}{\delta'}$. Then:
a) If D is a subset of X and $T : D \to X$ is a k-set-contractive mapping, then $\frac{T}{\gamma}$ is a k-ball-contractive mapping.
b) If D is a subset of X and $T : D \to X$ is a k-ball-contractive mapping, then $\frac{T}{\delta}$ is a k-set-contractive mapping.
c) If D is a subset of X and $T : D \to X$ is a k-δ-contractive mapping, then $\frac{T}{\delta}$ is a k-set-contractive mapping.
d) If D is a subset of X and $T : D \to X$ is a k-δ-contractive mapping, then $\frac{T}{\gamma}$ is a k-ball-contractive mapping.
e) If D is a subset of X and $T : D \to X$ is a k-ball-contractive mapping, then $\frac{T}{\gamma}$ is a k-δ-contractive mapping.
f) If D is a subset of X and $T : D \to X$ is a k-set-contractive mapping, then T is a k-δ-contractive mapping.
Furthermore these costants γ and δ are the best possible.

Let us calculate finally γ, δ and δ' in some classes of separable metric spaces.

1) Let X be a separable metric space which has the β-property with constant μ. By lemma 1.1 and lemma 2.3 in [4] we has that $\beta(A) = \mu\alpha(A)$ for every bounded, α-minimal and non precompact subset A of X. Thus $\delta = \delta' = \mu$ and $\gamma(X) = 1$.
Two important examples of separable Banach spaces with the β-property are the followings:
a) Every separable Hilbert space has the β-property with constant $\mu = \sqrt{2}$ by theorem 4.4 in [3] and theorem 2.5 in [4].
b) Every l^p-space, $1 \leq p < +\infty$, has the β-property with constant $\mu = 2^{\frac{p-1}{p}}$ by theorem 3.4 and proposition 3.5 in [4].

2) In [4, remark 3.8] is proved that $L^p([0,1])$, $p \neq 2$, $1 \leq p < +\infty$, has not the β-property. However, we prove in [1] that if $(\Omega, \mu)$is a σ-finite measure space and $L^p(\Omega)$, $1 \leq p < +\infty$ is separable, then

$$\min\{2^{\frac{1}{p}}, 2^{\frac{p-1}{p}}\} \leq \frac{\beta(A)}{\alpha(A)} \leq \max\{2^{\frac{1}{p}}, 2^{\frac{p-1}{p}}\}$$

for every bounded, α-minimal and non precompact subset A of $L^p(\Omega)$. Moreover, these bounds are the best possible if Ω satisfies the following property:
(R): There exists a subset of Ω of positive and finite measure each of whose measurable subsets F_1 contains a measurable subset F_2 such that $2\mu(F_2) = \mu(F_1)$.
Thus it follows that $\gamma(L^p(\Omega)) \leq 2^{\frac{|p-2|}{p}}$ if (Ω, μ) is a σ-finite measure space and $L^p(\Omega)$ is separable, and the equality holds if the property (R) is satisfied.

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