

JACOBI-SOBOLEV-TYPE ORTHOGONAL POLYNOMIALS: SECOND ORDER DIFFERENTIAL EQUATION AND ZEROS.

J. Arvesú, R. Álvarez-Nodarse, F. Marcellán, and K. Pan

Preprint MA/UC3M/7/1997 *



Dedicated to Professor Mario Rosario Occorsio on his 65-th birthday.

Key words and phrases: Orthogonal polynomials, Jacobi polynomials, hypergeometric function, Sobolev-type orthogonal polynomials, WKB method.

AMS (MOS) subject classification: **33C45**, **33A65**, **42C05**.

Abstract

We obtain an explicit expression for the Sobolev-type orthogonal polynomials $Q_n(x)$ associated with the inner product

$$\langle p, q \rangle = \int_{-1}^1 p(x)q(x)\rho(x)dx + A_1p(1)q(1) + B_1p(-1)q(-1) + A_2p'(1)q'(1) + B_2p'(-1)q'(-1),$$

where $\rho(x) = (1-x)^\alpha(1+x)^\beta$ is the Jacobi weight function, $\alpha, \beta > -1$, $A_1, B_1, A_2, B_2 \geq 0$ and $p, q \in \mathbb{P}$, the linear space of polynomials with real coefficients. The hypergeometric representation (${}_6F_5$) and the second order linear differential equation that such polynomials satisfy are also obtained. The asymptotic behaviour of such polynomials in $[-1, 1]$ is studied. Furthermore, we obtain some estimates for the largest zero of $Q_n(x)$. Such a zero is located outside the interval $[-1, 1]$. We deduce his dependence of the masses. Finally, the WKB analysis for the distribution of zeros is presented.

1 Introduction.

The study of some particular cases of orthogonal polynomials in Sobolev spaces has attracted the interest of several authors [1], [9], [15], [20], [21] and [25]. Particular emphasis was given to the so-called classical Sobolev polynomials of discrete type, i.e., polynomials orthogonal with respect to an inner product

$$\langle p, q \rangle = \int p(x)q(x)d\mu(x) + \sum_{k=0}^N \int p^{(k)}(x)q^{(k)}(x)d\mu_k(x),$$

where $d\mu(x)$ is a classical measure (Jacobi [1], Gegenbauer [8], Laguerre [16], Bessel [21]) and $d\mu_k(x)$ are Dirac measures.

*December 14, 1997

$$\langle p, q \rangle = \int_{-1}^1 p(x)q(x)\rho(x)dx + A_1p(1)q(1) + B_1p(-1)q(-1) + A_2p'(1)q'(1) + B_2p'(-1)q'(-1),$$

where $\rho(x) = (1-x)^\alpha(1+x)^\beta$ is the Jacobi weight function, $\alpha, \beta > -1$, $A_1, B_1, A_2, B_2 \geq 0$ and $p, q \in \mathbb{P}$, the linear space of polynomials with real coefficients. Some estimates concerning to this kind of polynomials have been obtained in [3]. However, the explicit form of these polynomials in the general case remains as an open question as well as the study of their zeros. We are trying in this paper to cover this lack. Moreover, some of the usual properties of classical orthogonal polynomials – symmetry property, their representation as hypergeometric series and the second order linear differential equation – are translated to the context of Sobolev-type orthogonality.

The structure of the paper is the following. In Section 2 we give some results concerning to classical Jacobi polynomials. Using these results, in Section 3 we obtain an explicit formula for the Jacobi-Sobolev-type orthogonal polynomials in terms of the classical ones and their first and second derivatives which allows us to deduce a symmetry property. In Section 4 we establish the recurrence relation that the Jacobi-Sobolev-type orthogonal polynomials satisfy, when the masses A_2 and B_2 are both different from zero. In Section 5 a representation of our polynomials as a ${}_6F_5$ hypergeometric function is deduced. Finally, in Section 6 a general algorithm in order to generate the second order linear differential equations that such polynomials satisfy is given. This result is basic for the development of the Section 8, more precisely for the WKB method, in order to obtain the distribution of their zeros. In Section 7, some asymptotic formulas, useful in the study of the zeros, are presented. Finally, in Section 8 we obtain the speed of convergence of those zeros located outside $[-1, 1]$. On the other hand, we show some graphics concerning the WKB density as well as the analytic behaviour of the distribution of zeros for Jacobi-Sobolev-type orthogonal polynomials.

2 Classical Jacobi polynomials.

In this section we have enclosed some formulas for the classical Jacobi polynomials which will be useful to obtain some properties of the Sobolev-type orthogonal polynomials. All the formulas as well as some special properties for the classical Jacobi polynomials can be found in the literature [23, Chapter 1-2], [27]. In this work we will use monic polynomials, i.e., polynomials with leading coefficient equal to 1.

The classical Jacobi polynomials $P_n^{\alpha, \beta}(x)$ satisfy the orthogonality relation

$$\int_{-1}^1 P_n^{\alpha, \beta}(x)P_m^{\alpha, \beta}(x)(1-x)^\alpha(1+x)^\beta dx = \delta_{nm}d_n^2, \quad (1)$$

where

$$d_n^2 = \|P_n^{\alpha, \beta}(x)\|^2 = \frac{2^{2n+\alpha+\beta+1}n!\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+2)}.$$

They are the polynomial solution of the second order linear differential equation of hypergeometric type

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0, \quad (2)$$

where

$$\sigma(x) = (1-x^2), \quad \tau(x) = \beta - \alpha - (\alpha + \beta + 2)x, \quad \lambda_n = n(n + \alpha + \beta + 1),$$

respectively. Notice that $\deg \sigma = 2$ and $\deg \tau = 1$. Also they verify the symmetry property

$$P_n^{\alpha, \beta}(x) = (-1)^n P_n^{\beta, \alpha}(-x), \quad (3)$$

$$\frac{d^\nu}{dx^\nu} P_n^{\alpha,\beta}(x) \equiv (P_n^{\alpha,\beta}(x))^{(\nu)} = \frac{n!}{(n-\nu)!} P_{n-\nu}^{\alpha+\nu,\beta+\nu}(x), \quad \text{with } \nu \leq n \quad \text{and } n = 0, 1, 2, \dots, \quad (4)$$

as well as the three-term recurrence relation

$$xP_n(x) = P_{n+1}^{\alpha,\beta}(x) + \beta_n^{\alpha,\beta} P_n^{\alpha,\beta}(x) + \gamma_n^{\alpha,\beta} P_{n-1}^{\alpha,\beta}(x), \quad (5)$$

where

$$\beta_n^{\alpha,\beta} = \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + 2 + \alpha + \beta)}, \quad (6)$$

$$\gamma_n^{\alpha,\beta} = \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)}.$$

They are represented as the hypergeometric series

$$P_n^{\alpha,\beta}(x) = \frac{2^n(\alpha + 1)_n}{(n + \alpha + \beta + 1)_n} {}_2F_1 \left(\begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix} \middle| \frac{1-x}{2} \right), \quad (7)$$

where

$${}_pF_q \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}, \quad (8)$$

and $(a)_k$ is the Pochhammer symbol or shifted factorial $(a)_0 := 1$, $(a)_k := a(a+1)(a+2)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$, $k = 1, 2, 3, \dots$. As a consequence of this representation we get

$$P_n^{\alpha,\beta}(1) = \frac{2^n(\alpha + 1)_n}{(n + \alpha + \beta + 1)_n}, \quad P_n^{\alpha,\beta}(-1) = \frac{(-1)^n 2^n (\beta + 1)_n}{(n + \alpha + \beta + 1)_n}. \quad (9)$$

The Christoffel-Darboux formula is

$$\sum_{m=0}^{n-1} \frac{P_m^{\alpha,\beta}(x)P_m^{\alpha,\beta}(y)}{d_m^2} = \frac{1}{x-y} \frac{P_n^{\alpha,\beta}(x)P_{n-1}^{\alpha,\beta}(y) - P_{n-1}^{\alpha,\beta}(x)P_n^{\alpha,\beta}(y)}{d_{n-1}^2}, \quad n = 1, 2, 3, \dots \quad (10)$$

Throughout the work we will denote

$$K_n^{\alpha,\beta(p,q)}(x, y) = \sum_{m=0}^n \frac{(P_m^{\alpha,\beta})^{(p)}(x)(P_m^{\alpha,\beta})^{(q)}(y)}{d_m^2} = \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_n^{\alpha,\beta}(x, y), \quad (11)$$

the kernels of the Jacobi polynomials, as well as their derivatives with respect to x and y , respectively. By using the symmetry property (3) and (11) it is straightforward to prove that the following symmetry properties for the Jacobi kernels

$$\begin{aligned} K_n^{\alpha,\beta}(x, y) &= K_n^{\beta,\alpha}(-x, -y), \\ K_n^{\alpha,\beta(0,1)}(x, y) &= -K_n^{\beta,\alpha(0,1)}(-x, -y), \\ K_n^{\alpha,\beta(1,1)}(x, y) &= K_n^{\beta,\alpha(1,1)}(-x, -y), \end{aligned} \quad (12)$$

hold. In our work we need the explicit expressions of the kernels $K_{n-1}^{\alpha,\beta}(x, 1)$, $K_{n-1}^{\alpha,\beta(0,1)}(x, 1)$, $K_{n-1}^{\alpha,\beta}(x, -1)$ and $K_{n-1}^{\alpha,\beta(0,1)}(x, -1)$, respectively. To obtain these kernels we can use the Christoffel-Darboux formula, the structure relation, the three-term recurrence relation and the differentiation formula for classical monic Jacobi polynomials, respectively. The detailed computation can be found in [5]. We will provide

notation $\kappa_n^{\alpha,\beta} = (2n + \alpha + \beta + 1)\gamma_n^{\alpha,\beta}$

$$K_{n-1}^{\alpha,\beta}(x, 1) = \frac{P_n^{\alpha,\beta}(1)}{d_{n-1}^2 \kappa_n^{\alpha,\beta}} \left[(1+x)(P_n^{\alpha,\beta})'(x) - nP_n^{\alpha,\beta}(x) \right], \quad (13)$$

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,1)}(x, 1) &= \frac{(P_n^{\alpha,\beta})'(1)}{d_{n-1}^2 \kappa_n^{\alpha,\beta}} \left[(1+x)(P_n^{\alpha,\beta})'(x) - nP_n^{\alpha,\beta}(x) \right] - \\ &\quad - \frac{P_n^{\alpha,\beta}(1)}{d_{n-1}^2 \kappa_n^{\alpha,\beta}(\alpha+1)} \left[(1+\beta)(P_n^{\alpha,\beta})'(x) + (x+1)(P_n^{\alpha,\beta})''(x) \right]. \end{aligned} \quad (14)$$

From the two previous formulas and using the symmetry properties (12), we find

$$\begin{aligned} K_{n-1}^{\alpha,\beta}(x, -1) &= -\frac{P_n^{\alpha,\beta}(-1) \left[(1-x)(P_n^{\alpha,\beta})'(x) + nP_n^{\alpha,\beta}(x) \right]}{d_{n-1}^2 \kappa_n^{\alpha,\beta}}, \\ K_{n-1}^{\alpha,\beta(0,1)}(x, -1) &= -\frac{(P_n^{\alpha,\beta})'(-1) \left[(1-x)(P_n^{\alpha,\beta})'(x) + nP_n^{\alpha,\beta}(x) \right]}{d_{n-1}^2 \kappa_n^{\alpha,\beta}} + \\ &\quad + \frac{P_n^{\alpha,\beta}(-1) \left[(1-x)(P_n^{\alpha,\beta})''(x) - (\alpha+1)(P_n^{\alpha,\beta})'(x) \right]}{d_{n-1}^2 \kappa_n^{\alpha,\beta}(\beta+1)}. \end{aligned} \quad (15)$$

Also the following values are needed [5]

$$\begin{aligned} K_{n-1}^{\alpha,\beta}(1, 1) &= \frac{(P_n^{\alpha,\beta}(1))^2 n(n+\beta)}{d_{n-1}^2 \kappa_n^{\alpha,\beta}(\alpha+1)}, & K_{n-1}^{\alpha,\beta(0,1)}(1, 1) &= \frac{(P_n^{\alpha,\beta})'(1) P_n^{\alpha,\beta}(1)(n+\beta)}{d_{n-1}^2 \kappa_n^{\alpha,\beta}(\alpha+2)(n-1)^{-1}}, \\ K_{n-1}^{\alpha,\beta}(1, -1) &= -\frac{n P_n^{\alpha,\beta}(-1) P_n^{\alpha,\beta}(1)}{d_{n-1}^2 \kappa_n^{\alpha,\beta}}, & K_{n-1}^{\alpha,\beta(0,1)}(1, -1) &= \frac{(P_n^{\alpha,\beta})'(-1) P_n^{\alpha,\beta}(1)(1-n)}{d_{n-1}^2 \kappa_n^{\alpha,\beta}}, \\ K_{n-1}^{\alpha,\beta(1,1)}(1, 1) &= \frac{P_n^{\alpha,\beta}(1)(P_n^{\alpha,\beta})'(1)(n+\beta) \left[(\alpha+2)(n^2+n\alpha+n\beta) - (\alpha+1)(\alpha+\beta+2) \right]}{2d_{n-1}^2 \kappa_n^{\alpha,\beta}(\alpha+1)(\alpha+2)(\alpha+3)(n-1)^{-1}}, \\ K_{n-1}^{\alpha,\beta(1,1)}(1, -1) &= \frac{(P_n^{\alpha,\beta})'(-1) P_n^{\alpha,\beta}(1)(1-n) \left[n^2+n\alpha+n\beta - \alpha - \beta - 2 \right]}{2d_{n-1}^2 \kappa_n^{\alpha,\beta}(\alpha+1)}. \end{aligned} \quad (16)$$

3 Jacobi-Sobolev-type orthogonal polynomials.

Consider the inner product in the linear space of polynomials with real coefficients

$$\langle p, q \rangle = \langle p, q \rangle_c + A_1 p(1)q(1) + B_1 p(-1)q(-1) + A_2 p'(1)q'(1) + B_2 p'(-1)q'(-1), \quad (17)$$

where $\langle p, q \rangle_c$ is the Jacobi inner product

$$\langle p, q \rangle_c = \int_{-1}^1 p(x)q(x)(1-x)^\alpha(1+x)^\beta dx, \quad \alpha > -1, \quad \beta > -1, \quad (18)$$

We will denote $\{Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)\}_n$ the monic orthogonal polynomial sequence with respect to the inner product (17). They will be called *Jacobi-Sobolev-type orthogonal polynomials*. Let us now to find an explicit representation of the polynomials $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$ in terms of the classical ones. To obtain this we write the Fourier expansion of the Jacobi-Sobolev-type polynomials in terms of the Jacobi polynomials

$$\tilde{Q}_n(x) \equiv Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) = P_n^{\alpha,\beta}(x) + \sum_{k=0}^{n-1} a_{n,k} P_k^{\alpha,\beta}(x), \quad (19)$$

where $P_n^{\alpha,\beta}(x)$ is the classical Jacobi monic polynomial of degree n . To find the coefficients $a_{n,k}$ we can use the orthogonality of the polynomials $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$ with respect to \langle, \rangle , i.e.,

$$\langle Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x), P_k^{\alpha,\beta}(x) \rangle = 0 \quad 0 \leq k < n. \quad (20)$$

Thus, according to (17) we find

$$\begin{aligned} & \langle Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x), P_k^{\alpha,\beta}(x) \rangle = \langle Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x), P_k^{\alpha,\beta}(x) \rangle_c + \\ & + A_1 Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(1) P_k^{\alpha,\beta}(1) + B_1 Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(-1) P_k^{\alpha,\beta}(-1) + \\ & + A_2 (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1) (P_k^{\alpha,\beta})'(1) + B_2 (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(-1) (P_k^{\alpha,\beta})'(-1), \end{aligned} \quad (21)$$

If we use the decomposition (19) and taking into account (20) we find the following expression for the coefficients $a_{n,k}$

$$\begin{aligned} a_{n,k} = & - \frac{A_1 Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(1) P_k^{\alpha,\beta}(1) + B_1 Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(-1) P_k^{\alpha,\beta}(-1)}{d_k^2} \\ & - \frac{A_2 (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1) (P_k^{\alpha,\beta})'(1) + B_2 (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(-1) (P_k^{\alpha,\beta})'(-1)}{d_k^2}, \end{aligned} \quad k < n \quad (22)$$

where d_k^2 denotes the square norm of the classical Jacobi polynomials (1). Finally, the equation (19) becomes

$$\begin{aligned} Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) = & P_n^{\alpha,\beta}(x) - A_1 Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(1) K_{n-1}^{\alpha,\beta}(x, 1) - \\ & - B_1 Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(-1) K_{n-1}^{\alpha,\beta}(x, -1) - A_2 (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1) K_{n-1}^{\alpha,\beta(0,1)}(x, 1) - \\ & - B_2 (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(-1) K_{n-1}^{\alpha,\beta(0,1)}(x, -1). \end{aligned} \quad (23)$$

In order to find the unknowns $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(1)$, $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(-1)$, $(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1)$ and $(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(-1)$ we can take derivatives in (23) and evaluate the resulting equation, as well as (23), at $x = 1$ and $x = -1$. This leads to a linear system of equations

$$\mathbb{K} \cdot \tilde{\mathbf{Q}}_n = \mathbf{Q}_n, \quad (24)$$

$$k_1 = \begin{pmatrix} 1 + A_1 K_{n-1}^{\alpha,\beta}(1, 1) \\ A_1 K_{n-1}^{\alpha,\beta}(1, -1) \\ A_1 K_{n-1}^{\alpha,\beta(0,1)}(1, 1) \\ A_1 K_{n-1}^{\alpha,\beta(0,1)}(1, -1) \end{pmatrix}, \quad k_2 = \begin{pmatrix} B_1 K_{n-1}^{\alpha,\beta}(1, -1) \\ 1 + B_1 K_{n-1}^{\alpha,\beta}(-1, -1) \\ B_1 K_{n-1}^{\alpha,\beta(0,1)}(-1, 1) \\ B_1 K_{n-1}^{\alpha,\beta(0,1)}(-1, -1) \end{pmatrix},$$

$$k_3 = \begin{pmatrix} A_2 K_{n-1}^{\alpha,\beta(0,1)}(1, 1) \\ A_2 K_{n-1}^{\alpha,\beta(0,1)}(-1, 1) \\ 1 + A_2 K_{n-1}^{\alpha,\beta(1,1)}(1, 1) \\ A_2 K_{n-1}^{\alpha,\beta(1,1)}(1, -1) \end{pmatrix}, \quad k_4 = \begin{pmatrix} B_2 K_{n-1}^{\alpha,\beta(0,1)}(1, -1) \\ B_2 K_{n-1}^{\alpha,\beta(0,1)}(-1, -1) \\ B_2 K_{n-1}^{\alpha,\beta(1,1)}(1, -1) \\ 1 + B_2 K_{n-1}^{\alpha,\beta(1,1)}(-1, -1) \end{pmatrix},$$

and $\tilde{\mathbf{Q}}_n$ and \mathbf{Q}_n are the column vectors

$$\tilde{\mathbf{Q}}_n = \begin{pmatrix} Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(1) \\ Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(-1) \\ (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1) \\ (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(-1) \end{pmatrix}, \quad \mathbf{Q}_n = \begin{pmatrix} P_n^{\alpha,\beta}(1) \\ P_n^{\alpha,\beta}(-1) \\ (P_n^{\alpha,\beta})'(1) \\ (P_n^{\alpha,\beta})'(-1) \end{pmatrix},$$

respectively. Let us denote $\mathbb{K}_j(\mathbf{Q}_n)$ the matrix obtained substituting the j column in \mathbb{K} by \mathbf{Q}_n . Then, from the Cramer's, rule the system (24) has a unique solution if and only if the determinant of \mathbb{K} does not vanish. Moreover, the solution is given by

$$Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(1) = \frac{\det \mathbb{K}_1(\mathbf{Q}_n)}{\det \mathbb{K}}, \quad Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(-1) = \frac{\det \mathbb{K}_2(\mathbf{Q}_n)}{\det \mathbb{K}},$$

$$(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1) = \frac{\det \mathbb{K}_3(\mathbf{Q}_n)}{\det \mathbb{K}}, \quad (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(-1) = \frac{\det \mathbb{K}_4(\mathbf{Q}_n)}{\det \mathbb{K}}.$$
(25)

Here we want to remark that, since our polynomials are orthogonal with respect to (17), then the polynomials $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$ exist for all values of the nonnegative masses A_1 , B_1 , A_2 and B_2 . In particular this implies that $\det \mathbb{K} \neq 0$. This situation is very different from one studied in [5] where the polynomials are orthogonal with respect to a linear functional which is not positive definite (in general it is not a quasi-definite linear functional).

Proposition 1 *The following symmetry property for the Jacobi-Sobolev polynomials holds*

$$Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(-x) = (-1)^n Q_n^{\beta,\alpha,B_1,A_1,B_2,A_2}(x). \quad (26)$$

Proof: Let us denote the determinant of \mathbb{K} by $\Delta_n^{\alpha,\beta,A_1,B_1,A_2,B_2}$ and the determinant of $\mathbb{K}_j(\mathbf{Q}_n)$ by $\Delta_{n,j}^{\alpha,\beta,A_1,B_1,A_2,B_2}(\mathbf{Q}_n)$. If we interchange in $\mathbb{K}_2(\mathbf{Q}_n)$ the first and second columns and the first and second rows, the third and fourth columns and the third and fourth rows, respectively, and then we use the symmetry property of Jacobi polynomials (9) and their kernels (12) we find the following relation for the determinants

$$\Delta_{n,2}^{\beta,\alpha,B_1,A_1,B_2,A_2}(\mathbf{Q}_n) = (-1)^n \Delta_{n,1}^{\alpha,\beta,A_1,B_1,A_2,B_2}(\mathbf{Q}_n).$$

If we handle with the same rows and columns but in \mathbb{K} we get

$$\Delta_n^{\alpha,\beta,A_1,B_1,A_2,B_2} = \Delta_n^{\beta,\alpha,B_1,A_1,B_2,A_2}.$$

Then, from (25) we obtain

$$Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(-1) = (-1)^n Q_n^{\beta,\alpha,B_1,A_1,B_2,A_2}(1). \quad (27)$$

$$(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(-1) = (-1)^{n-1} (Q_n^{\beta,\alpha,B_1,A_1,B_2,A_2})'(1). \quad (28)$$

Now, if we provide the change of parameters $\alpha \leftrightarrow \beta$, $A_1 \leftrightarrow B_1$ and $A_2 \leftrightarrow B_2$ in (23) and then use the symmetry properties for the Jacobi kernels (12) and (27)-(28) the proposition holds. \blacksquare

Let us now to obtain an explicit formula of $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$ in terms of the classical Jacobi polynomials and their first and second derivatives. We start from formula (23) where we substitute the kernels by their explicit expressions (13)-(15) and use the formulas (25). This leads to the following.

Proposition 2 *The Jacobi-Sobolev orthogonal polynomials $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$ can be given in terms of the classical Jacobi polynomials and their first and second derivatives*

$$\begin{aligned} Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) &= (1 + n\zeta_n + n\eta_n)P_n^{\alpha,\beta}(x) + [\zeta_n(1-x) - \eta_n(1+x) + \\ &+ (\beta+1)\chi_n + (\alpha+1)\omega_n](P_n^{\alpha,\beta}(x))' + [\chi_n(1+x) - \omega_n(1-x)](P_n^{\alpha,\beta}(x))'', \end{aligned} \quad (29)$$

where

$$\zeta_n = B_1 C_n^{\beta,\alpha,B_1,A_1,B_2,A_2} + B_2 D_n^{\beta,\alpha,B_1,A_1,B_2,A_2}, \quad (30)$$

$$\eta_n = A_1 C_n^{\alpha,\beta,A_1,B_1,A_2,B_2} + A_2 D_n^{\alpha,\beta,A_1,B_1,A_2,B_2},$$

$$\chi_n = A_2 E_n^{\alpha,\beta,A_1,B_1,A_2,B_2}, \quad \omega_n = B_2 E_n^{\beta,\alpha,B_1,A_1,B_2,A_2}, \quad (31)$$

and

$$\begin{aligned} C_n^{\alpha,\beta,A_1,B_1,A_2,B_2} &= \frac{Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(1)P_n^{\alpha,\beta}(1)}{d_{n-1}^2 \kappa_n^{\alpha,\beta}}, \\ D_n^{\alpha,\beta,A_1,B_1,A_2,B_2} &= \frac{(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1)(P_n^{\alpha,\beta})'(1)}{d_{n-1}^2 \kappa_n^{\alpha,\beta}}, \\ E_n^{\alpha,\beta,A_1,B_1,A_2,B_2} &= \frac{(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1)P_n^{\alpha,\beta}(1)}{d_{n-1}^2 \kappa_n^{\alpha,\beta} (1+\alpha)}. \end{aligned} \quad (32)$$

Notice that the constants ζ_n, η_n, χ_n and ω_n depend on n, α, β and the masses A_1, B_1, A_2 and B_2 .

In the next Section we will establish the recurrence relation that the polynomials $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$ satisfy. Notice that, since the matrix of the moments of the inner product defined by (17) is not of Hankel type because $\langle x, x \rangle \neq \langle 1, x^2 \rangle$, then the Sobolev-type orthogonal polynomials $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$ don't satisfy a three-term recurrence relation. In fact they will satisfy a seven-term recurrence relation (see [15]).

4 The seven-term recurrence relation for $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$.

Here we will prove that the polynomials $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$ satisfy a seven-term recurrence relation. In fact, it's straightforward to prove that the multiplication operator by $(x^2 - 1)^2$ is symmetric with respect to (17). The problem is to find a polynomial operator of the lowest degree which be symmetric with respect to the Sobolev inner product (17).

CASES:

1. If $A_2 = B_2 = 0$ we have a standard inner product, hence the multiplication operator by x is symmetric.

obtain a five-term recurrence relation.

3. If $A_2 \neq 0$ and $B_2 = 0$ then the multiplication operator by $(x - 1)^2$ is symmetric. Hence we obtain a five-term recurrence relation.

There is another interesting case, when the masses A_2 and B_2 are both different from zero. This situation will be considered below.

We assume that $A_2 \neq 0$ and $B_2 \neq 0$. In particular, from (17) we get

$$\langle hp, q \rangle = \langle p, hq \rangle \quad p, q \in \mathbb{P}, \quad (33)$$

for some polynomial $h(x)$ of degree less than or equal to four. This implies that

$$A_2(hp)'(1)q'(1) + B_2(hp)'(-1)q'(-1) = A_2(hq)'(1)p'(1) + B_2(hq)'(-1)p'(-1), \quad \forall p, q \in \mathbb{P}. \quad (34)$$

Therefore

$$A_2h'(1)p(1)q'(1) + B_2h'(-1)p(-1)q'(-1) = A_2h'(1)q(1)p'(1) + B_2h'(-1)q(-1)p'(-1), \quad (35)$$

or, equivalently,

$$A_2h'(1)[p(1)q'(1) - p'(1)q(1)] + B_2h'(-1)[p(-1)q'(-1) - q(-1)p'(-1)] = 0, \quad \forall p, q \in \mathbb{P}. \quad (36)$$

If $p(x) = 1$ and $q(x) = x$ the equation (36) yields

$$A_2h'(1) + B_2h'(-1) = 0. \quad (37)$$

If $p(x) = 1$ and $q(x) = x^2$ the equation (36) leads

$$2A_2h'(1) - 2B_2h'(-1) = 0. \quad (38)$$

Thus, from (37)-(38) we get

$$\begin{cases} A_2h'(1) + B_2h'(-1) = 0, \\ A_2h'(1) - B_2h'(-1) = 0. \end{cases} \quad (39)$$

As $A_2 \neq 0$ and $B_2 \neq 0 \implies h'(1) = h'(-1) = 0$, hence $h'(x) = (x^2 - 1)r(x)$. The minimal choice of $r(x)$ is, in this situation, $r(x) \equiv 1$. Therefore

$$h(x) = \frac{x^3}{3} - x + a \quad (40)$$

or, equivalently,

$$h(x) = x^3 - 3x + b. \quad (41)$$

In order to operate with $h(x)$ we put $b = 0$. In such a way we can guarantee that $h(x) = x^3 - 3x$ leads to the searched symmetric operator on \mathbb{P} , when $A_2 \neq 0$ and $B_2 \neq 0$. This fact allows to write a seven-term recurrence relation for $Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(x)$. In fact, from

$$(x^3 - 3x)Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(x) = \sum_{j=0}^{n+3} \alpha_{nj} Q_j^{\alpha, \beta, A_1, B_1, A_2, B_2}(x). \quad (42)$$

and taking into account that

$$\begin{aligned} \alpha_{nj} &= \frac{\langle (x^3 - 3x)Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(x), Q_j^{\alpha, \beta, A_1, B_1, A_2, B_2}(x) \rangle}{\langle Q_j^{\alpha, \beta, A_1, B_1, A_2, B_2}(x), Q_j^{\alpha, \beta, A_1, B_1, A_2, B_2}(x) \rangle} = \\ &= \frac{\langle Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(x), (x^3 - 3x)Q_j^{\alpha, \beta, A_1, B_1, A_2, B_2}(x) \rangle}{\langle Q_j^{\alpha, \beta, A_1, B_1, A_2, B_2}(x), Q_j^{\alpha, \beta, A_1, B_1, A_2, B_2}(x) \rangle} = 0, \quad \text{if } j < n - 3, \end{aligned} \quad (43)$$

$$(x^3 - 3x)Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) = \sum_{j=n-3}^{n+3} \alpha_{nj} Q_j^{\alpha,\beta,A_1,B_1,A_2,B_2}(x), \quad (44)$$

where

$$\begin{aligned} \alpha_{n,n-3} &= \frac{\langle (x^3 - 3x)Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x), Q_{n-3}^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) \rangle}{\langle Q_{n-3}^{\alpha,\beta,A_1,B_1,A_2,B_2}(x), Q_{n-3}^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) \rangle} = \\ &= \frac{\langle Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x), (x^3 - 3x)Q_{n-3}^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) \rangle}{\langle Q_{n-3}^{\alpha,\beta,A_1,B_1,A_2,B_2}(x), Q_{n-3}^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) \rangle} = \\ &= \frac{\langle Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x), Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) \rangle}{\langle Q_{n-3}^{\alpha,\beta,A_1,B_1,A_2,B_2}(x), Q_{n-3}^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) \rangle} > 0. \end{aligned} \quad (45)$$

5 Representation as hypergeometric series.

Here we will prove the following proposition

Proposition 3 *The orthogonal polynomial $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$ is, up to a constant factor, a generalized hypergeometric series. More precisely,*

$$Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) = \frac{2^{n-3}(\alpha+3)_{n-3}\pi_4(0)}{(n+\alpha+\beta+1)_n} {}_6F_5 \left(\begin{matrix} -n, n+\alpha+\beta+1, \beta_0+1, \beta_1+1, \beta_2+1, \beta_3+1 \\ \alpha+3, \beta_0, \beta_1, \beta_2, \beta_3 \end{matrix} \middle| \frac{1-x}{2} \right), \quad (46)$$

where $\pi_4(0)$ is given in (52) and the coefficients $-\beta_0, -\beta_1, -\beta_2$ and $-\beta_3$ are the zeros of a polynomial of fourth degree at k (see formula (49) from below). In general, they are complex numbers. If for some $i = 0, 1, 2, 3$, $-\beta_i$ is a negative integer number we need to take the analytic continuation of the hypergeometric series (46).

The representation (46) can be considered as a generalization of the representation as hypergeometric series of the Jacobi polynomials.

Proof: Using (4)-(5) we can rewrite (29) as follows

$$\begin{aligned} Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) &= A_n P_n^{\alpha,\beta}(x) + nB_n P_{n-1}^{\alpha+1,\beta+1}(x) + nC_n P_n^{\alpha+1,\beta+1}(x) + nD_n P_{n-2}^{\alpha+1,\beta+1}(x) + \\ &+ n(n-1)E_n P_{n-2}^{\alpha+2,\beta+2}(x) + n(n-1)F_n P_{n-1}^{\alpha+2,\beta+2}(x) + n(n-1)G_n P_{n-3}^{\alpha+2,\beta+2}(x), \end{aligned} \quad (47)$$

where

$$\begin{aligned} A_n &= 1 - nC_n, \quad B_n = \zeta_n - \eta_n + C_n \beta_{n-1}^{\alpha+1,\beta+1}, \quad C_n = -(\zeta_n + \eta_n), \quad D_n = C_n \gamma_{n-1}^{\alpha+1,\beta+1}, \\ E_n &= \chi_n - \omega_n + F_n \beta_{n-2}^{\alpha+2,\beta+2}, \quad F_n = \chi_n + \omega_n, \quad G_n = F_n \gamma_{n-2}^{\alpha+2,\beta+2}. \end{aligned} \quad (48)$$

Substituting the hypergeometric representation of the Jacobi polynomials (7) in (47) we find

$$\begin{aligned} Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) &= \frac{2^{n-3}(\alpha+3)_{n-3}}{(n+\alpha+\beta+1)_n} \sum_{k=0}^{\infty} \left[8A_n(n+\alpha)(k+\alpha+1)(k+\alpha+2) - \right. \\ &\quad \left. - 4B_n(n+\alpha)(k-n)(k+n+\alpha+\beta+1)(k+\alpha+2) + \right. \\ &\quad \left. + \frac{8nC_n(n+\alpha)(n+\alpha+1)(k+n+\alpha+\beta+1)(k+n+\alpha+\beta+2)(k+\alpha+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \right] \end{aligned}$$

$$\begin{aligned}
& +2nE_n(n+\alpha)(k-n)(k-n+1)(k+n+\alpha+\beta+1)(k+n+\alpha+\beta+2)- \\
& -\frac{4nF_n(n-1)(n+\alpha)(k-n)(k+n+\alpha+\beta+1)(k+n+\alpha+\beta+2)(k+n+\alpha+\beta+3)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)(n+\alpha+1)^{-1}} \\
& -\left[\frac{G_n(k-n)(k-n+1)(k-n+2)(k+n+\alpha+\beta+1)(2n+\alpha+\beta)(2n+\alpha+\beta-1)}{n-2}\right] \times \\
& \times \frac{(-n)_k(n+\alpha+\beta+1)_k}{k!(\alpha+3)_k} \left(\frac{1-x}{2}\right)^k.
\end{aligned}$$

Taking into account that the expression inside the quadratic brackets is a polynomial in k of degree 4, denoted $\pi_4(k)$, we can write

$$\begin{aligned}
Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) &= \frac{2^{n-3}(\alpha+3)_{n-3}v_n}{(n+\alpha+\beta+1)_n} \times \\
& \times \sum_{k=0}^{\infty} \frac{(-n)_k(n+\alpha+\beta+1)_k(k+\beta_0)(k+\beta_1)(k+\beta_2)(k+\beta_3)}{k!(\alpha+3)_k} \left(\frac{1-x}{2}\right)^k, \tag{49}
\end{aligned}$$

where v_n is the leading coefficient of $\pi_4(k)$

$$v_n = 2n(n+\alpha)E_n - \frac{4n(n-1)(n+\alpha)(n+\alpha+1)F_n}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} - \frac{(2n+\alpha+\beta)(2n+\alpha+\beta-1)G_n}{n-2}, \tag{50}$$

and $-\beta_i = -\beta_i(n, \alpha, \beta, A_1, A_2, B_1, B_2)$ with $i = 0, 1, 2, 3$, are the zeros of $\pi_4(k)$. Since $(k+\beta_i) = \frac{\beta_i(\beta_i+1)_k}{(\beta_i)_k}$, $i = 0, 1, 2, 3$, then (49) becomes

$$\begin{aligned}
Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) &= \frac{2^{n-3}(\alpha+3)_{n-3}\pi_4(0)}{(n+\alpha+\beta+1)_n} \times \\
& \times \sum_{k=0}^{\infty} \frac{(-n)_k(n+\alpha+\beta+1)_k(1+\beta_0)_k(1+\beta_1)_k(1+\beta_2)_k(1+\beta_3)_k}{k!(\alpha+3)_k(\beta_0)_k(\beta_1)_k(\beta_2)_k(\beta_3)_k} \left(\frac{1-x}{2}\right)^k, \tag{51}
\end{aligned}$$

where

$$\begin{aligned}
\pi_4(0) &= 8A_n(n+\alpha)(\alpha+1)(\alpha+2) + 4B_n(n+\alpha)(n+\alpha+\beta+1)(\alpha+2)n + \\
& + \frac{8C_n(n+\alpha)(n+\alpha+1)(n+\alpha+\beta+1)(n+\alpha+\beta+2)(\alpha+2)n}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} + \\
& + 2D_n(\alpha+2)n + 2E_n(n+\alpha)(n+\alpha+\beta+1)(n+\alpha+\beta+2)n(n-1) + \\
& + \frac{4F_n(n+\alpha)(n+\alpha+1)(n+\alpha+\beta+1)(n+\alpha+\beta+2)(n+\alpha+\beta+3)n(n-1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} + \\
& + G_n(n+\alpha+\beta+1)(2n+\alpha+\beta-1)(2n+\alpha+\beta)n(n-1), \tag{52}
\end{aligned}$$

It is straightforward to see that in the case A_2, B_2 equal to zero, from (46) we recover the monic Jacobi-Koornwinder polynomials [17], and when $A_1 = A_2 = B_1 = B_2 = 0$, (46) becomes the representation (8), i.e., we recover the classical Jacobi polynomials.

6 Second order differential equation.

In this section we will show that the Jacobi-Sobolev polynomials satisfy a second order linear differential equation (SODE). We want to remark here that the main result that allows us to obtain the SODE is, as in the case considered in [5], the fact that the polynomials $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$ admit a representation in terms of the classical Jacobi polynomials and their first and second derivatives of the form (29).

First of all, we will rewrite (29) in terms of the polynomials and their first derivatives. In order to do this we will use the SODE for the Jacobi polynomials (2). Then, if we multiply (29) by $\sigma(x)$ and use the SODE (2) we find the following equivalent representation formula

$$\sigma(x)Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) = a(x;n)P_n^{\alpha,\beta}(x) + b(x;n)\frac{d}{dx}P_n^{\alpha,\beta}(x), \quad (53)$$

where $a(x;n), b(x;n)$ are polynomials of bounded degree in x with coefficients depending on n

$$a(x;n) = (1 + n\zeta_n + n\eta_n)\sigma(x) - \lambda_n [\chi_n(1+x) - \omega_n(1-x)], \quad (54)$$

$$b(x;n) = [\zeta_n(1-x) - \eta_n(1+x) + (\beta+1)\chi_n + (\alpha+1)\omega_n]\sigma(x) - \tau(x) [\chi_n(1+x) - \omega_n(1-x)].$$

Taking derivatives in the above expression and using (2) and (53)

$$\sigma^2(x)\frac{d}{dx}Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) = c(x;n)P_n^{\alpha,\beta}(x) + d(x;n)\frac{d}{dx}P_n^{\alpha,\beta}(x), \quad (55)$$

where

$$c(x;n) = \sigma(x)a'(x;n) - \sigma'(x)a(x;n) - \lambda_nb(x;n), \quad (56)$$

$$d(x;n) = \sigma(x)[a(x;n) + b'(x;n)] - \sigma'(x)b(x;n) - \tau(x)b(x;n).$$

Analogously, if we take derivatives in (55) and use (2) and (53), we obtain

$$\sigma^3(x)\frac{d^2}{dx^2}Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) = e(x;n)P_n^{\alpha,\beta}(x) + f(x;n)\frac{d}{dx}P_n^{\alpha,\beta}(x), \quad (57)$$

where

$$e(x;n) = \sigma(x)c'(x;n) - 2\sigma'(x)c(x;n) - \lambda_nd(x;n), \quad (58)$$

$$f(x;n) = \sigma(x)[c(x;n) + d'(x;n)] - 2\sigma'(x)d(x;n) - \tau(x)d(x;n).$$

The above expressions (53), (55) and (57) lead to

$$\begin{vmatrix} Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) & a(x;n) & b(x;n) \\ \sigma(x)(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x))' & c(x;n) & d(x;n) \\ \sigma^2(x)(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x))'' & e(x;n) & f(x;n) \end{vmatrix} = 0. \quad (59)$$

This yields

$$\tilde{\sigma}_n(x)\frac{d^2}{dx^2}Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) + \tilde{\tau}_n(x)\frac{d}{dx}Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) + \tilde{\lambda}_n(x)Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) = 0, \quad (60)$$

$$\tilde{\sigma}(x; n) = \sigma^2(x) [a(x; n)d(x; n) - c(x; n)b(x; n)],$$

$$\tilde{\tau}(x; n) = \sigma(x)[e(x; n)b(x; n) - a(x; n)f(x; n)], \quad (61)$$

$$\tilde{\lambda}(x; n) = c(x; n)f(x; n) - e(x; n)d(x; n).$$

The explicit formulas for the coefficients (61) of the SODE (60) are cumbersome and we will not provide it here. We have obtained explicitly the coefficients $\tilde{\sigma}(x; n)$, $\tilde{\tau}(x; n)$ and $\tilde{\lambda}(x; n)$ by using the algorithm described in [6] and implemented in *Mathematica* [28]. From (61) one can see that the coefficients $\tilde{\sigma}$, $\tilde{\tau}$, $\tilde{\lambda}$ are polynomials on x . Some straightforward calculations with *Mathematica* show that the degree of polynomials $\tilde{\sigma}(x; n)$, $\tilde{\tau}(x; n)$ and $\tilde{\lambda}(x; n)$ is, at most, 6, 5 and 4, respectively.

7 Asymptotic formulas.

Some general asymptotic formulas for Sobolev-type orthogonal polynomials have been obtained in [18]. In order to complete the present work we will provide some of them. We will use them later on when we analyze the distribution of their zeros.

Using the asymptotic formula for the Gamma function [26, formula 8.16 page 88]

$$\Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}}, \quad x \gg 1, x \in \mathbb{R}, \quad (62)$$

we can find the following estimates

$$P_n^{\alpha, \beta}(1) \sim \frac{\sqrt{\pi} n^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+1)2^{n+\alpha+\beta}}, \quad (P_n^{\alpha, \beta})'(1) \sim \frac{\sqrt{\pi} n^{\alpha+\frac{5}{2}}}{\Gamma(\alpha+2)2^{n+\alpha+\beta+1}}, \quad d_{n-1}^2 \sim \frac{\pi}{2^{2n+\alpha+\beta-2}}. \quad (63)$$

From these formulas and (16) we can deduce

$$K_{n-1}^{\alpha, \beta}(1, 1) \sim \frac{n^{2\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+2)2^{\alpha+\beta+1}}, \quad K_{n-1}^{\alpha, \beta}(1, -1) \sim \frac{(-1)^{n+1} n^{\alpha+\beta+1}}{\Gamma(\alpha+1)\Gamma(\beta+1)2^{\alpha+\beta+1}}, \quad (64)$$

$$K_{n-1}^{\alpha, \beta(0,1)}(1, 1) \sim \frac{n^{2\alpha+4}}{\Gamma(\alpha+1)\Gamma(\alpha+3)2^{\alpha+\beta+2}}, \quad K_{n-1}^{\alpha, \beta(0,1)}(1, -1) \sim \frac{(-1)^{n+1} n^{\alpha+\beta+3}}{\Gamma(\beta+2)\Gamma(\alpha+1)2^{\alpha+\beta+2}}, \quad (65)$$

$$K_{n-1}^{\alpha, \beta(1,1)}(1, 1) \sim \frac{(\alpha+2)n^{2\alpha+6}}{\Gamma(\alpha+2)\Gamma(\alpha+4)2^{\alpha+\beta+3}}, \quad K_{n-1}^{\alpha, \beta(1,1)}(1, -1) \sim \frac{(-1)^{n+1} n^{\alpha+\beta+5}}{\Gamma(\alpha+2)\Gamma(\beta+2)2^{\alpha+\beta+3}}, \quad (66)$$

where $x_n \sim y_n$ means $\lim_{n \rightarrow \infty} x_n/y_n = 1$. Using the above estimates and doing some straightforward calculations in (25) we find the following asymptotic expressions for $Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(1)$ and $Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(1)$

$$Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(1) \sim -\frac{\sqrt{\pi}\Gamma(\alpha+3)}{2^{n-1}A_1 n^{\alpha+\frac{3}{2}}}, \quad (Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2})'(1) \sim \frac{\sqrt{\pi}\Gamma(\alpha+4)}{2^{n-2}A_2 n^{\alpha+\frac{7}{2}}}, \quad (67)$$

as well as for the constants defined by (30)-(32)

$$C_n^{\alpha, \beta, A_1, B_1, A_2, B_2} \sim -\frac{(\alpha+1)(\alpha+2)}{A_1 n^2}, \quad D_n^{\alpha, \beta, A_1, B_1, A_2, B_2} \sim \frac{(\alpha+2)(\alpha+3)}{A_2 n^2}, \quad E_n^{\alpha, \beta, A_1, B_1, A_2, B_2} \sim \frac{2(\alpha+2)(\alpha+3)}{A_2 n^4}, \quad (68)$$

$$C_n^{\beta, \alpha, B_1, A_1, B_2, A_2} \sim -\frac{(\beta+1)(\beta+2)}{B_1 n^2}, \quad D_n^{\beta, \alpha, B_1, A_1, B_2, A_2} \sim \frac{(\beta+2)(\beta+3)}{B_2 n^2}, \quad E_n^{\beta, \alpha, B_1, A_1, B_2, A_2} \sim -\frac{2(\beta+2)(\beta+3)}{A_1 n^4}.$$

$$\zeta_n \sim \frac{1}{n^2}, \quad \eta_n \sim \frac{1}{n^2}, \quad \chi_n \sim \frac{1}{n^4}, \quad \omega_n \sim \frac{1}{n^4}. \quad (65)$$

Using (29), (63) and the following asymptotic formula for the Jacobi polynomials

$$\left(P_n^{\alpha,\beta}\right)^{(k)}(1) \sim \frac{\sqrt{\pi}}{\Gamma(\alpha+k+1)} \frac{n^{\alpha+2k+\frac{1}{2}}}{2^{n+\alpha+\beta+k}}, \quad k \geq 2,$$

from (67)-(69), we can give the asymptotic behaviour for $(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})''(1)$,

$$(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})''(1) \sim \frac{\sqrt{\pi}n^{\alpha+\frac{9}{2}}}{2^{n+\alpha+\beta+1}\Gamma(\alpha+5)}. \quad (70)$$

Taking into account the Darboux formula for the asymptotics of the Jacobi polynomials on the interval $\theta \in [\varepsilon, \pi - \varepsilon], 0 < \varepsilon \ll 1$ [27, equation 8.21.10 page 196]

$$a_n P_n^{\alpha,\beta}(\cos \theta) = \frac{\left(\sin \frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}} \left(\cos \frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}}{\sqrt{n\pi}} \cos \left[n\theta + \frac{1}{2}(\alpha + \beta + 1)\theta - \frac{1}{2}\left(\alpha + \frac{1}{2}\right)\pi \right] + O\left(n^{-\frac{3}{2}}\right),$$

where $a_n = \frac{(n+\alpha+\beta+1)}{2^n n!}$, and the explicit expression (29), we can find an explicit expression for the difference between the new polynomials and the classical ones. Thus, we have the following asymptotics on $[-1, 1]$

$$\begin{aligned} \frac{n 2^{n+\alpha+\beta}}{2} \left[Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(\cos \theta) - P_n^{\alpha,\beta}(\cos \theta) \right] &= \frac{1}{\sqrt{\pi n}} \left(\sin \frac{\theta}{2}\right)^{-\alpha-\frac{3}{2}} \left(\cos \frac{\theta}{2}\right)^{-\beta-\frac{1}{2}} \times \\ &\times \left[\frac{1}{2}(\alpha + \beta + 2) \sin \theta \cos(N\theta + \Gamma_1) + \left((\beta + 2) \sin^2 \frac{\theta}{2} - \cos^2 \frac{\theta}{2} (\alpha + 2) \right) \cos(N\theta + \Gamma_2) \right] + O\left(\frac{1}{n^{\frac{3}{2}}}\right), \end{aligned} \quad (71)$$

where $N = n + \frac{1}{2}(\alpha + \beta + 1)$, $\Gamma_1 = -(\alpha + \frac{1}{2})\frac{\pi}{2}$ and $\Gamma_2 = -(\alpha + \frac{3}{2})\frac{\pi}{2}$.

On the other hand, using the well known result

$$\frac{1}{n} \frac{(P_n^{\alpha,\beta})'(z)}{P_n^{\alpha,\beta}(z)} = \frac{1}{\sqrt{z^2 - 1}} + o(1),$$

which is a simple consequence of the Darboux formula (see [27, Eq. 8.21.10 page 196]), the relative asymptotics of the new polynomials is

$$\frac{Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(z)}{P_n^{\alpha,\beta}(z)} \sim 1 + \frac{2(\beta+2)}{n} \left[1 - \sqrt{\frac{x-1}{x+1}} \right] + \frac{2(\alpha+2)}{n} \left[1 - \sqrt{\frac{x+1}{x-1}} \right] + o\left(\frac{1}{n}\right). \quad (72)$$

which holds uniformly outside any closed contour containing the interval $[-1, 1]$.

8 The zeros of $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$.

In this section we will study the properties of the zeros of the Jacobi-Sobolev-type orthogonal polynomials when the masses A_1, B_1, A_2 and B_2 are positive. It is known that for polynomials which are orthogonal on an interval with respect to a positive weight function their zeros are real and simple and lie inside the interval (see [11], [27] and [24]). The study of zeros for Sobolev-type orthogonal polynomials in the non diagonal case was presented in [2]. Since the situation here is very different from one studied in [2] – the polynomials that we have investigated are orthogonal with respect to the inner product (17), i.e., a diagonal case – we need to search their algebraic properties. In fact, we will prove the following theorem.

different, real and simple zeros belonging to the interval $(-1,1)$. For n large enough the two remainder zeros are outside of the interval being one positive and the other one negative.

Proof: Let $x_1, x_2, x_3, \dots, x_k$ be the different real zeros of odd multiplicity of $Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(x)$ on the interval $(-1, 1)$ and $q(x)$ be a polynomial such that

$$q(x) = (x - x_1)(x - x_2)\dots(x - x_k), \quad \text{hence } Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(x)q(x) \geq 0, \quad \forall x \in [-1, 1]. \quad (73)$$

Now define $h(x)$

$$h(x) = (x + a)(x + b)q(x) = (x + a)(x + b)(x - x_1)(x - x_2)\dots(x - x_k), \quad (74)$$

in such a way that $h'(1) = h'(-1) = 0$. Then

$$\begin{cases} a + b + 2 + (a + 1)(b + 1) \frac{q'(1)}{q(1)} = 0, \\ a + b - 2 + (a - 1)(b - 1) \frac{q'(-1)}{q(-1)} = 0, \end{cases} \quad (75)$$

where

$$\begin{cases} \frac{q'(1)}{q(1)} = \frac{1}{1 - x_1} + \frac{1}{1 - x_2} + \dots + \frac{1}{1 - x_k} > 0, \\ \frac{q'(-1)}{q(-1)} = - \left(\frac{1}{1 + x_1} + \frac{1}{1 + x_2} + \dots + \frac{1}{1 + x_k} \right) < 0. \end{cases} \quad (76)$$

From (75)-(76) between all the choices of a and b

$$\begin{aligned} & \begin{cases} -1 < a < 1, \\ -1 < b < 1, \end{cases} \quad \begin{cases} a > 1, \\ b > 1, \end{cases} \quad \begin{cases} a < -1, \\ b < -1, \end{cases} \\ & \begin{cases} -1 < a < 1, \\ b < -1, \quad \text{or} \quad b > 1 \end{cases} \quad (a \leftrightarrow b) \quad \text{and} \quad \begin{cases} a > 1, \\ b < -1, \end{cases} \quad (a \leftrightarrow b) \end{aligned} \quad (77)$$

only the last one holds.

Hence, from

$$\langle h(x), Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(x) \rangle < 0, \quad \text{then } \deg h(x) \geq n, \quad \text{i.e., } k \geq n - 2. \quad (78)$$

To obtain the above one writes in (78) the inner product (17) and makes use of (77).

To prove that $Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(x)$ has one real simple negative zero and one real simple positive zero outside $[-1, 1]$ we use the fact that for n large enough $Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(1) < 0$, $(Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2})'(1) > 0$ (see formula (67)) and the polynomial $Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(x) = x^n + \text{lower degree terms}$ is a continuous convex upward function for $x > 1$, then in some positive value $x > 1$ the polynomial changes its sign. Using the symmetry property (26) and the same argument we prove that the polynomial has one simple real negative zero outside $[-1, 1]$. This immediately implies that $k = n - 2$, hence the proposition holds. \blacksquare

We will denote the zeros of $Q_n^{\alpha, \beta, A_1, B_1, A_2, B_2}(x)$ as $x_{n,1} < -1 < x_{n,2} < \dots < x_{n,n-2} < 1 < x_{n,n}$. Let us study the zeros $x_{n,1}$ and $x_{n,n}$ in more detail.

$$\begin{aligned}
1 < x_{n,n} < 1 + \frac{\sqrt{2^{\alpha+\beta+3}\Gamma(\alpha+3)\Gamma(\alpha+5)}}{n^{\alpha+3}\sqrt{A_1}} - \frac{2^{\alpha+\beta+3}\Gamma(\alpha+4)\Gamma(\alpha+5)}{A_2 n^{2\alpha+8}} + O\left(n^{-3\alpha-13}\right), \\
-1 > x_{n,1} > -1 - \frac{\sqrt{2^{\alpha+\beta+3}\Gamma(\beta+3)\Gamma(\beta+5)}}{n^{\beta+3}\sqrt{B_1}} + \frac{2^{\alpha+\beta+3}\Gamma(\beta+4)\Gamma(\beta+5)}{B_2 n^{2\beta+8}} + O\left(n^{-3\beta-13}\right).
\end{aligned} \tag{79}$$

Proof: Using Taylor's Theorem we have for $x > 1$,

$$\begin{aligned}
Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) &= Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(1) + (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1)(x-1) + \\
&\quad + (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})''(1)\frac{(x-1)^2}{2} + (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'''(\xi)\frac{(x-1)^3}{6},
\end{aligned} \tag{80}$$

where $1 < \xi < x$. From (67) for n large enough, $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(1)$ is negative while $(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1)$ is positive. Moreover, $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$ is a convex upward function for $x > 1$ and has its first inflection point (from the right) somewhere at $x < 1$. Then for all $x > 1$, $(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'''(x) \geq 0$. Hence

$$\begin{aligned}
Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) &\geq \frac{(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})''(1)}{2}x^2 + \left[(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1) - (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})''(1) \right]x + \\
&\quad + \frac{(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})''(1)}{2} - (Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1) + Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(1),
\end{aligned}$$

and the zero $x_{n,n}$ is located between the zeros of the quadratic polynomial on the right hand of the previous expression. If we denote

$$x_{1,2} = 1 - \frac{(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1)}{(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})''(1)} \pm \sqrt{\left[\frac{(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1)}{(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})''(1)} \right]^2 - 2\frac{Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(1)}{(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})''(1)}} \tag{81}$$

and taking into account (67) and (70) we get

$$\begin{aligned}
\frac{(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(1)}{(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})''(1)} &\sim \frac{2^{\alpha+\beta+3}\Gamma(\alpha+4)\Gamma(\alpha+5)}{A_2 n^{2\alpha+8}}, \\
\frac{Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(1)}{(Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2})''(1)} &\sim -\frac{2^{\alpha+\beta+2}\Gamma(\alpha+3)\Gamma(\alpha+5)}{A_1 n^{2\alpha+6}}.
\end{aligned} \tag{82}$$

Thus, when $(n \rightarrow \infty)$ x_2 behaves asymptotically as follows

$$x_2 \sim 1 + \frac{\sqrt{2^{\alpha+\beta+3}\Gamma(\alpha+3)\Gamma(\alpha+5)}}{n^{\alpha+3}\sqrt{A_1}} - \frac{2^{\alpha+\beta+3}\Gamma(\alpha+4)\Gamma(\alpha+5)}{A_2 n^{2\alpha+8}} + O\left(n^{-3\alpha-13}\right). \tag{83}$$

In the same way, using the symmetry property (26) we find the speed of convergence for $x_{n,1}$, then (79) holds. ■

In this section we will apply the so-called semiclassical or WKB approximation (see [7], [29] and references therein) to find the WKB density of zeros of the polynomials $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$. Let us denote these zeros by $\{x_{n,i}\}_{i=1}^n$. Then the corresponding distribution function of zeros is given by

$$\nu_n(x) = \frac{1}{n} \sum_{i=1}^n \delta(x - x_{n,i}). \quad (84)$$

Here we will use the method presented in [29] in order to obtain the WKB density of zeros, which gives an approximate analytic expression for the density of zeros of the solutions of any linear second order differential equation with polynomial coefficients. In particular we will consider (60)

$$\tilde{\sigma}(x)y'' + \tilde{\tau}(x)y' + \tilde{\lambda}(x)y = 0. \quad (85)$$

The key step is the following

Theorem 3 ([29])

Let $S(x)$ and $\epsilon(x)$ be the functions

$$S(x) = \frac{1}{4\tilde{\sigma}(x)^2} \left[2\tilde{\sigma}(x) \left(2\tilde{\lambda}(x) - \tilde{\tau}'(x) \right) + \tilde{\tau}(x) \left(2\tilde{\sigma}'(x) - \tilde{\tau}(x) \right) \right], \quad (86)$$

$$\epsilon(x) = \frac{1}{4[S(x)]^2} \left\{ \frac{5[S'(x)]^2}{4[S(x)]} - S''(x) \right\} = \frac{P(x, n)}{Q(x, n)}, \quad (87)$$

where $P(x, n)$ and $Q(x, n)$ are polynomials in x as well as in n . If the condition $\sup_{x \in X} |\epsilon(x)| \ll 1$ holds, then the semiclassical or WKB density of zeros of the solutions of (85) is given by

$$\rho_{WKB}(x) = \frac{1}{\pi} \sqrt{S(x)}, \quad x \in X \subseteq \mathbb{R}, \quad (88)$$

in every interval X where the function $S(x)$ is positive.

Using the above algorithm, the computations have been performed by using the symbolic computer algebra package *Mathematica* [28]. First of all we check the conditions of the Theorem finding that in the considered case $\epsilon \sim n^{-1}$, so the Theorem can be applied for n large enough. The explicit expression for $\rho_{WKB}(x)$ given by (88) is extremely large and we will omit it here. It is straightforward to see that if we take the limit $A_1, A_2, B_1, B_2 \rightarrow 0$ in the resulting expression for $\rho_{WKB}(x)$ we recover the classical expression for the Jacobi polynomials [29]. We will provide here some graphics for the normalized $\rho_{WKB}(x)$ function. In Figure 1 the WKB density of zeros for the Jacobi-Sobolev-type orthogonal polynomials appears. We have used the formulas (60), (69), (86) and (88) and plotted the normalized Density function for $n = 10^4$ in four different cases with several values of the parameters α and β ($\alpha = \beta = 0$, $\alpha = \beta = -\frac{1}{2}$, $\alpha = \beta = 5$ and nonsymmetric case $\alpha = 0$ and $\beta = 1$). In Figure 2 appears the WKB density of zeros for the same values of α and β and $n = 10^5$. In Figure 3 we represent the WKB density of zeros for $n = 10^6$ and the same values of the parameters α and β . Finally, in Figure 4 is shown $\rho_{WKB}(x)$, for $n = 10^7$ with the above values of α and β . Clear, in each Figure from the bottom to the top, is distinguishible the case $\alpha = \beta = 0$, while the remaining cases behave almost equal. Some numerical tests based on the computation of the number N of zeros in the interval $(-\frac{1}{10}, \frac{1}{10})$ by using the expression $N \approx \int_{-1/10}^{1/10} \rho_{WKB}(x) dx$ for both families of orthogonal polynomials $P_n^{\alpha,\beta}(x)$ and $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$ show that their *global* spectral properties are the same. This result is in accordance with the next one.

zero of $Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)$. Then

$$\nu_n \xrightarrow{*} \frac{1}{\pi\sqrt{1-x^2}},$$

in the weak star topology.

Proof: From (6) and (69), we get

$$\begin{aligned} A_n &= 1 - \frac{\alpha + \beta + 4}{n} + o\left(\frac{1}{n}\right), & E_n &= \frac{1 + 5\alpha + \alpha^2 - 5\beta - \beta^2}{n^4} + o\left(\frac{1}{n^4}\right), \\ B_n &= \frac{2(\beta - \alpha)}{n^2} + \frac{2(\alpha(19 + \alpha(7 + \alpha)) + 2\alpha(5 + \alpha)\beta - 5\beta^2 - \beta^3 + 3(4 + \beta))}{n^4} + o\left(\frac{1}{n^4}\right) \\ C_n &= \frac{2}{n^2}(\alpha + \beta + 4) + o\left(\frac{1}{n^2}\right), & D_n &= \frac{1}{2n^2}(\alpha + \beta + 6) + o\left(\frac{1}{n^2}\right), \\ F_n &= \frac{12 + \alpha(5 + \alpha) + \beta(5 + \beta)}{2n^4} + o\left(\frac{1}{n^4}\right), & G_n &= \frac{2(12 + \alpha(5 + \alpha) + \beta(5 + \beta))}{n^4} + o\left(\frac{1}{n^4}\right). \end{aligned}$$

Using (47),

$$\begin{aligned} \|Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)\|_{[-1,1]} &\leq A_n \|P_n^{\alpha,\beta}(x)\|_{[-1,1]} + nB_n \|P_{n-1}^{\alpha+1,\beta+1}(x)\|_{[-1,1]} \\ &+ nC_n \|P_n^{\alpha+1,\beta+1}(x)\|_{[-1,1]} + nD_n \|P_{n-2}^{\alpha+1,\beta+1}(x)\|_{[-1,1]} + n(n-1)E_n \|P_{n-2}^{\alpha+2,\beta+2}(x)\|_{[-1,1]} \\ &+ n(n-1)F_n \|P_{n-1}^{\alpha+2,\beta+2}(x)\|_{[-1,1]} + n(n-1)G_n \|P_{n-3}^{\alpha+2,\beta+2}(x)\|_{[-1,1]}, \end{aligned} \quad (89)$$

where $\|\cdot\|_{[-1,1]}$ denotes the sup-norm in the interval $[-1, 1]$.

Because of $\|P_n^{\alpha,\beta}(x)\|_{[-1,1]}^{\frac{1}{n}} \leq \frac{1}{2}$ (see [27]), we deduce

$$\overline{\lim}_{n \rightarrow \infty} \|Q_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x)\|_{[-1,1]}^{\frac{1}{n}} \leq \frac{1}{2}. \quad (90)$$

Thus, from Theorem 2.1 in [10]

$$\nu_n \xrightarrow{*} \frac{1}{\pi\sqrt{1-x^2}}. \quad (91)$$

■

ACKNOWLEDGEMENTS

Part of this work was provided during the stay of the second author in the University of Amsterdam. He is very grateful to the Department of Mathematics of the University of Amsterdam for his kind hospitality. The research of the first author (JA) was supported by a grant of Ministerio de Educación y Cultura (MEC) of Spain. The research of the three first authors (JA, RAN and FM) was supported by Dirección General de Enseñanza Superior (DGES) of Spain under grant PB 96-0120-C03-01. The authors are very grateful to the unknown referees for their helpful remarks and for help us to correct some missprints and errors and significantly improve the paper.

- [1] M. Alfaro, F. Marcellán, M. Rezola, and A. Ronveaux: *On orthogonal polynomials of Sobolev type: Algebraic properties and zeros*. SIAM J. Math. Anal. **23** (1992), 737-757.
- [2] M. Alfaro, F. Marcellán, M. Rezola, and A. Ronveaux: *Sobolev-type orthogonal polynomials: The nondiagonal case*. J. of Approx. Theory. **83** (1995), 266-287.
- [3] M. Alfaro, F. Marcellán, and M. Rezola: *Estimates for Jacobi-Sobolev type orthogonal polynomials* (1996). Publicaciones del Seminario Matemático García Galdeano. Universidad de Zaragoza. Serie II.
- [4] R. Álvarez-Nodarse and F. Marcellán: *On the Modifications of Classical Orthogonal Polynomials: The Symmetric Case*. Approx. Th. and Appl. (1997). (In press)
- [5] R. Álvarez-Nodarse, J. Arvesú, and F. Marcellán : *A Generalization of the Jacobi-Koornwinder Polynomials*. (Submitted) Preprint Dept. Matemáticas (Univ. Carlos III de Madrid) MA/UC3M/6/1997 (1997)
- [6] R. Álvarez-Nodarse and A. Zarzo: *On some modifications of classical orthogonal polynomials: Differential and spectral properties*. Preprint 1997.
- [7] E.R. Arriola, A. Zarzo, and J.S. Dehesa: *Spectral Properties of the biconfluent Heun differential equation*. J. Comput. Appl. Math. **37** (1991), 161-169.
- [8] H. Bavinck and H. G. Meijer: *Orthogonal Polynomials with Respect to a Symmetric Inner Product Involving Derivatives*. Appl. Analysis **33**, (1989), 103-117.
- [9] H. Bavinck and H. G. Meijer: *On Orthogonal Polynomials with Respect to an Inner Product Involving Derivatives: Zeros and Recurrence Relations*. Indag. Math. (N.S) **1**, (1990), 7-14.
- [10] H. P. Blatt, E. B. Saff and M. Simkani: *Jentzsch-Szegő type Theorems for the zeros of best approximants*. J. London Math. Soc. **38**, (1988), 307-316.
- [11] T. S. Chihara: *Orthogonal polynomials and measures with end point masses*. Rocky Mount. J. of Math. **15**,(1985), 705-719.
- [12] T.S. Chihara: **An Introduction to Orthogonal Polynomials**. (Gordon and Breach, New York, 1978).
- [13] N. Draïdi: *Sur l'adjonction de deux masses de Dirac à une forme linéaire régulière quelconque*. Thèse Doctorat de l'Université Pierre et Marie Curie. Paris, 1990.
- [14] N. Draïdi and P. Maroni: *Sur l'adjonction de deux masses de Dirac à une forme régulière quelconque*. In **Polinomios ortogonales y sus aplicaciones**. A. Cachafeiro and E. Godoy Eds. Actas del V Simposium. Universidad de Santiago. Vigo 1988, 83-90.
- [15] W.D. Evans, L.L. Littlejohn, F. Marcellán, C. Markett, and A. Ronveaux: *On recurrence relations for Sobolev orthogonal polynomials*. SIAM J. Math. Anal. **26**, (1995), 446-467.
- [16] R. Koekoek and H.G. Meijer: *A generalization of Laguerre polynomials*. SIAM J. Math. Anal. **24**, (1993), 768-782.
- [17] T. H. Koornwinder: *Orthogonal polynomials with weight function $(1 - x)^\alpha(1 + x)^\beta + M\delta(x + 1) + N\delta(x - 1)$* . Canad. Math. Bull, **27**, (1984), 205-214.

- with respect to a discrete Sobolev inner product. Constr. Approx. **11**, (1995), 107-137.
- [19] F. Marcellán and P. Maroni: *Sur l'adjonction d'une masse de Dirac à une forme régulière et semi-classique*. Ann. Mat. Pura ed Appl., **IV**, CLXII, (1992), 1-22.
- [20] F. Marcellán and A. Ronveaux : *On a class of polynomials orthogonal with respect to a Sobolev inner product*. Indag. Math. (N.S.) **1**, (1990), 451-464.
- [21] F. Marcellán, T.E. Pérez, and M.A. Piñar: *Regular Sobolev type orthogonal polynomials: The Bessel case*. Rocky Mount. J. of Math. **25**, (1995), 1431-1457.
- [22] P. Nevai: *Orthogonal Polynomials*. Memoirs of the Amer. Math. Soc. **213**, Amer. Math. Soc. Providence, Rhode Island, 1979.
- [23] A. F. Nikiforov and V. B. Uvarov: **Special Functions of Mathematical Physics**. Birkhäuser Verlag, Basel, 1988.
- [24] E. M. Nikishin and V. N. Sorokin: **Rational Approximations and Orthogonality**. Trans. of Math. Monographs, **92**, Amer. Math. Soc., Providence, Rhode Island, 1991.
- [25] M.A. Piñar: *Polinomios ortogonales tipo Sobolev. Aplicaciones*. Doctoral Dissertation, University of Granada, Spain, 1992.
- [26] F. W. J. Olver: **Asymptotics and Special Functions**. Academic Press Inc., New York, 1974.
- [27] G. Szegő: **Orthogonal Polynomials**. Amer. Math. Soc. Colloq. Publ., **23**, Amer. Math. Soc., Providence, Rhode Island, 1975 (4th edition).
- [28] S. Wolfram: **MATHEMATICA. A system for doing Mathematics by Computer**. Addison-Wesley Publishing Co., New York, 1991.
- [29] A. Zarzo and J.S. Dehesa: *Spectral Properties of solutions of hypergeometric-type differential equations*. J. Comput. Appl. Math. **50** (1994), 613-623.

J. Arvesú[†] E-mail: jarvesu@math.uc3m.es

R. Álvarez-Nodarse^{†,*} E-mail: nodar@math.uc3m.es

F. Marcellán[†] E-mail: pacomarc@ing.uc3m.es

K. Pan^{**} E-mail: pan@dominic.barry.edu

[†] Departamento de Matemáticas. Escuela Politécnica Superior.
Universidad Carlos III de Madrid. Butarque 15, 28911, Leganés, Madrid.

* Instituto Carlos I de Física Teórica y Computacional
Universidad de Granada E-18071, Granada

** Department of Mathematics and Computer Science,
Barry University, Miami Shores, Florida 33161-6695. USA.

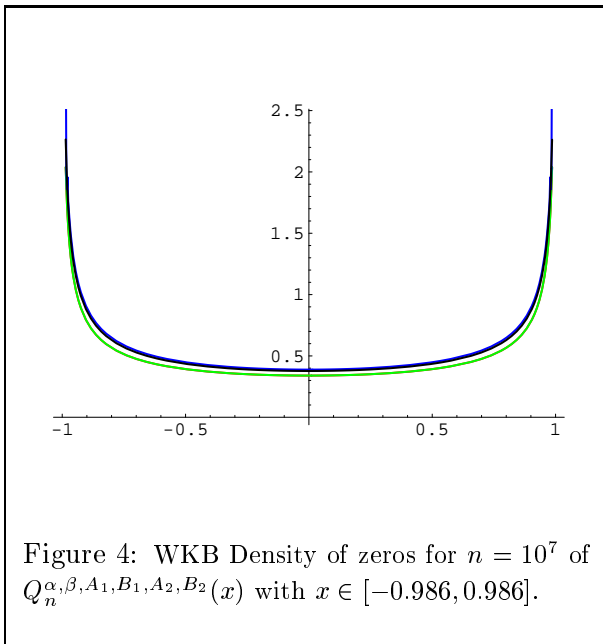
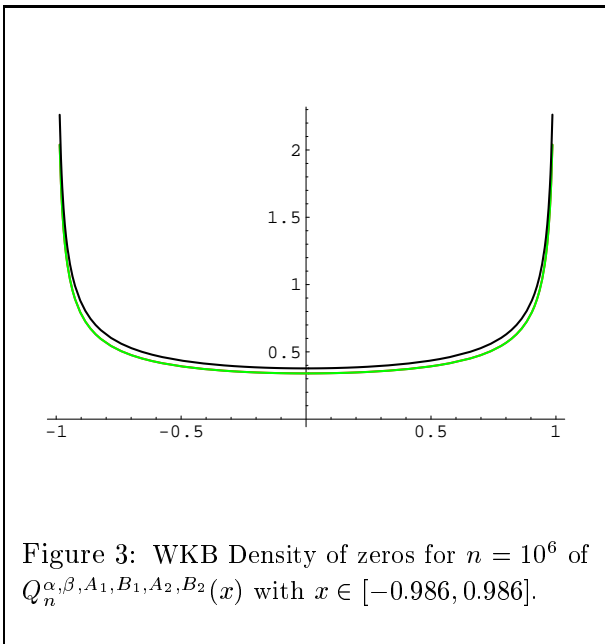
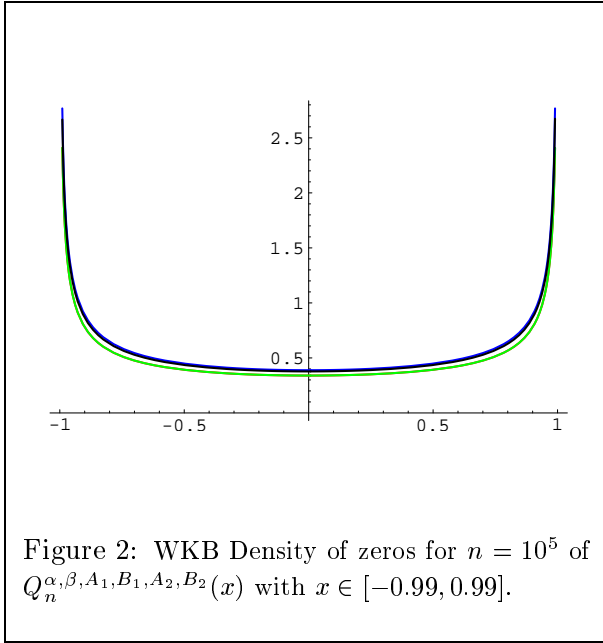
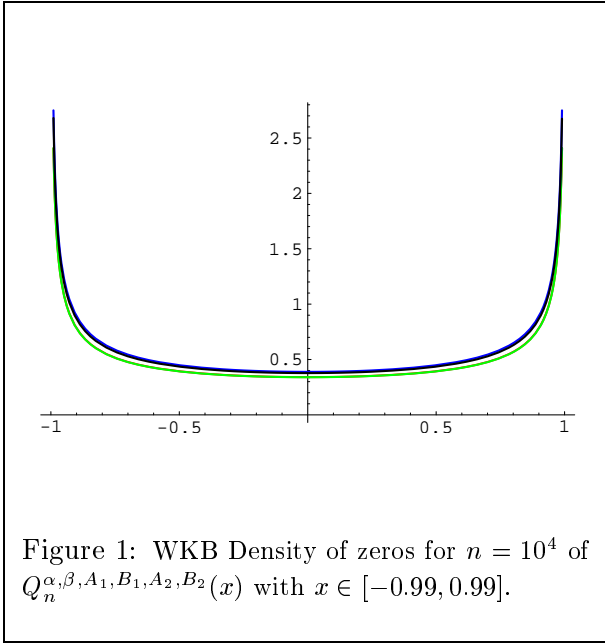


Figure 5: Comparison of the numerical computation results.