

Existence for the α -patch model and the QG sharp front in Sobolev spaces

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Abstract

We consider a family of contour dynamics equations depending on a parameter α with $0 < \alpha \leq 1$. The vortex patch problem of the 2-D Euler equation is obtained taking $\alpha \rightarrow 0$, and the case $\alpha = 1$ corresponds to a sharp front of the QG equation. We prove local-in-time existence for the family of equations in Sobolev spaces.

1 Introduction

The 2-D QG equation provides particular solutions of the evolution of the temperature from a general quasi-geostrophic system for atmospheric and oceanic flows. This equation is derived considering small Rossby and Ekman numbers and constant potential vorticity (see [12] for more details). It reads

$$\begin{aligned}\theta_t(x, t) + u(x, t) \cdot \nabla \theta(x, t) &= 0, \quad x \in \mathbb{R}^2, \\ \theta(x, 0) &= \theta_0(x).\end{aligned}\tag{1}$$

Here θ is the temperature of the fluid, the incompressible velocity u is expressed by means of the stream function as follows

$$u = \nabla^\perp \psi = (-\partial_{x_2} \psi, \partial_{x_1} \psi),$$

and the relation between the stream function and the temperature is given by

$$\theta = -(-\Delta)^{1/2} \psi.$$

This system have been considered in frontogenesis, where the dynamics of hot and cold fluids is studied together with the formation and the evolution of fronts (see [4], [5], [8], [11]).

From a mathematical point of view, this equation have been presented as a two dimensional model of the 3-D Euler equation due to their strong analogies (see [4]), being the formation of singularities for a regular initial data an open problem (see [4], [6], [7]). Nevertheless the QG equation has global in time weak solutions due to an extra cancellation (see [13]). A few sparse results are known about weak solutions of the 2-D and 3-D Euler equation in its primitive-variable form.

An outstanding kind of weak solutions for the QG equation are those in which the temperature takes two different values in complementary domains, modelling the evolution of a sharp front as follows

$$\theta(x_1, x_2, t) = \begin{cases} \theta_1, & \Omega(t) \\ \theta_2, & \mathbb{R}^2 \setminus \Omega(t). \end{cases} \quad (2)$$

In this work we study a problem similar to the 2-D vortex patch problem, where the vorticity of the 2-D Euler equation is given by a characteristic function of a domain, and it is considered the regularity of the free boundary of such domain. For this equation the vorticity satisfies

$$\begin{aligned} w_t(x, t) + u(x, t) \cdot \nabla w(x, t) &= 0, & x \in \mathbb{R}^2, \\ w(x, 0) &= w_0(x), \end{aligned} \quad (3)$$

in a weak sense, and the velocity is given by the Biot-Savart law or analogously

$$u = \nabla^\perp \psi, \quad \text{and} \quad w = \Delta \psi.$$

Chemin [3] proved global-in-time regularity for the free boundary using paradifferential calculus. A simpler proof can be found in [1] due to Bertozzi and Constantin.

We point out that in the QG equation, the velocity is determined from the temperature by singular integral operators (see [15]) as follows

$$u = (-R_2 \theta, R_1 \theta), \quad (4)$$

where R_1 and R_2 are the Riesz transforms, making the system more singular than (3).

Rodrigo [14] proposed the problem of the evolution of a sharp front for the QG equation. He derived the velocity on the free boundary in the normal direction, and proved local-existence and uniqueness for a periodic C^∞ front, i.e.

$$\theta(x_1, x_2, t) = \begin{cases} \theta_1, & \{f(x_1, t) > x_2\} \\ \theta_2, & \{f(x_1, t) \leq x_2\}, \end{cases}$$

with $f(x_1, t)$ periodic, using the Nash-Moser iteration.

In this paper we study a family of contour dynamics equation given by weak solutions of the following system

$$\begin{aligned} \theta_t + u \cdot \nabla \theta &= 0, & x \in \mathbb{R}^2, \\ u &= \nabla^\perp \psi, & \theta = -(-\Delta)^{1-\alpha/2} \psi, & 0 < \alpha \leq 1, \end{aligned} \quad (5)$$

where the active scalar $\theta(x, t)$ satisfies (2). We notice that the case $\alpha = 0$ is the 2-D vortex patch problem, and $\alpha = 1$ correspond to the sharp front for the QG equation.

This system was introduced by Córdoba, Fontelos, Mancho and Rodrigo in [9], where they present a proof of local-existence for a periodic C^∞ front, and show evidence of singularities in finite time. The singular scenario is due to two patches collapse point-wise.

Here we give a proof of local-existence of the system (5) where the solution satisfies (2), with the boundary $\partial\Omega(t)$ given by the curve

$$\partial\Omega(t) = \{x(\gamma, t) = (x_1(\gamma, t), x_2(\gamma, t)) : \gamma \in [-\pi, \pi]\},$$

and $x(\gamma, t)$ belongs to a Sobolev space. In the cases $0 < \alpha < 1$ we show uniqueness.

It is well-known (see [10] and [14]) that in this kind of contour dynamics equations, the velocity in the tangential direction only moves the particles on the boundary. Therefore we do not alter the shape of the contour if we change the tangential component of the velocity; i.e., we are making a change on the parametrization. In the most singular case, $\alpha = 1$ or the QG equation, we need to change the velocity in the tangential direction in order to get existence in the Sobolev spaces. We take a tangential velocity in such a way that $|\partial_\gamma x(\gamma, t)|$ satisfies

$$|\partial_\gamma x(\gamma, t)|^2 = A(t),$$

and does not depend on γ . We would like to cite the work of Hou, Lowengrub and Shelley [10] in which this idea was used to study a contour dynamics problem.

We notice that in order to get a non-singular normal velocity of the curve for $0 < \alpha \leq 1$ (see [9] and [14]), we need a one to one curve, and parameterized in such a way that

$$|\partial_\gamma x(\gamma, t)|^2 > 0.$$

Rigorously, we need that

$$\frac{|x(\gamma, t) - x(\gamma - \eta, t)|}{|\eta|} > 0, \quad \forall \gamma, \eta \in [-\pi, \pi], \quad (6)$$

therefore we give an initial data satisfying this property, and we prove that this condition is satisfied locally in time. We point out the importance to take into account the evolution of this quantity due to the numerical simulations in [9].

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2 The Contour Equation

In this section we deduce the family of contour equations in term of the free boundary $x(\gamma, t)$. We consider the equations given by the system (1), with a velocity satisfying

$$u(x, t) = \nabla^\perp \psi(x, t), \quad (7)$$

for the stream function it follows

$$\theta = -(-\Delta)^{1-\alpha/2} \psi, \quad (8)$$

and the active scalar fulfills

$$\theta(x_1, x_2, t) = \begin{cases} \theta_1, & \Omega(t) \\ \theta_2, & \mathbb{R}^2 \setminus \Omega(t). \end{cases} \quad (9)$$

The boundary of $\Omega(t)$ is given by the curve

$$\partial\Omega(t) = \{x(\gamma, t) = (x_1(\gamma, t), x_2(\gamma, t)) : \gamma \in [-\pi, \pi] = \mathbb{T}\},$$

with $x(\gamma, t)$ one to one. Due to the identity (9), we find that

$$\nabla^\perp \theta = (\theta_1 - \theta_2) \partial_\gamma x(\gamma, t) \delta(x - x(\gamma, t)),$$

where δ is the Dirac distribution. Using (7) and (8), we got that

$$u = -(-\Delta)^{\alpha/2-1} \nabla^\perp \theta.$$

Due to the integral operators $-(-\Delta)^{\alpha/2-1}$ are Riesz potentials (see [15]), using the last to identities we obtain that

$$u(x, t) = -\frac{\Theta_\alpha}{2\pi} \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma - \eta, t)}{|x - x(\gamma - \eta, t)|^\alpha} d\eta, \quad (10)$$

for $x \neq x(\gamma, t)$, and $\Theta_\alpha = (\theta_1 - \theta_2)\Gamma(\alpha/2)/2^{1-\alpha}\Gamma(2 - \alpha/2)$. We notice that for $\alpha = 1$, if $x \rightarrow x(\gamma, t)$ the integral in (10) is divergent. As we have showed before, we are interested in the normal velocity of the systems. Then we have that using the identity (10), and taking the limit as follows

$$u(x, t) \cdot \partial_\gamma^\perp x(\gamma, t), \quad x \rightarrow x(\gamma, t), \quad (11)$$

we obtain

$$u(x(\gamma, t), t) \cdot \partial_\gamma^\perp x(\gamma, t) = -\frac{\Theta_\alpha}{2\pi} \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma - \eta, t) \cdot \partial_\gamma^\perp x(\gamma, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|^\alpha} d\eta. \quad (12)$$

This identity is well defined for $0 < \alpha \leq 1$ and a one to one curve $x(\gamma, t)$. Due to the fact that tangential velocity does not change the shape of the boundary, we fix the contour α -patch equations as follows

$$\begin{aligned} x_t(\gamma, t) &= \frac{\Theta_\alpha}{2\pi} \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|^\alpha} d\eta, \quad 0 < \alpha \leq 1, \\ x(\gamma, 0) &= x_0(\gamma). \end{aligned} \quad (13)$$

Seeing the equation (10), we show that the velocity in QG presents a logarithmic divergence in the tangential direction on the boundary. Nevertheless it belongs to $L^p(\mathbb{R}^2)$ for $1 < p < \infty$, and to the bounded mean oscillation space (see [15] for the definition of the BMO space). In QG the velocity is given by (4), and writing the temperature in the following way

$$\theta(x, t) = (\theta_1 - \theta_2)X_{\Omega(t)}(x) + \theta_2,$$

we find that

$$u(x, t) = (\theta_1 - \theta_2)(-R_2(X_{\Omega(t)}), R_1(X_{\Omega(t)})).$$

Using that $X_{\Omega(t)} \in L^p(\mathbb{R}^2)$ for $1 \leq p \leq \infty$, we conclude de argument. In particular the energy of the system is conserved due to $\|u\|_{L^2}(t) = |\theta_1 - \theta_2| |\Omega(t)|^{1/2}$, and the area of $\Omega(t)$ is constant in time.

3 Weak solutions for the α -system

In this section we show that if $\theta(x, t)$ is defined by (9) and the curve $x(\gamma, t)$ is convected by the normal velocity (12), then $\theta(x, t)$ is a weak solution of the system (5) and conversely. We give the definition of weak solution below.

Definition 3.1 *The active scalar θ is a weak solution of the α -system if for any function $\varphi \in C_c^\infty(\mathbb{R}^2 \times (0, T))$, we have*

$$\int_0^T \int_{\mathbb{R}^2} \theta(x, t) (\partial_t \varphi(x, t) + u(x, t) \cdot \nabla \varphi(x, t)) dx dt = 0, \quad (14)$$

where the incompressible velocity u is given by (7), and the stream function satisfies (8).

Then

Proposition 3.2 *If $\theta(x, t)$ is defined by (9), and the curve $x(\gamma, t)$ satisfies (6) and (12), then $\theta(x, t)$ is a weak solution of the α -system. Furthermore, if $\theta(x, t)$ is a weak solution of the α -system given by (9), and $x(\gamma, t)$ satisfies (6), then $x(\gamma, t)$ verifies (12).*

Proof: Let $\theta(x, t)$ be a weak solution of the α -system defined by (9). Integrating by parts we have

$$\begin{aligned} I &= \int_0^T \int_{\mathbb{R}^2} \theta(x, t) \partial_t \varphi(x, t) dx dt = \theta_1 \int_0^T \int_{\Omega(t)} \partial_t \varphi(x, t) dx dt + \theta_2 \int_0^T \int_{\Omega(t) \setminus \mathbb{R}^2} \partial_t \varphi(x, t) dx dt \\ &= -(\theta_1 - \theta_2) \int_0^T \int_{\mathbb{T}} \varphi(x(\gamma, t), t) x_t(\gamma, t) \cdot \partial_\gamma^\perp x(\gamma, t) d\gamma dt. \end{aligned}$$

On the other hand, we obtain

$$J = \int_0^T \int_{\mathbb{R}^2} \theta u \cdot \nabla \varphi dx dt = \theta_1 \int_0^T \int_{\Omega} u \cdot \nabla \varphi dx dt + \theta_2 \int_0^T \int_{\mathbb{R}^2 \setminus \Omega} u \cdot \nabla \varphi dx dt.$$

Taking

$$\Omega_1^\varepsilon(t) = \{x \in \Omega : \text{dist}(x, \Omega(t)) \geq \varepsilon\},$$

and

$$\Omega_2^\varepsilon(t) = \{x \in \mathbb{R}^2 \setminus \Omega : \text{dist}(x, \mathbb{R}^2 \setminus \Omega(t)) \geq \varepsilon\},$$

we have that $J^\varepsilon \rightarrow J$ if $\varepsilon \rightarrow 0$, where J^ε is given by

$$J^\varepsilon = \theta_1 \int_0^T \int_{\Omega_1^\varepsilon(t)} u \cdot \nabla \varphi dx dt + \theta_2 \int_0^T \int_{\Omega_2^\varepsilon(t)} u \cdot \nabla \varphi dx dt.$$

Integrating by part in J^ε , using that the velocity is divergence free, and taking the limit as in (11), we obtain

$$\begin{aligned} J &= (\theta_1 - \theta_2) \int_0^T \int_{\mathbb{T}} \varphi(x(\gamma, t), t) u(x(\gamma, t), t) \cdot \partial_\gamma^\perp x(\gamma, t) d\gamma dt \\ &= -(\theta_1 - \theta_2) \frac{\Theta_\alpha}{2\pi} \int_0^T \int_{\mathbb{T}} \varphi(x(\gamma, t), t) \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma - \eta, t) \cdot \partial_\gamma^\perp x(\gamma, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|^\alpha} d\eta \right) d\gamma dt. \end{aligned}$$

We have that $I + J = 0$ using (14), and it follows

$$\int_0^T \int_{\mathbb{T}} f(\gamma, t) \left(x_t(\gamma, t) \cdot \partial_\gamma^\perp x(\gamma, t) + \frac{\Theta_\alpha}{2\pi} \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma - \eta, t) \cdot \partial_\gamma^\perp x(\gamma, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|^\alpha} d\eta \right) d\gamma dt = 0,$$

for $f(\gamma, t)$ periodic in γ . We find that (12) is satisfied. Following the same arguments it is easy to check that if $x(\gamma, t)$ satisfies (12), then θ is a weak solution given by (9).

4 Local well-posedness for $0 < \alpha < 1$

In this section we prove existence and uniqueness for the contour equation in the cases $0 < \alpha < 1$. We denote the Sobolev spaces by $H^k(\mathbb{T})$, with norms

$$\|x\|_{H^k}^2 = \|x\|_{L^2}^2 + \|\partial_\gamma^k x\|_{L^2}^2,$$

and the spaces $C^k(\mathbb{T})$ with

$$\|x\|_{C^k} = \max_{j \leq k} \|\partial_\gamma^j x\|_{L^\infty}.$$

We need that the curve satisfies

$$\frac{|x(\gamma, t) - x(\gamma - \eta, t)|}{|\eta|} > 0, \quad \forall \gamma, \eta \in [-\pi, \pi], \quad (15)$$

then we define

$$F(x)(\gamma, \eta, t) = \frac{|\eta|}{|x(\gamma, t) - x(\gamma - \eta, t)|} \quad \forall \gamma, \eta \in [-\pi, \pi], \quad (16)$$

with

$$F(x)(\gamma, 0, t) = \frac{1}{|\partial_\gamma x(\gamma, t)|}.$$

The main theorem in this section is the following

Theorem 4.1 *Let $x_0(\gamma) \in H^k(\mathbb{T})$ for $k \geq 3$ with $F(x_0)(\gamma, \eta) < \infty$. Then there exists a time $T > 0$ so that there is a unique solution to (13) for $0 < \alpha < 1$ in $C^1([0, T]; H^k(\mathbb{T}))$ with $x(\gamma, 0) = x_0(\gamma)$.*

Proof: We can choose $\Theta_\alpha = 2\pi$ without loss of generality, obtaining the following equation

$$\begin{aligned} x_t(\gamma, t) &= \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|^\alpha} d\eta, \quad 0 < \alpha < 1, \\ x(\gamma, 0) &= x_0(\gamma). \end{aligned} \quad (17)$$

We present the proof for $k = 3$, being analogous for $k > 3$, using energy estimates (see [2] for more details). We ignore the time dependence to simplify the notation in some terms. Considering the quantity

$$\begin{aligned}
\int_{\mathbb{T}} x(\gamma) \cdot x_t(\gamma) d\gamma &= \int_{\mathbb{T}} \int_{\mathbb{T}} x(\gamma) \cdot \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\eta)}{|x(\gamma) - x(\eta)|^\alpha} d\eta d\gamma \\
&= - \int_{\mathbb{T}} \int_{\mathbb{T}} x(\eta) \cdot \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\eta)}{|x(\gamma) - x(\eta)|^\alpha} d\eta d\gamma \\
&= \frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{(x(\gamma) - x(\eta)) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\eta))}{|x(\gamma) - x(\eta)|^\alpha} d\eta d\gamma \\
&= \frac{1}{2(2-\alpha)} \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma |x(\gamma) - x(\gamma - \eta)|^{2-\alpha} d\gamma d\eta \\
&= 0,
\end{aligned} \tag{18}$$

we obtain

$$\frac{d}{dt} \|x\|_{L^2}(t) = 0. \tag{19}$$

We decompose as follows

$$\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^3 x_t(\gamma) d\gamma = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &= \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \frac{\partial_\gamma^4 x(\gamma) - \partial_\gamma^4 x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|^\alpha} d\eta d\gamma, \\
I_2 &= 3 \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)) \partial_\gamma (|x(\gamma) - x(\gamma - \eta)|^{-\alpha}) d\eta d\gamma, \\
I_3 &= 3 \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)) \partial_\gamma^2 (|x(\gamma) - x(\gamma - \eta)|^{-\alpha}) d\eta d\gamma, \\
I_4 &= \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \partial_\gamma^3 (|x(\gamma) - x(\gamma - \eta)|^{-\alpha}) d\eta d\gamma.
\end{aligned}$$

Operating as in (18), the term I_1 becomes

$$\begin{aligned}
I_1 &= \frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} (\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)) \cdot \frac{\partial_\gamma^4 x(\gamma) - \partial_\gamma^4 x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|^\alpha} d\eta d\gamma \\
&= \frac{1}{4} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{\partial_\gamma |\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)|^2}{|x(\gamma) - x(\gamma - \eta)|^\alpha} d\eta d\gamma \\
&= \frac{\alpha}{4} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)|^2 (x(\gamma) - x(\gamma - \eta)) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta))}{|x(\gamma) - x(\gamma - \eta)|^{\alpha+2}} d\eta d\gamma.
\end{aligned}$$

One finds that

$$I_1 \leq \frac{\alpha}{4} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)|^2 |\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)|}{|x(\gamma) - x(\gamma - \eta)|^{\alpha+1}} d\eta d\gamma,$$

and due to the inequality $|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)| |\eta|^{-1} \leq \|x\|_{C^2}$, it follows

$$\begin{aligned} I_1 &\leq \frac{\alpha}{4} \|x\|_{C^2} \int_{\mathbb{T}} \int_{\mathbb{T}} |\eta|^{-\alpha} |F(x)(\gamma, \eta)|^{1+\alpha} |\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)|^2 d\eta d\gamma \\ &\leq \frac{1}{2} \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} \int_{\mathbb{T}} |\eta|^{-\alpha} \int_{\mathbb{T}} (|\partial_\gamma^3 x(\gamma)|^2 + |\partial_\gamma^3 x(\gamma - \eta)|^2) d\gamma d\eta \\ &\leq \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} \|\partial_\gamma^3 x\|_{L^2}^2 \int_{\mathbb{T}} |\eta|^{-\alpha} d\eta \\ &\leq C_\alpha \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} \|\partial_\gamma^3 x\|_{L^2}^2. \end{aligned} \tag{20}$$

As before, we can obtain $I_2 = -6I_1$, and it yields

$$I_2 \leq C_\alpha \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} \|\partial_\gamma^3 x\|_{L^2}^2. \tag{21}$$

In order to estimate the term I_3 , we consider $I_3 = J_1 + J_2 + J_3$, where

$$\begin{aligned} J_1 &= -3\alpha \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)) \frac{A(\gamma, \eta)}{|x(\gamma) - x(\gamma - \eta)|^{\alpha+2}} d\eta d\gamma, \\ J_2 &= -3\alpha \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)) \frac{|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)|^2}{|x(\gamma) - x(\gamma - \eta)|^{\alpha+2}} d\eta d\gamma, \\ J_3 &= 3\alpha(2 + \alpha) \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)) \frac{(B(\gamma, \eta))^2}{|x(\gamma) - x(\gamma - \eta)|^{\alpha+4}} d\eta d\gamma, \end{aligned}$$

with

$$A(\gamma, \eta) = (x(\gamma) - x(\gamma - \eta)) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)),$$

and

$$B(\gamma, \eta) = (x(\gamma) - x(\gamma - \eta)) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)).$$

The identity

$$\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta) = \eta \int_0^1 \partial_\gamma^3 x(\gamma + (s-1)\eta) ds, \tag{22}$$

yields

$$\begin{aligned} J_1 &\leq 3 \int_0^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |\eta| \frac{(|\partial_\gamma^2 x(\gamma)| + |\partial_\gamma^2 x(\gamma - \eta)|) |\partial_\gamma^3 x(\gamma)| |\partial_\gamma^3 x(\gamma + (s-1)\eta)|}{|x(\gamma) - x(\gamma - \eta)|^{\alpha+1}} d\gamma d\eta ds \\ &\leq 3 \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} \int_0^1 \int_{\mathbb{T}} |\eta|^{-\alpha} \int_{\mathbb{T}} (|\partial_\gamma^3 x(\gamma)|^2 + |\partial_\gamma^3 x(\gamma + (s-1)\eta)|^2) d\gamma d\eta ds \\ &\leq C_\alpha \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} \|\partial_\gamma^3 x\|_{L^2}^2. \end{aligned}$$

Using (22), we have for J_2

$$\begin{aligned}
J_2 &= -3\alpha \int_0^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |F(x)(\gamma, \eta)|^{2+\alpha} \frac{|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)|^2}{\eta} \frac{\partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^3 x(\gamma + (s-1)\eta)}{|\eta|^\alpha} d\gamma d\eta ds \\
&\leq 3 \|F(x)\|_{L^\infty}^{2+\alpha} \|x\|_{C^2}^2 \int_0^1 \int_{\mathbb{T}} |\eta|^{-\alpha} \int_{\mathbb{T}} (|\partial_\gamma^3 x(\gamma)|^2 + |\partial_\gamma^3 x(\gamma + (s-1)\eta)|^2) d\gamma d\eta ds \\
&\leq C_\alpha \|F(x)\|_{L^\infty}^{2+\alpha} \|x\|_{C^2}^2 \|\partial_\gamma^3 x\|_{L^2}^2.
\end{aligned}$$

The term J_3 is estimated by

$$\begin{aligned}
J_3 &\leq 9 \int_0^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |\eta| \frac{|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)|^2 |\partial_\gamma^3 x(\gamma)| |\partial_\gamma^3 x(\gamma + (s-1)\eta)|}{|x(\gamma) - x(\gamma - \eta)|^{\alpha+2}} d\gamma d\eta ds \\
&\leq C_\alpha \|F(x)\|_{L^\infty}^{2+\alpha} \|x\|_{C^2}^2 \|\partial_\gamma^3 x\|_{L^2}^2.
\end{aligned}$$

We get finally

$$I_3 \leq C_\alpha (\|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} + \|F(x)\|_{L^\infty}^{2+\alpha} \|x\|_{C^2}^2) \|\partial_\gamma^3 x\|_{L^2}^2. \quad (23)$$

We decompose the term $I_4 = J_4 + J_5 + J_6 + J_7 + J_8$ as follows

$$\begin{aligned}
J_4 &= -\alpha \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \frac{C(\gamma, \eta)}{|x(\gamma) - x(\gamma - \eta)|^{\alpha+2}} d\eta d\gamma, \\
J_5 &= -3\alpha \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \frac{D(\gamma, \eta)}{|x(\gamma) - x(\gamma - \eta)|^{\alpha+2}} d\eta d\gamma, \\
J_6 &= 5\alpha(\alpha + 2) \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \frac{A(\gamma, \eta)B(\gamma, \eta)}{|x(\gamma) - x(\gamma - \eta)|^{\alpha+4}} d\eta d\gamma, \\
J_7 &= 5\alpha(\alpha + 2) \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \frac{B(\gamma, \eta) |\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)|^2}{|x(\gamma) - x(\gamma - \eta)|^{\alpha+4}} d\eta d\gamma, \\
J_8 &= -2\alpha(\alpha + 2)(\alpha + 4) \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \frac{(B(\gamma, \eta))^3}{|x(\gamma) - x(\gamma - \eta)|^{\alpha+6}} d\eta d\gamma,
\end{aligned}$$

with

$$\begin{aligned}
C(\gamma, \eta) &= (x(\gamma) - x(\gamma - \eta)) \cdot (\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)), \\
D(\gamma, \eta) &= (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)).
\end{aligned}$$

The most singular term is J_4 , in such a way that

$$\begin{aligned}
J_4 &\leq \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} \int_{\mathbb{T}} |\eta|^{-\alpha} \int_{\mathbb{T}} |\partial_\gamma^3 x(\gamma)| |\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)| d\gamma d\eta \\
&\leq C_\alpha \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} \|\partial_\gamma^3 x\|_{L^2}^2.
\end{aligned}$$

For J_5 , we have

$$\begin{aligned}
J_5 &\leq 3 \|F(x)\|_{L^\infty}^{2+\alpha} \|x\|_{C^2}^2 \int_{\mathbb{T}} |\eta|^{-\alpha} \int_{\mathbb{T}} |\partial_\gamma^3 x(\gamma)| |\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)| d\gamma d\eta \\
&\leq C_\alpha \|F(x)\|_{L^\infty}^{2+\alpha} \|x\|_{C^2}^2 \|\partial_\gamma^2 x\|_{L^2} \|\partial_\gamma^3 x\|_{L^2}.
\end{aligned}$$

In a similar way, we obtain

$$\begin{aligned} J_6 &\leq 15\|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2 \int_{\mathbb{T}} |\eta|^{-\alpha} \int_{\mathbb{T}} |\partial_\gamma^3 x(\gamma)| |\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)| d\gamma d\eta \\ &\leq C_\alpha \|F(x)\|_{L^\infty}^{2+\alpha} \|x\|_{C^2}^2 \|\partial_\gamma^2 x\|_{L^2} \|\partial_\gamma^3 x\|_{L^2}, \end{aligned}$$

and

$$\begin{aligned} J_7 &\leq 15\|F(x)\|_{L^\infty}^{3+\alpha}\|x\|_{C^2}^3 \int_{\mathbb{T}} |\eta|^{-\alpha} \int_{\mathbb{T}} |\partial_\gamma^3 x(\gamma)| |\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)| d\gamma d\eta \\ &\leq C_\alpha \|F(x)\|_{L^\infty}^{3+\alpha} \|x\|_{C^2}^3 \|\partial_\gamma x\|_{L^2} \|\partial_\gamma^3 x\|_{L^2}. \end{aligned}$$

For the term J_8 , we get

$$\begin{aligned} J_8 &\leq 30\|F(x)\|_{L^\infty}^{3+\alpha}\|x\|_{C^2}^3 \int_{\mathbb{T}} |\eta|^{-\alpha} \int_{\mathbb{T}} |\partial_\gamma^3 x(\gamma)| |\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)| d\gamma d\eta \\ &\leq C_\alpha \|F(x)\|_{L^\infty}^{3+\alpha} \|x\|_{C^2}^3 \|\partial_\gamma x\|_{L^2} \|\partial_\gamma^3 x\|_{L^2}, \end{aligned}$$

and finally it follows

$$I_4 \leq C_\alpha (\|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} + \|F(x)\|_{L^\infty}^{2+\alpha} \|x\|_{C^2}^2 + \|F(x)\|_{L^\infty}^{3+\alpha} \|x\|_{C^2}^3) \|x\|_{H^3}^2. \quad (24)$$

The inequalities (20), (21), (23) and (24) yield

$$\frac{d}{dt} \|\partial_\gamma^3 x\|_{L^2}^2(t) \leq C_\alpha \|F(x)\|_{L^\infty}^{3+\alpha}(t) \|x\|_{C^2}^3(t) \|x\|_{H^3}^2(t).$$

Due to the identity $\|x\|_{H^3}^2 = \|x\|_{L^2}^2 + \|\partial_\gamma^3 x\|_{L^2}^2$ and (19), we have

$$\frac{d}{dt} \|x\|_{H^3}(t) \leq C_\alpha \|F(x)\|_{L^\infty}^{3+\alpha}(t) \|x\|_{C^2}^3(t) \|x\|_{H^3}(t).$$

Finally, using Sobolev inequalities, we obtain

$$\frac{d}{dt} \|x\|_{H^3}(t) \leq C_\alpha \|F(x)\|_{L^\infty}^{3+\alpha}(t) \|x\|_{H^3}^4(t). \quad (25)$$

Notice that if we use energy methods at this point of the proof (see [2] to get the comprehensive argument), we need to regularize the equation (17) as follows

$$\begin{aligned} x_t^\varepsilon(\gamma, t) &= \phi_\varepsilon * \int_{\mathbb{T}} \frac{\partial_\gamma(\phi_\varepsilon * x^\varepsilon(\gamma, t) - \phi_\varepsilon * x^\varepsilon(\gamma - \eta, t))}{|x^\varepsilon(\gamma, t) - x^\varepsilon(\gamma - \eta, t)|^\alpha} d\eta, \\ x^\varepsilon(\gamma, 0) &= x_0(\gamma), \end{aligned} \quad (26)$$

where ϕ_ε is a regular approximation to the identity. If the inequality (15) is satisfied initially, due to the properties of the regular approximations to the identity, we get a Picard system as follows

$$\begin{aligned} x_t^\varepsilon(\gamma, t) &= G^\varepsilon(x^\varepsilon(\gamma, t)), \\ x^\varepsilon(\gamma, 0) &= x_0(\gamma), \end{aligned}$$

where G^ε is Lipschitz. Therefore, for any $\varepsilon > 0$, we obtain a time of existence t_ε where (15) is fulfilled. The way to have a time of existence of the system (26) independent of ε is to find energy estimates with bounds independent of ε . Next, by taking $\varepsilon \rightarrow 0$, we get solutions of the original equation. In this particular case, we obtain

$$\frac{d}{dt} \|x^\varepsilon\|_{H^3}(t) \leq C_\alpha \|F(x^\varepsilon)\|_{L^\infty}^{3+\alpha}(t) \|x^\varepsilon\|_{H^3}^4(t),$$

and if we take $\varepsilon \rightarrow 0$, it is possible that $\|F(x^\varepsilon)\|_{L^\infty} \rightarrow \infty$. In fact, we have an energy estimate that depends on ε and then the argument fails. We can not suppose that if the initial data fulfils (15), there exist a time $t > 0$ independent of ε in which (15) is satisfied, because just at this moment of the proof we do not have a well-posed system when $\varepsilon \rightarrow 0$ (the Lipschitz constant of G^ε goes to infinity when $\varepsilon \rightarrow 0$).

In order to solve this problem, we consider the evolution of the quantity $\|F(x)\|_{L^\infty}$. Taking $p > 2$, it follows

$$\begin{aligned} \frac{d}{dt} \|F(x)\|_{L^p}^p(t) &= \frac{d}{dt} \int_{\mathbb{T}} \int_{\mathbb{T}} \left(\frac{|\eta|}{|x(\gamma, t) - x(\gamma - \eta, t)|} \right)^p d\gamma d\eta \\ &= -p \int_{\mathbb{T}} \int_{\mathbb{T}} |\eta|^p \frac{(x(\gamma, t) - x(\gamma - \eta, t)) \cdot (x_t(\gamma, t) - x_t(\gamma - \eta, t))}{|x(\gamma, t) - x(\gamma - \eta, t)|^{p+2}} d\gamma d\eta \\ &\leq p \int_{\mathbb{T}} \int_{\mathbb{T}} \left(\frac{|\eta|}{|x(\gamma, t) - x(\gamma - \eta, t)|} \right)^{p+1} \frac{|x_t(\gamma, t) - x_t(\gamma - \eta, t)|}{|\eta|} d\gamma d\eta. \end{aligned}$$

We have

$$\begin{aligned} x_t(\gamma) - x_t(\gamma - \eta) &= \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \xi)}{|x(\gamma) - x(\gamma - \xi)|^\alpha} d\xi - \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma - \eta) - \partial_\gamma x(\gamma - \eta - \xi)}{|x(\gamma - \eta) - x(\gamma - \eta - \xi)|^\alpha} d\xi \\ &= \int_{\mathbb{T}} \left(\frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \xi)}{|x(\gamma) - x(\gamma - \xi)|^\alpha} - \frac{\partial_\gamma x(\gamma - \eta) - \partial_\gamma x(\gamma - \eta - \xi)}{|x(\gamma - \eta) - x(\gamma - \eta - \xi)|^\alpha} \right) d\xi \\ &\quad + \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta) + \partial_\gamma x(\gamma - \eta - \xi) - \partial_\gamma x(\gamma - \xi)}{|x(\gamma - \eta) - x(\gamma - \eta - \xi)|^\alpha} d\xi \\ &= I_5 + I_6. \end{aligned}$$

In order to estimate the term I_5 , we consider the function $f(a) = a^\alpha$. For $a, b > 0$, we obtain that

$$|a^\alpha - b^\alpha| = \alpha \left| \int_0^1 (sa + (1-s)b)^{\alpha-1} (a-b) ds \right| \leq \alpha (\min\{a, b\})^{\alpha-1} |a-b|. \quad (27)$$

One finds

$$\begin{aligned} I_5 &\leq \int_{\mathbb{T}} \frac{|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \xi)| \left| |x(\gamma) - x(\gamma - \xi)|^\alpha - |x(\gamma - \eta) - x(\gamma - \eta - \xi)|^\alpha \right|}{|x(\gamma) - x(\gamma - \xi)|^\alpha |x(\gamma - \eta) - x(\gamma - \eta - \xi)|^\alpha} d\xi \\ &\leq \|F(x)\|_{L^\infty}^{2\alpha} \|x\|_{C^2} \int_{\mathbb{T}} |\xi|^{1-\alpha} \left| \left| \frac{x(\gamma) - x(\gamma - \xi)}{\xi} \right|^\alpha - \left| \frac{x(\gamma - \eta) - x(\gamma - \eta - \xi)}{\xi} \right|^\alpha \right| d\xi. \end{aligned}$$

Using (27), we get

$$\begin{aligned}
I_5 &\leq \alpha \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} \int_{\mathbb{T}} |\xi|^{1-\alpha} \left| \frac{x(\gamma) - x(\gamma - \xi)}{\xi} \right| - \left| \frac{x(\gamma - \eta) - x(\gamma - \eta - \xi)}{\xi} \right| d\xi \\
&\leq \alpha \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} \int_{\mathbb{T}} |\xi|^{-\alpha} (|x(\gamma) - x(\gamma - \eta)| + |x(\gamma - \xi) - x(\gamma - \eta - \xi)|) d\xi \\
&\leq 2\alpha \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2}^2 |\eta| \int_{\mathbb{T}} |\xi|^{-\alpha} d\xi \\
&\leq C_\alpha \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2}^2 |\eta|.
\end{aligned}$$

We obtain for I_6 that

$$\begin{aligned}
I_6 &\leq \int_{\mathbb{T}} \frac{|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)| + |\partial_\gamma x(\gamma - \eta - \xi) - \partial_\gamma x(\gamma - \xi)|}{|x(\gamma - \eta) - x(\gamma - \eta - \xi)|^\alpha} d\xi \\
&\leq C_\alpha \|F(x)\|_{L^\infty}^\alpha \|x\|_{C^2} |\eta|.
\end{aligned}$$

The last two estimates show that

$$\begin{aligned}
\frac{d}{dt} \|F(x)\|_{L^p}^p(t) &\leq pC_\alpha \|x\|_{C^2}^2(t) \|F(x)\|_{L^\infty}^{1+\alpha}(t) \int_{\mathbb{T}^2} (F(x)(\gamma, \eta, t))^{p+1} d\gamma d\eta \\
&\leq pC_\alpha \|x\|_{C^2}^2(t) \|F(x)\|_{L^\infty}^{2+\alpha}(t) \|F(x)\|_{L^p}^p(t),
\end{aligned}$$

and therefore

$$\frac{d}{dt} \|F(x)\|_{L^p}(t) \leq C_\alpha \|x\|_{C^2}^2(t) \|F(x)\|_{L^\infty}^{2+\alpha}(t) \|F(x)\|_{L^p}(t).$$

Integrating in time it follows

$$\|F(x)\|_{L^p}(t+h) \leq \|F(x)\|_{L^p}(t) \exp\left(C_\alpha \int_t^{t+h} \|x\|_{C^2}^2(s) \|F(x)\|_{L^\infty}^{2+\alpha}(s) ds\right),$$

and taking $p \rightarrow \infty$ we obtain

$$\|F(x)\|_{L^\infty}(t+h) \leq \|F(x)\|_{L^\infty}(t) \exp\left(C_\alpha \int_t^{t+h} \|x\|_{C^2}^2(s) \|F(x)\|_{L^\infty}^{2+\alpha}(s) ds\right).$$

In order to estimate the derivative of the quantity $\|F(x)\|_{L^\infty}(t)$, we use the last inequality getting

$$\begin{aligned}
\frac{d}{dt} \|F(x)\|_{L^\infty}(t) &= \lim_{h \rightarrow 0} (\|F(x)\|_{L^\infty}(t+h) - \|F(x)\|_{L^\infty}(t)) h^{-1} \\
&\leq \|F(x)\|_{L^\infty}(t) \lim_{h \rightarrow 0} (\exp\left(C_\alpha \int_t^{t+h} \|x\|_{C^2}^2(s) \|F(x)\|_{L^\infty}^{2+\alpha}(s) ds\right) - 1) h^{-1} \\
&\leq C_\alpha \|x\|_{C^2}^2(t) \|F(x)\|_{L^\infty}^{3+\alpha}(t).
\end{aligned}$$

Applying Sobolev inequalities we conclude that

$$\frac{d}{dt} \|F(x)\|_{L^\infty}(t) \leq C_\alpha \|x\|_{H^3}^2(t) \|F(x)\|_{L^\infty}^{3+\alpha}(t). \tag{28}$$

This estimate does not give a global in time bound for $\|F(x)\|_{L^\infty}(t)$ in terms of norms of $x(\gamma, t)$. Then, adding the estimate (28) to (25), we have

$$\frac{d}{dt}(\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t)) \leq C_\alpha \|F(x)\|_{L^\infty}^{3+\alpha}(t) \|x\|_{H^3}^4(t),$$

and finally

$$\frac{d}{dt}(\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t)) \leq C_\alpha (\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t))^{7+\alpha}. \quad (29)$$

Integrating, we get

$$\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t) \leq \frac{\|x_0\|_{H^3} + \|F(x_0)\|_{L^\infty}}{(1 - tC_\alpha (\|x_0\|_{H^3} + \|F(x_0)\|_{L^\infty})^{6+\alpha})^{\frac{1}{6+\alpha}}},$$

with C_α depending on α . Then, using the regularized problem (26), the same estimate is obtained for x^ε instead of x . Therefore we get to find a time of existence independent of ε , and taking $\varepsilon \rightarrow 0$, the existence result follows.

Let x and y be two solutions of the equation (17) with $x(\gamma, 0) = y(\gamma, 0)$, and $z = x - y$. One has that

$$\begin{aligned} \int_{\mathbb{T}} z(\gamma) \cdot z_t(\gamma) d\gamma &= \int_{\mathbb{T}} \int_{\mathbb{T}} z(\gamma) \cdot \left(\frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|^\alpha} - \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)}{|y(\gamma) - y(\gamma - \eta)|^\alpha} \right) d\eta d\gamma \\ &\quad + \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{z(\gamma) \cdot (\partial_\gamma z(\gamma) - \partial_\gamma z(\gamma - \eta))}{|y(\gamma) - y(\gamma - \eta)|^\alpha} d\eta d\gamma \\ &= I_7 + I_8. \end{aligned}$$

The term I_7 is estimated using (27) by

$$\begin{aligned} I_7 &\leq \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|z(\gamma)| |\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)| \left| |x(\gamma) - x(\gamma - \eta)|^\alpha - |y(\gamma) - y(\gamma - \eta)|^\alpha \right|}{|x(\gamma) - x(\gamma - \eta)|^\alpha |y(\gamma) - y(\gamma - \eta)|^\alpha} d\eta d\gamma \\ &\leq \|F(x)\|_{L^\infty}^\alpha \|F(y)\|_{L^\infty}^\alpha \|x\|_{C^2} \int_{\mathbb{T}} \int_{\mathbb{T}} |\eta|^{1-\alpha} |z(\gamma)| \left| \left| \frac{x(\gamma) - x(\gamma - \eta)}{\eta} \right|^\alpha - \left| \frac{y(\gamma) - y(\gamma - \eta)}{\eta} \right|^\alpha \right| d\eta d\gamma \\ &\leq \|F(x)\|_{L^\infty} \|F(y)\|_{L^\infty} \|x\|_{C^2} \int_{\mathbb{T}} \int_{\mathbb{T}} |\eta|^{1-\alpha} |z(\gamma)| \left| \left| \frac{x(\gamma) - x(\gamma - \eta)}{\eta} \right| - \left| \frac{y(\gamma) - y(\gamma - \eta)}{\eta} \right| \right| d\eta d\gamma \\ &\leq \|F(x)\|_{L^\infty} \|F(y)\|_{L^\infty} \|x\|_{C^2} \int_{\mathbb{T}} \int_{\mathbb{T}} |\eta|^{-\alpha} |z(\gamma)| |z(\gamma) - z(\gamma - \eta)| d\eta d\gamma \\ &\leq C_\alpha \|F(x)\|_{L^\infty} \|F(y)\|_{L^\infty} \|x\|_{C^2} \|z\|_{L^2}^2. \end{aligned}$$

Integrating by parts in I_8 yields

$$\begin{aligned}
I_8 &= \frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{(z(\gamma) - z(\gamma - \eta)) \cdot (\partial_\gamma z(\gamma) - \partial_\gamma z(\gamma - \eta))}{|y(\gamma) - y(\gamma - \eta)|^\alpha} d\eta d\gamma \\
&= \frac{1}{4} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{\partial_\gamma (|z(\gamma) - z(\gamma - \eta)|^2)}{|y(\gamma) - y(\gamma - \eta)|^\alpha} d\eta d\gamma \\
&= \frac{\alpha}{4} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|z(\gamma) - z(\gamma - \eta)|^2 (y(\gamma) - y(\gamma - \eta)) \cdot (\partial_\gamma y(\gamma) - \partial_\gamma y(\gamma - \eta))}{|y(\gamma) - y(\gamma - \eta)|^{\alpha+2}} d\eta d\gamma \\
&\leq C_\alpha \|F(y)\|_{L^\infty}^{1+\alpha} \|y\|_{C^2} \|z\|_{L^2}^2.
\end{aligned}$$

Finally we obtain

$$\frac{d}{dt} \|z\|_{L^2}^2(t) \leq C(\alpha, x, F(x), y, F(y)) \|z\|_{L^2}^2(t),$$

and using Gronwall inequality we conclude that $z = 0$.

5 Existence for $\alpha = 1$; the QG sharp front

In this section we prove existence for the QG sharp front in Sobolev spaces. We give the norm of the Holder space $C^{k, \frac{1}{2}}(\mathbb{T})$ by

$$\|x\|_{C^{k, \frac{1}{2}}} = \|x\|_{C^k} + \max_{\gamma, \eta \in \mathbb{T}} \frac{|\partial_\gamma^k x(\gamma) - \partial_\gamma^k x(\gamma - \eta)|}{|\eta|^{1/2}}.$$

In the case of $\alpha = 1$, we have the following equation

$$\begin{aligned}
x_t(\gamma, t) &= \frac{\theta_2 - \theta_1}{2\pi} \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta, \\
x(\gamma, 0) &= x_0(\gamma).
\end{aligned} \tag{30}$$

We take $\theta_2 - \theta_1 = 2\pi$ without lost of generality. This equation loses two derivatives, therefore the technique applied in the last section does not work. Recalling that we are trying to solve the QG equation in a weak sense, we can modify the system (30) in the tangential direction without changing the shape of the front, as far as the curve satisfies

$$x_t(\gamma, t) \cdot \partial_\gamma^\perp x(\gamma, t) = - \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma - \eta, t) \cdot \partial_\gamma^\perp x(\gamma, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta.$$

We showed in section 3 that the temperature $\theta(x, t)$ given by (9) is a weak solution of the QG equation. Then we propose to modify the equation (30) as follows

$$\begin{aligned}
x_t(\gamma, t) &= \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta + \lambda(\gamma, t) \partial_\gamma x(\gamma, t), \\
x(\gamma, 0) &= x_0(\gamma).
\end{aligned} \tag{31}$$

The parameter $\lambda(\gamma, t)$ is to get an extra cancellation in such a way that

$$\partial_\gamma x(\gamma, t) \cdot \partial_\gamma^2 x(\gamma, t) = 0. \quad (32)$$

Given an initial data satisfying (15), we can reparameterize it obtaining that $|\partial_\gamma x(\gamma, 0)|^2 = 1$, and therefore (32) is fulfilled at $t = 0$. We can not have $|\partial_\gamma x(\gamma, t)|^2 = 1$ for all time, but

$$|\partial_\gamma x(\gamma, t)|^2 = A(t). \quad (33)$$

We have

$$\begin{aligned} A'(t) &= 2\partial_\gamma x(\gamma, t) \cdot \partial_\gamma x_t(\gamma, t) \\ &= 2\partial_\gamma x(\gamma, t) \cdot \partial_\gamma \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta \right) + 2\partial_\gamma \lambda(\gamma, t) A(t), \end{aligned}$$

and therefore

$$\partial_\gamma \lambda(\gamma, t) = \frac{A'(t)}{2A(t)} - \frac{1}{A(t)} \partial_\gamma x(\gamma, t) \cdot \partial_\gamma \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta \right). \quad (34)$$

Because $\lambda(\gamma, t)$ has to be periodic, we obtain

$$\frac{A'(t)}{2A(t)} = \frac{1}{2\pi A(t)} \int_{\mathbb{T}} \partial_\gamma x(\gamma, t) \cdot \partial_\gamma \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta \right) d\gamma. \quad (35)$$

Using (35) in (34), and integrating in γ , one gets the following formula for $\lambda(\gamma, t)$

$$\begin{aligned} \lambda(\gamma, t) &= \frac{\gamma + \pi}{2\pi} \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t)}{|\partial_\gamma x(\gamma, t)|^2} \cdot \partial_\gamma \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta \right) d\gamma \\ &\quad - \int_{-\pi}^{\gamma} \frac{\partial_\gamma x(\eta, t)}{|\partial_\gamma x(\eta, t)|^2} \cdot \partial_\eta \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\eta, t) - \partial_\gamma x(\eta - \xi, t)}{|x(\eta, t) - x(\eta - \xi, t)|} d\xi \right) d\eta, \end{aligned} \quad (36)$$

taking $\lambda(-\pi, t) = \lambda(\pi, t) = 0$. If we consider solutions of the equation (31) with $\lambda(\gamma, t)$ given by (36), it is easy to check that

$$\frac{d}{dt} |\partial_\gamma x(\gamma, t)|^2 = \lambda(\gamma, t) \partial_\gamma |\partial_\gamma x(\gamma, t)|^2 + \mu(t) |\partial_\gamma x(\gamma, t)|^2,$$

with

$$\mu(t) = \frac{1}{\pi} \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t)}{|\partial_\gamma x(\gamma, t)|^2} \cdot \partial_\gamma \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta \right) d\gamma.$$

Solving this linear partial differential equation, if (32) is satisfied initially, one finds that the unique solution is given by

$$|\partial_\gamma x(\gamma, t)|^2 = |\partial_\gamma x(\gamma, 0)|^2 + \frac{1}{\pi} \int_0^t \int_{\mathbb{T}} \partial_\gamma x(\gamma, s) \cdot \partial_\gamma \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, s) - \partial_\gamma x(\gamma - \eta, s)}{|x(\gamma, s) - x(\gamma - \eta, s)|} d\eta \right) d\gamma ds.$$

Therefore we obtain (33).

The main theorem of this section is

Theorem 5.1 *Let $x_0(\gamma) \in H^k(\mathbb{T})$ for $k \geq 3$ with $F(x_0)(\gamma, \eta) < \infty$. Then there exists a time $T > 0$ so that there is a solution to (31) in $C^1([0, T]; H^k(\mathbb{T}))$ with $x(\gamma, 0) = x_0(\gamma)$ and $\lambda(\gamma, t)$ given by (36).*

Proof: Being analogous for $k > 3$, we give the proof for $k = 3$. We have showed before that (33) is satisfied if $x(\gamma, t)$ is a solution of (31). Then we can rewrite $\lambda(\gamma, t)$ as follows

$$\begin{aligned} \lambda(\gamma, t) &= \frac{\gamma + \pi}{2\pi A(t)} \int_{\mathbb{T}} \partial_\gamma x(\gamma, t) \cdot \partial_\gamma \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta \right) d\gamma \\ &\quad - \frac{1}{A(t)} \int_{-\pi}^{\gamma} \partial_\gamma x(\eta, t) \cdot \partial_\eta \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\eta, t) - \partial_\gamma x(\eta - \xi, t)}{|x(\eta, t) - x(\eta - \xi, t)|} d\xi \right) d\eta. \end{aligned} \quad (37)$$

We obtain

$$\begin{aligned} \int_{\mathbb{T}} x(\gamma) \cdot x_t(\gamma) d\gamma &= \int_{\mathbb{T}} \int_{\mathbb{T}} x(\gamma) \cdot \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} d\eta d\gamma + \int_{\mathbb{T}} \lambda(\gamma) x(\gamma) \cdot \partial_\gamma x(\gamma) d\gamma \\ &= I_1 + I_2, \end{aligned}$$

One finds that $I_1 = 0$, since

$$\begin{aligned} I_1 &= \int_{\mathbb{T}} \int_{\mathbb{T}} x(\gamma) \cdot \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\eta)}{|x(\gamma) - x(\eta)|} d\eta d\gamma = - \int_{\mathbb{T}} \int_{\mathbb{T}} x(\eta) \cdot \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\eta)}{|x(\gamma) - x(\eta)|} d\eta d\gamma \\ &= \frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{(x(\gamma) - x(\eta)) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\eta))}{|x(\gamma) - x(\eta)|} d\eta d\gamma = \frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma |x(\gamma) - x(\gamma - \eta)| d\gamma d\eta \\ &= 0. \end{aligned}$$

For the term I_2 , one obtains that $I_2 \leq \|\lambda\|_{L^\infty} \|x\|_{L^2} \|\partial_\gamma x\|_{L^2}$, and

$$\begin{aligned} \|\lambda\|_{L^\infty} &\leq \frac{2}{A(t)} \int_{\mathbb{T}} |\partial_\gamma x(\gamma)| \left| \partial_\gamma \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} d\eta \right| d\gamma \\ &\leq \frac{2}{A(t)} \int_{\mathbb{T}} |\partial_\gamma x(\gamma)| \int_{\mathbb{T}} \frac{|\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)|}{|x(\gamma) - x(\gamma - \eta)|} d\eta d\gamma \\ &\quad + \frac{2}{A(t)} \int_{\mathbb{T}} |\partial_\gamma x(\gamma)| \int_{\mathbb{T}} \frac{|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)|^2}{|x(\gamma) - x(\gamma - \eta)|^2} d\eta d\gamma = J_1 + J_2. \end{aligned}$$

Due to $1/A(t) \leq \|F(x)\|_{L^\infty}^2(t)$, we have

$$J_1 \leq 2\|F(x)\|_{L^\infty}^3 \int_0^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |\partial_\gamma^3 x(\gamma + (s-1)\eta)| |\partial_\gamma x(\gamma)| d\gamma d\eta ds \leq 2\|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^2,$$

and

$$J_2 \leq 2\|F(x)\|_{L^\infty}^4 \|x\|_{C^1} \int_0^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |\partial_\gamma^2 x(\gamma + (s-1)\eta)|^2 d\gamma d\eta ds \leq 2\|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^3.$$

Therefore we obtain that

$$\frac{d}{dt}\|x\|_{L^2}^2(t) \leq C\|F(x)\|_{L^\infty}^4(t)\|x\|_{H^3}^5(t). \quad (38)$$

We decompose as follows

$$\begin{aligned} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^3 x_t(\gamma) d\gamma &= \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^3 \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} d\eta \right) d\gamma \\ &\quad + \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^3 (\lambda(\gamma) \partial_\gamma x(\gamma)) d\gamma \\ &= I_3 + I_4. \end{aligned}$$

We take $I_3 = J_3 + J_4 + J_5 + J_6$ where

$$\begin{aligned} J_3 &= \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \frac{\partial_\gamma^4 x(\gamma) - \partial_\gamma^4 x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} d\eta d\gamma, \\ J_4 &= 3 \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)) \partial_\gamma (|x(\gamma) - x(\gamma - \eta)|^{-1}) d\eta d\gamma, \\ J_5 &= 3 \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)) \partial_\gamma^2 (|x(\gamma) - x(\gamma - \eta)|^{-1}) d\eta d\gamma, \\ J_6 &= \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \partial_\gamma^3 (|x(\gamma) - x(\gamma - \eta)|^{-1}) d\eta d\gamma. \end{aligned}$$

The term J_3 can be written as

$$\begin{aligned} J_3 &= \frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} (\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)) \cdot \frac{\partial_\gamma^4 x(\gamma) - \partial_\gamma^4 x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} d\eta d\gamma \\ &= \frac{1}{4} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{\partial_\gamma |\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)|^2}{|x(\gamma) - x(\gamma - \eta)|} d\eta d\gamma \\ &= \frac{1}{4} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)|^2 (x(\gamma) - x(\gamma - \eta)) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta))}{|x(\gamma) - x(\gamma - \eta)|^3} d\eta d\gamma. \end{aligned}$$

If we define

$$B(\gamma, \eta) = (x(\gamma) - x(\gamma - \eta)) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)),$$

due to (32), we obtain that

$$J_3 = \frac{1}{4} \int_{\mathbb{T}} \int_{\mathbb{T}} |F(x)(\gamma, \eta)|^3 |\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)|^2 \frac{B(\gamma, \eta) \eta^{-2} - \partial_\gamma x(\gamma) \cdot \partial_\gamma^2 x(\gamma)}{|\eta|} d\eta d\gamma.$$

Using that

$$\left| \frac{B(\gamma, \eta)\eta^{-2} - \partial_\gamma x(\gamma) \cdot \partial_\gamma^2 x(\gamma)}{\eta} \right| \leq 2\|x\|_{C^{2, \frac{1}{2}}}^2 |\eta|^{-1/2},$$

we find

$$\begin{aligned} J_3 &\leq \|F(x)\|_{L^\infty}^3 \|x\|_{C^{2, \frac{1}{2}}}^2 \int_{\mathbb{T}} |\eta|^{-1/2} \int_{\mathbb{T}} (|\partial_\gamma^3 x(\gamma)|^2 + |\partial_\gamma^3 x(\gamma - \eta)|^2) d\gamma d\eta \\ &\leq C \|F(x)\|_{L^\infty}^3 \|x\|_{C^{2, \frac{1}{2}}}^2 \|\partial_\gamma^3 x\|_{L^2}^2 \\ &\leq C \|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4. \end{aligned} \quad (39)$$

We obtain that $J_4 = -6J_3$, and it yields

$$J_4 \leq C \|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4. \quad (40)$$

In order to estimate the term J_5 , we consider $J_5 = K_1 + K_2 + K_3$, where

$$\begin{aligned} K_1 &= -3 \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)) \frac{C(\gamma, \eta)}{|x(\gamma) - x(\gamma - \eta)|^3} d\eta d\gamma, \\ K_2 &= -3 \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)) \frac{|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)|^2}{|x(\gamma) - x(\gamma - \eta)|^3} d\eta d\gamma, \\ K_3 &= 9 \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)) \frac{(B(\gamma, \eta))^2}{|x(\gamma) - x(\gamma - \eta)|^5} d\eta d\gamma, \end{aligned}$$

with

$$C(\gamma, \eta) = (x(\gamma) - x(\gamma - \eta)) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)).$$

The inequality

$$|\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)| |\eta|^{-1/2} \leq \|x\|_{C^{2, \frac{1}{2}}}, \quad (41)$$

yields

$$\begin{aligned} K_1 &\leq 3 \|F(x)\|_{L^\infty}^2 \|x\|_{C^{2, \frac{1}{2}}}^2 \int_0^1 \int_{\mathbb{T}} |\eta|^{-1/2} \int_{\mathbb{T}} |\partial_\gamma^3 x(\gamma)| |\partial_\gamma^3 x(\gamma + (s-1)\eta)| d\gamma d\eta ds \\ &\leq C \|F(x)\|_{L^\infty}^2 \|x\|_{H^3}^3. \end{aligned}$$

As before, we have for K_2 that

$$K_2 \leq C \|F(x)\|_{L^\infty}^3 \|x\|_{C^2}^2 \|\partial_\gamma^3 x\|_{L^2}^2 \leq C \|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4.$$

The term K_3 is estimated by

$$K_3 \leq C \|F(x)\|_{L^\infty}^3 \|x\|_{C^2}^2 \|\partial_\gamma^3 x\|_{L^2}^2 \leq C \|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4.$$

We get finally

$$J_5 \leq C \|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4. \quad (42)$$

We decompose the term $J_6 = K_4 + K_5 + K_6 + K_7 + K_8$ as follows

$$\begin{aligned}
K_4 &= - \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \frac{D(\gamma, \eta)}{|x(\gamma) - x(\gamma - \eta)|^3} d\eta d\gamma, \\
K_5 &= -3 \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \frac{E(\gamma, \eta)}{|x(\gamma) - x(\gamma - \eta)|^3} d\eta d\gamma, \\
K_6 &= 15 \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \frac{B(\gamma, \eta)C(\gamma, \eta)}{|x(\gamma) - x(\gamma - \eta)|^5} d\eta d\gamma, \\
K_7 &= 15 \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \frac{B(\gamma, \eta)|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)|^2}{|x(\gamma) - x(\gamma - \eta)|^5} d\eta d\gamma, \\
K_8 &= -30 \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \frac{(B(\gamma, \eta))^3}{|x(\gamma) - x(\gamma - \eta)|^7} d\eta d\gamma,
\end{aligned}$$

with

$$\begin{aligned}
D(\gamma, \eta) &= (x(\gamma) - x(\gamma - \eta)) \cdot (\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)), \\
E(\gamma, \eta) &= (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)).
\end{aligned}$$

We obtain

$$K_5 \leq 3 \|F(x)\|_{L^\infty}^3 \|x\|_{C^2}^2 \|\partial_\gamma^3 x\|_{L^2}^2 \leq 3 \|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4,$$

$$K_6 \leq 15 \|F(x)\|_{L^\infty}^3 \|x\|_{C^2}^2 \|\partial_\gamma^3 x\|_{L^2}^2 \leq 15 \|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4,$$

$$K_7 \leq 15 \|F(x)\|_{L^\infty}^4 \|x\|_{C^2}^3 \|\partial_\gamma^3 x\|_{L^2} \|\partial_\gamma^2 x\|_{L^2} \leq 15 \|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^5,$$

and

$$K_8 \leq 30 \|F(x)\|_{L^\infty}^4 \|x\|_{C^2}^3 \|\partial_\gamma^3 x\|_{L^2} \|\partial_\gamma^2 x\|_{L^2} \leq 30 \|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^5.$$

For the most singular term, we have

$$\begin{aligned}
K_4 &= \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \frac{\eta \partial_\gamma x(\gamma) \cdot (\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)) - D(\gamma, \eta)}{|x(\gamma) - x(\gamma - \eta)|^3} d\eta d\gamma \\
&\quad - \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \frac{\eta \partial_\gamma x(\gamma) \cdot (\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta))}{|x(\gamma) - x(\gamma - \eta)|^3} d\eta d\gamma \\
&= L_1 + L_2.
\end{aligned}$$

One finds that

$$L_1 \leq \|F(x)\|_{L^\infty}^3 \|x\|_{C^2}^2 \int_{\mathbb{T}} \int_{\mathbb{T}} |\partial_\gamma^3 x(\gamma)| |\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)| d\gamma d\eta \leq C \|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4.$$

The term L_2 is decomposed, and it yields

$$\begin{aligned}
L_2 &= \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \frac{\eta (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \cdot \partial_\gamma^3 x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|^3} d\eta d\gamma \\
&\quad - \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \eta \frac{\partial_\gamma x(\gamma) \cdot \partial_\gamma^3 x(\gamma) - \partial_\gamma x(\gamma - \eta) \cdot \partial_\gamma^3 x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|^3} d\eta d\gamma \\
&= M_1 + M_2.
\end{aligned}$$

We estimate the term M_1 as follows

$$M_1 \leq \|F(x)\|_{L^\infty}^3 \|x\|_{C^2}^2 \int_{\mathbb{T}} \int_{\mathbb{T}} |\partial_\gamma^3 x(\gamma)| |\partial_\gamma^3 x(\gamma - \eta)| d\gamma d\eta \leq \|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4.$$

Taking the derivative in (32), we find that $\partial_\gamma x(\gamma) \cdot \partial_\gamma^3 x(\gamma) = -|\partial_\gamma^2 x(\gamma)|^2$, and we rewrite

$$M_2 = \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \eta \frac{|\partial_\gamma^2 x(\gamma)|^2 - |\partial_\gamma^2 x(\gamma - \eta)|^2}{|x(\gamma) - x(\gamma - \eta)|^3} d\eta d\gamma.$$

The inequality

$$||\partial_\gamma^2 x(\gamma)|^2 - |\partial_\gamma^2 x(\gamma - \eta)|^2| \leq 2\|x\|_{C^2} |\eta| \int_0^1 |\partial_\gamma^3 x(\gamma + (s-1)\eta)| ds, \quad (43)$$

yields

$$M_2 \leq 2\|F(x)\|_{L^\infty}^3 \|x\|_{C^2}^2 \int_0^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |\partial_\gamma^3 x(\gamma)| |\partial_\gamma^3 x(\gamma + (s-1)\eta)| d\gamma d\eta ds \leq C\|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4.$$

We recall that $K_4 = L_1 + L_2 = L_1 + M_1 + M_2 \leq C\|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4$, and finally it follows

$$J_6 \leq C\|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^5. \quad (44)$$

Due to (39), (40), (42) and (44), we obtain

$$I_3 \leq C\|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^5. \quad (45)$$

We take $I_4 = J_7 + J_8 + J_9 + J_{10}$, where

$$\begin{aligned}
J_7 &= \int_{\mathbb{T}} \lambda(\gamma) \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^4 x(\gamma) d\gamma, & J_8 &= 3 \int_{\mathbb{T}} \partial_\gamma \lambda(\gamma) |\partial_\gamma^3 x(\gamma)|^2 d\gamma, \\
J_9 &= 3 \int_{\mathbb{T}} \partial_\gamma^2 \lambda(\gamma) \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^2 x(\gamma) d\gamma, & J_{10} &= \int_{\mathbb{T}} \partial_\gamma^3 \lambda(\gamma) \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma x(\gamma) d\gamma.
\end{aligned}$$

We integrate by parts in the term J_7 , and we get

$$J_7 = -\frac{1}{2} \int_{\mathbb{T}} \partial_\gamma \lambda(\gamma) |\partial_\gamma^3 x(\gamma)|^2 d\gamma \leq \frac{1}{2} \|\partial_\gamma \lambda\|_{L^\infty} \|\partial_\gamma^3 x\|_{L^2}^2.$$

Using (37), we find that

$$\begin{aligned} \partial_\gamma \lambda(\gamma, t) &= \frac{1}{2\pi A(t)} \int_{\mathbb{T}} \partial_\gamma x(\gamma, t) \cdot \partial_\gamma \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta \right) d\gamma \\ &\quad - \frac{1}{A(t)} \partial_\gamma x(\gamma, t) \cdot \partial_\gamma \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta \right) \\ &= K_9 + K_{10}. \end{aligned} \tag{46}$$

The term K_9 is estimated as J_1 and J_2 , obtaining

$$K_9 \leq \|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^3.$$

We have for K_{10} that

$$\begin{aligned} K_{10} &\leq \frac{\|x\|_{C^2}}{A(t)} \int_{\mathbb{T}} \left(\frac{|\partial_\gamma^2 x(\gamma, t) - \partial_\gamma^2 x(\gamma - \eta, t)|}{|x(\gamma, t) - x(\gamma - \eta, t)|} + \frac{|\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)|^2}{|x(\gamma, t) - x(\gamma - \eta, t)|^2} \right) d\eta \\ &\leq 2 \|F(x)\|_{L^\infty}^4 \|x\|_{C^{2, \frac{1}{2}}}^3 \int_{\mathbb{T}} |\eta|^{-1/2} d\eta \\ &\leq C \|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^3, \end{aligned}$$

and therefore

$$J_7 \leq C \|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^5. \tag{47}$$

Due to the identity $J_8 = -6J_7$, one finds that

$$J_8 \leq C \|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^5. \tag{48}$$

Using that

$$\begin{aligned} \partial_\gamma^2 \lambda(\gamma, t) &= -\frac{1}{A(t)} \partial_\gamma^2 x(\gamma, t) \cdot \partial_\gamma \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta \right) \\ &\quad - \frac{1}{A(t)} \partial_\gamma x(\gamma, t) \cdot \partial_\gamma^2 \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta \right), \end{aligned}$$

one gets

$$\begin{aligned} J_9 &= -\frac{1}{A(t)} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^2 x(\gamma) \partial_\gamma^2 x(\gamma) \cdot \partial_\gamma \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} d\eta \right) d\gamma \\ &\quad - \frac{1}{A(t)} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^2 x(\gamma) \partial_\gamma x(\gamma) \cdot \partial_\gamma^2 \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} d\eta \right) d\gamma \\ &= L_3 + L_4. \end{aligned}$$

Therefore

$$\begin{aligned}
L_3 &\leq \frac{\|x\|_{C^2}^2}{A(t)} \int_{\mathbb{T}} \int_{\mathbb{T}} |\partial_\gamma^3 x(\gamma)| \left(\frac{|\partial_\gamma^2 x(\gamma, t) - \partial_\gamma^2 x(\gamma - \eta, t)|}{|x(\gamma, t) - x(\gamma - \eta, t)|} + \frac{|\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)|^2}{|x(\gamma, t) - x(\gamma - \eta, t)|^2} \right) d\eta d\gamma \\
&\leq \|F(x)\|_{L^\infty}^4 \|x\|_{C^2}^3 \int_0^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |\partial_\gamma^3 x(\gamma)| (|\partial_\gamma^3 x(\gamma + (t-1)\eta)| + |\partial_\gamma^2 x(\gamma + (t-1)\eta)|) d\gamma d\eta ds \\
&\leq C \|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^5.
\end{aligned}$$

Moreover

$$\begin{aligned}
L_4 &= -\frac{1}{A(t)} \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^2 x(\gamma) \partial_\gamma x(\gamma) \cdot \frac{\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} d\eta d\gamma \\
&\quad + \frac{2}{A(t)} \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^2 x(\gamma) \partial_\gamma x(\gamma) \cdot \frac{(\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta))B(\gamma, \eta)}{|x(\gamma) - x(\gamma - \eta)|^3} d\eta d\gamma \\
&\quad - \frac{1}{A(t)} \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^2 x(\gamma) \partial_\gamma x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \partial_\gamma^2 (|x(\gamma) - x(\gamma - \eta)|^{-1}) d\eta d\gamma \\
&= M_3 + M_4 + M_5.
\end{aligned}$$

The terms M_4 and M_5 are estimated as before, and we obtain

$$M_4 + M_5 \leq C \|F(x)\|_{L^\infty}^5 \|x\|_{H^3}^6.$$

The most singular term is M_3 , but we find that

$$\begin{aligned}
M_3 &= \frac{1}{A(t)} \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^2 x(\gamma) \partial_\gamma^3 x(\gamma - \eta) \cdot \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} d\eta d\gamma \\
&\quad - \frac{1}{A(t)} \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^2 x(\gamma) \frac{\partial_\gamma^3 x(\gamma) \cdot \partial_\gamma x(\gamma) - \partial_\gamma^3 x(\gamma - \eta) \cdot \partial_\gamma x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} d\eta d\gamma \\
&= N_1 + N_2.
\end{aligned}$$

We obtain

$$N_1 \leq \|F(x)\|_{L^\infty}^3 \|x\|_{C^2}^2 \|\partial_\gamma^3 x\|_{L^2}^2 \leq \|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4,$$

and using (32)

$$N_2 = \frac{1}{A(t)} \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^2 x(\gamma) \frac{|\partial_\gamma^2 x(\gamma)|^2 - |\partial_\gamma^2 x(\gamma - \eta)|^2}{|x(\gamma) - x(\gamma - \eta)|} d\eta d\gamma.$$

Due to (43), we conclude that

$$N_2 \leq 2 \|F(x)\|_{L^\infty}^3 \|x\|_{C^2}^2 \|\partial_\gamma^3 x\|_{L^2}^2 \leq 2 \|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4.$$

We have $J_9 = L_3 + L_4 = L_3 + M_3 + M_4 + M_5 = L_3 + N_1 + N_2 + M_4 + M_5$, and therefore

$$J_9 \leq \|F(x)\|_{L^\infty}^5 \|x\|_{H^3}^6. \quad (49)$$

The identity (32) yields

$$J_{10} = - \int_{\mathbb{T}} \partial_\gamma^3 \lambda(\gamma) |\partial_\gamma^2 x(\gamma)|^2 d\gamma = 2 \int_{\mathbb{T}} \partial_\gamma^2 \lambda(\gamma) \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^2 x(\gamma) d\gamma = \frac{2}{3} J_9,$$

and therefore

$$J_{10} \leq \|F(x)\|_{L^\infty}^5 \|x\|_{H^3}^6. \quad (50)$$

Due to the inequalities (47), (48), (49), and (50), we get

$$I_4 \leq C \|F(x)\|_{L^\infty}^5 \|x\|_{H^3}^6.$$

Using (45) and the last estimate, we have

$$\frac{d}{dt} \|\partial_\gamma^3 x\|_{L^2}^2(t) \leq C \|F(x)\|_{L^\infty}^5(t) \|x\|_{H^3}^6(t).$$

This inequality and (38) bound the evolution of the Sobolev norms of the curve as follows

$$\frac{d}{dt} \|x\|_{H^3}(t) \leq C \|F(x)\|_{L^\infty}^5(t) \|x\|_{H^3}^5(t). \quad (51)$$

We continue the argument considering the evolution of the quantity $\|F(x)\|_{L^\infty}(t)$. Taking $p > 2$, it yields

$$\frac{d}{dt} \|F(x)\|_{L^p}^p(t) \leq p \int_{\mathbb{T}} \int_{\mathbb{T}} \left(\frac{|\eta|}{|x(\gamma, t) - x(\gamma - \eta, t)|} \right)^{p+1} \frac{|x_t(\gamma, t) - x_t(\gamma - \eta, t)|}{|\eta|} d\gamma d\eta.$$

We have

$$\begin{aligned} x_t(\gamma) - x_t(\gamma - \eta) &= \int_{\mathbb{T}} \left(\frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \xi)}{|x(\gamma) - x(\gamma - \xi)|} - \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \xi)}{|x(\gamma - \eta) - x(\gamma - \eta - \xi)|} \right) d\xi \\ &\quad + \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta) + \partial_\gamma x(\gamma - \eta - \xi) - \partial_\gamma x(\gamma - \xi)}{|x(\gamma - \eta) - x(\gamma - \eta - \xi)|} d\xi \\ &\quad + (\lambda(\gamma) - \lambda(\gamma - \eta)) \partial_\gamma x(\gamma) + \lambda(\gamma - \eta) (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)) \\ &= I_5 + I_6 + I_7 + I_8. \end{aligned}$$

The term I_5 yields

$$\begin{aligned} I_5 &\leq \int_{\mathbb{T}} \frac{|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \xi)| \left| |x(\gamma) - x(\gamma - \xi)| - |x(\gamma - \eta) - x(\gamma - \eta - \xi)| \right|}{|x(\gamma) - x(\gamma - \xi)| |x(\gamma - \eta) - x(\gamma - \eta - \xi)|} d\xi \\ &\leq \|F(x)\|_{L^\infty}^2 \|x\|_{C^2} \int_{\mathbb{T}} |\xi|^{-1} |x(\gamma) - x(\gamma - \eta) - (x(\gamma - \xi) - x(\gamma - \eta - \xi))| d\xi \\ &\leq \|F(x)\|_{L^\infty}^2 \|x\|_{C^2} |\eta| \int_0^1 \int_{\mathbb{T}} \frac{|\partial_\gamma x(\gamma + (s-1)\eta) - \partial_\gamma x(\gamma + (s-1)\eta - \xi)|}{|\xi|} d\xi ds \\ &\leq 2\pi \|F(x)\|_{L^\infty}^2 \|x\|_{C^2}^2 |\eta|. \end{aligned}$$

For I_6 we take

$$\begin{aligned}
I_6 &\leq \|F(x)\|_{L^\infty} |\eta| \int_0^1 \int_{\mathbb{T}} \frac{|\partial_\gamma^2 x(\gamma + (s-1)\eta) - \partial_\gamma^2 x(\gamma + (s-1)\eta - \xi)|}{|\xi|} d\xi ds \\
&\leq \|F(x)\|_{L^\infty} \|x\|_{C^{2, \frac{1}{2}}} |\eta| \int_0^1 \int_{\mathbb{T}} |\xi|^{-1/2} d\xi ds \\
&\leq C \|F(x)\|_{L^\infty} \|x\|_{C^{2, \frac{1}{2}}} |\eta|
\end{aligned}$$

We have for I_7

$$\begin{aligned}
I_7 &\leq \frac{2\|x\|_{C^2}}{A(t)} |\eta| \max_\gamma |\partial_\gamma x(\gamma)| \left| \partial_\gamma \left(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} d\eta \right) \right| \\
&\leq 2\|F(x)\|_{L^\infty}^2 \|x\|_{C^2}^2 |\eta| \max_\gamma \left(\int_{\mathbb{T}} \frac{|\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)|}{|x(\gamma) - x(\gamma - \eta)|} d\eta + \int_{\mathbb{T}} \frac{|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta)|^2}{|x(\gamma) - x(\gamma - \eta)|^2} d\eta \right) \\
&\leq 4\|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^4 |\eta|.
\end{aligned}$$

Estimating $\|\lambda\|_{L^\infty}$ as before, easily we get

$$I_8 \leq \|\lambda\|_{L^\infty} \|x\|_{C^2} |\eta| \leq 4\|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^4 |\eta|.$$

The last four estimates show that

$$\frac{d}{dt} \|F(x)\|_{L^p}(t) \leq C \|x\|_{H^3}^4(t) \|F(x)\|_{L^\infty}^5(t) \|F(x)\|_{L^p}(t),$$

by integrating in time and taking $p \rightarrow \infty$, we obtain

$$\|F(x)\|_{L^\infty}(t+h) \leq \|F(x)\|_{L^\infty}(t) \exp \left(C \int_t^{t+h} \|x\|_{H^3}^4(s) \|F(x)\|_{L^\infty}^5(s) ds \right).$$

As in the previous section, it follows

$$\frac{d}{dt} \|F(x)\|_{L^\infty}(t) \leq C \|x\|_{H^3}^4(t) \|F(x)\|_{L^\infty}^6(t).$$

Then, due to (51) and the above estimate, we find finally that

$$\frac{d}{dt} (\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t)) \leq C (\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t))^{10}.$$

Integrating, we have

$$\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t) \leq \frac{\|x_0\|_{H^3} + \|F(x_0)\|_{L^\infty}}{(1 - tC(\|x_0\|_{H^3} + \|F(x_0)\|_{L^\infty})^9)^{\frac{1}{9}}},$$

where C is a constant.

We have used the equality (32) to obtain the a priori estimates. In order to get the solution of (31), we have to choose an appropriate regularized problem preserving (32). We propose the system

$$\begin{aligned} x_t^{\varepsilon,\delta}(\gamma, t) &= \phi_\varepsilon * \int_{\mathbb{T}} \frac{\partial_\gamma(\phi_\varepsilon * x^{\varepsilon,\delta}(\gamma, t) - \phi_\varepsilon * x^{\varepsilon,\delta}(\gamma - \eta, t))}{|x^{\varepsilon,\delta}(\gamma, t) - x^{\varepsilon,\delta}(\gamma - \eta, t)| + \delta} d\eta + \lambda^{\varepsilon,\delta}(\gamma, t) \partial_\gamma x^{\varepsilon,\delta}(\gamma, t), \\ x^{\varepsilon,\delta}(\gamma, 0) &= x_0(\gamma), \end{aligned} \quad (52)$$

with

$$\begin{aligned} \lambda^{\varepsilon,\delta}(\gamma, t) &= \frac{\gamma + \pi}{2\pi} \int_{\mathbb{T}} \frac{\partial_\gamma x^{\varepsilon,\delta}(\gamma, t)}{|\partial_\gamma x^{\varepsilon,\delta}(\gamma, t)|^2} \cdot \partial_\gamma \left(\phi_\varepsilon * \int_{\mathbb{T}} \frac{\partial_\gamma(\phi_\varepsilon * x^{\varepsilon,\delta}(\gamma, t) - \phi_\varepsilon * x^{\varepsilon,\delta}(\gamma - \eta, t))}{|x^{\varepsilon,\delta}(\gamma, t) - x^{\varepsilon,\delta}(\gamma - \eta, t)| + \delta} d\eta \right) d\gamma \\ &\quad - \int_{-\pi}^{\gamma} \frac{\partial_\gamma x^{\varepsilon,\delta}(\eta, t)}{|\partial_\gamma x^{\varepsilon,\delta}(\eta, t)|^2} \cdot \partial_\eta \left(\phi_\varepsilon * \int_{\mathbb{T}} \frac{\partial_\gamma(\phi_\varepsilon * x^{\varepsilon,\delta}(\eta, t) - \phi_\varepsilon * x^{\varepsilon,\delta}(\eta - \xi, t))}{|x^{\varepsilon,\delta}(\eta, t) - x^{\varepsilon,\delta}(\eta - \xi, t)| + \delta} d\xi \right) d\eta. \end{aligned}$$

We can obtain energy estimates of the system (52) depending on ε and δ , but without using (32), and therefore we obtain existence of (52). As long as the solution exists, we have that

$$\partial_\gamma x^{\varepsilon,\delta}(\gamma, t) \cdot \partial_\gamma^2 x^{\varepsilon,\delta}(\gamma, t) = 0.$$

Using this property of the solution, we obtain energy estimates that depend only on δ , and taking $\varepsilon \rightarrow 0$ we get a solution of the following equation

$$\begin{aligned} x_t^\delta(\gamma, t) &= \int_{\mathbb{T}} \frac{\partial_\gamma x^\delta(\gamma, t) - \partial_\gamma x^\delta(\gamma - \eta, t)}{|x^\delta(\gamma, t) - x^\delta(\gamma - \eta, t)| + \delta} d\eta + \lambda^\delta(\gamma, t) \partial_\gamma x^\delta(\gamma, t), \\ x^\delta(\gamma, 0) &= x_0(\gamma), \end{aligned} \quad (53)$$

with

$$\begin{aligned} \lambda^\delta(\gamma, t) &= \frac{\gamma + \pi}{2\pi} \int_{\mathbb{T}} \frac{\partial_\gamma x^\delta(\gamma, t)}{|\partial_\gamma x^\delta(\gamma, t)|^2} \cdot \partial_\gamma \left(\int_{\mathbb{T}} \frac{\partial_\gamma x^\delta(\gamma, t) - \partial_\gamma x^\delta(\gamma - \eta, t)}{|x^\delta(\gamma, t) - x^\delta(\gamma - \eta, t)| + \delta} d\eta \right) d\gamma \\ &\quad - \int_{-\pi}^{\gamma} \frac{\partial_\gamma x^\delta(\eta, t)}{|\partial_\gamma x^\delta(\eta, t)|^2} \cdot \partial_\eta \left(\int_{\mathbb{T}} \frac{\partial_\gamma x^\delta(\eta, t) - \partial_\gamma x^\delta(\eta - \xi, t)}{|x^\delta(\eta, t) - x^\delta(\eta - \xi, t)| + \delta} d\xi \right) d\eta. \end{aligned}$$

Again we have that the solutions of this system satisfy

$$\partial_\gamma x^\delta(\gamma, t) \cdot \partial_\gamma^2 x^\delta(\gamma, t) = 0,$$

and taking advantage of this, we find energy estimates independent of δ . If we tend δ to 0, we conclude the existence result.

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