Singularity Formation in a Surface Wave Model

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Abstract

In this paper we study the Burgers equation with a nonlocal term of the form Hu where H is the Hilbert transform. This system has been considered as a quadratic approximation for the dynamics of a free boundary of a vortex patch (see [6] and [2]). We prove blow up in finite time for a large class of initial data with finite energy. Considering a more general nonlocal term, of the form $\Lambda^\alpha Hu$ for $0 < \alpha < 1$, finite time singularity formation is also shown.

1 Introduction.

We shall study the formation of singularities for the equation

$$\begin{aligned} u_t + uu_x &= \Lambda^\alpha Hu, \\ u(x,0) &= u_0(x), \end{aligned} \tag{1}$$

with $0 \le \alpha < 1$, where H is the Hilbert transform [9] defined by

$$Hf(x) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{f(y)}{x-y} dy$$

and $\Lambda^\alpha \equiv (-\Delta)^{\alpha/2}$ is given by the following expression

$$\Lambda^\alpha f(x) = k_\alpha \int_{\mathbb{R}} \frac{f(x)-f(y)}{|x-y|^{1+\alpha}} dy, \qquad k_\alpha = \frac{\Gamma(1+\alpha)\cos((1-\alpha)\pi/2)}{\pi}.$$

The case $\alpha = 0$

$$u_t + uu_x = Hu \tag{2}$$

was introduced by J. Marsden and A. Weinstein [6] as a second order approximation for the dynamics of a free boundary of a vortex patch (see [3] and [1]). Recently J. Biello and J.K. Hunter [2] proposed it as a model for waves with constant nonzero linearized frequency. They gave a dimensional argument to show that it models nonlinear Hamiltonian waves with constant frequency. In addition, an asymptotic equation from (2) is derived, describing surface waves on a planar discontinuity in vorticity for a two-dimensional inviscid incompressible fluid. They also carried out numerical analysis showing evidence of singularity formation in finite time. Let us point out that the Hamiltonian structure of the equation (1) (in particular for $\alpha = 0$) comes from the representation

$$u_t + \partial_x \left[\frac{\delta \mathcal{H}}{\delta u}\right] = 0, \qquad \text{where} \qquad \mathcal{H}(u) = \int_{\mathbb{R}} \Big(\frac{1}{2} u \Lambda^{\alpha-1} u + \frac{1}{6} u^3\Big) dx. \tag{3}$$

In section 2 we show that the linear term in the equation (2) is too weak to prevent the singularity formation of the Burgers equation. In fact, we show that, if the L^∞ norm of the initial data is large enough compare with the L^2 norm, the maximum of the solution has a singular behavior during the time of existence. One of the ingredients in the proof is to use the following pointwise inequality

$$u(x)^4 \leq 16 ||u||^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2} dy, \tag{4}$$

(see lemma 2.2 below) which can be understood as the local version of the well-known bound

$$||u||^4_{L^4} \leq C ||u||^2_{L^2} ||\Lambda^{1/2} u||^2_{L^2} = \frac{C}{2\pi} ||u||^2_{L^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2} dy dx.$$

In the appendix we provide a generalized pointwise inequality ($n-$dimensional) in terms of fractional derivatives.

In section 3 we consider the more general family of equations, with a higher order term in derivatives, given by (1). By a different method, we prove that the blow up phenomena still arises. Let us note that, since $\Lambda H u = -u_x$, the case $\alpha = 1$ trivializes. Using the same approach as in section 2, it is possible to obtain blow up for $0 < \alpha < 1/3$. Inspired by the method used in [5], we check the evolution of the following quantity

$$J^p_q u(x) = \int_{\mathbb{R}} w^p_q(x-y) u(y) dy, \quad \text{where} \quad w^p_q(x) = \begin{cases} |x|^{-q} \text{sign}\,(x) & \text{if } |x| < 1 \\ |x|^{-p} \text{sign}\,(x) & \text{if } |x| > 1 \end{cases},$$

with $0 < q < 1$ and $p > 2$ to find a singular behavior. Let us note that a similar approach was used by H. Dong, D. Lu and D. Li (see [7]) to show blow up for the Burgers equation with fractional dissipation in the supercritical case ($0 < \alpha < 1$):

$$u_t + u u_x = -\Lambda^\alpha u. \tag{5}$$

A different method to show singularities can be found in [8].

It is well known that the L^p norms of the solutions of equation (5) are bounded for all $1 \leq p \leq \infty$. However, to the best to the authors knowledge, two quantities are conserved by equation (1). The orthogonality property of the Hilbert transform provides the conservation of the L^2 norm, i.e.

$$||u(\cdot, t)||_{L^2(\mathbb{R})} = ||u_0||_{L^2(\mathbb{R})}.$$

Since the equation is given by (3), we have that

$$\int_{\mathbb{R}} \Big(\frac{1}{3} u^3(x,t) + \Big(\Lambda^{\frac{\alpha-1}{2}} u(x,t)\Big)^2\Big) dx = \int_{\mathbb{R}} \Big(\frac{1}{3} u_0^3(x) + \Big(\Lambda^{\frac{\alpha-1}{2}} u_0(x)\Big)^2\Big) dx.$$

2 Blow up for the Burgers-Hilbert equation.

The purpose of this section is to show finite time singularity formation in solutions of the equation (2). The result we shall prove is the following:

Theorem 2.1 *Let $u_0 \in L^2(\mathbb{R}) \cap C^{1+\delta}(\mathbb{R})$, with $0 < \delta < 1$, satisfying the following condition:*
There exists a point $\beta_0 \in \mathbb{R}$ with

$$Hu_0(\beta_0) > 0, \tag{6}$$

such that

$$u_0(\beta_0) \geq \left(32\pi ||u_0||^2_{L^2(\mathbb{R})}\right)^{1/3}. \tag{7}$$

Then there is a finite time T such that

$$\lim_{t \to T} ||u(\cdot, t)||_{C^{1+\delta}(\mathbb{R})} = \infty,$$

where $u(x,t)$ is the solution to the equation (2).

Proof: Let us assume that there exist a solution of the equation (2)

$$u(x,t) \in C([0,T), C^{1+\delta}(\mathbb{R})),$$

for all time $T < \infty$ and with u_0 satisfying the hypotheses.

Now, we shall define the trajectories $x(\beta, t)$ by the equation

$$\begin{aligned} \frac{dx(\beta,t)}{dt} &= u(x(\beta,t),t), \\ x(\beta,0) &= \beta. \end{aligned}$$

Considering the evolution of the solution along trajectories, it is easy to get the identity

$$\frac{du(x(\beta,t),t)}{dt} = u_t(x(\beta,t),t) + \frac{dx(\beta,t)}{dt} u_x(x(\beta,t),t) = Hu(x(\beta,t),t),$$

and taking a derivative in time we obtain

$$\begin{aligned} \frac{d^2u(x(\beta,t),t)}{dt^2} &= Hu_t(x(\beta,t),t) + u(x(\beta,t),t)Hu_x(x(\beta,t),t) \\ &= -H(uu_x)(x(\beta,t),t) - u(x(\beta,t),t) + u(x(\beta,t),t)Hu_x(x(\beta,t),t). \end{aligned}$$

Since

$$H(uu_x)(x) = \frac{1}{2}H((u^2)_x) = \frac{1}{2}\Lambda(u^2)(x),$$

we can write

$$\frac{1}{2}\Lambda(u^2)(x) = \frac{1}{2\pi} P.V \int_{\mathbb{R}} \frac{u(x)^2 - u(y)^2}{(x-y)^2} dy = u(x)\Lambda u(x) - \frac{1}{2\pi} \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x-y)^2} dy,$$

and therefore it follows that

$$\frac{d^2u(x(\beta,t),t)}{dt^2} = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(u(x(\beta,t),t)) - u(y,t)))^2}{(x(\beta,t)-y)^2}dy - u(x(\beta,t),t). \tag{8}$$

In order to continue with the proof we will prove the lemma below (for similar approach see [4]):

Lemma 2.2 *Let $u \in L^2(\mathbb{R}) \cap C^{1+\delta}(\mathbb{R})$, for $0<\delta<1$. Then*

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2}dy \geq Cu(x)^4,$$

where

$$C = \frac{1}{32\pi E}$$

and

$$E = ||u||^2_{L^2(\mathbb{R})}.$$

Proof of lemma 2.2: Let us assume that $u(x)>0$ (a similar proof holds for $u(x)<0$). Let Ω be the set

$$\Omega = \{y \in \mathbb{R} \quad : \quad |x-y| < \Delta\},$$

where Δ will be given below. And let Ω^1 and Ω^2 be the subsets

$$\begin{aligned} \Omega^1 &= \{y \in \Omega \,:\, u(x)-u(y) \geq \frac{u(x)}{2}\}, \\ \Omega^2 &= \{y \in \Omega \,:\, u(x)-u(y) < \frac{u(x)}{2}\} = \{y \in \Omega \,:\, u(y) > \frac{u(x)}{2}\}. \end{aligned}$$

Then

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2}dy \geq \frac{u(x)^2}{8\pi\Delta^2}|\Omega^1|.$$

On the other hand

$$E = \int_{\mathbb{R}} u(y)^2 dy \geq \int_{\Omega^2} u(y)^2 dy \geq \frac{u(x)^2}{4}|\Omega^2|,$$

and therefore

$$|\Omega^2| \leq \frac{4E}{u(x)^2}.$$

Since $|\Omega^1| = |\Omega| - |\Omega^2|$ and $|\Omega| = 2\Delta$, we have that

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2}dy \geq \frac{u(x)^2}{8\pi\Delta^2}(2\Delta - \frac{4E}{u(x)^2}).$$

We achieve the conclusion of lemma 2.2 by taking $\Delta = \dfrac{4E}{u(x)^2}$. ∎

Next, let us define $J(t) = u(x(\beta_0, t), t)$. Thus, applying lemma 2.2 to the expression (8), we obtain the inequality

$$J_{tt}(t) \geq CJ(t)^4 - J(t). \qquad (9)$$

Since $Hu_0(\beta_0) > 0$ and $J_t(t) = Hu(x(\beta_0, t), t)$, we obtain that $J_t(t) > 0$ and $J(t) > J(0)$ for $t \in (0, t^*)$ and t^* small enough. Therefore, multiplying (9) by $J_t(t)$ we have that

$$\frac{1}{2}(J_t(t)^2)_t \geq \frac{C}{5}(J(t)^5)_t - \frac{1}{2}(J(t)^2)_t, \quad \forall t \in [0, t^*).$$

Integrating this inequality in time from 0 to t we get

$$J_t(t) \geq \left(J_t(0)^2 + \frac{2C}{5}(J(t)^5 - J(0)^5) - (J(t)^2 - J(0)^2)\right)^{\frac{1}{2}}, \quad \forall t \in [0, t^*). \qquad (10)$$

Now, since $CJ(0)^4 - J(0) \geq 0$, by the statements of the theorem we obtain that $J_{tt}(t) > J_{tt}(0) \geq 0$ for $t \in (0, t^*)$. Therefore, $J_t(t)$ is an increasing function $[0, t^*)$. Thus, the inequality (10) holds for all time t and we have a contradiction.

Remark 2.3 *It is easy to check that there exists a large class of functions satisfying the requirement of the theorem (2.1). For example, we can consider the function*

$$\begin{aligned} u_0(x) &= \frac{-ax}{1 + (bx)^2}, \\ Hu_0(x) &= \frac{a}{1 + (bx)^2}, \end{aligned}$$

where a, $b > 0$. Choosing a and b in a suitable way we can have the norm $||u_0||_{L^2(\mathbb{R})}$ as small as we want and the norm $||u_0||_{L^\infty(\mathbb{R})}$ as large as we want.

Remark 2.4 *We note that the requirements (6) and (7) in theorem 2.1 can be replaced by*

$$\begin{aligned} Hu_0(\beta_0) &\geq 0, \\ u_0(\beta_0) &> \left(32\pi ||u_0||^2_{L^2(\mathbb{R})}\right)^{1/3}, \end{aligned}$$

attaining the same conclusion.

3 Blow up for the whole range $0 < \alpha < 1$.

In this section we shall show formation of singularities for the equation (1), with $0 < \alpha < 1$. The aim is to prove the following result:

Theorem 3.1 *There exist initial data $u_0 \in L^2(\mathbb{R}) \cap C^{1+\delta}(\mathbb{R})$, with $0 < \delta < 1$, and a finite time T, depending on u_0, such that*

$$\lim_{t \to T} ||u(\cdot, t)||_{C^{1+\delta}(\mathbb{R})} = \infty$$

where $u(x, t)$ is the solution to the equation (1).

Proof: Let us assume that there exists a solution of the equation (1), $u(x,t) \in C([0,T), C^{1+\delta}(\mathbb{R}))$, for all time $T < \infty$. Let $J_q^p u$ be the convolution

$$J_q^p u(x) = \int_{\mathbb{R}} w_q^p(x-y)u(y)dy$$

where

$$w_q^p(x) = \begin{cases} \frac{1}{|x|^q}\text{sign}\,(x) & \text{if } |x| < 1 \\ \frac{1}{|x|^p}\text{sign}\,(x) & \text{if } |x| > 1 \end{cases},$$

with $0 < q < 1$ and $p > 2$. In order to prove theorem 3.1 we shall need the following two lemmas.

Lemma 3.2 *Let f in $C^{1+\delta}(\mathbb{R}) \cap L^2(\mathbb{R})$ and $0 < \alpha < 1$. Then*

$$\Lambda^\alpha H f(x) = k_\alpha \int_{\mathbb{R}} \frac{f(x)-f(y)}{|x-y|^{1+\alpha}}\text{sign}\,(x-y)dy$$

where

$$k_\alpha = -\frac{\Gamma(1+\alpha)\sin((1+\alpha)\pi/2)}{\pi}.$$

Proof: Let f be a function on the Schwartz class. The inverse Fourier transform formula yields

$$\Lambda^\alpha H f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} -i\text{sign}\,(k)|k|^\alpha \hat{f}(k)\exp(ikx)dk.$$

We will understand the above identity as the following limit

$$\begin{aligned}\Lambda^\alpha H f(x) &= \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{\mathbb{R}} -i\text{sign}\,(k)|k|^\alpha \exp(-\varepsilon|k|)\exp(ikx)\Big(\int_{\mathbb{R}} f(y)\exp(-iky)dy\Big)dk \\ &= \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{\mathbb{R}} f(y)\Big(\int_0^\infty k^\alpha \exp(-\varepsilon k)\sin(k(x-y))dk\Big)dy.\end{aligned}$$

Next, we can compute that

$$\begin{aligned}\Lambda^\alpha H f(x) &= \lim_{\varepsilon\to 0^+}\frac{\Gamma(1+\alpha)}{\pi}\int_{\mathbb{R}} \frac{f(y)}{(\varepsilon^2+(x-y)^2)^{(1+\alpha)/2}}\sin\Big((1+\alpha)\arctan\Big(\frac{x-y}{\varepsilon}\Big)\Big)dy \\ &= -\lim_{\varepsilon\to 0^+}\frac{\Gamma(1+\alpha)}{\pi}\int_{\mathbb{R}} \frac{f(x)-f(y)}{(\varepsilon^2+(x-y)^2)^{(1+\alpha)/2}}\sin\Big((1+\alpha)\arctan\Big(\frac{x-y}{\varepsilon}\Big)\Big)dy \\ &= -\frac{\Gamma(1+\alpha)\sin((1+\alpha)\pi/2)}{\pi}\int_{\mathbb{R}} \frac{f(x)-f(y)}{|x-y|^{1+\alpha}}\text{sign}\,(x-y)dy.\end{aligned}$$

We achieve the conclusion of lemma 3.2 by the classical density argument. ∎

Lemma 3.3 *Let $I_q^p(x)$ be the integral*

$$I_q^p(x) = \int_{\mathbb{R}} \frac{w_q^p(x)-w_q^p(y)}{|x-y|^{1+\alpha}}\text{sign}\,(x-y)dy$$

where $0<q<1$ and $p>2$. Then

$$|I_q^p(x)| \le \begin{cases} \frac{K^1}{|x|^{q+\alpha}} & \text{if } 0<|x|<\frac{1}{2} \\ \frac{K^2}{|x|^{2+\alpha}} & \text{if } 2<|x|<\infty \\ K^3 & \text{if } \frac{1}{2}\le |x| \le 2 \end{cases}$$

where K^1, K^2 and K^3 are universal constants depending on q and p.

Proof: Since the function $I_q^p(x)$ is even, we can assume that $x>0$. The constant values of K^1 and K^2 can be different along the estimates below.

First, let us consider the case $0<x<1/2$. We split as follows

$$I_q^p(x) = \int_{|y|<1} dy + \int_{|y|>1} dy = I_1(x) + I_2(x).$$

It yields

$$\begin{aligned} I_1(x) &= \int_{|y|<1} \frac{\frac{1}{x^q} - \text{sign}\,(y)\frac{1}{|y|^q}}{|x-y|^{1+\alpha}} \text{sign}\,(x-y) dy \\ &= \int_0^1 \Big(\frac{\frac{1}{x^q} - \frac{1}{y^q}}{|x-y|^{1+\alpha}} \text{sign}\,(x-y) + \frac{\frac{1}{x^q} + \frac{1}{y^q}}{|x+y|^{1+\alpha}} \Big) dy, \end{aligned}$$

and a change of variables allow us to split further

$$\begin{aligned} I_1(x) &= \frac{1}{x^{q+\alpha}} \int_0^{\frac{1}{x}} \Big(\frac{1-\frac{1}{\eta^q}}{|1-\eta|^{1+\alpha}} \text{sign}\,(1-\eta) + \frac{1+\frac{1}{\eta^q}}{|1+\eta|^{1+\alpha}} \Big) d\eta \\ &= \frac{1}{x^{q+\alpha}} \Big(\int_0^1 + \int_1^{\frac{1}{x}} \Big) = \frac{1}{x^{q+\alpha}} (F_1(x) + F_2(x)). \end{aligned}$$

For $F_1(x)$ we find the bound

$$|F_1(x)| \le \int_0^1 \Big| \frac{1-\frac{1}{\eta^q}}{|1-\eta|^{1+\alpha}} \Big| d\eta + \int_0^1 \Big| \frac{1+\frac{1}{\eta^q}}{|1+\eta|^{1+\alpha}} \Big| d\eta \le K_1.$$

On the other hand

$$F_2(x) = \int_1^{\frac{1}{x}} \Big(\frac{\frac{1}{\eta^q} - 1}{|1-\eta|^{1+\alpha}} + \frac{\frac{1}{\eta^q}+1}{|1+\eta|^{1+\alpha}} \Big) d\eta = \int_1^{\frac{3}{2}} + \int_{\frac{3}{2}}^{\frac{1}{x}} = j_1(x) + j_2(x).$$

For $j_1(x)$ it is easy to obtain

$$|j_1(x)| \le \int_1^{\frac{3}{2}} \Big| \frac{\frac{1}{\eta^q} - 1}{|1-\eta|^{1+\alpha}} \Big| d\eta + \int_1^{\frac{3}{2}} \Big| \frac{\frac{1}{\eta^q}+1}{|1+\eta|^{1+\alpha}} \Big| d\eta \le K_1.$$

For $j_2(x)$ we decompose as follows

$$j_2(x) = \int_{\frac{3}{2}}^{\frac{1}{x}} \frac{1}{\eta^q} \Big(\frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}} \Big) d\eta + \int_{\frac{3}{2}}^{\frac{1}{x}} \Big(\frac{1}{|1-\eta|^{1+\alpha}} - \frac{1}{|1+\eta|^{1+\alpha}} \Big) d\eta.$$

Thus, since $0 < q < 1$ and

$$\Big|\frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}}\Big| \le \frac{K^1}{|\eta|^{1+\alpha}}, \quad \text{for} \quad \eta \in [3/2, \infty)$$

we have that

$$|j_2(x)| \le K^1 \int_{\frac{3}{2}}^{\infty} \frac{1}{\eta^{q+1}} d\eta + \int_{\frac{3}{2}}^{\infty} \Big|\frac{1}{|1-\eta|^{1+\alpha}} - \frac{1}{|1+\eta|^{1+\alpha}}\Big| d\eta \le K^1.$$

Let us continue with I_2 which can be written in the form

$$\begin{aligned} I_2(x) &= \int_{|y|>1} \frac{\frac{1}{x^q} - \text{sign}\,(y)\frac{1}{|y|^p}}{|x-y|^{1+\alpha}} \text{sign}\,(x-y) dy = \int_1^{\infty} \Big(- \frac{\frac{1}{x^q} - \frac{1}{|y|^p}}{|x-y|^{1+\alpha}} + \frac{\frac{1}{x^q} + \frac{1}{|y|^p}}{|x+y|^{1+\alpha}} \Big) dy \\ &= \frac{1}{x^{q+\alpha}} \int_{\frac{1}{x}}^{\infty} \Big(\frac{\frac{x^{q-p}}{\eta^p} - 1}{|1-\eta|^{1+\alpha}} + \frac{1 + \frac{x^{q-p}}{\eta^p}}{|1-\eta|^{1+\alpha}} \Big) d\eta. \end{aligned}$$

The following decomposition

$$\begin{aligned} I_2(x) &= \frac{1}{x^{q+\alpha}} \int_{\frac{1}{x}}^{\infty} \Big(\frac{1}{|1+\eta|^{1+\alpha}} - \frac{1}{|1-\eta|^{1+\alpha}} \Big) d\eta \\ &\quad + \frac{1}{x^{p+\alpha}} \int_{\frac{1}{x}}^{\infty} \frac{1}{\eta^p} \Big(\frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}} \Big) d\eta \end{aligned}$$

yields

$$\begin{aligned} |I_2(x)| &\le \frac{K^1}{x^{q+\alpha}} + \frac{1}{x^{p+\alpha}} \int_{\frac{1}{x}}^{\infty} \frac{1}{\eta^p} \Big| \frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}} \Big| d\eta \\ &\le \frac{K^1}{x^{q+\alpha}} + \frac{K^1}{x^{p+\alpha}} \int_{\frac{1}{x}}^{\infty} \frac{1}{\eta^{p+1}} d\eta \le K^1 \Big(\frac{1}{|x|^{\alpha+q}} + \frac{1}{|x|^{\alpha}} \Big) \le \frac{K_1}{x^{q+\alpha}}. \end{aligned}$$

Next, we consider the case $2 < x < \infty$ taking

$$I_q^p(x) = \int_{\mathbb{R}} \frac{\frac{1}{x^p} - w(y)}{|x-y|^{1+\alpha}} \text{sign}\,(x-y) dy = \int_{|y|<1} dy + \int_{|y|>1} dy = J_1(x) + J_2(x).$$

For $J_2(x)$ we have that

$$\begin{aligned} J_2(x) &= \int_{|y|>1} \frac{\frac{1}{x^p} - \text{sign}\,(y)\frac{1}{|y|^p}}{|x-y|^{1+\alpha}} \text{sign}\,(x-y) dy \\ &= \int_1^{\infty} \Big(\frac{\frac{1}{x^p} - \frac{1}{|y|^p}}{|x-y|^{1+\alpha}} \text{sign}\,(x-y) + \frac{\frac{1}{x^p} + \frac{1}{|y|^p}}{|x+y|^{1+\alpha}} \Big) dy \end{aligned}$$

and a change of variables provides

$$J_2(x) = \frac{1}{x^{p+\alpha}} \int_{\frac{1}{x}}^{\infty} \Big(\frac{1-\frac{1}{\eta^p}}{|1-\eta|^{1+\alpha}} \mathrm{sign}\,(1-\eta) + \frac{1+\frac{1}{\eta^p}}{|1+\eta|^{1+\alpha}} \Big) d\eta$$
$$= \frac{1}{x^{\alpha+p}} \Big(\int_{\frac{1}{x}}^{1} + \int_{1}^{\infty} \Big) = \frac{1}{x^{\alpha+p}} (H_1(x) + H_2(x)).$$

For $H_2(x)$ one could bound as follow

$$|H_2(x)| \leq \int_1^\infty \Big| \frac{1-\frac{1}{\eta^p}}{|1-\eta|^{1+\alpha}} \Big| d\eta + \int_1^\infty \Big| \frac{1+\frac{1}{\eta^p}}{|1+\eta|^{1+\alpha}} \Big| d\eta \leq K^2.$$

On the other hand, in $H_1(x)$ we split further

$$H_1(x) = \int_{\frac{1}{x}}^{1} \Big(\frac{1-\frac{1}{\eta^p}}{|1-\eta|^{1+\alpha}} + \frac{1+\frac{1}{\eta^p}}{|1+\eta|^{1+\alpha}} \Big) d\eta = \int_{\frac{1}{x}}^{\frac{2}{3}} d\eta + \int_{\frac{2}{3}}^{1} d\eta = h_1(x) + h_2(x).$$

The term $h_2(x)$ is bounded by

$$|h_2(x)| \leq \int_{\frac{2}{3}}^{1} \Big| \frac{1-\frac{1}{\eta^p}}{|1-\eta|^{1+\alpha}} \Big| d\eta + \int_{\frac{2}{3}}^{1} \Big| \frac{1+\frac{1}{\eta^p}}{|1+\eta|^{1+\alpha}} \Big| d\eta \leq K^2.$$

We reorganize $h_1(x)$ so that

$$h_1(x) = \int_{\frac{1}{x}}^{\frac{2}{3}} \Big(\frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}} \Big) d\eta + \int_{\frac{1}{x}}^{\frac{2}{3}} \frac{1}{\eta^p} \Big(\frac{1}{|1-\eta|^{1+\alpha}} - \frac{1}{|1+\eta|^{1+\alpha}} \Big) d\eta.$$

Since $p > 2$ and

$$\Big| \frac{1}{|1-\eta|^{1+\alpha}} - \frac{1}{|1+\eta|^{1+\alpha}} \Big| \leq K^2 \eta \quad \text{for} \quad \eta \in [0, 2/3],$$

we obtain that

$$|h_1(x)| \leq \int_0^{\frac{2}{3}} \Big| \frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}} \Big| d\eta + K^2 \int_{\frac{1}{x}}^{\frac{2}{3}} \frac{1}{\eta^{p-1}} d\eta \leq K^2 (1 + x^{p-2}).$$

Therefore

$$|J_2(x)| \leq K^2 \Big(\frac{1}{x^{p+\alpha}} + \frac{1}{x^{2+\alpha}} \Big) \leq \frac{K^2}{x^{2+\alpha}}.$$

Next, we deal with J_1 given by

$$J_1(x) = \int_{|y|<1} \frac{\frac{1}{x^p} - \mathrm{sign}\,(y) \frac{1}{|y|^q}}{|x-y|^{1+\alpha}} dy = \int_0^1 \left(\frac{\frac{1}{x^p} - \frac{1}{|y|^q}}{|x-y|^{1+\alpha}} + \frac{\frac{1}{x^p} + \frac{1}{|y|^q}}{|x+y|^{1+\alpha}} \right) dy$$
$$= \frac{1}{x^{p+\alpha}} \int_0^{\frac{1}{x}} \left(\frac{1 - \frac{x^{p-q}}{\eta^q}}{|1-\eta|^{1+\alpha}} + \frac{1 + \frac{x^{p-q}}{\eta^q}}{|1+\eta|^{1+\alpha}} \right) d\eta.$$

Hence

$$J_1(x) = \frac{1}{x^{p+\alpha}} \int_0^{\frac{1}{x}} \Big(\frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}} \Big) d\eta + \frac{1}{x^{q+\alpha}} \int_0^{\frac{1}{x}} \frac{1}{\eta^q} \Big(\frac{1}{|1+\eta|^{1+\alpha}} - \frac{1}{|1-\eta|^{1+\alpha}} \Big) d\eta.$$

Since $p > 2$ and

$$\Big| \frac{1}{|1+\eta|^{1+\alpha}} - \frac{1}{|1-\eta|^{1+\alpha}} \Big| \leq K^2 \eta, \quad \text{for} \quad \eta \in [0, 1/2],$$

we obtain that

$$|J_1(x)| \leq \frac{K^2}{x^{p+\alpha}} + \frac{K^2}{x^{q+\alpha}} \int_0^{\frac{1}{x}} \eta^{1-q} d\eta \leq K^2 \Big(\frac{1}{x^{p+\alpha}} + \frac{1}{x^{2+\alpha}} \Big) \leq \frac{K^2}{x^{2+\alpha}}.$$

The bound for $1/2 < x < 2$ is obvious, which allow us to conclude the proof. ▪

In order to prove theorem 3.1, we shall study the evolution of $J(t) = J_q^p u(x_b(t), t)$, where x_b is the trajectory $x_b(t) = x(0, t)$. Hence

$$\frac{dJ(t)}{dt} = -\frac{1}{2} J_q^p((u^2)_x)(x_b(t), t) + J_q^p \Lambda^\alpha H u(x_b(t), t) + u(x_b(t), t)(\partial_x J_q^p u)(x_b(t), t).$$

We can write

$$J_q^p((u^2)_x) = \int_{\mathbb{R}} (u(x)^2 - u(y)^2) W_q^p(x-y) dy$$

and

$$\partial_x (J_q^p u)(x) = \int_{\mathbb{R}} (u(x) - u(y)) W_q^p(x-y) dy$$

where

$$W_q^p = \begin{cases} \frac{q}{|x|^{q+1}} & \text{if } |x| < 1 \\ \frac{p}{|x|^{p+1}} & \text{if } |x| > 1 \end{cases}.$$

Then, it is easy to check that

$$-\frac{1}{2} J_q^p((u^2)_x)(x) + u(x)(\partial_x J_q^p u)(x) = \frac{1}{2} \int_{\mathbb{R}} (u(x) - u(y))^2 W_q^p(x-y) dy,$$

and therefore

$$\frac{dJ(t)}{dt} = \frac{1}{2} \int_{\mathbb{R}} (u(x_b(t)) - u(y))^2 W_q^p(x_b(t) - y) dy + J_q^p \Lambda^\alpha H u(x_b(t), t). \tag{11}$$

Using lemma (3.2), the linear term becomes

$$J_q^p \Lambda^\alpha H u(x) = k_\alpha \int_{\mathbb{R}} w_q^p(x-y) \int_{\mathbb{R}} \frac{u(y) - u(s)}{|y-s|^{1+\alpha}} \text{sign}\,(y-s) ds dy,$$

and a wise use of the principal value provides

$$\begin{aligned}
J_q^p \Lambda^\alpha H u(x) &= k_\alpha \int_{\mathbb{R}} w_q^p(x-y) P.V. \int_{\mathbb{R}} \frac{-u(s)}{|y-s|^{1+\alpha}} \text{sign}\,(y-s) ds dy \\
&= k_\alpha \int_{\mathbb{R}} w_q^p(x-y) P.V. \int_{\mathbb{R}} \frac{u(x)-u(s)}{|y-s|^{1+\alpha}} \text{sign}\,(y-s) ds dy \\
&= k_\alpha \int_{\mathbb{R}} (u(x)-u(s)) P.V. \int_{\mathbb{R}} \frac{w_q^p(x-y)}{|y-s|^{1+\alpha}} \text{sign}\,(y-s) dy ds \\
&= k_\alpha \int_{\mathbb{R}} (u(x)-u(s)) \int_{\mathbb{R}} \frac{w_q^p(x-s)-w_q^p(x-y)}{|y-s|^{1+\alpha}} \text{sign}\,(s-y) dy ds
\end{aligned}$$

to find finally

$$J_q^p \Lambda^\alpha H u(x) = k_\alpha \int_{\mathbb{R}} (u(x)-u(s)) I_q^p(x-s) ds.$$

Therefore

$$\begin{aligned}
|J_q^p \Lambda^\alpha H u(x)| &\le |k_\alpha| \int_{\mathbb{R}} |u(x)-u(y)||I_q^p(x-y)| dy \\
&\le |k_\alpha| \left(\int_{\mathbb{R}} (u(x)-u(y))^2 W_q^p(x-y) dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{I_q^p(x)^2}{W_q^p(x)} dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Since

$$\frac{I_q^p(x)^2}{W_q^p(x)} \le \begin{cases} \frac{C}{|x|^{2\alpha+q-1}} & \text{when } |x| \to 0 \\ \frac{C}{|x|^{3+2\alpha-p}} & \text{when } |x| \to \infty \end{cases},$$

by taking $2 < p < 2+2\alpha$ and $q < 2(1-\alpha)$, we obtain that

$$\begin{aligned}
|J_q^p \Lambda^\alpha H u(x)| &\le C(p,q) \left(\int_{\mathbb{R}} (u(x)-u(y))^2 W_q^p(y) dy \right)^{\frac{1}{2}} \\
&\le \frac{1}{4} \int_{\mathbb{R}} (u(x)-u(y))^2 W_q^p(x-y) dy + C.
\end{aligned}$$

This inequality in the equation (11) yields

$$\frac{dJ(t)}{dt} \ge \frac{1}{4} \int_{\mathbb{R}} (u(x_b(t))-u(y))^2 W_q^p(x_b(t)-y) dy - C(p,q)$$

On the other hand

$$\begin{aligned}
J(t) = \int_{\mathbb{R}} u(y) w_q^p(x_b(t)-y) dy &= \int_{\mathbb{R}} (u(y)-u(x_b(t))) w_q^p(x_b(t)-y) dy \\
\le \left(\int_{\mathbb{R}} (u(x_b(t))-u(y))^2 W_q^p(x_b(t)-y) dy \right)^{\frac{1}{2}} &\left(\int_{\mathbb{R}} \frac{w_q^p(x)^2}{W_q^p(x)} dx \right)^{\frac{1}{2}}.
\end{aligned}$$

The following bound

$$\frac{w_q^p(x)^2}{W_q^p(x)} \le \begin{cases} \frac{C}{|x|^{q-1}} & \text{when } |x| \to 0 \\ \frac{C}{|x|^{p-1}} & \text{when } |x| \to \infty \end{cases}$$

allows us to obtain, for $2 < p < 2+2\alpha$ and $0 < q < 1$,

$$\int_{\mathbb{R}} (u(x_b(t)) - u(y))^2 W_q^p(x_b(t) - y)dy \geq c(q,p)J(t)^2.$$

Therefore we obtain a quadratic evolution equation

$$\frac{dJ(t)}{dt} \geq c(q,p)J(t)^2 - C(q,p)$$

and by taking $c(q,p)J(0)^2 - C(q,p) > 0$, we find a contradiction for the mere fact that

$$J(t) \leq C(q,p)||u||_{L^\infty}.$$

4 Appendix

In this section we generalize the pointwise inequality (4) evolving the nonlocal operator $2f\Lambda^\alpha f - \Lambda^\alpha(f^2)$. Some simple applications to Gagliardo-Nirenberg-Sobolev inequalities are also shown.

Lemma 4.1 *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$ in the Schwartz class, $0 < \alpha < 2$ and $0 < p < \infty$. Then*

$$|f(x)|^{2+\frac{\alpha p}{n}} \leq C(\alpha,p,n)||f||_{L^p(\mathbb{R}^n)}^{\frac{\alpha p}{n}}(2f(x)\Lambda^\alpha f(x) - \Lambda^\alpha(f^2)(x)) \qquad (12)$$

for any $x \in \mathbb{R}^n$.

Proof: The formula for the operator Λ^α in $\mathbb{R}^n$

$$\Lambda^\alpha f(x) = k_{\alpha,n}\int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+\alpha}}dy$$

and $0 < \alpha < 2$, allows us to find

$$2f(x)\Lambda^\alpha f(x) - \Lambda^\alpha(f^2)(x) = k_{\alpha,n}\int_{\mathbb{R}^n} \frac{(f(y) - f(x))^2}{|x-y|^{n+\alpha}}dy.$$

We consider $f(x) > 0$, being the case $f(x) < 0$ analogous. Let Ω, Ω^1 and Ω^2 be the sets

$$\begin{aligned}
\Omega\ &= \{y \in \mathbb{R} : |x-y| < \Delta\},\\
\Omega^1 &= \{y \in \Omega : f(x) - f(y) \geq f(x)/2\},\\
\Omega^2 &= \{y \in \Omega : f(x) - f(y) < f(x)/2\} = \{y \in \Omega : f(y) > f(x)/2\}.
\end{aligned}$$

Then

$$2f(x)\Lambda^\alpha f(x) - \Lambda^\alpha(f^2)(x) \geq k_{\alpha,n}\int_{\Omega} \frac{(f(y) - f(x))^2}{|x-y|^{n+\alpha}}dy \geq k_{\alpha,n}\frac{f(x)^2}{4\Delta^{n+\alpha}}|\Omega^1|.$$

On the other hand

$$||f||_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |f(y)|^p dy \geq \frac{f(x)^p}{2^p}|\Omega^2|,$$

therefore

$$2f(x)\Lambda^{\alpha}f(x)-\Lambda^{\alpha}(f^2)(x)\geq k_{\alpha,n}\frac{f(x)^2}{4\Delta^{n+\alpha}}(|\Omega|-|\Omega^2|)\geq k_{\alpha,n}\frac{f(x)^2}{4\Delta^{n+\alpha}}(c_n\Delta^n-\frac{2^p||f||^p_{L^p(\mathbb{R}^n)}}{f(x)^p}),$$

where $c_n=2\pi^{n/2}/(n\Gamma(n/2))$. By choosing

$$\Delta^n=\frac{(n+\alpha)2^p||f||^p_{L^p(\mathbb{R}^n)}}{\alpha c_n f(x)^p}$$

we obtain the desired estimate. ▪

Remark 4.2 *Inequality* (12) *allows us to get easily the following Gagliardo-Nirenberg-Sobolev estimate:*

$$||f||^{2+\frac{\alpha p}{n}}_{L^{2+\frac{\alpha p}{n}}}\leq 2C(\alpha,p,n)||f||^{\frac{\alpha p}{n}}_{L^p(\mathbb{R}^n)}||\Lambda^{\frac{\alpha}{2}}f||^2_{L^2(\mathbb{R}^n)},$$

for $0<\alpha<2$ *and* $0<p<\infty$.

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