Interface evolution: the Hele-Shaw and Muskat problems.

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Abstract

We study the dynamics of the interface between two incompressible 2-D flows where the evolution equation is obtained from Darcy's law. The free boundary is given by the discontinuity among the densities and viscosities of the fluids. This physical scenario is known as the two dimensional Muskat problem or the two-phase Hele-Shaw flow. We prove local-existence in Sobolev spaces when, initially, the difference of the gradients of the pressure in the normal direction has the proper sign, an assumption which is also known as the Rayleigh-Taylor condition.

1 Introduction

We consider the following evolution problem for the active scalar $\rho = \rho(x,t)$, $x \in \mathbb{R}^2$, and $t \geq 0$:

$$\rho_t + v \cdot \nabla \rho = 0,$$

with a velocity $v = (v_1, v_2)$ satisfying the momentum equation

$$\frac{\mu}{\kappa} v = -\nabla p - (0, \mathrm{g}\, \rho), \tag{1.1}$$

and the incompressibility condition $\nabla \cdot v = 0$.

In the following we achieve a rather complete local existence analysis of the dynamics of the interface between two incompressible 2-D flows with different characteristics (i.e. distinct values of μ and ρ) which are evolving under (1.1), also known as Darcy's law [2]. This system was studied by Muskat [15] in order to model the interface between two fluids in a porous media, where p is the pressure, μ is the dynamic viscosity, κ is the permeability of the medium, ρ is the liquid density and g is the acceleration due to gravity. Saffman and Taylor [16] made the observation that the one phase version (one of the fluids has zero viscosity) was also known as the Hele-Shaw cell equation [13], which, in turn, is the zero-specific heat case of the classical one-phase Stefan problem.

There is a vast literature about those problems (see [5] and [14] for references). In order to frame our result let us point out that in [17] is treated the case where both densities are equal, showing global existence for small data in the stable case and ill-possednes in the unstable case. In [1] the well-possednes in the stable case was considered under time dependent assumption of the arc-chord condition. Finally, in the case where the viscosities are the same, the character of the interphase as the graph of a function is preserved and in [8] [9] this fact has been used to prove local existence and a maximum principle, in the stable case, together with ill-possednes in the unstable situation.

Due to the direction of gravity, the horizontal and the vertical coordinates play different rolls. Here we shall assume spatial periodicity in the horizontal space variable, says $\rho(x_1 + 2k\pi, x_2, t) = \rho(x_1, x_2, t)$. The free boundary is given by the discontinuity on the densities and viscosities of the fluids, where (μ, ρ) are defined by

$$(\mu, \rho)(x_1, x_2, t) = \begin{cases} (\mu^1, \rho^1), & x \in \Omega^1(t) \\ (\mu^2, \rho^2), & x \in \Omega^2(t) = \mathbb{R}^2 - \Omega^1(t), \end{cases} \tag{1.2}$$

and $\mu^1 \neq \mu^2$, and $\rho^1 \neq \rho^2$ are constants.

Let the free boundary be parameterized by

$$\partial\Omega^j(t) = \{z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t)) : \alpha \in \mathbb{R}\}$$

such that

$$(z_1(\alpha + 2k\pi, t), z_2(\alpha + 2k\pi, t)) = (z_1(\alpha, t) + 2k\pi, z_2(\alpha, t)),$$

with the initial data $z(\alpha, 0) = z_0(\alpha)$.

Notice that each fluid is irrotational, i.e. $\omega = \nabla \times u = 0$, in the interior of each domain Ω^i $(i = 1, 2)$. Therefore the vorticity ω has its support on the curve $z(\alpha, t)$ and it can be shown easily to be of the form

$$\omega(x, t) = \varpi(\alpha, t)\delta(x - z(\alpha, t)).$$

Then $z(\alpha, t)$ evolves with a velocity field coming from Biot-Savart law, which can be explicitly computed and is given by the Birkhoff-Rott integral of the amplitude ϖ along the interface curve:

$$\begin{aligned} BR(z, \varpi)(\alpha, t) = \big(-\frac{1}{4\pi} PV \int_{\mathbb{T}} \varpi(\beta, t) \frac{\tanh(\frac{z_2(\alpha,t) - z_2(\beta,t)}{2})(1 + \tan^2(\frac{z_1(\alpha,t) - z_1(\beta,t)}{2}))}{\tan^2(\frac{z_1(\alpha,t) - z_1(\beta,t)}{2}) + \tanh^2(\frac{z_2(\alpha,t) - z_2(\beta,t)}{2})} d\beta, \\ \frac{1}{4\pi} PV \int_{\mathbb{T}} \varpi(\beta, t) \frac{\tan(\frac{z_1(\alpha,t) - z_1(\beta,t)}{2})(1 - \tanh^2(\frac{z_2(\alpha,t) - z_2(\beta,t)}{2}))}{\tan^2(\frac{z_1(\alpha,t) - z_1(\beta,t)}{2}) + \tanh^2(\frac{z_2(\alpha,t) - z_2(\beta,t)}{2})} d\beta\big), \end{aligned} \tag{1.3}$$

where PV denotes principal value [18]. It gives us the velocity field at the interface to which we can subtract any term in the tangential direction without modifying the geometric evolution of the curve

$$z_t(\alpha, t) = BR(z, \varpi)(\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t). \tag{1.4}$$

A wise choice of $c(\alpha, t)$ namely:

$$\begin{aligned} c(\alpha, t) = & \frac{\alpha + \pi}{2\pi} \int_{\mathbb{T}} \frac{\partial_\alpha z(\alpha, t)}{|\partial_\alpha z(\alpha, t)|^2} \cdot \partial_\alpha BR(z, \varpi)(\alpha, t) d\alpha \\ & - \int_{-\pi}^{\alpha} \frac{\partial_\alpha z(\beta, t)}{|\partial_\alpha z(\beta, t)|^2} \cdot \partial_\beta BR(z, \varpi)(\beta, t) d\beta, \end{aligned} \tag{1.5}$$

allows us to accomplish the fact that the length of the tangent vector to $z(\alpha,t)$ be just a function in the variable t only:

$$A(t) = |\partial_\alpha z(\alpha,t)|^2,$$

as will be shown in section 2 (see also [14] and [12]). Then we can close the system using Darcy's law with the equation:

$$\varpi(\alpha,t) = -2A_\mu BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) - 2\kappa \mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_\alpha z_2(\alpha,t), \tag{1.6}$$

where

$$A_\mu = \frac{\mu_1-\mu_2}{\mu_1+\mu_2}$$

is the Atwood number.

Finally we give the function which measures the arc-chord condition in the periodic case

$$\mathcal{F}(z)(\alpha,\beta,t) = \frac{\beta^2/4}{\tan^2(\frac{z_1(\alpha,t)-z_1(\alpha-\beta,t)}{2}) + \tanh^2(\frac{z_2(\alpha,t)-z_2(\alpha-\beta,t)}{2})} \qquad \forall\,\alpha,\beta\in(-\pi,\pi), \tag{1.7}$$

with

$$\mathcal{F}(z)(\alpha,0,t) = \frac{1}{|\partial_\alpha z(\alpha,t)|^2},$$

(see [12] for a closed curve).

Our main result consists on the existence of a positive time T (depending upon the initial condition) for which we have a solution of the periodic Muskat problem (equations (1.3)-(1.6)) during the time interval $[0,T]$ so long as the initial data satisfy $z_0(\alpha)\in H^k(\mathbb{T})$ for $k\geq 3$, $\mathcal{F}(z_0)(\alpha,\beta)<\infty$, and

$$\sigma_0(\alpha) = -(\nabla p^2(z_0(\alpha)) - \nabla p^1(z_0(\alpha)))\cdot\partial_\alpha^\perp z_0(\alpha) > 0,$$

where p^j denote the pressure in Ω^j.

It is interesting to remark that the equality of pressure at each side of the free boundary is obtained in section 2 directly from Darcy's law without any other assumption.

Theorem 1.1 *Let $z_0(\alpha)\in H^k(\mathbb{T})$ for $k\geq 3$, $\mathcal{F}(z_0)(\alpha,\beta)<\infty$, and*

$$\sigma_0(\alpha) = -(\nabla p^2(z_0(\alpha)) - \nabla p^1(z_0(\alpha)))\cdot\partial_\alpha^\perp z_0(\alpha) > 0.$$

Then there exists a time $T>0$ so that there is a solution to (1.3)-(1.6) in $C^1([0,T];H^k(\mathbb{T}))$ with $z(\alpha,0)=z_0(\alpha)$.

We devote the rest of the paper to the proof of theorem 1.1 which is organized as follows. In section 2 we derive the system of equations (1.3)-(1.6) with the corresponding choice of $c(\alpha,t)$ and we also obtain the properties of the pressure. In section 3 and 4 we present several crucial estimates on the operator $T(u)(\alpha) = 2BR(z,u)(\alpha)\cdot\partial_\alpha z(\alpha)$ and on the inverse operator $(I-\xi T)^{-1}$, $|\xi|\leq 1$. Our proofs rely upon the boundedness properties of the Hilbert transforms associated to $C^{1,\alpha}$ curves, for which we need precise estimates obtained with arguments involving conformal mappings, Hopf maximum principle and Harnack inequalities. We then provide upper bounds for the amplitude of the vorticity, the Birkhoff-Rott integral, the parametrization of the curve and the arc-chord condition, namely:

$$\|\varpi\|_{H^k} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}) \quad \text{(section 5)},$$

$$\|BR(z,\varpi)\|_{H^k} \leq \exp(C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}) \quad \text{(section 6)},$$

$$\begin{aligned} \frac{d}{dt}\|z\|^2_{H^k}(t) \leq & -\frac{\kappa}{2\pi(\mu_1+\mu_2)} \int_{\mathbb{T}} \frac{\sigma(\alpha,t)}{|\partial_\alpha z(\alpha)|^2} \partial^k_\alpha z(\alpha,t)\cdot\Lambda(\partial^k_\alpha z)(\alpha,t)d\alpha \\ & + \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^k}) \qquad \text{(section 7)}, \end{aligned}$$

and

$$\frac{d}{dt}\|\mathcal{F}(z)\|^2_{L^\infty}(t) \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)) \quad \text{(section 8)},$$

where the operator Λ is defined by the Fourier transform $\widehat{\Lambda f}(\xi) = |\xi|\widehat{f}(\xi)$ and $\sigma(\alpha,t)$ is the difference of the gradients of the pressure in the normal direction. In section 9 we study the evolution of $m(t) = \min_{\alpha\in\mathbb{T}} \sigma(\alpha,t)$, which satisfies the following lower bound

$$m(t) \geq m(0) - \int_0^t \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(s) + \|z\|^2_{H^3}(s))ds.$$

Finally, in section 10, we introduce a regularized evolution equation where we use the previous a priori estimates together with a pointwise inequality satisfied by the non-local operator Λ [6] to show local existence.

2 The evolution equation

Here (μ,ρ) are defined by

$$(\mu,\rho)(x_1,x_2,t) = \begin{cases} (\mu^1,\rho^1), & x\in\Omega^1(t) \\ (\mu^2,\rho^2), & x\in\Omega^2(t), \end{cases}$$

where $\mu^1 \neq \mu^2$, and $\rho^1 \neq \rho^2$. Then using the Biot-Savart law we get

$$v(x,t) = \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(x - z(\beta,t))^\perp}{|x - z(\beta,t)|^2} \varpi(\beta,t)d\beta$$

for $x \neq z(\alpha,t)$ where the principal value is taken at infinity.

It is convenient to introduce the complex notation $z = x_1 + ix_2$, then the complex conjugate $\bar{v}$ of the velocity field is given by

$$\bar{v}(z,t) = \frac{1}{2\pi i} PV \int_{\mathbb{R}} \frac{\varpi(\beta,t)}{z - z(\beta,t)} d\beta.$$

In our case of periodic interface, $z(\alpha+2\pi k,t)=z(\alpha,t)+2\pi k$, the following classical identity

$$\frac{1}{\pi}\big(\frac{1}{z}+\sum_{k\geq 1}\frac{2z}{z^2-(2\pi k)^2}\big)=\frac{1}{2\pi\tan(z/2)},$$

yields

$$\begin{aligned} v(x,t)=\Big(&-\frac{1}{4\pi}\int_{\mathbb{T}}\varpi(\beta,t)\frac{\tanh(\frac{x_2-z_2(\beta,t)}{2})(1+\tan^2(\frac{x_1-z_1(\beta,t)}{2}))}{\tan^2(\frac{x_1-z_1(\beta,t)}{2})+\tanh^2(\frac{x_2-z_2(\beta,t)}{2})}d\beta,\\ &\frac{1}{4\pi}\int_{\mathbb{T}}\varpi(\beta,t)\frac{\tan(\frac{x_1-z_1(\beta,t)}{2})(1-\tanh^2(\frac{x_2-z_2(\beta,t)}{2}))}{\tan^2(\frac{x_1-z_1(\beta,t)}{2})+\tanh^2(\frac{x_2-z_2(\beta,t)}{2})}d\beta\Big)\end{aligned}$$

for $x\neq z(\alpha,t)$.

We have

$$v^2(z(\alpha,t),t)=BR(z,\varpi)(\alpha,t)+\frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t),$$

$$v^1(z(\alpha,t),t)=BR(z,\varpi)(\alpha,t)-\frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t),$$

where $v^j(z(\alpha,t),t)$ denotes the limit velocity field obtained approaching the boundary in the normal direction inside Ω^j and $BR(z,\varpi)(\alpha,t)$ is given by (1.3).

Darcy's law implies

$$\Delta p(x,t)=-\mathrm{div}\,(\frac{\mu(x,t)}{\kappa}v(x,t))-\mathrm{g}\,\partial_{x_2}\rho(x,t),$$

therefore

$$\Delta p(x,t)=\Pi(\alpha,t)\delta(x-z(\alpha,t)),$$

where $\Pi(\alpha,t)$ is given by

$$\Pi(\alpha,t)=\frac{\mu^2-\mu^1}{\kappa}v(z(\alpha,t),t)\cdot\partial_\alpha^\perp z(\alpha,t)+\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t).$$

It follows that:

$$p(x,t)=-\frac{1}{2\pi}\int_{\mathbb{T}}\ln\big(\cosh(x_2-z_2(\alpha,t))-\cos(x_1-z_1(\alpha,t))\big)\Pi(\alpha,t)d\alpha,$$

for $x\neq z(\alpha,t)$, implying the important identity

$$p^2(z(\alpha,t),t)=p^1(z(\alpha,t),t),$$

which is just a mathematical consequence of Darcy's law, making unnecessary to impose it as a physical assumption.

Let us introduce the following notation:

$$[\mu v](\alpha,t) = (\mu^2 v^2(z(\alpha,t),t) - \mu^1 v^1(z(\alpha,t),t)) \cdot \partial_\alpha z(\alpha,t).$$

Then taking the limit in Darcy's law we obtain

$$\begin{aligned}\frac{[\mu v](\alpha,t)}{\kappa} &= -(\nabla p^2(z(\alpha,t),t) - \nabla p^1(z^1(\alpha,t),t)) \cdot \partial_\alpha z(\alpha,t) - \mathrm{g}(\rho^2-\rho^1)\,\partial_\alpha z_2(\alpha,t)\\ &= -\partial_\alpha(p^2(z(\alpha,t),t) - p^1(z(\alpha,t),t)) - \mathrm{g}(\rho^2-\rho^1)\,\partial_\alpha z_2(\alpha,t)\\ &= -\mathrm{g}(\rho^2-\rho^1)\,\partial_\alpha z_2(\alpha,t),\end{aligned}$$

which gives us

$$\frac{\mu^2+\mu^1}{2\kappa}\varpi(\alpha,t) + \frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) = -\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_2(\alpha,t),$$

so that

$$\varpi(\alpha,t) = -A_\mu 2BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) - 2\kappa\mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_\alpha z_2(\alpha,t).$$

Next we modify the velocity of the curve in the tangential direction:

$$z_t(\alpha,t) = BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t), \tag{2.1}$$

where the scalar $c(\alpha,t)$ is chosen in such a way that the tangent vector only depends on the variable t as follows:

$$|\partial_\alpha z(\alpha,t)|^2 = A(t). \tag{2.2}$$

To find such a $c(\alpha,t)$ let us differentiate the identity (2.2)

$$A'(t) = 2\partial_\alpha z(\alpha,t)\cdot\partial_\alpha z_t(\alpha,t) = 2\partial_\alpha z(\alpha,t)\cdot\partial_\alpha BR(z,\varpi)(\alpha,t) + 2\partial_\alpha c(\alpha,t)A(t),$$

so that

$$\partial_\alpha c(\alpha,t) = \frac{A'(t)}{2A(t)} - \frac{1}{A(t)}\partial_\alpha z(\alpha,t)\cdot\partial_\alpha BR(z,\varpi)(\alpha,t). \tag{2.3}$$

Because $c(\alpha,t)$ has to be periodic, we obtain

$$\frac{A'(t)}{2A(t)} = \frac{1}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z(\alpha,t)\cdot\partial_\alpha BR(z,\varpi)(\alpha,t)d\alpha. \tag{2.4}$$

Using (2.4) in (2.3), and integrating in α, one gets the following formula

$$\begin{aligned}c(\alpha,t) = \frac{\alpha+\pi}{2\pi}&\int_{\mathbb{T}}\frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta\\ &-\int_{-\pi}^{\alpha}\frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta,\end{aligned} \tag{2.5}$$

where we have chosen $c(-\pi,t)=c(\pi,t)=0$.

Let us consider now the solutions of equation (2.1) with $c(\alpha,t)$ given by (2.5). It is easy to check that

$$\frac{d}{dt}|\partial_\alpha z(\alpha,t)|^2 = c(\alpha,t)\partial_\alpha|\partial_\alpha z(\alpha,t)|^2 + b(t)|\partial_\alpha z(\alpha,t)|^2,$$

where

$$b(t)=\frac{1}{\pi}\int_{\mathbb{T}}\frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta.$$

Next we solve this linear partial differential equation, assuming that (2.2) is satisfied initially, to find that the unique solution is given by

$$|\partial_\alpha z(\alpha,t)|^2 = |\partial_\alpha z(\alpha,0)|^2 + \frac{1}{\pi}\int_0^t\int_{\mathbb{T}}\partial_\alpha z(\alpha,t)\cdot\partial_\beta BR(z,\varpi)(\alpha,t)d\alpha ds,$$

which proves (2.2).

Our next step is to find the formula for the difference of the gradients of the pressure in the normal direction:

$$-(\nabla p^2(z(\alpha,t),t)-\nabla p^1(z(\alpha,t),t))\cdot\partial_\alpha^\perp z(\alpha,t),$$

which we denote by $\sigma(\alpha,t)$. Approaching the boundary in Darcy's law, we get

$$\sigma(\alpha,t)=\frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t)+\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t).$$

It is easy to check that

$$\frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t)=\frac{1}{4\pi}\partial_\alpha\int_{\mathbb{T}}\varpi(\beta,t)\log G(\alpha,\beta,t)d\beta,$$

with

$$\begin{aligned}G(\alpha,\beta,t)=&\sin^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2})\cosh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2})\\&+\cos^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2})\sinh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2}),\end{aligned}$$

and therefore

$$\int_{\mathbb{T}}\frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t)d\alpha=0.$$

This shows that the condition $\rho^2\neq\rho^1$ is crucial in order to have a constant sign in the normal direction of the difference of the gradient. Furthermore, since $z_1(\alpha,t)-\alpha$ is periodic we have

$$\int_{\mathbb{T}}\partial_\alpha z_1(\alpha,t)d\alpha=2\pi.$$

Remark 2.1 *If we consider a closed contour, then it is easy to check that*

$$\int_{\mathbb{T}}\sigma(\alpha,t)d\alpha=0,$$

which makes impossible the task of prescribing a sign to σ along a closed curve.

3 The basic operator

Let us consider the operator T defined by the formula

$$T(u)(\alpha) = 2BR(z,u)(\alpha) \cdot \partial_\alpha z(\alpha). \tag{3.1}$$

Lemma 3.1 *Suppose that $\|\mathcal{F}(z)\|_{L^\infty} < \infty$ and $z \in C^{2,\delta}$ with $0 < \delta$. Then $T : L^2 \to H^1$ and*

$$\|T\|_{L^2 \to H^1} \le \|\mathcal{F}(z)\|^2_{L^\infty} \|z\|^4_{C^{2,\delta}}.$$

Remark 3.2 *In section 5, lemma 5.2. there is a proof showing that T also maps H^k into H^{k+1}, $k \ge 1$.*

Proof: Since the formula (1.3) yields

$$T(u)(\alpha) = \frac{1}{\pi}\partial_\alpha \int_{\mathbb{T}} u(\beta) \arctan\Big(\frac{\tanh(\frac{z_2(\alpha)-z_2(\beta)}{2})}{\tan(\frac{z_1(\alpha)-z_1(\beta)}{2})}\Big) d\beta,$$

we have

$$\int_{\mathbb{T}} T(u)(\alpha) d\alpha = 0,$$

which implies $\|T(u)\|_{L^2} \le \|\partial_\alpha T(u)\|_{L^2}$.

Let us denote

$$V(\alpha,\beta) = (V_1(\alpha,\beta), V_2(\alpha,\beta)) = (\tan(\frac{z_1(\alpha)-z_1(\beta)}{2}), \tanh(\frac{z_2(\alpha)-z_2(\beta)}{2})).$$

In the following we shall refer to the Appendix for the definition of V_j, A_j and their properties.

We write first:

$$\partial_\alpha T(u) = 2BR(z,u)(\alpha) \cdot \partial^2_\alpha z(\alpha) + 2\partial_\alpha z(\alpha) \cdot \partial_\alpha BR(z,u)(\alpha) = I_1 + I_2.$$

For I_1 we have the expression

$$\begin{aligned} I_1 &= 2(BR(z,u)(\alpha) - \frac{\partial^\perp_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2} H(u)(\alpha)) \cdot \partial^2_\alpha z(\alpha) + 2H(u)(\alpha)\frac{\partial^\perp_\alpha z(\alpha) \cdot \partial^2_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2} \\ &= J_1 + J_2, \end{aligned}$$

where $H(u)$ is the (periodic) Hilbert transform of the function u.

Then

$$\begin{aligned} J_1 = &-\frac{1}{2\pi}\partial^2_\alpha z_1(\alpha) \int_{\mathbb{T}} u(\beta) \frac{V_2(\alpha,\beta)V_1^2(\alpha,\beta)}{|V(\alpha,\beta)|^2} d\beta \\ &- \frac{1}{2\pi}\partial^2_\alpha z_2(\alpha) \int_{\mathbb{T}} u(\beta) \frac{V_1(\alpha,\beta)V_2^2(\alpha,\beta)}{|V(\alpha,\beta)|^2} d\beta \\ &- \frac{1}{2\pi}\partial^2_\alpha z_1(\alpha) \int_{\mathbb{T}} u(\alpha-\beta)[\frac{V_2(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_2(\alpha)}{\tan(\beta/2)}] d\beta \\ &+ \frac{1}{2\pi}\partial^2_\alpha z_2(\alpha) \int_{\mathbb{T}} u(\alpha-\beta)[\frac{V_1(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_1(\alpha)}{\tan(\beta/2)}] d\beta \\ = &\quad K_1 + K_2 + K_3 + K_4. \end{aligned}$$

And we may use that $|V_2(\alpha,\beta)| \leq 1$, to get $|K_1| + |K_2| \leq C\|z\|_{C^2}\|u\|_{L^2}$.

To estimate K_3 let us observe that the following term

$$A_1(\alpha, \alpha-\beta) = \frac{V_2(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_2(\alpha)}{\tan(\frac{\beta}{2})}$$

satisfies $\|A_1\|_{L^\infty} \leq \|\mathcal{F}(z)\|_{L^\infty}\|z\|^2_{C^2}$ (see appendix lemma 11.1).

In K_4 we have the term

$$A_2(\alpha, \alpha-\beta) = \frac{V_1(\alpha,\alpha-\beta)}{|V(\alpha,\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_1(\alpha)}{\tan(\frac{\beta}{2})},$$

which satisfies $\|A_2\|_{L^\infty} \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|^2_{C^2}$.

Then we obtain $|K_3|+|K_4| \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|^3_{C^2}\|u\|_{L^2}$, and therefore $J_1 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|^3_{C^2}\|u\|_{L^2}$. Since the estimate $J_2 \leq C\|\mathcal{F}(z)\|^{1/2}_{L^\infty}\|z\|_{C^2}|H(u)(\alpha)|$ is immediate, we finally have

$$|I_1| \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|^3_{C^2}(\|u\|_{L^2} + |H(u)(\alpha)|). \tag{3.2}$$

Next we write $2BR(z,u)(\alpha)$ as follows:

$$\begin{aligned}2BR(z,u)(\alpha) &= \frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)(1-V_2^2(\alpha,\beta))\frac{V^\perp(\alpha,\beta)}{|V(\alpha,\beta)|^2}d\beta - \frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)V_2(\alpha,\beta)(1,0)d\beta \\ &= J_3(\alpha) + J_4(\alpha).\end{aligned}$$

Easily we have $|\partial_\alpha J_4(\alpha)\cdot\partial_\alpha z(\alpha)| \leq C\|z\|^2_{C^1}\|u\|_{L^2}$.Taking one derivative in $J_3(\alpha)$, and using the cancellation $\partial_\alpha z(\alpha)\cdot\partial_\alpha^\perp z(\alpha) = 0$, we get

$$\partial_\alpha J_3(\alpha)\cdot\partial_\alpha z(\alpha) = K_5 + K_6 + K_7 + K_8 + K_9,$$

where

$$K_5 = -\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)(1-V_2^2(\alpha,\beta))V_2(\alpha,\beta)\partial_\alpha z_2(\alpha)\frac{V^\perp(\alpha,\beta)\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\beta)|^2}d\beta,$$

$$K_6 = \frac{1}{4\pi}\int_{\mathbb{T}} u(\beta)(1-V_2^2(\alpha,\beta))\partial_\alpha z_1(\alpha)\partial_\alpha z_2(\alpha)d\beta,$$

$$K_7 = -\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)(1-V_2^2(\alpha,\beta))(\partial_\alpha z_1(\alpha)V_1^3(\alpha,\beta)-\partial_\alpha z_2(\alpha)V_2^3(\alpha,\beta)))\frac{V^\perp(\alpha,\beta)\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\beta)|^4}d\beta,$$

$$K_8 = \frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)V_2^2(\alpha,\beta)(\partial_\alpha z_1(\alpha)V_1(\alpha,\beta)+\partial_\alpha z_2(\alpha)V_2(\alpha,\beta))\frac{V^\perp(\alpha,\beta)\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\beta)|^4}d\beta,$$

and

$$K_9 = -\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)(\partial_\alpha z_1(\alpha)V_1(\alpha,\beta)+\partial_\alpha z_2(\alpha)V_2(\alpha,\beta))\frac{V^\perp(\alpha,\beta)\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\beta)|^4}d\beta.$$

We have $|K_5| + |K_6| + |K_7| + |K_8| \leq C\|z\|^2_{C^1}\|u\|_{L^2}$.

Next we split $K_9 = -L_1 - L_2$, where

$$L_1 = \frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\partial_\alpha z_1(\alpha)V_1(\alpha,\beta)\frac{V^\perp(\alpha,\beta)\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\beta)|^4}d\beta,$$

can be rewritten as follows

$$L_1 = \frac{1}{2\pi}\int_{\mathbb{T}} u(\alpha-\beta)\partial_\alpha z_1(\alpha)V_1(\alpha,\alpha-\beta)\frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}d\beta.$$

We have $L_1 = M_1 + M_2$ where

$$M_1 = \frac{(\partial^2_\alpha z(\alpha))^\perp\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4}|\partial_\alpha z_1(\alpha)|^2 H(u)(\alpha),$$

and

$$M_2 = \frac{1}{2\pi}\int_{\mathbb{T}} u(\alpha-\beta)\partial_\alpha z_1(\alpha)B(\alpha,\alpha-\beta)d\beta,$$

for

$$B(\alpha,\alpha-\beta) = V_1(\alpha,\alpha-\beta)\frac{V(\alpha,\alpha-\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4} - \partial_\alpha z_1(\alpha)\frac{(\partial^2_\alpha z(\alpha))^\perp\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4\tan(\beta/2)}.$$

The term M_1 satisfies $|M_1| \leq C\|\mathcal{F}(z)\|^{1/2}_{L^\infty}\|z\|_{C^2}|H(u)(\alpha)|$. We claim that

$$|M_2| \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^4_{C^{2,\delta}}\int_{\mathbb{T}}|\beta|^{\delta-1}|u(\alpha-\beta)|d\beta$$

(see the appendix lemma 11.2 for the proof).

A similar estimate can be obtained for L_2. Finally we have

$$|I_2| \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^4_{C^{2,\delta}}(\|u\|_{L^2} + |H(u)(\alpha)| + \int_{\mathbb{T}}|\beta|^{\delta-1}|u(\alpha-\beta)|d\beta).$$

This inequality together with (3.2) yields

$$|\partial_\alpha T(u)(\alpha)| \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^4_{C^{2,\delta}}(\|u\|_{L^2} + |H(u)(\alpha)| + \int_{\mathbb{T}}|\beta|^{\delta-1}|u(\alpha-\beta)|d\beta).$$

To finish we use the L^2 boundedness of H and Minkowski's inequality to obtain the estimate

$$\|\partial_\alpha T(u)\|_{L^2} \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^4_{C^{2,\delta}}\|u\|_{L^2}.$$

q.e.d.

4 Estimates on the inverse operator $(I-\xi T)^{-1}$

In 3.1 we have considered the operator $T : L^2 \to H^1$

$$T(u) = 2BR(z,u)(\alpha)\cdot\partial_\alpha z(\alpha),$$

for $\mathcal{F}(z)(\alpha,\beta)<\infty$. Then T is a compact operator from Sobolev space L^2 to itself whose adjoint is given by the formula

$$\begin{aligned}
T^*(u)(\alpha) = &\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_2(\beta)\tan(\frac{z_1(\alpha)-z_1(\beta)}{2}) - \partial_\alpha z_1(\beta)\tanh(\frac{z_2(\alpha)-z_2(\beta)}{2})}{\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2})+\tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2})}d\beta\\
&-\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_2(\beta)\tan(\frac{z_1(\alpha)-z_1(\beta)}{2})\tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2})}{\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2})+\tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2})}d\beta\\
&-\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_1(\beta)\tanh(\frac{z_2(\alpha)-z_2(\beta)}{2})\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2})}{\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2})+\tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2})}d\beta.
\end{aligned}$$

We will show that, in $H^{\frac{1}{2}}$, $I-\xi T$ has a bounded inverse $(I-\xi T)^{-1}$ for $|\xi|\leq 1$, whose norm grows at most like $\exp(C|||z|||^2)$ with $|||z||| = \|z\|_{H^3}+\|\mathcal{F}(z)\|_{L^\infty}$.

Let z be outside the curve $z(\alpha)$, then we define

$$\begin{aligned}
f(z) = &\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_2(\beta)\tan(\frac{z_1-z_1(\beta)}{2}) - \partial_\alpha z_1(\beta)\tanh(\frac{z_2-z_2(\beta)}{2})}{\tan^2(\frac{z_1-z_1(\beta)}{2})+\tanh^2(\frac{z_2-z_2(\beta)}{2})}d\beta\\
&-\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_2(\beta)\tan(\frac{z_1-z_1(\beta)}{2})\tanh^2(\frac{z_2-z_2(\beta)}{2})}{\tan^2(\frac{z_1-z_1(\beta)}{2})+\tanh^2(\frac{z_2-z_2(\beta)}{2})}d\beta.\\
&-\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_1(\beta)\tanh(\frac{z_2-z_2(\beta)}{2})\tan^2(\frac{z_1-z_1(\beta)}{2})}{\tan^2(\frac{z_1-z_1(\beta)}{2})+\tanh^2(\frac{z_2-z_2(\beta)}{2})}d\beta.\\
=&\frac{1}{2\pi}\Im\int_{\mathbb{T}}\frac{u(\beta)\partial_\alpha z(\beta)}{\tan(\frac{z-z(\beta)}{2})}d\beta
\end{aligned}$$

that is f is the real part of the Cauchy integral

$$F(z) = f(z)+ig(z) = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{u(\beta)\partial_\alpha z(\beta)}{\tan(\frac{z-z(\beta)}{2})}d\beta$$

which is defined in both periodic domains Ω_1,Ω_2, placed above and below, respectively, of the curve $z(\alpha)$. In the following we shall denote by $\widetilde{\Omega}_j$ a corresponding fundamental domain i.e. $\Omega_j = \bigcup\{\widetilde{\Omega}_j + 2\pi n\}$.

Taking $z = z(\alpha)+\varepsilon\partial_\alpha^\perp z(\alpha)$ we obtain

$$f(z(\alpha)+\varepsilon\partial_\alpha^\perp z(\alpha)) = \frac{1}{2\pi}\Im\int_{\mathbb{T}}\frac{u(\beta)\partial_\alpha z(\beta)}{\tan(\frac{z(\alpha)-z(\beta)+\varepsilon\partial_\alpha^\perp z(\alpha)}{2})}d\beta$$

and letting $\varepsilon \to 0$ we get

$$f(z(\alpha)) = T^*(u) - \text{sign}\,(\varepsilon)u(\alpha). \tag{4.1}$$

On the other hand we have

$$\lim_{\varepsilon\to 0} g(z(\alpha) + \varepsilon\partial_\alpha^\perp z(\alpha)) = \lim_{\varepsilon\to 0} \Im F(z(\alpha) + \varepsilon\partial_\alpha^\perp z(\alpha)) = \mathcal{G}(u)(\alpha)$$

where

$$\begin{aligned}\mathcal{G}(u)(\alpha) = &-\frac{1}{2\pi}PV\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_1(\beta)\tan(\frac{z_1(\alpha)-z_1(\beta)}{2})(1-\tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2}))}{\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2})+\tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2})}d\beta \\ &-\frac{1}{2\pi}PV\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_2(\beta)\tanh(\frac{z_2(\alpha)-z_2(\beta)}{2})(1+\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2}))}{\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2})+\tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2})}d\beta.\end{aligned}$$

independent of the sign of $\varepsilon \to 0$.

First we will show that $T^*u = \lambda u \Rightarrow |\lambda| < 1$, and since T^* is a compact operator (of Hilbert-Schmidt type) we can then conclude the existence of $(I - \xi T^*)^{-1}$ with $|\xi| \leq 1$ (see also [3]). To do that let us compute the value of $\nabla f(z(\alpha))$. Denoting $z = x_1 + ix_2$, the identity

$$\begin{aligned}f(z) &= \frac{1}{2\pi}\Im\int_{\mathbb{T}}\frac{u(\beta)\partial_\alpha z(\beta)}{\tan(\frac{z-z(\beta)}{2})}d\beta = -\frac{1}{2\pi}\Im\int_{\mathbb{T}} u(\beta)\partial_\beta \ln(\sin(\frac{z-z(\beta)}{2}))d\beta \\ &= \frac{1}{2\pi}\Im\int_{\mathbb{T}}\partial_\beta u(\beta)\ln(\sin(\frac{z-z(\beta)}{2}))d\beta,\end{aligned}$$

yields

$$\nabla f(z) = \frac{1}{2\pi}\Im\int_{\mathbb{T}}\partial_\beta u(\beta)\nabla\ln(\sin(\frac{z-z(\beta)}{2}))d\beta.$$

That is

$$\begin{aligned}\nabla f(x) = \big(&-\frac{1}{4\pi}\int_{\mathbb{T}}\partial_\beta u(\beta)(\beta,t)\frac{\tanh(\frac{x_2-z_2(\beta,t)}{2})(1+\tan^2(\frac{x_1-z_1(\beta,t)}{2}))}{\tan^2(\frac{x_1-z_1(\beta,t)}{2})+\tanh^2(\frac{x_2-z_2(\beta,t)}{2})}d\beta, \\ &\frac{1}{4\pi}\int_{\mathbb{T}}\partial_\beta u(\beta)\frac{\tan(\frac{x_1-z_1(\beta,t)}{2})(1-\tanh^2(\frac{x_2-z_2(\beta,t)}{2}))}{\tan^2(\frac{x_1-z_1(\beta,t)}{2})+\tanh^2(\frac{x_2-z_2(\beta,t)}{2})}d\beta\big).\end{aligned}$$

Taking the limit as before we get

$$\nabla f(z(\alpha)) = 2BR(z,\partial_\alpha u)(\alpha) + \text{sign}\,(\varepsilon)\frac{\partial_\alpha u(\alpha)}{2|\partial_\alpha z(\alpha)|^2}\partial_\alpha z(\alpha) \tag{4.2}$$

Assuming now that $T^*u = \lambda u$, the analyticity of $F(z)$ allows us to obtain:

$$\begin{aligned}0 < \int_{\tilde{\Omega}_1}|F'(z)|^2 dx &= -\int_{\mathbb{T}} f(z(\alpha))\nabla f(z(\alpha))\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}d\alpha \\ &= \int_{\mathbb{T}}(1-\lambda)u(\alpha)2BR(z,\partial_\alpha u)(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}d\alpha\end{aligned} \tag{4.3}$$

and

$$0 < \int_{\widetilde{\Omega}_2} |F'(z)|^2 dx = \int_{\mathbb{T}} f(z(\alpha))\nabla f(z(\alpha)) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} d\alpha$$
$$= \int_{\mathbb{T}} (\lambda+1) u(\alpha) 2BR(z, \partial_\alpha u)(\alpha) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} d\alpha \tag{4.4}$$

where we have used (4.1) and (4.2). Multiplying together both inequalities we get

$$0 < (1-\lambda^2)\Big(\int_{\mathbb{T}} u(\alpha) 2BR(z, \partial_\alpha u)(\alpha) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} d\alpha\Big)^2,$$

and therefore $|\lambda| < 1$.

Proposition 4.1 *The norms* $||(I \mp T^*)^{-1}||_{L^2_0}$ *are bounded from above by* $\exp(C|||z|||^2)$ *for some universal constant* C *where the space* L^2_0 *is the usual* L^2 *with the extra condition of mean value zero i.e. the subspace orthogonal to the constants.*

Proof: With the notation introduced before we have

$$F_1 = F/\Omega_1 = f_1 + ig_1$$
$$F_2 = F/\Omega_2 = f_2 + ig_2$$
$$f_1/\partial\Omega = T^*u - u$$
$$f_2/\partial\Omega = T^*u + u$$
$$g_1/\partial\Omega = g_2/\partial\Omega = \mathcal{G}(u).$$

The proof follows easily from the estimate

$$e^{-C|||z|||^2} \le \frac{||u - T^*u||_{L^2_0}}{||u + T^*u||_{L^2_0}} \le e^{C|||z|||^2} \tag{4.5}$$

valid for every nonzero $u \in L^2_0(\partial\Omega)$.

This is because if we assume $||u - T^*u||_{L^2_0} \le e^{-2C|||z|||^2}$ for some $||u||_{L^2_0} = 1$ then we obtain $||u + T^*u||_{L^2_0} \ge 2||u||_{L^2_0} - e^{-2C|||z|||^2} \ge 1$ which contradicts (4.5). Therefore we must have $||u - T^*u||_{L^2_0} \ge e^{-2C|||z|||^2}$ for all $||u||_{L^2_0} = 1$ i.e. $||(I - T^*)^{-1}||_{L^2_0} \le e^{2C|||z|||^2}$. Similarly we also have $||(I + T^*)^{-1}||_{L^2_0} \le e^{2C|||z|||^2}$.

Since $u - T^*u = \mathcal{H}_1(\mathcal{G}(u))$ and $u + T^*u = \mathcal{H}_2(\mathcal{G}(u))$ where $\mathcal{H}_j$ denotes the Hilbert transforms corresponding to each domain Ω_j, then (4.5) is a consequence of the estimate

$$||\mathcal{H}_j||_{L^2(\partial\Omega_j)} \le e^{C|||z|||^2}, \tag{4.6}$$

where C denotes a universal constant not necessarily the same at each occurrence.

This is because the identity $\mathcal{H}_j^2 = -I$ implies

$$||u - T^*u||_{L^2_0} = ||\mathcal{H}_1(\mathcal{G}(u))||_{L^2_0} \le e^{C|||z|||^2} ||\mathcal{G}(u)||_{L^2_0}$$
$$\le e^{2C|||z|||^2} ||\mathcal{H}_2(\mathcal{G}(u))||_{L^2_0} = e^{2C|||z|||^2} ||u + T^*u||_{L^2_0}$$

and similarly we have $||u+T^*u||_{L^2_0} \leq e^{2C|||z|||^2}||u-T^*u||_{L^2_0}$.

It is enough to prove (4.6) for Ω_1 (the case Ω_2 will follows by symmetry) and we will rely on the following geometric fact whose elementary proof is left to the reader.

Lemma 4.2 *Let Ω be a domain in R^2 whose boundary is a $C^{2,\delta}$ parameterized curve $z(\alpha)$ satisfying the arc-chord condition $||\mathcal{F}(z)||_{L^\infty} < \infty$. Then we have tangent balls to the boundary contained in both Ω and R^2/Ω. Furthermore, we can estimate from below the radius of those balls by $C|||z|||^{-1}$, for some universal constant $C > 0$.*

Let $\phi = u + iv$ be the conformal mapping from Ω_1 to the upper half-plane $\mathbb{R}^2_+$. Then v is a non-negative harmonic function vanishing only on $\partial\Omega_1$. Let ϕ^{-1} be the inverse transformation.

Lemma 4.3 *Since Ω_1 is 2π periodic in the horizontal direction we have $\phi(z+2\pi) = \phi(z)+\alpha$ for a certain fixed real number α.*

Proof: Let us define $\psi(w) = \phi(\phi^{-1}(w) + 2\pi)$. Then ψ is a conformal mapping from $\mathbb{R}^2_+$ to itself and, therefore, given by a linear fractional transformation $\psi(w) = \frac{aw+b}{cw+d}$ satisfying $ad - bc = 1$, where a, b, c and d are real numbers. Since ψ can not have a fixed point in $\mathbb{R}^2_+$ then it follows that $c = 0$ and $a = d$. Therefore taking $z = \phi^{-1}(w)$ we get the formula $\phi(z + 2\pi) = \phi(z) + \alpha$ with $\alpha = \frac{b}{d}$, proving lemma 4.3.

Next we observe that $\phi'(z+2k\pi) = \phi'(z)$ for every $z \in \Omega_1$ and since $\partial\Omega_1$ is smooth enough we know from general theory that ϕ and ϕ' extend continuously to $\partial\Omega_1$. Furthermore, in order to estimate the size of $\phi'|_{\partial\Omega_1}$ it will be enough to consider the compact part of that boundary corresponding to a full period.

Composing with ϕ, ϕ^{-1} one easily gets the formula

$$\mathcal{H}_1 f = H(f \circ \phi^{-1}) \circ \phi$$
$$H(f \circ \phi^{-1}) = \mathcal{H}_1(f) \circ \phi^{-1}$$

therefore our problem is reduced to a weighted estimate for the Hilbert transform with respect to the weight $|(\phi'(x, 0))^{-1}| = w(x)$ for which we have to show that w belongs to the Muckenhoupt class A_2 (see [11]). Now it turns out that for general C^1 chord-arc curves that statement is false, but we will take advantage of the fact $\partial\Omega_1$ is of class C^2 (in fact $C^{1,\alpha}$ will suffice) to show that in our case $w(x)$ trivializes i.e. it is bounded above and below, more precisely:

Lemma 4.4 *Let $w(x) = |(\phi'(x, 0))^{-1}|$ then we have*

$$w(x_0)e^{-C|||z|||^2} \leq w(x) \leq w(x_0)e^{C|||z|||^2}$$

where C is a universal constant, $|||z|||$ is our usual norm in the curve $\partial\Omega_1$ and x_0 is any point. Normalizing our conformal mapping ϕ one may take $w(x_0) = 1$.

Proof: From the geometric lemma 4.3 we know the existence of tangent balls to $\partial\Omega_1$ contained inside $\overline{\Omega}_1$ of radius $r = O(1/|||z|||)$ and such that each one of those balls touches the boundary $\partial\Omega_1$ at a single point and its centers describe a parallel curve Γ to $\partial\Omega_1$ which is also of class C^2 with norms $O(|||z|||)$. In the following we shall concentrate our attention to the band B of those points in Ω_1 whose distance to $\partial\Omega_1$ is less that r. Then the boundary of B consists of two parts, namely $\partial\Omega_1$ and its parallel curve Γ at distance r which can also be parameterized throughout $z(\alpha)$ in an obvious manner.

The length of the part of Γ corresponding to a full period $0 \leq \alpha \leq 2\pi$, is clearly $O(|||z|||)$. Then, after several applications of Harnack inequality in steps of length $O(r)$, we obtain

$$e^{-C|||z|||^2} \leq \frac{v(z_1)}{v(z_2)} \leq e^{C|||z|||^2}$$

for any $z_1, z_2 \in \Gamma$. Let us consider a point $P \in \partial\Omega_1$ and $Q \in \Gamma$ to be the center of the circle of radius r tangent to $\partial\Omega_1$ at P, furthermore let us denote by ν the inner normal vector. Then the non-negative harmonic function v takes its strict minimum at the point P and by Hopf principle we get the estimate

$$\frac{\partial v}{\partial \nu}(P) \geq \frac{C}{r} v(Q) \tag{4.7}$$

for some absolute constant $C > 0$. On the other hand we may consider a domain D contained in the band B in such a way that its boundary consists of a piece of $\partial\Omega_1$ of length $2r$ containing P at its medium point, then the corresponding portion of Γ, says L_2, obtained by vertical translation of the points of L_1 and finally two arcs of C^2 curves smoothly connecting L_1 and L_2 in such a way that ∂D becomes a C^2 curve with norm $O(|||z|||)$.

Let ψ be conformal mapping from the unit ball B_r to D with standard normalization. By the Kellogg-Warschawski theorem it follows that ψ extends continuously to the boundary and its derivative is bounded from above and below by universal constants. We also have the Poisson's kernel K in D obtained by conformal mapping of the kernel for the ball of radius r. Then we may represent the harmonic function v as the integral of its boundary values against the Poisson kernel:

$$v(x) = \int_{\partial D} K(x, y) v(y) d\sigma(y)$$

and

$$\frac{\partial v}{\partial \nu}(x) = \int_{\partial D} \frac{\partial K}{\partial \nu_x}(x, y) v(y) d\sigma(y)$$

which is a legitimate integral. We can take the limit (when $x \to P \in \partial\Omega_1$) because v vanishes identically in L_1 and the points $y \in \partial D - L_1$ are at distance at least r from P to obtain the estimate

$$\frac{\partial v}{\partial \nu}(P) \leq \frac{C}{r} sup_{x \in D} v(x)$$

To finish we can invoke Dahlberg-Harnack principle up to the boundary for the positive harmonic function v (see [4] and [10]), which gives us the inequality

$$\frac{\partial v}{\partial \nu}(P) \leq \frac{C}{r} v(Q) \tag{4.8}$$

for some fixed constant C. Then the estimates (4.7) and (4.8) yield

$$C^{-1}\frac{v(Q_1)}{v(Q_2)} \leq \frac{\frac{\partial v}{\partial \nu}(P_1)}{\frac{\partial v}{\partial \nu}(P_2)} \leq C\frac{v(Q_1)}{v(Q_2)}, \tag{4.9}$$

but we know from Harnack that

$$e^{-C|||z|||^2} \leq \frac{v(Q_1)}{v(Q_2)} \leq e^{C|||z|||^2}$$

for two arbitrary points Q_1, Q_2 in Γ, and that ends the proofs of lemma 4.4 and proposition 4.1 because $|\phi'(z(\alpha))| = |\nabla v(z(\alpha))| = \frac{\partial v}{\partial \nu}(z(\alpha))$ since $\partial\Omega_1$ is the level set $v = 0$ of the positive harmonic function v ($\phi = u + iv$) q.e.d.

The identity $I + \xi T^* = \xi(I + T^*) + (1 - \xi)I$ allows us to conclude that

$$||(u + \xi T^* u)^{-1}||_{L_0^2} \leq e^{C|||z|||^2}$$

for $1 - e^{-C_1|||z|||^2} \leq |\xi| \leq 1$ with an appropriate constant C_1, but for general ξ ($|\xi| \leq 1$) we have

Proposition 4.5 *For $|\xi| \leq 1$ the following estimate holds*

$$||(I + \xi T)^{-1}||_{H_0^{\frac{1}{2}}} = ||(I + \xi T^*)^{-1}||_{H_0^{\frac{1}{2}}} \leq e^{C|||z|||^2}$$

for a universal constant C.

Proof: First let us consider the inequality

$$\int_{\Omega_j} |\nabla f_j|^2 \geq e^{-C_2|||z|||^2} ||u||^2_{H^{\frac{1}{2}}} \tag{4.10}$$

where $F_j = f_j + ig_j$ is the Cauchy integral of u in Ω_j which follows easily from estimate (4.7) for the derivative of the conformal mapping ϕ:

$$\begin{aligned}\int_{\Omega_j} |\nabla f_j|^2 &= \int_{\Omega_j} \Delta f_j^2 = \int_{\mathbb{R}_+^2} \Delta f_j^2(\phi^{-1})|(\phi^{-1})'|^2 \\ &= \int_{\mathbb{R}_+^2} \Delta(f_j \circ \phi^{-1})^2 = \int_{\partial\mathbb{R}_+^2} f_j \circ \phi^{-1} \frac{\partial}{\partial \nu} f_j \circ \phi^{-1}\end{aligned}$$

where $\frac{\partial}{\partial \nu}$ is the derivative in the normal direction

$$\frac{\partial f_j}{\partial y}(x, 0) = lim_{y \to 0} \frac{1}{\pi} \int \frac{u(x - t) - u(x)}{t^2 + y^2} dt = \Lambda u(x).$$

Therefore we can conclude that

$$\int_{\Omega_j} |\nabla f_j|^2 = \int_{-\infty}^{+\infty} f_j \circ \phi^{-1} \Lambda(f_j \circ \phi^{-1}) = \int_{-\infty}^{+\infty} |\Lambda^{\frac{1}{2}}(f_j \circ \phi^{-1})|^2 \geq e^{-C_2|||z|||^2} ||u||^2_{H^{\frac{1}{2}}}$$

for a certain positive constant C_2 as a consequence of the following lemma:

Lemma 4.6 *Let ψ be a diffeomorphism of the real line such that $0 < C^{-1} \leq |\psi'(x)| \leq C$ then we have the equivalence of Sobolev norms*

$$C^{-(3+2s)}||f||_{H^s} \leq ||f \circ \psi||_{H^s} \leq C^{3+2s}||f||_{H^s}$$

for $0 \leq s \leq \frac{1}{2}$.

Proof: Given f in H^s we have

$$\begin{aligned} ||\Lambda^s(f\circ\psi)||_{L^2}^2 &= \int (f\circ\psi)(x)\Lambda^{2s}(f\circ\psi)(x)dx = \int f(\psi(x)) \int \frac{f(\psi(x)) - f(\psi(y))}{|x-y|^{1+2s}}dydx \\ &= \frac{1}{2}\int\int \frac{(f(\psi(x)) - f(\psi(y)))^2}{|x-y|^{1+2s}}dydx \\ &= \frac{1}{2}\int\int \frac{(f(\bar{x}) - f(\bar{y}))^2}{|(\psi^{-1})'(\bar{\bar{x}})|^{1+2s}|\bar{x}-\bar{y}|^{1+2s}}(\psi^{-1})'(\bar{x})(\psi^{-1})'(\bar{y})d\bar{y}d\bar{x} \end{aligned}$$

where $\bar{\bar{x}}$ comes from the application of the mean value theorem. From our hypothesis we have

$$C^{-(3+2s)} \leq \frac{(\psi^{-1})'(\bar{x})(\psi^{-1})'(\bar{y})}{|(\psi^{-1})'(\bar{\bar{x}})|^{1+2s}} \leq C^{3+2s}$$

which together with the equality

$$||\Lambda^s f||_{L^2}^2 = \frac{1}{2}\int\int \frac{(f(x)-f(y))^2}{|x-y|^{1+2s}}dydx$$

allows us to finish the proof of the lemma.

Remark 4.7 *In our case the diffeomorphism is given by $\Psi(\alpha) = \Phi(z(\alpha))$ and we will use the periodic version of (4.10) i.e.*

$$\int_{\tilde{\Omega}_j} |\nabla f_j|^2 \geq e^{-C_2|||z|||^2}||u||^2_{H^{\frac{1}{2}}(\mathbb{T})}.$$

To continue, let us assume that proposition 4.5 is false, then there exist $u \in H_0^{\frac{1}{2}}$, $||u||_{H^{\frac{1}{2}}} = 1$ and $|\eta| > 1$ such that $||\eta u - T^* u||_{H^{\frac{1}{2}}} \leq e^{-C_3|||z|||^2}$, where C_3 will be fixed later to be big enough for our purposes.

Let us also assume that the following estimate holds

$$\begin{aligned} &|\int_{\mathbb{T}} (\eta u - T^* u)2BR(z, \partial_\alpha u)(\alpha)\cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}d\alpha| \leq \\ &\quad \|\eta u - T^* u\|_{H^{\frac{1}{2}}} \|2BR(z, \partial_\alpha u)(\alpha)\cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}\|_{H^{-\frac{1}{2}}} \leq e^{-50C_2|||z|||^2} \end{aligned}$$

Then from identities (4.3) and (4.4) we get

$$e^{-C_2|||z|||^2} \leq \int_{\mathbb{T}} (1-\eta)u(\alpha)2BR(z,\partial_\alpha u)(\alpha)\cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}d\alpha + e^{-50C_2|||z|||^2} \qquad (4.11)$$

$$e^{-C_2|||z|||^2} \leq \int_{\mathbb{T}} (1+\eta)u(\alpha)2BR(z,\partial_\alpha u)(\alpha) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} d\alpha + e^{-50C_2|||z|||^2}.$$

Adding these two inequalities together we obtain the positivity of

$$\int_{\mathbb{T}} u(\alpha)2BR(z,\partial_\alpha u)(\alpha) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} d\alpha > 0$$

and then we get a contradiction when we substitute the value of η in the first inequality of (4.11), if $\eta \geq 1$, or in the second one, if $\eta \leq -1$. Therefore the hypothesis $\|\eta u - T^* u\|_{H^{\frac{1}{2}}} \leq e^{-C_3|||z|||^2}$ is false for every u in $H_0^{\frac{1}{2}}$ and $\|u\|_{H^{\frac{1}{2}}} = 1$ and that gives us the desired estimate.

To finish the proof we need to show the following inequality

$$\|2BR(z,\partial_\alpha u)(\alpha) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}\|_{H^{-\frac{1}{2}}} \leq e^{C|||z|||^2}\|u\|_{H^{\frac{1}{2}}}$$

for a universal constant C.

In order to prove it let us first observe that

$$\partial_\alpha(BR(z,u)(\alpha) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}) = BR(z,\partial_\alpha u)(\alpha) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} + O_p(z)u$$

where $O_p(z)$ is a bounded operator in L^2 whose norm is controlled by $e^{C|||z|||^2}$ for a convenient value of C. Therefore our task is equivalent to show the estimate

$$\|2BR(z,u)(\alpha) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}\|_{H^{\frac{1}{2}}} \leq e^{C|||z|||^2}\|u\|_{H^{\frac{1}{2}}}.$$

Decoding the notation we have to consider the operators $\partial_\alpha^\perp z_k(\alpha) \cdot T_j u(\alpha)$ where

$$T_j u(\alpha) = PV \int \frac{z_j(\alpha) - z_j(\alpha-\beta)}{|z(\alpha) - z(\alpha-\beta)|^2} u(\alpha-\beta) d\beta.$$

Let ϕ be a C^∞ cut-off function supported on $|x| \leq r$ such that $\phi \equiv 1$ on $|x| \leq \frac{r}{2}$ where $r = \frac{|||z|||}{2}$. Then $T_j u(\alpha) = T_j^1 u + T_j^2 u$ for

$$\begin{aligned} T_j^1 u &= PV \int \phi(\beta) \frac{z_j(\alpha) - z_j(\alpha-\beta)}{|z(\alpha) - z(\alpha-\beta)|^2} u(\alpha-\beta) d\beta, \\ T_j^2 u &= PV \int (1-\phi(\beta)) \frac{z_j(\alpha) - z_j(\alpha-\beta)}{|z(\alpha) - z(\alpha-\beta)|^2} u(\alpha-\beta) d\beta. \end{aligned}$$

It is straightforward to check that T_j^2 is a smoothing operator for which the desired estimate trivializes. Furthermore a convenient Taylor expansion allows us to write $T_j^1 u(\alpha) = m(\alpha) H u(\alpha) + R(u)$ where R is a smoothing operator, H is the Hilbert transform and the bounded smooth function m depends upon the curve z in such a way that $\|\frac{\partial m}{\partial \alpha}\|_{L^\infty} \leq e^{C|||z|||^2}$. Finally we may invoke the following commutator estimate

$$\|\Lambda^{\frac{1}{2}}(bv) - b\Lambda^{\frac{1}{2}} v\|_{L^2(\mathbb{T})} \leq C\|\nabla b\|_{L^\infty}\|v\|_{L^2(\mathbb{T})}$$

to complete our task. q.e.d.

Remark 4.8 *Although it will not be needed to establish our main theorem we will improve the estimate on the eigenvalues of T^*, T by showing the existence of a constant $C_0 = C_0(z)$ whose inverse C_0^{-1} grows at most as a polynomial in $|||z|||$ and such that the eigenvalues of T^* must satisfy the estimate $|\lambda| \leq 1 - C_0$. To see that let us consider the identities*

$$\int_{\Omega_1} |\nabla f_1|^2 dx + \int_{\Omega_2} |\nabla f_2|^2 dx = 2 \int_{\mathbb{T}} u(\alpha) 2BR(z, \partial_\alpha u)(\alpha) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} d\alpha$$

$$|\int_{\Omega_1} |\nabla f_1|^2 dx - \int_{\Omega_2} |\nabla f_2|^2 dx| = 2|\lambda| \int_{\mathbb{T}} u(\alpha) 2BR(z, \partial_\alpha u)(\alpha) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} d\alpha.$$

Then it will be enough to show that both integrals $\int_{\Omega_j} |\nabla f|^2 dx$ are comparable i.e. there exists a constant $1 \leq C = C(|||z|||) < \infty$ such that

$$\frac{1}{C} \int_{\Omega_1} |\nabla f_1|^2 \leq \int_{\Omega_2} |\nabla f_2|^2 \leq C \int_{\Omega_1} |\nabla f_1|^2.$$

Observe that the Cauchy-Riemman equations imply that this is equivalent to show the analogous estimate for g in place of f.

The existence of such C depending continuously upon $|||z|||$ follows easily by a standard compactness argument. Nevertheless it is convenient to have a control of the growth of the constants. In the following we present an argument to show that $C(|||z|||)$ grows polynomially with $|||z|||$.

Proposition 4.9 *We shall consider periodic curves $z(\alpha)$ (period 2π). Because of the smoothness and arc-chord conditions such a curve divide the cylinder $\mathbb{R}/2\pi\mathbb{Z} \times (-\infty, \infty)$ in two regions Ω_j ($j = 1, 2$, above and below the curve respectively) containing tangent balls as in the previous lemma. Then there exist a constant $C = P(||z||_{C^{k,\delta}}, ||\mathcal{F}(z)||_{L^\infty})$, polynomial in $|||z|||$, such that*

$$\frac{1}{C} \int_{\Omega_1} |F_1'|^2 \leq \int_{\Omega_2} |F_2'|^2 \leq C \int_{\Omega_1} |F_1'|^2$$

for any pair of periodic (in x_1) holomorphic functions $F_j = f_j + ig_j$ ($j = 1, 2$), with f_j, g_j in Sobolev space $H^1(\Omega_j)$ and such that the imaginary parts g_j, $j = 1, 2$ (or respectively the real part f_j $j = 1, 2$) take the same boundary values.

Proof: *In the following we shall use the expression $P(\gamma)$ for different constants, to denote that they grow at most polynomially with γ.*

For $\frac{1}{r} = P(|||z|||)$ there exists two tangent circles to the curve z of radius r and contained respectively in Ω_1 and Ω_2. Therefore we can foliate the plane near z by parallel curves z_ϵ^j ($z_0^j = Z$), these curves are the locus of points in Ω_j whose distance to z is ϵ, in such a way that $|||z_\epsilon^j||| \leq C|||z|||$ uniformly on $0 \leq \epsilon \leq \frac{1}{10} r$ for some universal finite constant C.

The Cauchy-Riemann equations for the holomorphic functions $F_j = f_j + ig_j$ yields

$$\int_{\Omega_j} |F_j'|^2 = \int_{\Omega_j} |\nabla f_j|^2 = \int_{\Omega_j} |\nabla g_j|^2$$

Let us assume (without loss of generality) that

$$\int_{\Omega_1} |\nabla g_1|^2 \geq \int_{\Omega_2} |\nabla g_2|^2$$

then we want to show the estimate

$$\int_{\Omega_1} |\nabla g_1|^2 \leq P(|||z|||) \int_{\Omega_2} |\nabla g_2|^2$$

and that will finish the proof.

Let ϕ be a C^∞ cut-off function such that $\phi(t) \equiv 1$ when $|t| \leq \frac{1}{20} r$ and $\phi \equiv 0$ when $|t| \geq \frac{1}{10} r$, then we reflect the values of g_1 near $z(\alpha)$ by the formula

$$\tilde{g}_1(P) = g_2(Q)\phi(dist(P, z))$$

where $Q \in \Omega_2$ is obtained reflecting $P \in \Omega_1$ with respect to z, that is $dist(P, z) = dist(Q, z)$, and the line segment connecting them is normal to z at its medium point.

By Dirichlet principle

$$\int_{\Omega_1} |\nabla g_1|^2 \leq \int_{\Omega_1} |\nabla \tilde{g}_1|^2 \leq P(|||z|||)(\int_{\Omega_2} |\nabla g_2|^2 |\phi|^2 + \int_{\Omega_2} |g_2|^2 |\nabla \phi|^2)$$

Since F_2 is holomorphic we have the equalities

$$\int_0^{2\pi} F_2(x, y_1) dx = \int_0^{2\pi} F_2(x, y_2) dx$$

for $|y_j|$ big enough so that the horizontal lines (x, y_j) do not meet the curve z. The hypothesis that $f_j \in L^2(\Omega_j)$ implies that

$$\int_0^{2\pi} g_2(x, y) dx = 0$$

for those y which can be taken at distance $P(|||z|||)$ of the curve z. For such a y we get the estimate

$$|g_2(x, y)| \leq \int_0^{2\pi} |\nabla g_2(t, y)| dt$$

which implies

$$\int_0^{2\pi} \int_{-y-1}^{-y} |g_2(x, s)|^2 ds dx \leq (2\pi)^2 \int_0^{2\pi} \int_{-y-1}^{-y} |\nabla g_2(t, s)|^2 ds dt$$

therefore

$$m\{|g_2(x, s)| \geq 10 \cdot 2\pi ||\nabla g_2||_{L^2(\Omega_2)} | 0 \leq x \leq 2\pi, -y-1 \leq s \leq -y\} \leq \frac{1}{100}$$

where m denotes the Lebesgue measure.

Let (x_m, y_m) be in the curve z such that y_m has a minimum value. Then for all points Q in Ω_2 inside the band $1/(20r) \leq dist(Q,z) \leq 1/(20r)$ whose distance to (x_m, y_m) is less than $1/P(|||z|||)$ (we shall denote by N the set of such Q) the segments connecting its points to those of $\{(x,t), -y \leq t \leq -y-1\}$ are completely contained in Ω_2. For each $(x_0, y_0) \in N$ let us consider the line segment connecting (x_0, y_0) with the set

$$E = \{(x,s) || g_2(x,s)| < 10 \cdot 2\pi ||\nabla g_2||_{L^2(\Omega_2)} | 0 \leq x \leq 2\pi, -y-1 \leq s \leq -y\},$$

then given $(x,s) \in E$ we have the estimate

$$|g_2(x_0,y_0)| \leq 10 \cdot 2\pi ||\nabla g_2||_{L^2(\Omega_2)} + \int_0^L |\nabla g_2((x_0,y_0) + t\omega)| dt$$

where $\omega = \frac{(x-x_0, s-y_0)}{((x-x_0)^2+(s-y_0)^2)^{\frac{1}{2}}}$ and $0 \leq L \leq P(|||z|||)$.

Since the measure of E is big enough ($\geq \pi$) the measure of the region described in the unit circle by those ω's is also big enough ($\geq 1/P(|||z|||)$). Therefore

$$|g_2(x_0,y_0)| \leq P(|||z|||)\big(||\nabla g_2||_{L^2(\Omega_2)} + \int\int \frac{|\nabla g_2((x_0,y_0)-(x,y))|}{||(x,y)||} dxdy\big).$$

This inequality implies that

$$\int_N |g_2(x_0,y_0)|^2 dx_0 dy_0 \leq P(|||z|||) \int_{\Omega_2} |\nabla g_2(x,y)|^2 dxdy$$

To conclude the argument we just observe that because the parallel curves have tangent vector whose lengths are uniformly bounded by $P(|||z|||)$, the integral $\int_{\Omega_2} |g_2|^2 |\nabla\phi|^2$ is bounded by $P(|||z|||)(\int_N |g_2(x_0,y_0)|^2 dx_0 dy_0 + \int_{\Omega_2} |\nabla g_2(x,y)|^2 dxdy)$, q.e.d.

5 Estimates on ϖ

In this section we show that the amplitude of the vorticity ϖ is at the same level than $\partial_\alpha z$. We prove the following result:

Lemma 5.1 *Let ϖ be a function given by*

$$\varpi(\alpha) = -\frac{\mu^2-\mu^1}{\mu^2+\mu^1} 2BR(z,\varpi)(\alpha) \cdot \partial_\alpha z(\alpha) - 2\kappa g \frac{\rho^2-\rho^1}{\mu^2+\mu^1} \partial_\alpha z_2(\alpha). \tag{5.1}$$

Then

$$\|\varpi\|_{H^k} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}). \tag{5.2}$$

for $k \geq 2$.

Proof: We have $|A_\mu| \leq 1$, then the formula (5.1) is equivalent to

$$\varpi(\alpha) + A_\mu T(\varpi)(\alpha) = -2\kappa g \frac{\rho^2-\rho^1}{\mu^2+\mu^1} \partial_\alpha z_2(\alpha), \tag{5.3}$$

or

$$\varpi(\alpha)=-2\kappa \mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}(I+A_\mu T)^{-1}(\partial_\alpha z_2)(\alpha).$$

It yields

$$\|\varpi\|_{H^{\frac12}}\leq C\|(I+A_\mu T)^{-1}\|_{H^{\frac12}}\|\partial_\alpha z_2\|_{H^{\frac12}},$$

and proposition (4.1) gives

$$\|\varpi\|_{H^{\frac12}}\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}). \tag{5.4}$$

Taking the k derivative of (5.3) we get:

$$\partial^k_\alpha\varpi(\alpha)+A_\mu T(\partial^k_\alpha\varpi)(\alpha)=\Omega_k(\varpi)+C\partial^{k+1}_\alpha z_2(\alpha),\quad C=-2\kappa\mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}$$

and using Leibniz's rule we can write

$$\Omega_k(\varpi)(\alpha)=\sum_{j=1}^k C_j\int\Phi(\beta)\partial^j_\alpha(\frac{(z(\alpha)-z(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2})\partial^{k-j}_\alpha\varpi(\alpha-\beta)d\beta+S(\varpi)(\alpha),$$

where S is a smoothing operator, C_j are suitable constants and Φ is a C^∞ cut-off such that $\Phi\equiv 0$ outside the ball $B(0,r)$ of radius $r=\frac{1}{2|||z|||}$ and $\Phi\equiv 1$ in $B(0,\frac{r}{2})$.

Next let us consider

$$\begin{aligned}\Lambda^{\frac12}\partial^k_\alpha\varpi(\alpha)+A_\mu T(\Lambda^{\frac12}\partial^k_\alpha\varpi)(\alpha)&=A_\mu T(\Lambda^{\frac12}\partial^k_\alpha\varpi)(\alpha)-A_\mu\Lambda^{\frac12}T(\partial^k_\alpha\varpi)(\alpha)\\&\quad+\Lambda^{\frac12}\Omega_k(\varpi)+C\Lambda^{\frac12}\partial^{k+1}_\alpha z_2(\alpha),\end{aligned}$$

using the estimate for the inverse $(I+A_\mu T)^{-1}$ in the space $H^{\frac12}$ we get

$$\begin{aligned}\|\varpi\|_{H^{k+1}}&=\|\Lambda^{\frac12}\partial^k_\alpha\varpi\|_{H^{\frac12}}\\&\leq\exp(C|||z|||^2)(A_\mu\|T\Lambda^{\frac12}\partial^k_\alpha\varpi\|_{H^{\frac12}}+A_\mu\|T\partial^k_\alpha\varpi\|_{H^1}+\|\Omega_k\|_{H^1}+\|z\|_{H^{k+2}}).\end{aligned}$$

Then we have

$$\|T\partial^k_\alpha\varpi\|_{H^1}\leq C|||z|||^4\|\varpi\|_{H^k},$$

by Lemma 3.1, and

$$||\Omega_k||_{H^1}\leq C\|\mathcal{F}(z)\|^2_{L^\infty}||\varpi||_{H^k}||z||^2_{H^{k+2}},$$

by Lemma 5.2. (see below). Finally

$$\begin{aligned}\|T\Lambda^{\frac12}\partial^k_\alpha\varpi\|_{H^{\frac12}}&\leq\|T\Lambda^{\frac12}\partial^k_\alpha\varpi\|_{H^1}\leq C|||z|||^4\|\varpi\|_{H^{k+\frac12}}\\&\leq e^{C|||z|||^2}(\|\Omega_k\|_{H^{\frac12}}+\|z\|_{H^{k+\frac32}})\\&\leq e^{C|||z|||^2}(||\mathcal{F}(z)||_{L^\infty}||\varpi||_{H^k}||z||_{H^{k+2}}+||z||_{H^{k+\frac32}})\end{aligned}$$

where we have used $\partial_\alpha^k \varpi(\alpha) = (I - A_\mu T)^{-1}(\Omega_k + C\partial_\alpha^{k+1} z_2)$ and the estimate of the norm in $H^{\frac{1}{2}}$ of the inverse operator $(I - A_\mu T)^{-1}$.

A straightforward induction on $k \geq 2$ allows us to finish the proof. The estimates for $k = 1, \frac{3}{2}$ are obtained similarly, but in all of them the norm $||z||_{H^3}$ has to appear i.e. we have

$$\|\varpi\|_{H^{\frac{3}{2}}} \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2).$$

Lemma 5.2 *The operator T maps Sobolev space H^k, $k \geq 1$, into H^{k+1} (so long as $\|z\|_{H^{k+2}} < \infty$) and satisfies the estimate*

$$\|T\|_{H^k \to H^{k+1}} \leq C|||z|||^2 \|z\|_{H^{k+2}}^2$$

Proof: We have

$$T(u)(\alpha) = 2BR(z,u)(\alpha) \cdot \partial_\alpha z(\alpha) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{(z(\alpha) - z(\alpha-\beta))^\perp \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha-\beta)|^2} u(\alpha-\beta) d\beta$$

where, as usual and to simplify notation, we have dropped the time dependence of all functions.

Let ψ be a C^∞ cut-off such that $\psi \equiv 0$ outside the ball $B(0,r)$ of radius $r = \frac{1}{2|||z|||}$ and $\psi \equiv 1$ in $B(0, \frac{r}{2})$. Then

$$\begin{aligned}
T(u)(\alpha) &= \frac{1}{\pi} PV \int_{-\infty}^{\infty} \psi(\beta) \frac{(z(\alpha) - z(\alpha-\beta))^\perp \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha-\beta)|^2} u(\alpha-\beta) d\beta \\
&+ \frac{1}{\pi} PV \int_{-\infty}^{\infty} (1 - \psi(\beta)) \frac{(z(\alpha) - z(\alpha-\beta))^\perp \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha-\beta)|^2} u(\alpha-\beta) d\beta \\
&= T_1 u(\alpha) + T_2 u(\alpha).
\end{aligned}$$

i) Estimate of $T_2 u(\alpha)$: Leibniz's rule gives us

$$\begin{aligned}
\partial_\alpha^{k+1} T_2 u(\alpha) &= \frac{1}{\pi} \int_{-\infty}^{\infty} (1 - \psi(\beta)) \frac{(z(\alpha) - z(\alpha-\beta))^\perp \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha-\beta)|^2} \partial_\alpha^{k+1} u(\alpha-\beta) d\beta \\
&+ \frac{1}{\pi} \int_{-\infty}^{\infty} (1 - \psi(\beta)) \frac{(z(\alpha) - z(\alpha-\beta))^\perp \partial_\alpha^{k+2} z(\alpha)}{|z(\alpha) - z(\alpha-\beta)|^2} u(\alpha-\beta) d\beta \\
&+ \text{¨other terms¨} \\
&= I_1 + I_2 + \text{¨ other terms ¨}
\end{aligned}$$

The estimate for ¨other terms¨ is straightforward. For I_1 we integrate by parts:

$$\begin{aligned}
I_1 &= \frac{1}{\pi} \int_{-\infty}^{\infty} \psi'(\beta) \frac{(z(\alpha) - z(\alpha-\beta))^\perp \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha-\beta)|^2} \partial_\alpha^k u(\alpha-\beta) d\beta \\
&- \frac{1}{\pi} \int_{-\infty}^{\infty} (1 - \psi(\beta)) \partial_\beta \left(\frac{(z(\alpha) - z(\alpha-\beta))^\perp \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha-\beta)|^2} \right) \partial_\alpha^k u(\alpha-\beta) d\beta.
\end{aligned}$$

Then clearly we have:

$$||I_1||_{L^2} \le C|||z|||^2||z||^2_{H^{k+2}}||u||_{H^k}$$

Regarding I_2 we have

$$I_2(\alpha) = \sum_{j=1}^{2} \partial_\alpha^{k+2} z_j(\alpha) \cdot L_j u(\alpha)$$

and clearly $||L_j u||_{L^\infty} \le C|||z|||^2||u||_{H^k}$. Therefore

$$||I_2||_{L^2} \le C|||z|||^2||z||_{H^{k+2}}||u||_{H^k}$$

ii)Estimate of $T_1 u(\alpha)$: We have

$$\begin{aligned}
\partial_\alpha^{k+1} T_1 u(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \psi(\beta) \frac{(z(\alpha)-z(\alpha-\beta))^\perp \partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2} \partial_\alpha^{k+1} u(\alpha-\beta) d\beta \\
&+ \frac{1}{\pi}\int_{-\infty}^{\infty} \psi(\beta) \frac{(z(\alpha)-z(\alpha-\beta))^\perp \partial_\alpha^{k+2} z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2} u(\alpha-\beta) d\beta \\
&+ \frac{1}{\pi}\int_{-\infty}^{\infty} \psi(\beta) \frac{(\partial_\alpha^{k+1} z(\alpha)-\partial_\alpha^{k+1} z(\alpha-\beta))^\perp \partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2} u(\alpha-\beta) d\beta \\
&+ \text{¨other terms¨} \\
&= J_1 + J_2 + J_3 + \text{¨other terms¨}.
\end{aligned}$$

As in the previous case the ¨other terms¨ are easy to handle and we shall show how to estimate the remainder two cases.

We can write J_2 in the form

$$J_2(\alpha) = \sum \partial_\alpha^{k+2} z_j(\alpha) \int \psi(\beta) K_j(\alpha, \alpha-\beta) u(\alpha-\beta) d\beta = \sum_{j=1}^{2} \partial_\alpha^{k+2} z_j(\alpha) \cdot L_j u(\alpha)$$

and observe that

$$||L_j u||_{L^\infty} \le ||L_j u||_{H^1} \le C|||z|||^2||z||_{H^2}||u||_{H^1}$$

which yields

$$||J_2||_{L^2} \le C|||z|||^2||z||^2_{H^{k+2}}||u||_{H^k}.$$

To estimate J_1 we integrate by parts

$$\begin{aligned}
J_1(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \psi'(\beta) \frac{(z(\alpha)-z(\alpha-\beta))^\perp \partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2} \partial_\alpha^{k} u(\alpha-\beta) d\beta \\
&+ \frac{1}{\pi}\int_{-\infty}^{\infty} \psi(\beta) \partial_\beta\Big(\frac{(z(\alpha)-z(\alpha-\beta))^\perp \partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}\Big) \partial_\alpha^{k} u(\alpha-\beta) d\beta \\
&= J_1^1 + J_1^2.
\end{aligned}$$

For the first part J_1^1 we have

$$||J_1^1||_{L^2} \leq C|||z|||^2||z||^2_{H^3}||u||_{H^k}.$$

And

$$\begin{aligned} J_1^2(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\psi(\beta)\frac{(\partial_\alpha z(\alpha-\beta))^\perp\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}\partial_\alpha^k u(\alpha-\beta)d\beta \\ &+ \frac{2}{\pi}\int_{-\infty}^{\infty}\psi(\beta)\frac{[(z(\alpha)-z(\alpha-\beta))^\perp\partial_\alpha z(\alpha)][\partial_\alpha z(\alpha-\beta)(z(\alpha)-z(\alpha-\beta))]}{|z(\alpha)-z(\alpha-\beta)|^4}\partial_\alpha^k u(\alpha-\beta)d\beta \\ &= J_1^{2,1}+J_1^{2,2}. \end{aligned}$$

For $J_1^{2,1}$

$$(\partial_\alpha z(\alpha-\beta))^\perp\partial_\alpha z(\alpha) = (\partial_\alpha z(\alpha-\beta)-\partial_\alpha z(\alpha))^\perp\partial_\alpha z(\alpha) = -\partial_\alpha^2 z^\perp(\alpha)\partial_\alpha z(\alpha)\beta + O(\beta^2),$$

and

$$|z(\alpha)-z(\alpha-\beta)|^2 = |\partial_\alpha z|^2\beta^2 + O(\beta^3)$$

where the constants in the "O" terms (and in theirs first derivatives) are properly bounded in terms of $||z||_{H^3}$.

That is

$$J_1^{2,1}(\alpha) = -\frac{\partial_\alpha^2 z^\perp(\alpha)\partial_\alpha z(\alpha)}{|z_\alpha(\alpha)|^2}H\partial_\alpha^k u(\alpha) + \text{" bounded terms "}$$

where H denotes the Hilbert transform. Therefore for the first integral we get

$$||J_1^{2,1}||_{L^2} \leq C|||z|||^2||z||^2_{H^3}||u||_{H^k}.$$

Finally for $J_1^{2,2}$ we have

$$[(z(\alpha)-z(\alpha-\beta))^\perp\partial_\alpha z(\alpha)][\partial_\alpha z(\alpha-\beta)(z(\alpha)-z(\alpha-\beta))] = \frac{1}{2}\partial_\alpha^2 z^\perp(\alpha)\partial_\alpha z(\alpha)|\partial_\alpha z(\alpha)|^2\beta^3 + O(\beta^4)$$

and

$$|z(\alpha)-z(\alpha-\beta)|^4 = |\partial_\alpha z|^4\beta^4 + O(\beta^4).$$

By a similar approach we obtain

$$J_1^{2,1}(\alpha) = \frac{\partial_\alpha^2 z^\perp(\alpha)\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2}H\partial_\alpha^k u(\alpha) + \text{" bounded terms "},$$

and it yields

$$||J_1^{2,2}||_{L^2} \leq C|||z|||^2||z||^2_{H^3}||u||_{H^k}.$$

To estimate J_3 we observe first that the substitution of $u(\alpha-\beta)$ by $u(\alpha)-\partial_\alpha u(\alpha)\beta$ produces an error bounded by $||z||^2_{H^{k+2}}|||z|||^2||u||_{H^k}$.

Using the expansions

$$\frac{\psi(\beta)}{|z(\alpha)-z(\alpha-\beta)|^2}=\frac{\psi(\beta)}{|\partial_\alpha z(\alpha)|^2}\frac{1}{|\beta|^2}+O(1)\psi(\beta)$$
$$\partial_\alpha^{k+1}z(\alpha)-\partial_\alpha^{k+1}z(\alpha-\beta)=\beta\int_0^1\partial_\alpha^{k+2}z(\alpha-t\beta)dt$$

and since the term corresponding to $\partial_\alpha u(\alpha)\beta$ can be handled very easily, it remains to estimate:

$$\frac{u(\alpha)}{\pi}\int_{-\infty}^{\infty}\psi(\beta)\frac{(\partial_\alpha^{k+1}z(\alpha)-\partial_\alpha^{k+1}z(\alpha-\beta))^\perp\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}d\beta=\int_0^1 K_t(\alpha)dt,$$

where

$$K_t(\alpha)=\frac{u(\alpha)}{\pi|\partial_\alpha z(\alpha)|^2}\int_{-\infty}^{\infty}\psi(\frac{\beta}{t})\frac{\partial_\alpha^{k+2}z^\perp(\alpha-\beta)\cdot\partial_\alpha z(\alpha)}{\beta}d\beta.$$

Finally, the L^2 boundedness of the Hilbert transform yields

$$||K_t||_{L^2}\leq||z||^2_{H^{k+2}}||u||_{H^k}|||z|||$$

uniformly on t, allowing us to finish the proof.

6 Estimates on $BR(z,\varpi)$

This section is devoted to show that the Birkhoff-Rott integral is as regular as $\partial_\alpha z$.

Lemma 6.1 *The following estimate holds*

$$\|BR(z,\varpi)\|_{H^k}\leq\exp(C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^{k+1}}),\tag{6.1}$$

for $k\geq 2$.

Remark 6.2 *Using this estimate for* $k=2$ *we find easily that*

$$\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}\leq\exp(C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}),\tag{6.2}$$

which shall be used through out the paper.

Proof: We show the proof for $k=2$, being the rest of the cases analogous. We have

$$BR(z,\varpi)(\alpha)=\frac{1}{4\pi}PV\!\!\int_{\mathbb{T}}\varpi(\alpha-\beta)\big(-\frac{V_2(\alpha,\beta)(1+V_1^2(\alpha,\beta))}{|V(\alpha,\beta)|^2},\frac{V_1(\alpha,\beta)(1-V_2^2(\alpha,\beta))}{|V(\alpha,\beta)|^2}\big)d\beta$$

which is decomposed as follows:

$$
\begin{aligned}
BR(z,\varpi)(\alpha) &= \frac{1}{4\pi} PV\int_{\mathbb{T}} \varpi(\alpha-\beta)\frac{V^{\perp}(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta \\
&\quad - \frac{1}{4\pi}\int_{\mathbb{T}}\varpi(\beta)V_2^2(\alpha,\beta)\frac{V^{\perp}(\alpha,\beta)}{|V(\alpha,\beta)|^2}d\beta - \frac{1}{4\pi}\int_{\mathbb{T}}\varpi(\beta)V_2(\alpha,\beta)d\beta(1,0) \\
&= P_1(\alpha)+P_2(\alpha)+P_3(\alpha).
\end{aligned}
\tag{6.3}
$$

Using that $|V_2(\alpha,\beta)|\leq 1$, we get $|P_2(\alpha)|+|P_3(\alpha)|\leq C\|\varpi\|_{L^2}$, and lemma 5.1 yields $\|P_2\|_{L^2}+\|P_3\|_{L^2}\leq \exp(C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3})$.

Let us write

$$
\begin{aligned}
P_1(\alpha) &= \frac{1}{4\pi}\int_{\mathbb{T}}(-A_1(\alpha,\alpha-\beta),A_2(\alpha,\alpha-\beta))\varpi(\alpha-\beta)d\beta + \frac{\partial_\alpha^{\perp}z(\alpha)}{|\partial_\alpha z(\alpha)|^2}H\varpi(\alpha)d\alpha \\
&= J_1+J_2,
\end{aligned}
$$

where as before

$$
A_1(\alpha,\alpha-\beta)=\frac{V_2(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}-\frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_2(\alpha)}{\tan(\frac{\beta}{2})},
$$

and

$$
A_2(\alpha,\alpha-\beta)=\frac{V_1(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}-\frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_1(\alpha)}{\tan(\frac{\beta}{2})}.
$$

For J_1 since $\|A_1\|_{L^\infty}\leq \|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$ and $\|A_2\|_{L^\infty}\leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|^2_{C^2}$ (see appendix) one gets $\|J_1\|_{L^2}\leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|z\|_{L^2}\|\varpi\|_{L^2}$. The inequality $|\partial_\alpha z(\alpha)|^{-1}\leq \|\mathcal{F}(z)\|^{1/2}_{L^\infty}$ give us $\|J_2\|_{L^2}\leq C\|\mathcal{F}(z)\|^{1/2}_{L^\infty}\|z\|_{L^2}\|\varpi\|_{L^2}$.

Next it is easy to check that $|\partial_\alpha^2 P_3(\alpha)|\leq C\|\varpi\|_{L^2}(|\partial^2_\alpha z(\alpha)|+\|z\|^2_{C^2})$ and to estimate $\|\partial^2_\alpha P_3\|_{L^2}$. The kernel in the integral $P_2(\alpha)$ has order 1 in β, and taking two derivatives in α we get integrals as in P_3 and kernels of degree -1 which can be estimated as before. Similar terms of lower order are obtained in $\partial^2_\alpha P_1(\alpha)$ which are controlled analogously. The most singular terms are given by

$$
Q_1(\alpha)=\frac{1}{4\pi}PV\int_{\mathbb{T}}\partial^2_\alpha\varpi(\alpha-\beta)\frac{V^{\perp}(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta,
$$

$$
Q_2(\alpha)=\frac{1}{8\pi}PV\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{\partial^2_\alpha z(\alpha)-\partial^2_\alpha z(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta,
$$

$$
Q_3(\alpha)=-\frac{1}{4\pi}PV\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{V^{\perp}(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}\big(V(\alpha,\alpha-\beta)\cdot(\partial^2_\alpha z(\alpha)-\partial^2_\alpha z(\alpha-\beta))\big)d\beta.
$$

We have

$$
Q_1=\frac{1}{4\pi}PV\!\int_{\mathbb{T}}\partial^2_\alpha\varpi(\alpha-\beta)\big(\frac{V^{\perp}(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}-\frac{\partial^{\perp}_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2\tan(\beta/2)}\big)d\beta+\frac{\partial^{\perp}_\alpha z(\alpha)}{2|\partial_\alpha z(\alpha)|^2}H(\partial^2_\alpha\varpi)(\alpha),
$$

giving us

$$\begin{aligned}|Q_1(\alpha)| &\leq C\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^2}\|\partial^2_\alpha\varpi\|_{L^2} + \|\mathcal{F}(z)\|^{1/2}_{L^\infty}|H(\partial^2_\alpha\varpi)(\alpha)|\\ &\leq (1+|H(\partial^2_\alpha\varpi)(\alpha)|)\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t)).\end{aligned} \tag{6.4}$$

Next we write $Q_2 = R_1 + R_2 + R_3$ where

$$R_1(\alpha) = \frac{1}{8\pi}\int_{\mathbb{T}}(\varpi(\alpha-\beta)-\varpi(\alpha))\frac{\partial^2_\alpha z(\alpha)-\partial^2_\alpha z(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

$$R_2(\alpha) = \frac{\varpi(\alpha)}{8\pi}\int_{\mathbb{T}}(\partial^2_\alpha z(\alpha)-\partial^2_\alpha z(\alpha-\beta))\big(\frac{1}{|V(\alpha,\alpha-\beta)|^2}-\frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}\big)d\beta,$$

$$R_3(\alpha) = \frac{1}{8\pi}\frac{\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2}\int_{\mathbb{T}}(\partial^2_\alpha z(\alpha)-\partial^2_\alpha z(\alpha-\beta))\big(\frac{4}{|\beta|^2}-\frac{1}{\sin^2(\beta/2)}\big)d\beta+\frac{1}{2}\frac{\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2}\Lambda(\partial^2_\alpha z)(\alpha).$$

Using that

$$|\partial^2_\alpha z(\alpha)-\partial^2_\alpha z(\alpha-\beta)| \leq |\beta|^\delta\|z\|_{C^{2,\delta}},$$

we get

$$|R_1(\alpha)|+|R_2(\alpha)| \leq \|\varpi\|_{C^1}\|\mathcal{F}(z)\|^k\|z\|^k_{C^{2,\delta}} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}).$$

While for R_3 we have

$$\begin{aligned}|R_3(\alpha)| &\leq C\|\varpi\|_{L^\infty}\|\mathcal{F}(z)\|_{L^\infty}(\|z\|_{C^2}+|\Lambda(\partial^2_\alpha z)(\alpha)|)\\ &\leq (1+|\Lambda(\partial^2_\alpha z)(\alpha)|)\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}),\end{aligned}$$

that is

$$|Q_2(\alpha)| \leq (1+|\Lambda(\partial^2_\alpha z)(\alpha)|)\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}). \tag{6.5}$$

Let us consider $Q_3 = R_4 + R_5 + R_6 + R_7 + R_8 + R_9$, where

$$R_4 = -\frac{1}{4\pi}\int_{\mathbb{T}}(\varpi(\alpha-\beta)-\varpi(\alpha))\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}\big(V(\alpha,\alpha-\beta)\cdot(\partial^2_\alpha z(\alpha)-\partial^2_\alpha z(\alpha-\beta))\big)d\beta,$$

$$R_5 = -\frac{\varpi(\alpha)}{4\pi}\int_{\mathbb{T}}\frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^\perp}{|V(\alpha,\alpha-\beta)|^4}\big(V(\alpha,\alpha-\beta)\cdot(\partial^2_\alpha z(\alpha)-\partial^2_\alpha z(\alpha-\beta))\big)d\beta,$$

$$R_6 = -\frac{\varpi(\alpha)(\partial_\alpha z(\alpha))^\perp}{8\pi}\int_{\mathbb{T}}\frac{\beta(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)\cdot(\partial^2_\alpha z(\alpha)-\partial^2_\alpha z(\alpha-\beta))}{|V(\alpha,\alpha-\beta)|^4}d\beta,$$

$$R_7 = -\frac{\varpi(\alpha)(\partial_\alpha z(\alpha))^\perp}{16\pi}\partial_\alpha z(\alpha)\cdot\int_{\mathbb{T}}\beta^2(\partial^2_\alpha z(\alpha)-\partial^2_\alpha z(\alpha-\beta))\big(\frac{1}{|V(\alpha,\alpha-\beta)|^4}-\frac{16}{|\partial_\alpha z(\alpha)|^4|\beta|^4}\big)d\beta,$$

$$R_8 = -\frac{\varpi(\alpha)(\partial_\alpha z(\alpha))^\perp}{4\pi|\partial_\alpha z(\alpha)|^4}\partial_\alpha z(\alpha)\cdot\int_{\mathbb{T}}(\partial^2_\alpha z(\alpha)-\partial^2_\alpha z(\alpha-\beta))\big(\frac{4}{|\beta|^2}-\frac{1}{\sin^2(\beta/2)}\big)d\beta,$$

and

$$R_9 = -\frac{\varpi(\alpha)(\partial_\alpha z(\alpha))^\perp}{|\partial_\alpha z(\alpha)|^4}\partial_\alpha z(\alpha)\cdot\Lambda(\partial^2_\alpha z(\alpha)).$$

Proceeding as before we get

$$|Q_3(\alpha)| \leq (1 + |\Lambda(\partial_\alpha^2 z)(\alpha)|) \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2),$$

which together with (6.4) and (6.5) gives us the estimate

$$|\partial_\alpha^2 P_1(\alpha)| \leq (1 + |\Lambda(\partial_\alpha^2 z)(\alpha)| + |H(\partial_\alpha^2 \varpi)(\alpha)|) \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2),$$

and $\|\partial_\alpha^2 P_1\|_{L^2} \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2)$.

Finally we get

$$\|\partial_\alpha^2 BR(z,\varpi)\|_{L^2} \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2). \tag{6.6}$$

7 Estimates on $z(\alpha,t)$

In this section we give the proof of the below lemma for $k = 3$. The case $k > 3$ is left to the reader.

Lemma 7.1 *Let $z(\alpha,t)$ be a solution of 2DM. Then, the following a priori estimate holds:*

$$\begin{aligned}\frac{d}{dt}\|z\|_{H^k}^2(t) \leq &-\frac{\kappa}{2\pi(\mu_1+\mu_2)}\int_{\mathbb{T}} \frac{\sigma(\alpha,t)}{|\partial_\alpha z(\alpha)|^2}\partial_\alpha^k z(\alpha,t)\cdot\Lambda(\partial_\alpha^k z)(\alpha,t)d\alpha \\ &+ \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2(t) + \|z\|_{H^k}^2),\end{aligned} \tag{7.1}$$

for $k \geq 3$.

We split the proof in the following four parts.

7.1 Estimates for the L^2 norm of the curve

We have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |z(\alpha)|^2 d\alpha = \int_{\mathbb{T}} z(\alpha)\cdot z_t(\alpha)d\alpha &= \int_{\mathbb{T}} z(\alpha)\cdot BR(z,\varpi)(\alpha)d\alpha + \int_{\mathbb{T}} c(\alpha)z(\alpha)\cdot\partial_\alpha z(\alpha)d\alpha \\ &= I_1 + I_2.\end{aligned}$$

Taking $I_1 \leq \|z\|_{L^2}\|BR(z,\varpi)\|_{L^2}$ and the inequality (6.1) let us estimate I_1.

Next we get

$$I_2 \leq A^{1/2}(t)\|c\|_{L^\infty}\int_{\mathbb{T}} |z(\alpha)|d\alpha \leq 2\int_{\mathbb{T}} |\partial_\alpha BR(z,\varpi)(\alpha)|d\alpha \int_{\mathbb{T}} |z(\alpha)|d\alpha$$

which yields

$$I_2 \leq \exp(C\|F(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2)$$

if we consider the estimate (6.2). We conclude that

$$\frac{d}{dt}\|z\|_{L^2}^2(t) \leq \exp(C|||z|||^2) \tag{7.2}$$

for an appropriate finite constant C, where as before $|||z|||^2 = \|F(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2$.

7.2 The integrable terms in $\partial_\alpha^3 BR(z,\varpi)$

Since $z_t(\alpha) = BR(z,\varpi)(\alpha) + c(\alpha)\cdot\partial_\alpha z(\alpha)$ we have

$$\begin{aligned}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^3 z_t(\alpha)d\alpha &= \int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^3 BR(z,\varpi)(\alpha)d\alpha + \int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^3(c(\alpha)\partial_\alpha z(\alpha))d\alpha\\ &= I_1 + I_2.\end{aligned}$$

Here and in 7.3 we study I_1. We shall estimate I_2 in 7.4.

Let us write $BR(z,\varpi)(\alpha) = P_1(\alpha) + P_2(\alpha) + P_3(\alpha)$ as in (6.3). Then it is easy to check that

$$|\partial_\alpha^3 P_3(\alpha)| \leq C\|\varpi\|_{L^2}(|\partial_\alpha^3 z_2(\alpha)| + \|z\|_{C^2}^3),$$

giving us a term controlled by the energy estimate. The kernel in the integral $P_2(\alpha)$ has order 1 in β, therefore taking two derivatives in α produces regular integrals as in P_3 and kernels of degree -1 in β, for which we first exchange β by $\alpha-\beta$ and then take one more derivative. We obtain kernels of grade -1 in β acting in ϖ or ϖ_α which can be estimated as before. For the most singular term $P_1(\alpha)$, we have

$$\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^3 P_1(\alpha)d\alpha = I_3 + I_4 + I_5 + I_6,$$

where

$$I_3 = \frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\partial_\alpha^3\Big(\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}\Big)\varpi(\alpha-\beta)d\beta,$$

$$I_4 = \frac{3}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\partial_\alpha^2\Big(\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}\Big)\partial_\alpha\varpi(\alpha-\beta)d\beta,$$

$$I_5 = \frac{3}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\partial_\alpha\Big(\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}\Big)\partial_\alpha^2\varpi(\alpha-\beta)d\beta,$$

$$I_6 = \frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\Big(\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}\Big)\partial_\alpha^3\varpi(\alpha-\beta)d\beta,$$

The most singular terms for I_3 are those in which three derivatives appear and the kernels have degree -1. The rest of the terms have kernels with degree $k > -1$ and can be estimated as before. One of the two singular terms of I_3 is given by

$$J_1 = \frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\frac{(\partial_\alpha^3 z(\alpha) - \partial_\alpha^3 z(\alpha-\beta))^\perp}{|V(\alpha,\alpha-\beta)|^2}\varpi(\alpha-\beta)d\beta d\alpha,$$

which we decompose as follows:

$$
\begin{aligned}
J_1 &= \frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta))^\perp}{|V(\alpha,\beta)|^2}\varpi(\beta)d\beta d\alpha\\
&= \frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta))^\perp}{|V(\alpha,\beta)|^2}\frac{\varpi(\beta)+\varpi(\alpha)}{2}d\beta d\alpha\\
&\quad+\frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta))^\perp}{|V(\alpha,\beta)|^2}\frac{\varpi(\beta)-\varpi(\alpha)}{2}d\beta d\alpha\\
&= K_1+K_2.
\end{aligned}
$$

That is we have made a kind of integration by parts in J_1, allowing us to show that the most singular term K_1 vanishes:

$$
\begin{aligned}
K_1 &= -\frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\beta)\cdot\frac{(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta))^\perp}{|V(\alpha,\beta)|^2}\frac{\varpi(\beta)+\varpi(\alpha)}{2}d\beta d\alpha\\
&= \frac{1}{16\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} (\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta))\cdot\frac{(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta))^\perp}{|V(\alpha,\beta)|^2}\frac{\varpi(\beta)+\varpi(\alpha)}{2}d\beta d\alpha\\
&= 0,
\end{aligned}
$$

whether for K_2 we have

$$
\begin{aligned}
K_2 &= \frac{1}{16\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot(\partial_\alpha^3 z(\beta))^\perp\frac{\varpi(\alpha)-\varpi(\beta)}{|V(\alpha,\beta)|^2}d\beta d\alpha\\
&= \int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot(\partial_\alpha^3 Hz(\alpha))^\perp\frac{\partial_\alpha\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2}d\alpha+\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot(\partial_\alpha^3 z(\alpha-\beta))^\perp B_1(\alpha,\beta)d\beta d\alpha,
\end{aligned}
$$

where $|B_1(\alpha,\beta)|\leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|\varpi\|_{C^{1,\delta}}|\beta|^{\delta-1}$. The other singular term with three derivatives in $z(\alpha)$ and kernel of degree -1 inside I_3 is given by

$$
J_2 = -\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}V(\alpha,\alpha-\beta)\cdot(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\alpha-\beta))\varpi(\alpha-\beta)d\beta d\alpha
$$

Here we take $J_2 = K_3+K_4+K_5$ where

$$
K_3 = -\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{V^\perp(\alpha,\beta)}{|V(\alpha,\beta)|^4}(V(\alpha,\beta)-W(\alpha,\beta))\cdot(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta))\varpi(\beta)d\beta d\alpha,
$$

$$
K_4 = -\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}B_2(\alpha,\alpha-\beta)\cdot(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\alpha-\beta))\varpi(\alpha-\beta)d\beta d\alpha,
$$

with

$$
B_2(\alpha,\alpha-\beta) = W(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2,
$$

and

$$
W(\alpha,\beta) = ((\frac{z_1(\alpha)-z_1(\beta)}{2})_p,(\frac{z_2(\alpha)-z_2(\beta)}{2})_p)
$$

is defined in the appendix. Finally we have:

$$K_5 = -\frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot \frac{V^\perp(\alpha,\alpha-\beta)\beta}{|V(\alpha,\alpha-\beta)|^4}\partial_\alpha z(\alpha)\cdot(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\alpha-\beta))\varpi(\alpha-\beta)d\beta d\alpha.$$

The L^∞ norm of

$$\frac{V^\perp(\alpha,\beta)}{|V(\alpha,\beta)|^4}(V(\alpha,\beta)-W(\alpha,\beta))$$

is given in the appendix, allowing us to estimate the term K_3 as before.

Next we split $K_4 = L_1 + L_2$, where

$$L_1 = -\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot \frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}B_2(\alpha,\alpha-\beta)\cdot\partial_\alpha^3 z(\alpha)\varpi(\alpha-\beta)d\beta d\alpha,$$

and

$$L_2 = \frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot \frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}B_2(\alpha,\alpha-\beta)\cdot\partial_\alpha^3 z(\alpha-\beta)\varpi(\alpha-\beta)d\beta d\alpha,$$

We have

$$\begin{aligned}|L_1| \le C\Big|\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^4}\partial_\alpha^2 z(\alpha)\cdot\partial_\alpha^3 z(\alpha)H\varpi(\alpha)d\beta d\alpha\Big| \\ + \|\partial_\alpha^3 z\|_{L^2}^2\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^{2,\delta}}\|\varpi\|_{L^\infty}\end{aligned}$$

$$\begin{aligned}|L_2| \le C\Big|\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^4}\varpi(\alpha)\partial_\alpha^2 z(\alpha)\cdot H(\partial_\alpha^3 z)(\alpha)d\beta d\alpha\Big| \\ + \|\partial_\alpha^3 z\|_{L^2}^2\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^{2,\delta}}\|\varpi\|_{C^1}\end{aligned}$$

and the term K_4 is controlled.

For K_5 we split

$$\begin{aligned}K_5 &= \frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot \frac{V^\perp(\alpha,\alpha-\beta)\beta}{|V(\alpha,\alpha-\beta)|^4}(\partial_\alpha z(\alpha)-\partial_\alpha z(\alpha-\beta))\cdot\partial_\alpha^3 z(\alpha-\beta)\varpi(\alpha-\beta)d\beta d\alpha, \\ &\quad -\frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot B_3(\alpha,\alpha-\beta)(\partial_\alpha z(\alpha)\cdot\partial_\alpha^3 z(\alpha)-\partial_\alpha z(\alpha-\beta)\cdot\partial_\alpha^3 z(\alpha-\beta))d\beta d\alpha \\ &= L_3 + L_4\end{aligned}$$

where

$$B_3(\alpha,\alpha-\beta) = \frac{V^\perp(\alpha,\alpha-\beta)\varpi(\alpha-\beta)\beta}{|V(\alpha,\alpha-\beta)|^4}.$$

Then we have

$$\begin{aligned}|L_3| \le C\Big|\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^4}\partial_\alpha^2 z(\alpha)\cdot H(\partial_\alpha^3 z\,\varpi)(\alpha)d\alpha\Big| \\ + \|\partial_\alpha^3 z\|_{L^2}^2\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^{2,\delta}}\|\varpi\|_{L^\infty}.\end{aligned}$$

For L_4 we use an appropriated integration by part:

$$\partial_\alpha z(\alpha)\cdot\partial_\alpha^3 z(\alpha)=-|\partial_\alpha^2 z(\alpha)|^2,$$

to obtain

$$L_4=\frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot B_3(\alpha,\alpha-\beta)(|\partial_\alpha^2 z(\alpha)|^2-|\partial_\alpha^2 z(\alpha-\beta)|^2)d\beta d\alpha.$$

Next we write $L_4=M_1+M_2$, with

$$M_1=\frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot C(\alpha,\alpha-\beta)(|\partial_\alpha^2 z(\alpha)|^2-|\partial_\alpha^2 z(\alpha-\beta)|^2)d\beta d\alpha$$

for

$$\begin{aligned}C(\alpha,\alpha-\beta)&=B_3(\alpha,\alpha-\beta)-\frac{2\partial_\alpha^\perp z(\alpha)\varpi(\alpha)}{|\partial_\alpha^2 z(\alpha)|^4\sin^2(\beta/2)}\\&=\frac{V^\perp(\alpha,\alpha-\beta)\varpi(\alpha-\beta)\beta}{|V(\alpha,\alpha-\beta)|^4}-\frac{2\partial_\alpha^\perp z(\alpha)\varpi(\alpha)}{|\partial_\alpha^2 z(\alpha)|^4\sin^2(\beta/2)},\end{aligned}$$

and

$$M_2=C\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^\perp z(\alpha)\frac{\varpi(\alpha)}{|\partial_\alpha^2 z(\alpha)|^4}\Lambda(|\partial_\alpha^2 z|^2)d\alpha.$$

Since

$$|C(\alpha,\alpha-\beta)|\le\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|\varpi\|_{C^1}\frac{1}{|\beta|}$$

(see lemma 11.3 in the appendix for more details) and

$$||\partial_\alpha^2 z(\alpha)|^2-|\partial_\alpha^2 z(\alpha-\beta)|^2|\le 2\|z\|_{C^1}|\beta|\int_0^1|\partial_\alpha^3 z(\alpha+(s-1)\beta)|ds$$

we get $|M_1|\le\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|\varpi\|_{C^1}\|\partial_\alpha^3 z\|_{L^2}^2$.

For the term M_2 we use the estimate

$$\|\Lambda(|\partial_\alpha^2 z|^2)\|_{L^2}=\|\partial_\alpha(|\partial_\alpha^2 z|^2)\|_{L^2}\le 2\|\partial_\alpha^2 z\|_{L^\infty}\|\partial_\alpha^3 z\|_{L^2},$$

to obtain $|M_2|\le C\|\mathcal{F}(z)\|_{L^\infty}\|\varpi\|_{L^\infty}\|\partial_\alpha^2 z\|_{L^\infty}\|\partial_\alpha^3 z\|_{L^2}^2$.

For I_4 the most singular terms are those for which two derivatives are applied to $z(\alpha)$. One of those is J_3

$$J_3=C\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot(\partial_\alpha^2 z(\alpha)-\partial_\alpha^2 z(\alpha-\beta))\frac{\partial_\alpha\varpi(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta.$$

We split $J_3=K_6+K_7+K_8$

$$K_6=C\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot(\partial_\alpha^2 z(\alpha)-\partial_\alpha^2 z(\alpha-\beta))\frac{\partial_\alpha\varpi(\alpha-\beta)-\partial_\alpha\varpi(\alpha)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

$$K_7 = C\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha\varpi(\alpha)\partial_\alpha^3 z(\alpha)\cdot(\partial_\alpha^2 z(\alpha)-\partial_\alpha^2 z(\alpha-\beta))\Big(\frac{1}{|V(\alpha,\alpha-\beta)|^2}-\frac{4}{|\partial_\alpha z(\alpha)|^2\beta^2}\Big)d\beta,$$

$$K_8 = C\int_{\mathbb{T}} \partial_\alpha\varpi(\alpha)\partial_\alpha^3 z(\alpha)\cdot\int_{\mathbb{T}}\frac{\partial_\alpha^2 z(\alpha)-\partial_\alpha^2 z(\alpha-\beta)}{\beta^2}d\beta,$$

Using that

$$\partial_\alpha^2 z(\alpha)-\partial_\alpha^2 z(\alpha-\beta)=\beta\int_0^1 \partial_\alpha^3 z(\alpha+(s-1)\beta)ds \tag{7.3}$$

and $|\partial_\alpha\varpi(\alpha-\beta)-\partial_\alpha\varpi(\alpha)|\leq\|w\|_{C^{1,\delta}}|\beta|^\delta$ we have

$$\begin{aligned}|K_6| &\leq C\|\mathcal{F}(z)\|_{L^\infty}\|w\|_{C^{1,\delta}}\int_0^1\int_{\mathbb{T}}|\beta|^{\delta-1}\int_{\mathbb{T}}|\partial_\alpha^3 z(\alpha)||\partial_\alpha^3 z(\alpha+(s-1)\beta)|d\alpha d\beta ds\\ &\leq C\|\mathcal{F}(z)\|_{L^\infty}\|w\|_{C^{1,\delta}}\int_0^1\int_{\mathbb{T}}|\beta|^{\delta-1}\int_{\mathbb{T}}(|\partial_\alpha^3 z(\alpha)|^2+|\partial_\alpha^3 z(\alpha+(s-1)\beta)|^2)d\alpha d\beta ds\\ &\leq C\|\mathcal{F}(z)\|_{L^\infty}\|w\|_{C^{1,\delta}}\|\partial_\alpha^3 z\|_{L^3}^2.\end{aligned}$$

Due to (7.3) and the estimates obtained in the appendix we have

$$|K_7|\leq C\|\mathcal{F}(z)\|_{L^\infty}\|w\|_{C^1}\|z\|_{C^2}\|\partial_\alpha^3 z\|_{L^2}^2.$$

Then using that $1/\beta - 1/2\sin(\beta/2)$ is bounded, we get

$$K_8\leq C\|\mathcal{F}(z)\|_{L^\infty}\|w\|_{C^1}\|\partial_\alpha^3 z\|_{L^3}^2.$$

Regarding I_5, its most singular term is giving by

$$J_4 = C\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot(\partial_\alpha z(\alpha)-\partial_\alpha z(\alpha-\beta))\frac{\partial_\alpha^2\varpi(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

which after being decomposed in the form

$$\begin{aligned}J_4 = C&\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\Big(\frac{\partial_\alpha z(\alpha)-\partial_\alpha z(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}-\frac{2\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^2\tan(\beta/2)}\Big)\partial_\alpha^2\varpi(\alpha-\beta)d\beta\\ &+C\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\frac{\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^2}H(\partial_\alpha^2\varpi)(\alpha)d\alpha.\end{aligned}$$

can be estimated as before $|J_4|\leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^{2,\delta}}\|\partial_\alpha^2 w\|_{L^2}\|\partial_\alpha^3 z\|_{L^3}$.

7.3 Looking for $\sigma(\alpha)$

The term I_6 will gives us the proper sign (Rayleigh-Taylor condition) that has to be imposed upon $\sigma(\alpha)$. Let us recall the formula

$$\sigma(\alpha,t)=\frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t)+\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t)$$

We write I_6 in the form $I_6 = J_1 + J_2$ where

$$J_1 = \frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\Big(\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{(\partial_\alpha z(\alpha))^\perp}{|\partial_\alpha z(\alpha)|^2\tan(\beta/2)}\Big)\partial_\alpha^3\varpi(\alpha-\beta)d\beta d\alpha,$$

and

$$J_2 = \frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{(\partial_\alpha z(\alpha))^\perp}{|\partial_\alpha z(\alpha)|^2} H(\partial_\alpha^3\varpi)(\alpha)d\beta d\alpha.$$

Let us denote the kernel of J_1 by $\Sigma(\alpha,\alpha-\beta)$, which is of degree 0 in β. After an integration by parts we obtain:

$$\begin{aligned}
J_1 &= -\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\Sigma(\alpha,\alpha-\beta)\partial_\beta(\partial_\alpha^2\varpi(\alpha-\beta))d\beta d\alpha\\
&= \frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\partial_\beta\Sigma(\alpha,\alpha-\beta)\partial_\alpha^2\varpi(\alpha-\beta)d\beta d\alpha.
\end{aligned}$$

Then $\partial_\beta\Sigma(\alpha,\alpha-\beta)$ has terms of degree 0 which are estimated easily. The term with degree -1 is given by

$$\frac{(\partial_\alpha z(\alpha-\beta))^\perp}{2|V(\alpha,\alpha-\beta)|^2} + \frac{(\partial_\alpha z(\alpha))^\perp}{2|\partial_\alpha z(\alpha)|^2\sin^2(\beta/2)} - \frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}V(\alpha,\alpha-\beta)\cdot\partial_\alpha z(\alpha-\beta),$$

and we decompose it as a sum of a kernel of degree 0 (easy to estimate)

$$-\frac{(\partial_\alpha z(\alpha))^\perp}{|\partial_\alpha z(\alpha)|^2}\Big(\frac{2}{|\beta|^2} - \frac{1}{2\sin^2(\beta/2)}\Big),$$

and six kernels of degree -1, (P_1,..., P_6) given by

$$P_1(\alpha,\alpha-\beta) = \frac{(\partial_\alpha z(\alpha-\beta)-\partial_\alpha z(\alpha))^\perp}{2|V(\alpha,\alpha-\beta)|^2},$$

$$P_2(\alpha,\alpha-\beta) = -\frac{(\partial_\alpha z(\alpha))^\perp}{2}\Big(\frac{1}{|V(\alpha,\alpha-\beta)|^2} - \frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}\Big),$$

$$P_3(\alpha,\alpha-\beta) = \frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}V(\alpha,\alpha-\beta)\cdot(\partial_\alpha z(\alpha)-\partial_\alpha z(\alpha-\beta)),$$

$$P_4(\alpha,\alpha-\beta) = -\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)\cdot\partial_\alpha z(\alpha),$$

$$P_5(\alpha,\alpha-\beta) = -\frac{|\partial_\alpha z(\alpha)|^2\beta}{2}\frac{V^\perp(\alpha,\alpha-\beta)-\partial_\alpha^\perp z(\alpha)\beta/2}{|V(\alpha,\alpha-\beta)|^4},$$

$$P_6(\alpha,\alpha-\beta) = -\frac{\partial_\alpha^\perp z(\alpha)|\partial_\alpha z(\alpha)|^2|\beta|^2}{4|V(\alpha,\alpha-\beta)|^2}\Big(\frac{1}{|V(\alpha,\alpha-\beta)|^2} - \frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}\Big).$$

To control the term with kernel P_2 we consider $P_2 = Q_1 + Q_2$

$$Q_1(\alpha,\alpha-\beta) = P_2(\alpha,\alpha-\beta) - \frac{(\partial_\alpha z(\alpha))^\perp}{2}\frac{2\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^4\beta},$$

$$Q_2(\alpha,\alpha-\beta)=-\partial_\alpha^\perp z(\alpha)\Big(\frac{\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^4}\big(\frac{1}{\beta}-\frac{1}{2\tan(\beta/2)}\big)+\frac{\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^4 2\tan(\beta/2)}\Big).$$

It is shown in the appendix that $\|Q_1\|_{L^\infty}\leq\|\mathcal{F}(z)\|_{L^\infty}^k\|z\|_{C^{2,\delta}}^k|\beta|^{\delta-1}$, (see lemma 11.4) giving us

$$\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot Q_1(\alpha,\alpha-\beta)\partial_\alpha^2\varpi(\alpha-\beta)d\beta d\alpha\leq C\|\mathcal{F}(z)\|_{L^\infty}^k\|z\|_{C^{2,\delta}}^k\|\partial_\alpha^3 z\|_{L^2}\|\partial_\alpha^2\varpi\|_{L^2}.$$

The integral

$$K_1=\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot Q_2(\alpha,\alpha-\beta)\partial_\alpha^2\varpi(\alpha-\beta)d\beta d\alpha$$

is bounded by

$$\begin{aligned}|K_1|&\leq C\|\mathcal{F}(z)\|_{L^\infty}^{3/2}\|z\|_{C^2}\int_{\mathbb{T}}|\partial_\alpha^3 z(\alpha)|(\|\partial_\alpha^2\varpi\|_{L^2}+|H(\partial_\alpha^2\varpi)(\alpha)|)d\alpha\\&\leq C\|\mathcal{F}(z)\|_{L^\infty}^{3/2}\|z\|_{C^2}\|\partial_\alpha^3 z\|_{L^2}\|\partial_\alpha^2\varpi\|_{L^2}.\end{aligned}$$

It is now very clear that the other P_i terms can be estimated as above or as before i.e. we finally have

$$J_1\leq\exp C(\|\mathcal{F}(z)\|_{L^\infty}^2+\|z\|_{H^3}^2)$$

We consider now the J_2 term which can be written as follows

$$J_2=\frac{1}{4\pi A(t)}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^\perp z(\alpha)\Lambda(\partial_\alpha^2\varpi)(\alpha)d\alpha=\frac{1}{4\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\partial_\alpha^2\varpi(\alpha)d\alpha$$

and using the formula (5.3) we separate J_2 as a sum of two parts, K_2 and K_3, where

$$K_2=-\frac{\kappa\mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\partial_\alpha^3 z_2(\alpha)d\alpha,$$

and

$$K_3=-\frac{A_\mu}{4\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\partial_\alpha^2 T(\varpi)(\alpha)d\alpha.$$

For K_2 we decompose further $K_2=L_1+L_2$, where

$$L_1=\frac{\kappa\mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z_1\partial_\alpha z_2)(\alpha)\partial_\alpha^3 z_2(\alpha)d\alpha,$$

and

$$L_2=-\frac{\kappa\mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z_2\partial_\alpha z_1)(\alpha)\partial_\alpha^3 z_2(\alpha)d\alpha.$$

Then L_1 is written as $L_1=M_1+M_2$ with

$$M_1=\frac{\kappa\mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}(\Lambda(\partial_\alpha^3 z_1\partial_\alpha z_2)(\alpha)-\Lambda(\partial_\alpha^3 z_1)(\alpha)\partial_\alpha z_2(\alpha))\partial_\alpha^3 z_2(\alpha)d\alpha,$$

and

$$M_2 = \frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)} \int_{\mathbb{T}} \Lambda(\partial_\alpha^3 z_1)(\alpha)\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_2(\alpha) d\alpha.$$

Using the commutator estimate, we get

$$M_1 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^{2,\delta}}\|\partial_\alpha^3 z\|_{L^2}^2 \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2)$$

The identity

$$\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_2(\alpha) = -\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha) - |\partial_\alpha^2 z(\alpha)|^2,$$

lets us write M_2 as the sum of N_1 and N_2 where

$$N_1 = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)} \int_{\mathbb{T}} \Lambda(\partial_\alpha^3 z_1)(\alpha)|\partial_\alpha^2 z(\alpha)|^2 d\alpha,$$

and

$$N_2 = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)} \int_{\mathbb{T}} \partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha)\Lambda(\partial_\alpha^3 z_1)(\alpha) d\alpha.$$

Integration by parts shows that

$$N_1 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|\partial_\alpha^3 z\|_{L^2}^2 \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2).$$

Writing L_2 in the form:

$$L_2 = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)} \int_{\mathbb{T}} \partial_\alpha z_1(\alpha)\partial_\alpha^3 z_2(\alpha)\Lambda(\partial_\alpha^3 z_2)(\alpha) d\alpha,$$

we obtain finally

$$K_2 \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2) - \frac{2\kappa \mathrm{g}(\rho^2-\rho^1)}{4\pi(\mu^2+\mu^1)A(t)} \int_{\mathbb{T}} \partial_\alpha z_1(\alpha)\partial_\alpha^3 z(\alpha)\cdot\Lambda(\partial_\alpha^3 z)(\alpha) d\alpha.$$

In the estimate above we can observe how a part of $\sigma(\alpha)$ appears in the non-integrable terms. Let us now return to $K_3 = L_3 + L_4 + L_5$, where

$$L_3 = -\frac{A_\mu}{4\pi A(t)} \int_{\mathbb{T}} \Lambda(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)(2\partial_\alpha^2 BR(z,\varpi)(\alpha))\cdot\partial_\alpha z(\alpha) d\alpha,$$

$$L_4 = -\frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} \Lambda(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\partial_\alpha BR(z,\varpi)(\alpha))\cdot\partial_\alpha^2 z(\alpha) d\alpha,$$

and

$$L_5 = -\frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} \Lambda(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha) BR(z,\varpi)(\alpha)\cdot\partial_\alpha^3 z(\alpha) d\alpha.$$

We will control first the terms L_3 and L_4 and then we will show how the rest of $\sigma(\alpha)$ appears in L_5. Integrating by parts in L_4 and we obtain

$$L_4 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)\|_{L^2}(\|\partial_\alpha^2 BR(z,\varpi)\|_{L^2}\|\partial_\alpha^2 z\|_{L^\infty} + \|\partial_\alpha BR(z,\varpi)\|_{L^\infty}\|\partial_\alpha^3 z\|_{L^2})$$

and using the estimates for $\|\partial_\alpha^2 BR(z,\varpi)\|_{L^2}$, we get $L_4 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3})$ With L_3 we also integrate by parts to obtain $L_3 = M_3 + M_4$ where

$$M_3 = -\frac{A_\mu}{4\pi A(t)} \int_{\mathbb{T}} H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)(\alpha)(2\partial_\alpha^2 BR(z,\varpi)(\alpha)) \cdot \partial_\alpha^2 z(\alpha) d\alpha,$$

and

$$M_4 = -\frac{A_\mu}{4\pi A(t)} \int_{\mathbb{T}} H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)(\alpha)(2\partial_\alpha^3 BR(z,\varpi)(\alpha)) \cdot \partial_\alpha z(\alpha) d\alpha.$$

Easily we have

$$M_3 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)\|_{L^2}\|\partial_\alpha^2 BR(z,\varpi)\|_{L^2}\|\partial_\alpha^2 z\|_{L^\infty} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+ \|z\|^2_{H^3}).$$

In M_4 the application of Leibniz's rule to $\partial_\alpha^3 BR(z,\varpi)$ produces many terms which can be estimated with the same tools used before with I_4 and I_5. For the most singular terms we have the expressions:

$$N_3 = -\frac{A_\mu}{4\pi A(t)} \int_{\mathbb{T}} H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)(\alpha) 2\partial_\alpha(BR(z,\partial_\alpha^2\varpi)(\alpha)) \cdot \partial_\alpha z(\alpha) d\alpha,$$

$$N_4 = -\frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)(\alpha) \int_{\mathbb{T}} \varpi(\alpha-\beta)\frac{(\partial_\alpha^3 z(\alpha) - \partial_\alpha^3 z(\alpha-\beta))^\perp}{|V(\alpha,\alpha-\beta)|^2} d\beta \cdot \partial_\alpha z(\alpha) d\alpha,$$

$$N_5 = \frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)(\alpha) \int_{\mathbb{T}} \varpi(\alpha-\beta) B(\alpha,\alpha-\beta) \cdot (\partial_\alpha^3 z(\alpha) - \partial_\alpha^3 z(\alpha-\beta))) d\beta d\alpha,$$

where

$$B(\alpha,\alpha-\beta) = \frac{V^\perp(\alpha,\alpha-\beta)\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4} V(\alpha,\alpha-\beta).$$

Let us consider

$$\begin{aligned}\partial_\alpha(BR(z,\partial_\alpha^2\varpi)(\alpha)) \cdot \partial_\alpha z(\alpha) &= \partial_\alpha(BR(z,\partial_\alpha^2\varpi)(\alpha)) \cdot \partial_\alpha z(\alpha)) - BR(z,\partial_\alpha^2\varpi)(\alpha) \cdot \partial_\alpha^2 z(\alpha)\\ &= \frac{1}{2}\partial_\alpha(T(\partial_\alpha^2\varpi)(\alpha)) - BR(z,\partial_\alpha^2\varpi)(\alpha) \cdot \partial_\alpha^2 z(\alpha)\end{aligned}$$

which yields

$$N_3 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)\|_{L^2}(\|T(\partial_\alpha^2\varpi)\|_{H^1} + \|BR(z,\partial_\alpha^2\varpi)\|_{L^2}\|\partial_\alpha^2 z\|_{L^\infty})$$

and therefore

$$N_3 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+ \|z\|^2_{H^3}).$$

Next we write $N_4 = O_1 + O_2 + O_3 + O_4 + O_5$,

$$O_1 = -\frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)(\partial_\alpha^3 z(\alpha))^\perp\cdot\partial_\alpha z(\alpha) \int_{\mathbb{T}} (\frac{\varpi(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{4\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2|\beta|^2}) d\beta d\alpha,$$

$$O_2 = \frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha) \int_{\mathbb{T}} (\partial_\alpha^3 z(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha) (\frac{\varpi(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{4\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2|\beta|^2}) d\beta d\alpha,$$

$$O_3 = -\frac{A_\mu}{2\pi A^2(t)}\int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\varpi(\alpha)(\partial_\alpha^3 z(\alpha))^\perp\cdot\partial_\alpha z(\alpha)\int_{\mathbb{T}}\Big(\frac{4}{|\beta|^2}-\frac{1}{\sin^2(\beta/2)}\Big)d\beta d\alpha,$$

$$O_4 = \frac{A_\mu}{2\pi A^2(t)}\int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\varpi(\alpha)\int_{\mathbb{T}}(\partial_\alpha^3 z(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)\Big(\frac{4}{|\beta|^2}-\frac{1}{\sin^2(\beta/2)}\Big)d\beta d\alpha,$$

$$O_5 = -\frac{A_\mu}{2\pi A^2(t)}\int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\varpi(\alpha)\partial_\alpha z(\alpha)\cdot\Lambda((\partial_\alpha^3 z)^\perp)(\alpha)d\alpha,$$

The terms O_1, O_2, O_3 and O_4 can be estimated as before. Then we split $O_5 = R_1 + R_2$ where

$$R_1 = \frac{A_\mu}{2\pi A^2(t)}\int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)(\Lambda(\varpi\partial_\alpha z\cdot(\partial_\alpha^3 z)^\perp)(\alpha)-\varpi(\alpha)\partial_\alpha z(\alpha)\cdot\Lambda((\partial_\alpha^3 z)^\perp)(\alpha))d\alpha,$$

and

$$R_2 = -\frac{A_\mu}{2\pi A^2(t)}\int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\Lambda(\varpi\partial_\alpha z\cdot(\partial_\alpha^3 z)^\perp)(\alpha)d\alpha.$$

Using the commutator estimate, we obtain

$$R_1 \leq C\|\mathcal{F}(z)\|_{L^\infty}^2\|H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)\|_{L^2}\|\varpi\partial_\alpha z\|_{C^{1,\delta}}\|\partial_\alpha^3 z\|_{L^2} \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2).$$

The identity $\Lambda(H) = -\partial_\alpha$ gives

$$\begin{aligned} R_2 &= \frac{A_\mu}{2\pi A^2(t)}\int_{\mathbb{T}} \partial_\alpha(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\varpi(\alpha)\partial_\alpha z(\alpha)\cdot(\partial_\alpha^3 z(\alpha))^\perp d\alpha \\ &= -\frac{A_\mu}{2\pi A^2(t)}\int_{\mathbb{T}} \partial_\alpha(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\varpi(\alpha)\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^\perp z(\alpha)d\alpha \end{aligned}$$

and integrating by parts we get

$$R_2 \leq C\|\mathcal{F}(z)\|_{L^\infty}^2\|\partial_\alpha^3 z\cdot\partial_\alpha^\perp z\|_{L^2}^2\|\partial_\alpha\varpi\|_{L^\infty} \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2).$$

Regarding the term N_5 we have the expression

$$B(\alpha,\alpha-\beta) = \frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}V(\alpha,\alpha-\beta)$$

which shows that $B(\alpha,\alpha-\beta)$ has order -1 and, therefore, the term N_5 can be estimated as before.

Finally we have to find $\sigma(\alpha)$ in L_5 to finish the proof of the lemma. To do that let us split $L_5 = M_5 + M_6 + M_7 + M_8$ where

$$M_5 = \frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}} \Lambda(\partial_\alpha^3 z_1\partial_\alpha z_2)(\alpha)BR_1(z,\varpi)(\alpha)\partial_\alpha^3 z_1(\alpha)d\alpha,$$

$$M_6 = \frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}} \Lambda(\partial_\alpha^3 z_1\partial_\alpha z_2)(\alpha)BR_2(z,\varpi)(\alpha)\partial_\alpha^3 z_2(\alpha)d\alpha,$$

$$M_7 = -\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}} \Lambda(\partial_\alpha^3 z_2\partial_\alpha z_1)(\alpha)BR_1(z,\varpi)(\alpha)\partial_\alpha^3 z_1(\alpha)d\alpha,$$

$$M_8 = -\frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} \Lambda(\partial_\alpha^3 z_2 \partial_\alpha z_1)(\alpha) BR_2(z,\varpi)(\alpha) \partial_\alpha^3 z_2(\alpha) d\alpha.$$

and BR_j, $j = 1, 2$, is the jth-component of the Birkhoff-Rott integral.

Then

$$\begin{aligned} M_5 = & \frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} (\Lambda(\partial_\alpha z_2 \partial_\alpha^3 z_1)(\alpha) - \partial_\alpha z_2(\alpha)\Lambda(\partial_\alpha^3 z_1)(\alpha)) BR_1(z,\varpi)(\alpha)\partial_\alpha^3 z_1(\alpha) d\alpha \\ & + \frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} \partial_\alpha z_2(\alpha) BR_1(z,\varpi)(\alpha) \partial_\alpha^3 z_1(\alpha) \Lambda(\partial_\alpha^3 z_1)(\alpha) d\alpha, \end{aligned}$$

and the commutator estimates yields

$$M_5 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3}) + \frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} BR_1(z,\varpi)(\alpha) \partial_\alpha z_2(\alpha) \partial_\alpha^3 z_1(\alpha) \Lambda(\partial_\alpha^3 z_1)(\alpha) d\alpha. \quad (7.4)$$

In a similar way we have

$$M_6 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3}) + \frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} BR_2(z,\varpi)(\alpha) \partial_\alpha z_2(\alpha) \partial_\alpha^3 z_2(\alpha) \Lambda(\partial_\alpha^3 z_1)(\alpha) d\alpha.$$

Let us introduce the notation

$$N_4 = \frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} BR_2(z,\varpi)(\alpha) \partial_\alpha z_2(\alpha) \partial_\alpha^3 z_2(\alpha) \Lambda(\partial_\alpha^3 z_1)(\alpha) d\alpha,$$

the equality $\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_2(\alpha) = -\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha) - |\partial_\alpha^2 z(\alpha)|^2$ gives $N_4 = O_6 + O_7$ where

$$O_6 = -\frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} BR_2(z,\varpi)(\alpha) |\partial_\alpha^2 z(\alpha)|^2 \Lambda(\partial_\alpha^3 z_1)(\alpha) d\alpha,$$

$$O_7 = -\frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} BR_2(z,\varpi)(\alpha) \partial_\alpha z_1(\alpha) \partial_\alpha^3 z_1(\alpha) \Lambda(\partial_\alpha^3 z_1)(\alpha) d\alpha.$$

Integrating by parts in O_6 we get

$$\begin{aligned} O_6 \le & C\|\mathcal{F}(z)\|_{L^\infty} (\|\partial_\alpha BR(z,\varpi)\|_{L^\infty} \|\partial_\alpha^2 z\|^2_{L^\infty} + \|BR(z,\varpi)\|_{L^\infty} \|\partial_\alpha^2 z\|_{L^\infty} \|\partial_\alpha^3 z\|_{L^2}) \|H(\partial_\alpha^3 z_1)\|_{L^2} \\ & \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3}). \end{aligned}$$

Finally we get the estimate

$$M_6 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3}) - \frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} BR_2(z,\varpi)(\alpha) \partial_\alpha z_1(\alpha) \partial_\alpha^3 z_1(\alpha) \Lambda(\partial_\alpha^3 z_1)(\alpha) d\alpha.$$

Which together with (7.4) yields

$$M_5 + M_6 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3}) - \frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} BR(z,\varpi)(\alpha) \cdot \partial_\alpha^\perp z(\alpha)\, \partial_\alpha^3 z_1(\alpha) \Lambda(\partial_\alpha^3 z_1)(\alpha) d\alpha.$$

With M_7 and M_8 we use the equality $\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha) = -\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_2(\alpha) - |\partial_\alpha^2 z(\alpha)|^2$. Then operating similarly as we did with M_5 and M_6, we get

$$M_7 + M_8 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3}) - \frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} BR(z,\varpi)(\alpha) \cdot \partial_\alpha^\perp z(\alpha)\, \partial_\alpha^3 z_2(\alpha) \Lambda(\partial_\alpha^3 z_2)(\alpha) d\alpha.$$

The addition of both inequalities produces

$$L_5 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}) - \frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}} BR(z,\varpi)(\alpha)\cdot\partial^\perp_\alpha z(\alpha)\,\partial^3_\alpha z(\alpha)\cdot\Lambda(\partial^3_\alpha z)(\alpha)d\alpha.$$

and all the previous discussion shows that I_5 satisfies identical estimates than L_5.

7.4 Estimates on $\partial^3_\alpha(c(\alpha,t)\partial_\alpha z(\alpha,t))$.

In the evolution of the L^2 norm of $\partial^3_\alpha z(\alpha)$ it remains to control the term

$$I_2 = \int_{\mathbb{T}} \partial^3_\alpha z(\alpha)\cdot\partial^3_\alpha(c(\alpha)\partial_\alpha z(\alpha))d\alpha.$$

Let us recall the formula

$$\begin{aligned} c(\alpha,t) = \frac{\alpha+\pi}{2\pi A(t)}\int_{\mathbb{T}}\partial_\beta z(\beta,t)\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta \\ -\frac{1}{A(t)}\int_{-\pi}^{\alpha}\partial_\beta z(\beta,t)\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta. \end{aligned} \tag{7.5}$$

We take $I_2 = J_1+J_2+J_3+J_4$, where

$$J_1 = \int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial^4_\alpha z(\alpha)\,c(\alpha)d\alpha, \quad J_2 = 3\int_{\mathbb{T}}|\partial^3_\alpha z(\alpha)|^2\partial_\alpha c(\alpha)d\alpha,$$

$$J_3 = 3\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial^2_\alpha z(\alpha)\,\partial^2_\alpha c(\alpha)d\alpha, \quad J_4 = \int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial_\alpha z(\alpha)\,\partial^3_\alpha c(\alpha)d\alpha.$$

An integration by parts in J_1 yields

$$J_1 = -\frac{1}{2}\int_{\mathbb{T}}|\partial^3_\alpha z(\alpha)|^2\partial_\alpha c(\alpha)d\alpha \leq \|\partial_\alpha c\|_{L^\infty}\|\partial^3_\alpha z\|^2_{L^2} \leq 2\|\mathcal{F}(z)\|^{1/2}_{L^\infty}\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}\|\partial^3_\alpha z\|^2_{L^2},$$

and the estimate for $\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}$ obtained before gives us

$$J_1 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t)).$$

The term J_2 satisfies that $J_2 = -6J_1$, therefore

$$J_2 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t)).$$

Next we split $J_3 = K_1+K_2$, where

$$K_1 = -3\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial^2_\alpha z(\alpha)\frac{\partial^2_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2}\cdot\partial_\alpha BR(z,\varpi)d\alpha,$$

$$K_2 = -3\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial^2_\alpha z(\alpha)\frac{\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2}\cdot\partial^2_\alpha BR(z,\varpi)d\alpha.$$

We have

$$K_1 \leq 3\|\mathcal{F}(z)\|_{L^\infty}\|z\|^2_{C^2}\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}\|\partial^3_\alpha z\|_{L^2} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)),$$

$$K_2 \leq \|\mathcal{F}(z)\|^{1/2}_{L^\infty}\|z\|_{C^2}\|\partial^3_\alpha z\|_{L^2}\|\partial^2_\alpha BR(z,\varpi)\|_{L^2},$$

and therefore

$$I_3 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)).$$

The equality $\partial^3_\alpha z(\alpha)\cdot\partial_\alpha z(\alpha) = -|\partial^2_\alpha z(\alpha)|^2$ yields

$$I_4 = -\int_{\mathbb{T}} |\partial^2_\alpha z(\alpha)|^2 \partial^3_\alpha c(\alpha) d\alpha = 2\int_{\mathbb{T}} \partial^3_\alpha z(\alpha)\cdot\partial^2_\alpha z(\alpha)\partial^2_\alpha c(\alpha) d\alpha = \frac{2}{3} I_3,$$

and finally

$$I_4 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)).$$

8 The arc-chord condition

In this section we analyze the evolution of the quantity $\|\mathcal{F}(z)\|_{L^\infty}(t)$, which gives the local control of the arc-chord condition.

Lemma 8.1 *The following estimate holds*

$$\frac{d}{dt}\|\mathcal{F}(z)\|^2_{L^\infty}(t) \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)) \tag{8.1}$$

Proof: Let take $p > 1$. It follows that

$$\begin{aligned}\frac{d}{dt}\|\mathcal{F}(z)\|^p_{L^p}(t) &= \frac{d}{dt}\int_{\mathbb{T}}\int_{\mathbb{T}} \Big(\frac{|\beta|^2/4}{|V(\alpha,\alpha-\beta,t)|^2}\Big)^p d\beta d\alpha\\ &= I_1 + I_2 + I_3,\end{aligned}$$

where

$$I_1 = -p\int_{\mathbb{T}}\int_{\mathbb{T}} (|\beta|/2)^{2p}\frac{V(\alpha,\alpha-\beta,t)\cdot(z_t(\alpha,t) - z_t(\alpha-\beta,t))}{|V(\alpha,\alpha-\beta,t)|^{2p+2}} d\beta d\alpha,$$

$$I_2 = -p\int_{\mathbb{T}}\int_{\mathbb{T}} (|\beta|/2)^{2p}\frac{V^3_1(\alpha,\alpha-\beta,t)(z_{1\,t}(\alpha,t) - z_{1\,t}(\alpha-\beta,t))}{|V(\alpha,\alpha-\beta,t)|^{2p+2}} d\beta d\alpha,$$

and

$$I_3 = p\int_{\mathbb{T}}\int_{\mathbb{T}} (|\beta|/2)^{2p}\frac{V^3_2(\alpha,\alpha-\beta,t)(z_{2\,t}(\alpha,t) - z_{2\,t}(\alpha-\beta,t))}{|V(\alpha,\alpha-\beta,t)|^{2p+2}} d\beta d\alpha.$$

For I_1 we have

$$I_1 \leq p\int_{\mathbb{T}}\int_{\mathbb{T}} \Big(\frac{|\beta|/2}{|V(\alpha,\alpha-\beta,t)|}\Big)^{2p+1}\frac{|z_t(\alpha,t) - z_t(\alpha-\beta,t)|}{|\beta|} d\beta d\alpha.$$

Let us consider

$$\begin{aligned} z_t(\alpha)-z_t(\alpha-\beta) &= (BR(z,\varpi)(\alpha)-BR(z,\varpi)(\alpha-\beta))+(c(\alpha)-c(\alpha-\beta))\partial_\alpha z(\alpha)\\ &\quad + c(\alpha-\beta)(\partial_\alpha z(\alpha)-\partial_\alpha z(\alpha-\beta))\\ &= J_1+J_2+J_3. \end{aligned}$$

Then for J_1 we get $|J_1|\leq \|\partial_\alpha BR(z,\varpi)\|_{L^\infty}|\beta|$, and the estimate (6.2) gives

$$|J_1|\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t))|\beta|.$$

Using the definition for $c(\alpha)$ easily we obtain that

$$|c(\alpha)-c(\alpha-\beta)|\leq \frac{\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}}{A(t)^{1/2}}|\beta|,$$

and again using (6.2) we get

$$|J_2|\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t))|\beta|.$$

For J_3 we have $|J_3|\leq \|c\|_{L^\infty}\|z\|_{C^2}|\beta|$ that is

$$|J_3|\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t))|\beta|.$$

Those estimates obtained for J_i allow us to write

$$\begin{aligned} I_1 &\leq p\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t))\int_{\mathbb{T}}\int_{\mathbb{T}}\Big(\frac{|\beta|/2}{|V(\alpha,\alpha-\beta,t)|}\Big)^{2p+1}d\beta d\alpha\\ &\leq p\|\mathcal{F}(z)\|^{1/2}_{L^\infty}(t)\big(\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t))\big)\|\mathcal{F}(z)\|^p_{L^p}(t). \end{aligned}$$

Hölder inequality implies

$$\begin{aligned} |I_2| &\leq pC\|z_{1\,t}\|_{L^\infty}\int_{\mathbb{T}}\int_{\mathbb{T}}\Big(\frac{|\beta|/2}{|V(\alpha,\alpha-\beta,t)|}\Big)^{2p-1}d\beta d\alpha \leq pC\|z_{1\,t}\|_{L^\infty}(1+\|\mathcal{F}(z)\|^p_{L^p})\\ &\leq p\big(\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t))\big)\|\mathcal{F}(z)\|^p_{L^p}. \end{aligned}$$

Since $|V_2(\alpha,\alpha-\beta)|\leq 1$ we have

$$\begin{aligned} |I_3| &\leq 2p\|z_{2\,t}\|_{L^\infty}\int_{\mathbb{T}}\int_{\mathbb{T}}\Big(\frac{|\beta|/2}{|V(\alpha,\alpha-\beta,t)|}\Big)^{2p}d\beta d\alpha\\ &\leq 2p\big(\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t))\big)\|\mathcal{F}(z)\|^p_{L^p}. \end{aligned}$$

Those estimates for I_1,I_2 and I_3 gives us

$$\frac{d}{dt}\|\mathcal{F}(z)\|^p_{L^p}(t)\leq p\big(\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t))\big)\|\mathcal{F}(z)\|^p_{L^p}(t),$$

therefore

$$\frac{d}{dt}\|\mathcal{F}(z)\|_{L^p}(t)\leq \big(\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t))\big)\|\mathcal{F}(z)\|_{L^p}(t).$$

After an integration in the time variable t we get

$$\|\mathcal{F}(z)\|_{L^p}(t+h) \le \|\mathcal{F}(z)\|_{L^p}(t) \exp\Big(\int_t^{t+h} e^{C(\|\mathcal{F}(z)\|^2_{L^\infty}(s)+\|z\|^2_{H^3}(s))} ds\Big),$$

and letting $p \to \infty$ we obtain

$$\|\mathcal{F}(z)\|_{L^\infty}(t+h) \le \|\mathcal{F}(z)\|_{L^\infty}(t) \exp\Big(\int_t^{t+h} e^{C(\|\mathcal{F}(z)\|^2_{L^\infty}(s)+\|z\|^2_{H^3}(s))} ds\Big).$$

Therefore

$$\begin{aligned}\frac{d}{dt}\|\mathcal{F}(z)\|_{L^\infty}(t) &= \lim_{h\to 0}(\|\mathcal{F}(z)\|_{L^\infty}(t+h) - \|\mathcal{F}(z)\|_{L^\infty}(t))h^{-1}\\ &\le \|\mathcal{F}(z)\|_{L^\infty}(t) \lim_{h\to 0}(\exp\Big(\int_t^{t+h} e^{C(\|\mathcal{F}(z)\|^2_{L^\infty}(s)+\|z\|^2_{H^3}(s))} ds\Big) - 1)h^{-1}\\ &\le \|\mathcal{F}(z)\|_{L^\infty}(t) e^{C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t))}.\end{aligned}$$

With this we finish the proof of lemma 8.1. q.e.d.

9 The evolution of the minimum of $\sigma(\alpha, t)$

In this section we get an a priori estimate for the evolution of the minimum of the difference of the gradients of the pressure in the normal direction to the interface. This quantity is given by

$$\sigma(\alpha,t) = \frac{\mu^2-\mu^1}{\kappa} BR(z,\varpi)(\alpha,t)\cdot \partial_\alpha^\perp z(\alpha,t) + \mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t). \tag{9.1}$$

Lemma 9.1 *Let $z(\alpha,t)$ be a solution of the system with $z(\alpha,t) \in C^1([0,T];H^3)$, and*

$$m(t) = \min_{\alpha\in\mathbb{T}} \sigma(\alpha,t).$$

Then

$$m(t) \ge m(0) - \int_0^t \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(s) + \|z\|^2_{H^3}(s)) ds.$$

Proof: Suppose that $z(\alpha,t) \in C^1([0,T];H^3)$ is a solution of the system, then the result obtained in the preceding sections together with Sobolev inequalities show that $\sigma(\alpha,t) \in C^1([0,T]\times\mathbb{T})$. Therefore we may consider $\alpha_t \in \mathbb{T}$ such that

$$m(t) = \min_{\alpha\in\mathbb{T}} \sigma(\alpha,t) = \sigma(\alpha_t,t),$$

which is a Lipschitz function differentiable almost everywhere. With an analogous argument to the one used in [7] and [9], we may calculate the derivative of $m(t)$, to obtain

$$m'(\alpha_t,t) = \sigma_t(\alpha_t,t).$$

The identity (9.1) yields

$$
\begin{aligned}
\sigma_t(\alpha,t) &= \frac{\mu^2-\mu^1}{\kappa}\partial_t(BR(z,\varpi))(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t)\\
&\quad+(\frac{\mu^2-\mu^1}{k}BR(z,\varpi)(\alpha)\cdot\partial_\alpha^\perp z_t(\alpha,t)+\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_{1\,t}(\alpha,t))\\
&= I_1+I_2.
\end{aligned}
$$

And we have

$$|I_2|\leq C(\|BR(z,\varpi)\|_{L^\infty}+1)\|\partial_\alpha z_t\|_{L^\infty}.$$

We can easily estimate $\|BR(z,\varpi)\|_{L^\infty}$, obtaining

$$|I_2|\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3})\|\partial_\alpha z_t\|_{L^\infty}.$$

Next we use equation (1.4) to get

$$
\begin{aligned}
\|\partial_\alpha z_t\|_{L^\infty} &\leq (\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}+\|\partial_\alpha c\|_{L^\infty}\|\partial_\alpha z\|_{L^\infty}+\|c\|_{L^\infty}\|\partial^2_\alpha z\|_{L^\infty})\\
&\leq C\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}(1+\|\mathcal{F}(z)\|^{1/2}_{L^\infty}\|z\|_{C^2}),
\end{aligned}
$$

and with the bound obtained before for $\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}$ (6.2), we have

$$|I_2|\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}).$$

Let us write $BR(z,\varpi)(\alpha,t)=P_1(\alpha,t)+P_2(\alpha,t)+P_3(\alpha,t)$, where the P_j were defined in (6.3). We have

$$|\partial_t P_2(\alpha)|+|\partial_t P_3(\alpha)|\leq C(\|\varpi_t\|_{L^2}+\|\varpi\|_{L^2}\|z_t\|_{L^\infty}).$$

The norm $\|z_t\|_{L^\infty}$ is bounded by (1.4), and the adequate estimates for $\|\varpi_t\|_{L^2}$ which will be introduced later. In $\partial_t P_1$ there are terms of lower order which can be estimated as $\partial_t P_2$ and $\partial_t P_3$, but the most singular ones are given by

$$J_1=\frac{1}{4\pi}PV\int_{\mathbb{T}}\varpi_t(\alpha-\beta)\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

$$J_2=\frac{1}{8\pi}PV\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{z_t(\alpha)-z_t(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

$$J_3=-\frac{1}{4\pi}PV\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}\big(V(\alpha,\alpha-\beta)\cdot(z_t(\alpha)-z_t(\alpha-\beta))\big)d\beta,$$

Let us now split J_1 in a similar way as we did before, to obtain

$$J_1=\frac{1}{4\pi}PV\!\int_{\mathbb{T}}\varpi_t(\alpha-\beta)\big(\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}-\frac{\partial^\perp_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2\tan(\beta/2)}\big)d\beta+\frac{\partial_\alpha^\perp z(\alpha)}{2|\partial_\alpha z(\alpha)|^2}H(\varpi_t)(\alpha),$$

and

$$|J_1|\leq C\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^2}\|\varpi_t\|_{L^2}+\|\mathcal{F}(z)\|^{1/2}_{L^\infty}\|\varpi_t\|_{C^\delta}.$$

Next we divide $J_2 = K_1 + K_2 + K_3$ where

$$K_1 = \frac{1}{8\pi} \int_{\mathbb{T}} (\varpi(\alpha-\beta) - \varpi(\alpha)) \frac{z_t(\alpha) - z_t(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} d\beta,$$

$$K_2 = \frac{\varpi(\alpha)}{8\pi} \int_{\mathbb{T}} (z_t(\alpha) - z_t(\alpha-\beta)) \Big(\frac{1}{|V(\alpha,\alpha-\beta)|^2} - \frac{4}{|\partial_\alpha z(\alpha)|^2 |\beta|^2} \Big) d\beta,$$

$$K_3 = \frac{1}{8\pi} \frac{\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2} \int_{\mathbb{T}} (z_t(\alpha) - z_t(\alpha-\beta)) \Big(\frac{4}{|\beta|^2} - \frac{1}{\sin^2(\beta/2)} \Big) d\beta + \frac{1}{2} \frac{\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2} \Lambda(z_t)(\alpha).$$

The identity

$$z_t(\alpha) - z_t(\alpha-\beta) = \beta \int_0^1 \partial_\alpha z_t(\alpha + (s-1)\beta) ds,$$

gives us

$$|K_1| + |K_2| \le \|\varpi\|_{C^1} \|\mathcal{F}(z)\|^k \|z\|_{C^2}^k \|\partial_\alpha z_t\|_{L^\infty}.$$

And for K_3 we have

$$|K_3| \le C \|\varpi\|_{L^\infty} \|\mathcal{F}(z)\|_{L^\infty} \|z_t\|_{C^{1,\delta}}.$$

In order to control $\|\varpi_t\|_{C^\delta}$ we will use the inequality

$$\|f\|_{C^\delta} \le C(\|f\|_{L^2} + \|f\|_{\overline{C}^\delta}),$$

with

$$\|f\|_{\overline{C}^\delta} = \sup_{\alpha \neq \beta} \frac{|f(\alpha) - f(\beta)|}{|\alpha-\beta|^\delta}.$$

Let us now take the time derivative of the identity (5.1), we get

$$\varpi_t(\alpha) + A_\mu T(\varpi_t)(\alpha) = -\frac{A_\mu}{2\pi} R(\alpha) - 2\kappa g \frac{\rho^2 - \rho^1}{\mu^2 + \mu^1} \partial_\alpha z_{2\,t}(\alpha)$$

which yields

$$\|\varpi_t\|_{H^{\frac{1}{2}}} \le C \|(I + A_\mu T)^{-1}\|_{H^{\frac{1}{2}}} (\|R\|_{H^{\frac{1}{2}}} + \|\partial_\alpha z_t\|_{H^{\frac{1}{2}}})$$

and since we control $\|(I + A_\mu T)^{-1}\|_{H^{\frac{1}{2}}}$ it remains to estimate $\|R\|_{H^{\frac{1}{2}}}$.

Instead we will estimate $\|R\|_{H^1}$, and to do that we consider the splitting $R = S_1 + S_2 + S_3$ where

$$S_1(\alpha) = \int_{\mathbb{T}} \varpi(\alpha-\beta) \partial_t \Big(\frac{(V(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta/2)^\perp \cdot \partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^2} \Big) d\beta,$$

$$S_2(\alpha) = -\int_{\mathbb{T}} \varpi(\alpha-\beta) \partial_t \Big(V_2^2(\alpha,\alpha-\beta) \frac{V^\perp(\alpha,\alpha-\beta) \cdot \partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^2} \Big) d\beta,$$

$$S_3(\alpha) = -\int_{\mathbb{T}} \varpi(\alpha-\beta) \partial_t \Big(V_2(\alpha,\alpha-\beta) \partial_\alpha z_1(\alpha) \Big) d\beta.$$

The terms $S_2(\alpha)$ and $S_3(\alpha)$ are controlled as follows:

$$|S_2(\alpha)| + |S_3(\alpha)| \le C \|z_t\|_{C^1} \|\varpi\|_{L^2}.$$

For S_1 we split $S_1(\alpha) = Q_1(\alpha) + Q_2(\alpha) + Q_3(\alpha) + Q_4(\alpha) + Q_5(\alpha)$, where

$$Q_1(\alpha) = \int_{\mathbb{T}} \varpi(\alpha-\beta) \frac{(V(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta/2)^\perp \cdot \partial_\alpha z_t(\alpha)}{|V(\alpha,\alpha-\beta)|^2} d\beta,$$

$$Q_2(\alpha) = \frac{1}{2}\int_{\mathbb{T}} \varpi(\alpha-\beta) \frac{(z_t(\alpha) - z_t(\alpha-\beta) - \partial_\alpha z_t(\alpha)\beta)^\perp \cdot \partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^2} d\beta,$$

$$Q_3(\alpha) = \frac{1}{2}\int_{\mathbb{T}} \varpi(\alpha-\beta) \frac{(V(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta/2)^\perp \cdot \partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4} B(\alpha,\alpha-\beta) d\beta,$$

with

$$B(\alpha,\alpha-\beta) = V(\alpha,\alpha-\beta)\cdot(z_t(\alpha) - z_t(\alpha-\beta)), \tag{9.2}$$

$$Q_4(\alpha) = \frac{1}{2}\int_{\mathbb{T}} \varpi(\alpha-\beta)\frac{C(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} d\beta,$$

$$\begin{aligned} C(\alpha,\alpha-\beta) = {} & V_1^2(\alpha,\alpha-\beta)(z_{1\,t}(\alpha) - z_{1\,t}(\alpha-\beta))\partial_\alpha z_2(\alpha) \\ & + V_2^2(\alpha,\alpha-\beta)(z_{2\,t}(\alpha) - z_{2\,t}(\alpha-\beta))\partial_\alpha z_1(\alpha) \end{aligned}$$

$$Q_5(\alpha) = -\int_{\mathbb{T}} \varpi(\alpha-\beta) \frac{(V(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta/2)^\perp \cdot \partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4} D(\alpha,\alpha-\beta) d\beta,$$

$$D(\alpha,\alpha-\beta) = V_1^3(\alpha,\alpha-\beta)(z_{1\,t}(\alpha) - z_{1\,t}(\alpha-\beta)) - V_2^3(\alpha,\alpha-\beta)(z_{2\,t}(\alpha) - z_{2\,t}(\alpha-\beta)).$$

We have $|Q_4(\alpha)| \leq C\|z\|_{C^1}\|\varpi\|_{L^2}\|z_t\|_{L^\infty}$. In a similar way this estimates follows for Q_5:

$$|Q_5(\alpha)| \leq C(\|z\|_{C^1} + \|\mathcal{F}(z)\|_{L^\infty}^{1/2}\|z\|_{C^1}^2)\|\varpi\|_{L^2}\|z_t\|_{L^\infty}.$$

For Q_1 we proceed as before to obtain

$$|Q_1(\alpha)| \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|\varpi\|_{L^2}\|z_t\|_{C^1}.$$

The inequality

$$|z_t(\alpha) - z_t(\alpha-\beta) - \partial_\alpha z_t(\alpha)\beta| \leq \|z_t\|_{C^{1,\delta}}|\beta|^{1+\delta}$$

gives

$$|Q_2(\alpha)| \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^1}\|\varpi\|_{L^\infty}\|z_t\|_{C^{1,\delta}}.$$

And

$$|z_t(\alpha) - z_t(\alpha-\beta)| \leq \|z_t\|_{C^1}|\beta|,$$

yields

$$|Q_3(\alpha)| \leq \|\mathcal{F}(z)\|_{L^\infty}^{3/2}\|z\|_{C^2}^2\|\varpi\|_{L^2}\|z_t\|_{C^1}.$$

Finally we have

$$|R_1(\alpha)| \leq \|\mathcal{F}(z)\|_{L^\infty}^{3/2}\|z\|_{C^2}^2\|\varpi\|_{H^1}\|z_t\|_{C^{1,\delta}}.$$

Using (5.1) we obtain

$$\|\varpi_t\|_{\overline{C}^\delta} \leq C(\|T(\varpi_t)\|_{\overline{C}^\delta} + \|R\|_{\overline{C}^\delta} + \|\partial_\alpha z_t\|_{\overline{C}^\delta}).$$

For $\delta \leq 1/2$ we have

$$\|T(\varpi_t)\|_{\overline{C}^{\delta}} \leq \|T(\varpi_t)\|_{H^1} \leq 2\|\partial_\alpha T(\varpi_t)\|_{L^2} \leq \|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^4_{C^{2,\delta}}\|w_t\|_{L^2}.$$

Now to estimate $\|R\|_{\overline{C}^{\delta}} \leq \|R\|_{H^1}$ we consider $\|\partial_\alpha R\|_{L^2}$. The most singular terms for this quantity are those with two derivatives in α and one in time, or with one derivative in α, one in time and a principal value. Let us write:

$$Q_6(\alpha) = \int_{\mathbb{T}} \varpi(\alpha-\beta)\frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^{\perp}\cdot\partial^2_\alpha z_t(\alpha)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

$$Q_7(\alpha) = \frac{1}{2}\int_{\mathbb{T}} \varpi(\alpha-\beta)\frac{(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta)-\partial^2_\alpha z_t(\alpha)\beta)^{\perp}\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

$$Q_8(\alpha) = \frac{1}{2}\int_{\mathbb{T}} \varpi(\alpha-\beta)\frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^{\perp}\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}D(\alpha,\alpha-\beta)d\beta,$$

with

$$D(\alpha,\alpha-\beta) = V(\alpha,\alpha-\beta)\cdot(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta)). \tag{9.3}$$

We have

$$|Q_6(\alpha)| \leq \|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^2}\|w\|_{L^2}|\partial^2_\alpha z_t(\alpha)|,$$

and

$$|Q_8(\alpha)| \leq \|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^2}\|w\|_{L^\infty}\|z_t(\alpha)\|_{C^{1,\delta}}.$$

Let us split $Q_7(\alpha) = J_4 + J_5$ where

$$J_4 = \frac{1}{2}PV\int_{\mathbb{T}} \varpi(\alpha-\beta)\frac{(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta))^{\perp}\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

and

$$J_5 = -\frac{1}{2}(\partial^2_\alpha z_t(\alpha))^{\perp}\cdot\partial_\alpha z(\alpha)PV\int_{\mathbb{T}} \varpi(\alpha-\beta)\frac{\beta}{|V(\alpha,\alpha-\beta)|^2}d\beta.$$

For J_5 we have $|J_5| \leq \|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^2}\|w\|_{H^1}|\partial^2_\alpha z_t(\alpha)|$. Next we divide $J_4 = K_4+K_5+K_6+K_7$ where

$$K_4 = \frac{1}{2}\int_{\mathbb{T}}(\varpi(\alpha-\beta)-\varpi(\alpha))\frac{(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta))^{\perp}\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

$$K_5 = \frac{\varpi(\alpha)}{2}\int_{\mathbb{T}}(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta))^{\perp}\cdot\partial_\alpha z(\alpha)\Big(\frac{1}{|V(\alpha,\alpha-\beta)|^2}-\frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}\Big)d\beta,$$

$$K_6 = \frac{2\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2}\int_{\mathbb{T}}(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta))^{\perp}\cdot\partial_\alpha z(\alpha)\Big(\frac{1}{|\beta|^2}-\frac{1}{4\sin^2(\beta/2)}\Big)d\beta,$$

$$K_7 = 2\pi\frac{\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2}(\Lambda(\partial_\alpha z_t)(\alpha))^{\perp}\cdot\partial_\alpha z(\alpha).$$

We have $|K_4|+|K_5| \leq C\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^2}\|w\|_{H^1}\|z_t\|_{C^{1,\delta}}$, $|K_6| \leq \|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|w\|_{H^1}\|z_t\|_{C^1}$ and $|K_7| \leq \|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|w\|_{H^1}|\Lambda(\partial_\alpha z_t)(\alpha)|$.

Finally let us observe $\|z_t\|_{C^{1,\delta}} \leq \|z_t\|_{H^2}$, which provide us the control of $\|\partial_\alpha^2 z_t\|_{L^2}$. We consider now the terms of $\partial_\alpha^2 z_t(\alpha)$ given by

$$I_3 = \partial_\alpha^2 BR(z,\varpi)(\alpha), \quad I_4 = \partial_\alpha^2 (c(\alpha)\partial_\alpha z(\alpha)).$$

Easily we get

$$|I_4| \leq \|\mathcal{F}(z)\|_{L^\infty}^k \|z\|_{C^2}^k (1 + |\partial_\alpha^3 z(\alpha)| + |\partial_\alpha^2 BR(z,\varpi)(\alpha)|),$$

which yields

$$\|I_4\|_{L^2} \leq (\|\mathcal{F}(z)\|_{L^\infty}^k \|z\|_{C^2}^k (1 + \|z\|_{H^3} + \|\partial_\alpha^2 BR(z,\varpi)\|_{L^2}),$$

so that we can control $\|\partial_\alpha^2 BR(z,\varpi)\|_{L^2}$ as in (6.6), and finish the estimate of I_3.

The upper bound

$$|\sigma_t(\alpha,t)| \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2(t) + \|z\|_{H^3}^2(t)),$$

gives us

$$m'(t) \geq -\exp C(\|\mathcal{F}(z)\|_{L^\infty}^2(t) + \|z\|_{H^3}^2(t)),$$

for almost every t. And a further integration yields lemma 8.1

$$m(t) \geq m(0) - \int_0^t \exp(C|||z|||^2) ds.$$

10 Regularization and approximation

Our next step is to use the a priori estimates to get local-existence. For that purpose we introduce a regularized evolution equation having local-existence independently of the sign condition on $\sigma(\alpha,t)$ at $t=0$. But for $\sigma(\alpha,0) > 0$, we find a time of existence for the Muskat problem uniformly in the regularization, allowing us to take the limit.

Let $z^\varepsilon(\alpha,t)$ be a solution of the following system:

$$\begin{aligned} &z_t^{\varepsilon,\delta}(\alpha,t) = BR^\delta(z^{\varepsilon,\delta},\varpi^{\varepsilon,\delta})(\alpha,t) + c^{\varepsilon,\delta}(\alpha,t)\partial_\alpha z^{\varepsilon,\delta}(\alpha,t),\\ &z^{\varepsilon,\delta}(\alpha,0) = z_0(\alpha), \end{aligned}$$

where

$$\begin{aligned} BR^\delta(z,\varpi)(\alpha,t) = \Big(&-\frac{1}{4\pi}\int_{\mathbb{T}} \varpi(\beta,t) \frac{\tanh(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2})(1+\tan^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2}))}{\tan^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2}) + \tanh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2}) + \delta} d\beta,\\ &\frac{1}{4\pi}\int_{\mathbb{T}} \varpi(\beta,t) \frac{\tan(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2})(1-\tanh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2}))}{\tan^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2}) + \tanh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2}) + \delta} d\beta\Big), \end{aligned}$$

$$\varpi^{\varepsilon,\delta}(\alpha,t) = -A_\mu \phi_\varepsilon * \phi_\varepsilon * (2BR(z^{\varepsilon,\delta},\varpi^{\varepsilon,\delta})\cdot\partial_\alpha z^{\varepsilon,\delta})(\alpha) - 2\kappa \mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\phi_\varepsilon * \phi_\varepsilon * (\partial_\alpha z_2^{\varepsilon,\delta})(\alpha),$$

$$c^{\varepsilon,\delta}(\alpha,t) = \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\alpha z^{\varepsilon,\delta}(\alpha,t)}{|\partial_\alpha z^{\varepsilon,\delta}(\alpha,t)|^2}\cdot\partial_\alpha BR^\delta(z^{\varepsilon,\delta},\varpi^{\varepsilon,\delta})(\alpha,t)d\alpha$$
$$-\int_{-\pi}^{\alpha}\frac{\partial_\alpha z^{\varepsilon,\delta}(\beta,t)}{|\partial_\alpha z^{\varepsilon,\delta}(\alpha,t)|^2}\cdot\partial_\beta BR^\delta(z^{\varepsilon,\delta},\varpi^{\varepsilon,\delta})(\beta,t)d\beta,$$

$$\phi\in C_c^\infty(\mathbb{R}),\quad \phi(\alpha)\geq 0,\quad \phi(-\alpha)=\phi(\alpha),\quad \int_{\mathbb{R}}\phi(\alpha)d\alpha=1,\quad \phi_\varepsilon(\alpha)=\phi(\alpha/\varepsilon)/\varepsilon,$$

for $\varepsilon>0$ and $\delta>0$.

Then the operator $I+A_\mu\phi_\varepsilon*\phi_\varepsilon*T$ has a bounded inverse in $H^{\frac{1}{2}}$, for ε small enough, with a norm bounded independently of $\varepsilon>0$. For this system there is local-existence for initial data with $\mathcal{F}(z_0)(\alpha,\beta)<\infty$ even if $\sigma(\alpha,0)$ does not have the proper sign (see [12]). So that there exists a time $T^{\varepsilon,\delta}$ and a solution of the system $z^{\varepsilon,\delta}\in C^1([0,T^{\varepsilon,\delta}],H^k)$ for $k\leq 3$, and as long as the solution exists, we have $|\partial_\alpha z^{\varepsilon,\delta}(\alpha,t)|^2=A^{\varepsilon,\delta}(t)$. Taking advantage of this property, and using that $\varpi^{\varepsilon,\delta}$ is regular, we obtain estimates which are independent of δ. Letting now $\delta\to 0$ we get local-existence for the following system:

$$z_t^\varepsilon(\alpha,t)=BR(z^\varepsilon,\varpi^\varepsilon)(\alpha,t)+c^\varepsilon(\alpha,t)\partial_\alpha z^\varepsilon(\alpha,t),$$
$$z^\varepsilon(\alpha,0)=z_0(\alpha),$$

where

$$c^{\varepsilon}(\alpha,t) = \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\alpha z^{\varepsilon}(\alpha,t)}{|\partial_\alpha z^{\varepsilon}(\alpha,t)|^2}\cdot\partial_\alpha BR(z^{\varepsilon},\varpi^{\varepsilon})(\alpha,t)d\alpha$$
$$-\int_{-\pi}^{\alpha}\frac{\partial_\alpha z^{\varepsilon}(\beta,t)}{|\partial_\alpha z^{\varepsilon}(\alpha,t)|^2}\cdot\partial_\beta BR(z^{\varepsilon},\varpi^{\varepsilon})(\beta,t)d\beta,$$

$$\varpi^\varepsilon(\alpha,t)=-A_\mu\phi_\varepsilon*\phi_\varepsilon*(2BR(z^\varepsilon,\varpi^\varepsilon)\cdot\partial_\alpha z^\varepsilon)(\alpha)-2\kappa\mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\phi_\varepsilon*\phi_\varepsilon*(\partial_\alpha z_2^\varepsilon)(\alpha).$$

Next we will show that for this system we have

$$\begin{aligned}\frac{d}{dt}\|z^\varepsilon\|^2_{H^k}(t)\leq &-\frac{\kappa}{2\pi(\mu_1+\mu_2)}\int_{\mathbb{T}}\frac{\sigma^\varepsilon(\alpha,t)}{|\partial_\alpha z^\varepsilon(\alpha,t)|^2}\phi_\varepsilon*(\partial_\alpha^k z^\varepsilon)(\alpha,t)\cdot\Lambda(\phi_\varepsilon*(\partial_\alpha^k z^\varepsilon))(\alpha,t)d\alpha\\&+\exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)+\|z^\varepsilon\|^2_{H^k}(t)).\end{aligned}\tag{10.1}$$

where

$$\sigma^\varepsilon(\alpha,t)=\frac{\mu^2-\mu^1}{\kappa}BR(z^\varepsilon,\varpi^\varepsilon)(\alpha,t)\cdot\partial_\alpha^\perp z^\varepsilon(\alpha,t)+\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_1^\varepsilon(\alpha,t).$$

To do the task we have to repeat the arguments in our previous sections, with the exception of 7.3. (looking for $\sigma^\varepsilon(\alpha)$) where we proceed differently using the following well-known estimate for the commutator of the convolution:

$$\|\phi_\varepsilon*(gf)-g\phi_\varepsilon*(f)\|_{H^1}\leq C\|g\|_{C^1}\|f\|_{L^2}\tag{10.2}$$

regarding, where the constant C is independent of ε.

In the following we will present the details of the evolution of the L^2 norm of the third derivatives, being the case of the kth-derivative ($k>3$) completely analogous. Furthermore, with regards of the different decompositions introduced in the previous sections, in the following we shall select only the more singular terms, showing for them the corresponding uniform estimates and leaving to the reader the remainder easy cases.

If we consider the term corresponding to K_2 in section 7.3 we have

$$K_2^\varepsilon = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda\big(\partial_\alpha^3 z^\varepsilon\cdot\partial_\alpha^\perp z^\varepsilon\big)(\alpha)\phi_\varepsilon*\phi_\varepsilon*(\partial_\alpha^3 z_2^\varepsilon)(\alpha)d\alpha.$$

which, we write in the following manner

$$K_2^\varepsilon = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda\big(\phi_\varepsilon*(\partial_\alpha^3 z^\varepsilon\cdot\partial_\alpha^\perp z^\varepsilon)\big)(\alpha)\phi_\varepsilon*(\partial_\alpha^3 z_2^\varepsilon)(\alpha)d\alpha.$$

Then we have $K_2^\varepsilon = L_1^\varepsilon + L_2^\varepsilon$, where

$$L_1^\varepsilon = \frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda\big(\phi_\varepsilon*(\partial_\alpha^3 z_1^\varepsilon\partial_\alpha z_2^\varepsilon)\big)(\alpha)\phi_\varepsilon*(\partial_\alpha^3 z_2^\varepsilon)(\alpha)d\alpha,$$

$$L_2^\varepsilon = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda\big(\phi_\varepsilon*(\partial_\alpha^3 z_2^\varepsilon\partial_\alpha z_1^\varepsilon)\big)(\alpha)\phi_\varepsilon*(\partial_\alpha^3 z_2^\varepsilon)(\alpha)d\alpha.$$

Next we write $L_1^\varepsilon = M_1^\varepsilon + M_2^\varepsilon + M_3^\varepsilon + M_4^\varepsilon$ where

$$M_1^\varepsilon = \frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda\big(\phi_\varepsilon*(\partial_\alpha^3 z_1^\varepsilon\partial_\alpha z_2^\varepsilon)-\phi_\varepsilon*(\partial_\alpha^3 z_1^\varepsilon)\partial_\alpha z_2^\varepsilon\big)(\alpha)\phi_\varepsilon*(\partial_\alpha^3 z_2^\varepsilon)(\alpha)d\alpha,$$

$$M_2^\varepsilon = \frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}} [\Lambda\big(\phi_\varepsilon*(\partial_\alpha^3 z_1^\varepsilon)\partial_\alpha z_2^\varepsilon\big)(\alpha)-\Lambda\big(\phi_\varepsilon*(\partial_\alpha^3 z_1^\varepsilon)\big)(\alpha)\partial_\alpha z_2^\varepsilon(\alpha)]\phi_\varepsilon*(\partial_\alpha^3 z_2^\varepsilon)(\alpha)d\alpha,$$

$$M_3^\varepsilon = \frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda\big(\phi_\varepsilon*(\partial_\alpha^3 z_1^\varepsilon)\big)(\alpha)[\partial_\alpha z_2^\varepsilon(\alpha)\phi_\varepsilon*(\partial_\alpha^3 z_2^\varepsilon)(\alpha)-\phi_\varepsilon*(\partial_\alpha z_2^\varepsilon\partial_\alpha^3 z_2^\varepsilon)(\alpha)]d\alpha,$$

$$M_4^\varepsilon = \frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda\big(\phi_\varepsilon*(\partial_\alpha^3 z_1^\varepsilon)\big)(\alpha)\phi_\varepsilon*(\partial_\alpha z_2^\varepsilon\partial_\alpha^3 z_2^\varepsilon)(\alpha)d\alpha.$$

Using (10.2), we get

$$\begin{aligned}
M_1^\varepsilon &\leq C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|\Lambda\big(\phi_\varepsilon*(\partial_\alpha^3 z_1^\varepsilon\partial_\alpha z_2^\varepsilon)-\phi_\varepsilon*(\partial_\alpha^3 z_1^\varepsilon)\partial_\alpha z_2^\varepsilon\big)\|_{L^2}\|\phi_\varepsilon*(\partial_\alpha^3 z_2^\varepsilon)\|_{L^2}^2\\
&\leq C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|\phi_\varepsilon*(\partial_\alpha^3 z_1^\varepsilon\partial_\alpha z_2^\varepsilon)-\phi_\varepsilon*(\partial_\alpha^3 z_1^\varepsilon)\partial_\alpha z_2^\varepsilon\|_{H^1}\|\partial_\alpha^3 z_2^\varepsilon\|_{L^2}^2\\
&\leq C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|\partial_\alpha^3 z_1^\varepsilon\|_{L^2}\|\partial_\alpha z_2^\varepsilon\|_{C^1}\|\partial_\alpha^3 z_2^\varepsilon\|_{L^2}^2,
\end{aligned}$$

and therefore

$$M_1^\varepsilon \leq \exp C(\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}^2+\|z^\varepsilon\|_{H^3}^2).$$

For M_2^ε we use the commutator estimate for the operator Λ to obtain

$$M_2^\varepsilon \le C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)\|_{L^2}\|\partial_\alpha z_2^\varepsilon\|_{C^{1,\delta}}\|\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)\|_{L^2} \le \exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty} + \|z^\varepsilon\|^2_{H^3}).$$

Regarding M_3^ε we have

$$M_3^\varepsilon = \frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}}\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)(\alpha)\Lambda\big(\partial_\alpha z_2^\varepsilon \phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon) - \phi_\varepsilon * (\partial_\alpha z_2^\varepsilon \partial_\alpha^3 z_2^\varepsilon)\big)(\alpha)d\alpha,$$

showing that it can be estimated as M_1^ε.

The identity

$$\partial_\alpha z_2^\varepsilon(\alpha)\partial_\alpha^3 z_2^\varepsilon(\alpha) = -\partial_\alpha z_1^\varepsilon(\alpha)\partial_\alpha^3 z_1^\varepsilon(\alpha) - |\partial_\alpha^2 z^\varepsilon(\alpha)|^2,$$

allow us to write M_4^ε as the sum of N_1^ε and N_2^ε where

$$N_1^\varepsilon = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}}\Lambda\big(\phi_\varepsilon * \partial_\alpha^3 z_1^\varepsilon\big)(\alpha)\phi_\varepsilon * (|\partial_\alpha^2 z^\varepsilon|^2)(\alpha)d\alpha,$$

and

$$N_2^\varepsilon = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}}\phi_\varepsilon * (\partial_\alpha z_1^\varepsilon \partial_\alpha^3 z_1^\varepsilon)(\alpha)\Lambda\big(\phi_\varepsilon * \partial_\alpha^3 z_1^\varepsilon\big)(\alpha)d\alpha.$$

Then an integration by parts shows that

$$N_1^\varepsilon \le C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|z^\varepsilon\|_{C^2}\|\partial_\alpha^3 z^\varepsilon\|^2_{L^2} \le \exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty} + \|z^\varepsilon\|^2_{H^3}).$$

Using again the identity (10.2) in N_2^ε, we obtain finally

$$\begin{aligned} L_1^\varepsilon \le &-\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}}\partial_\alpha z_1^\varepsilon(\alpha)\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)(\alpha)\Lambda\big(\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)\big)(\alpha)d\alpha \\ &+ \exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty} + \|z^\varepsilon\|^2_{H^3}). \end{aligned}$$

In a similar way we get for L_2^ε

$$\begin{aligned} L_2^\varepsilon \le &-\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}}\partial_\alpha z_1^\varepsilon(\alpha)\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)(\alpha)\Lambda\big(\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)\big)(\alpha)d\alpha \\ &+ \exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty} + \|z^\varepsilon\|^2_{H^3}), \end{aligned}$$

giving us

$$\begin{aligned} K_2^\varepsilon \le &-\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}\partial_\alpha z_1^\varepsilon(\alpha)\phi_\varepsilon * (\partial_\alpha^3 z)(\alpha)\cdot\Lambda\big(\phi_\varepsilon * (\partial_\alpha^3 z)\big)(\alpha)d\alpha \\ &+ \exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty} + \|z^\varepsilon\|^2_{H^3}). \end{aligned}$$

The formula for $\sigma^\varepsilon(\alpha,t)$ begins to appear in the non-integrable terms. Using a similar method for the rest of the non-integrable terms we obtain the inequality (10.1) for $k=3$.

The next step is to integrate the system during a time T independent of ε. First let us observe that if $z_0(\alpha)\in H^k$, then we have the solution $z^\varepsilon \in C^1([0,T^\varepsilon];H^k)$. And if initially

$\sigma(\alpha,0)>0$, there is a time depending on ε, denoted by T^ε again, in which $\sigma^\varepsilon(\alpha,t)>0$. Now, for $t\leq T^\varepsilon$ we have (10.1), and then we use the following pointwise inequality (see [6]):

$$f(\alpha)\Lambda f(\alpha)-\frac{1}{2}\Lambda(f^2)(\alpha)\geq 0,$$

to obtain

$$\frac{d}{dt}\|z^\varepsilon\|^2_{H^k}(t)\leq I+\exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)+\|z^\varepsilon\|^2_{H^k}(t)),$$

where

$$I=-\frac{\kappa}{2\pi(\mu_1+\mu_2)A^\varepsilon(t)}\int_{\mathbb{T}}\sigma^\varepsilon(\alpha,t)\frac{1}{2}\Lambda(|\phi_\varepsilon*(\partial^k_\alpha z^\varepsilon)|^2)(\alpha,t)d\alpha.$$

We have

$$\|\Lambda(\sigma^\varepsilon)\|_{L^\infty}(t)\leq C\|\sigma^\varepsilon\|_{H^2}(t)\leq C(\|BR(z^\varepsilon,\varpi^\varepsilon)\|_{L^2}(t)+\|\partial^2_\alpha BR(z^\varepsilon,\varpi^\varepsilon)\|_{L^2}(t)+1)\|z\|_{H^3}(t),$$

and writing

$$I=-\frac{\kappa}{2\pi(\mu_1+\mu_2)A^\varepsilon(t)}\int_{\mathbb{T}}\Lambda(\sigma^\varepsilon)(\alpha,t)\frac{1}{2}|\phi_\varepsilon*(\partial^k_\alpha z^\varepsilon)|^2(\alpha,t)d\alpha,$$

we obtain

$$I\leq C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|\Lambda(\sigma^\varepsilon)\|_{L^\infty}\|\partial^k_\alpha z^\varepsilon\|^2_{L^2}\leq\exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}+\|z^\varepsilon\|^2_{H^k}).$$

Finally, for $t\leq T^\varepsilon$ we have

$$\frac{d}{dt}\|z^\varepsilon\|^2_{H^k}(t)\leq C\exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)+\|z^\varepsilon\|^2_{H^k}(t)). \tag{10.3}$$

We have also (see section 8):

$$\frac{d}{dt}\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)\leq C\exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)+\|z^\varepsilon\|^2_{H^3}(t)),$$

and from (10.3) it follows that

$$\frac{d}{dt}(\|z^\varepsilon\|^2_{H^k}(t)+\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t))\leq C\exp C(\|z^\varepsilon\|^2_{H^k}(t)+\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t))$$

for $t\leq T^\varepsilon$. Integrating

$$\|z^\varepsilon\|^2_{H^k}(t)+\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)\leq -\frac{1}{C}\ln\big(-t+\exp(-C(\|z_0\|^2_{H^k}+\|\mathcal{F}(z_0)\|^2_{L^\infty}))\big), \tag{10.4}$$

$t\leq T^\varepsilon$. Let us mention that at this point of the proof we can not assume local-existence, because we have the above estimate for $t\leq T^\varepsilon$, and if we let $\varepsilon\to 0$, it could be possible that $T^\varepsilon\to 0$ i.e. we cannot assume that if the initial data satisfy $\sigma(\alpha,0)>0$, there must be a time T, independent of ε, in which (10.4) is satisfied. In other words, at this stage of the proof we do not have local-existence when $\varepsilon\to 0$. But since in the evolution equation everything depends upon the sign of $\sigma^\varepsilon(\alpha,t)$, the following argument will allow us to continue the proof. First let us observe that as in section 9 we have

$$m^\varepsilon(t) \geq m(0) - \int_0^t \exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(s) + \|z^\varepsilon\|^2_{H^3}(s))ds, \tag{10.5}$$

where

$$m^\varepsilon(t) = \min_{\alpha\in\mathbb{T}} \sigma^\varepsilon(\alpha,t),$$

and $t \leq T^\varepsilon$. Using (10.4) in (10.5) we get

$$m^\varepsilon(t) \geq m(0) + C(\|z_0\|^2_{H^k} + \|\mathcal{F}(z_0)\|^2_{L^\infty}) + \ln\big(-t + \exp(-C(\|z_0\|^2_{H^k} + \|\mathcal{F}(z_0)\|^2_{L^\infty}))\big), \tag{10.6}$$

for $t \leq T^\varepsilon$. Using (10.6) and (10.4), now we find that if $\varepsilon \to 0$, then $T^\varepsilon \nrightarrow 0$, because if we take $T = \min(T_1, T_2)$ where T_1 satisfies

$$m(0) + C(\|z_0\|^2_{H^k} + \|\mathcal{F}(z_0)\|^2_{L^\infty}) + \ln\big(-T_1 + \exp(-C(\|z_0\|^2_{H^k} + \|\mathcal{F}(z_0)\|^2_{L^\infty}))\big) > 0,$$

and T_2

$$-\frac{1}{C}\ln\big(-T_2 + \exp(-C(\|z_0\|^2_{H^k} + \|\mathcal{F}(z_0)\|^2_{L^\infty}))\big) < \infty.$$

For $t \leq T$ we have $m^\varepsilon(t) > 0$ and

$$\|z^\varepsilon\|^2_{H^k}(t) + \|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t) \leq -\frac{1}{C}\ln\big(-T + \exp(-C(\|z_0\|^2_{H^k} + \|\mathcal{F}(z_0)\|^2_{L^\infty}))\big) < \infty,$$

and T only depends on the initial data z_0. Now we let ε tends to 0, to conclude the existence result.

11 Appendix

Let us denote

$$V(\alpha,\beta) = (V_1(\alpha,\beta), V_2(\alpha,\beta)) = (\tan(\frac{z_1(\alpha)-z_1(\beta)}{2}), \tanh(\frac{z_2(\alpha)-z_2(\beta)}{2})),$$

and

$$W(\alpha,\beta) = (W_1(\alpha,\beta), W_2(\alpha,\beta)) = ((\frac{z_1(\alpha)-z_1(\beta)}{2})_p, (\frac{z_2(\alpha)-z_2(\beta)}{2})_p),$$

where $(\alpha)_p$ is the periodic extension of the function α in $\mathbb{T}$. We give the following equalities for the hyperbolic tangent function:

$$(\tanh(\alpha) - (\alpha)_p)/\tanh^2(\alpha) = (\alpha)_p f(\alpha) \quad \text{with} \quad f \in L^\infty(\mathbb{R}), \tag{11.1}$$

$$(\tanh(\alpha) - (\alpha)_p)/\tanh^3(\alpha) = g(\alpha) \quad \text{with} \quad g \in L^\infty(\mathbb{R}). \tag{11.2}$$

For the tangent function it holds

$$(\tan(\alpha/2) - (\alpha/2)_p)/\tan(\alpha/2) = (\alpha/2)_p h(\alpha) \quad \text{with} \quad h \in L^\infty(\mathbb{R}), \tag{11.3}$$

$$(\tan(\alpha/2) - (\alpha/2)_p)/\tan^2(\alpha/2) = (\alpha/2)_p j(\alpha) \quad \text{with} \quad j \in L^\infty(\mathbb{R}), \tag{11.4}$$

$$(\tan(\alpha/2) - (\alpha/2)_p)/|\tan^3(\alpha/2)| = k(\alpha) \quad \text{with} \quad k \in L^\infty(\mathbb{R}). \tag{11.5}$$

Also we shall use that the below functions are bounded on $[-\pi, \pi]$

$$2/\alpha - 1/\tan(\alpha/2),\ 4/\alpha^2 - 1/\sin^2(\alpha/2) \in L^\infty(\mathbb{T}), \tag{11.6}$$

and the following estimates:

$$|W(\alpha, \alpha-\beta) - \partial_\alpha z(\alpha)\beta/2| \le \frac{1}{2}\|z\|_{C^2}|\beta|^2, \tag{11.7}$$

$$|W(\alpha, \alpha-\beta) - \partial_\alpha z(\alpha)\beta/2 - \partial_\alpha^2 z(\alpha)\beta^2/4| \le \frac{1}{2}\|z\|_{C^{2,\delta}}|\beta|^{2+\delta}. \tag{11.8}$$

Lemma 11.1 *Given*

$$A_1(\alpha, \alpha-\beta) = \frac{V_2(\alpha, \alpha-\beta)}{|V(\alpha, \alpha-\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_2(\alpha)}{\tan(\frac{\beta}{2})}$$

$$A_2(\alpha, \alpha-\beta) = \frac{V_1(\alpha, \alpha-\beta)}{|V(\alpha, \alpha-\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_1(\alpha)}{\tan(\frac{\beta}{2})},$$

we have

$$\|A_1(\alpha, \alpha-\beta)\|_{L^\infty} \le \|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2},$$

and

$$\|A_2(\alpha, \alpha-\beta)\|_{L^\infty} \le \|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}^2,$$

Proof: We introduce the splitting $A_1(\alpha, \alpha-\beta) = I_1 + I_2 + I_3 + I_4$ where

$$I_1 = \frac{\tanh(\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2}) - (\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2})_p}{V(\alpha, \alpha-\beta)},$$

$$I_2 = \mathcal{F}(z)(\alpha, \beta)\frac{((z_2(\alpha) - z_2(\alpha-\beta))/2)_p - \partial_\alpha z_2(\alpha)\beta/2}{\beta^2/4},$$

$$I_3 = \frac{\partial_\alpha z_2(\alpha)}{\beta/2}(\mathcal{F}(z)(\alpha, \beta) - \frac{1}{|\partial_\alpha z(\alpha)|^2}),$$

$$I_4 = \frac{\partial_\alpha z_2(\alpha)}{|\partial_\alpha z(\alpha)|^2}(\frac{2}{\beta} - \frac{1}{\tan(\frac{\beta}{2})}),$$

and $\mathcal{F}(z)(\alpha, \beta)$ was defined in (1.7).

Since

$$I_1 = \frac{1}{1+\frac{V_1^2(\alpha,\alpha-\beta)}{V_2^2(\alpha,\alpha-\beta)}} f(\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2})(\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2})_p$$

by (11.1), we get $I_1 \leq C$. Also $I_2 \leq \|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$ using (11.7), and $I_4 \leq C\|\mathcal{F}(z)\|^{1/2}_{L^\infty}$. We rewrite

$$I_3 = \frac{\partial_\alpha z_2(\alpha)}{\beta/2} \frac{(\partial_\alpha z(\alpha)\beta/2 + V(\alpha,\alpha-\beta)) \cdot (\partial_\alpha z(\alpha)\beta/2 - V(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2 |V(\alpha,\alpha-\beta)|^2},$$

and split further

$$I_3 = J_1 + J_2,$$

where

$$J_1 = \frac{\partial_\alpha z_2(\alpha)}{\beta/2} \frac{(\partial_\alpha z_1(\alpha)\beta/2 + V_1(\alpha,\alpha-\beta))(\partial_\alpha z_1(\alpha)\beta/2 - V_1(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2 |V(\alpha,\alpha-\beta)|^2},$$

$$J_2 = \frac{\partial_\alpha z_2(\alpha)}{\beta/2} \frac{(\partial_\alpha z_2(\alpha)\beta/2 + V_2(\alpha,\alpha-\beta))(\partial_\alpha z_2(\alpha)\beta/2 - V_2(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2 |V(\alpha,\alpha-\beta)|^2}.$$

We continue as follows

$$J_1 = K_1 + K_2,$$

for

$$K_1 = \frac{\partial_\alpha z_2(\alpha)\partial_\alpha z_1(\alpha)(\partial_\alpha z_1(\alpha)\beta/2 - V_1(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2 |V(\alpha,\alpha-\beta)|^2},$$

$$K_2 = \frac{\partial_\alpha z_2(\alpha)V_1(\alpha,\alpha-\beta)(\partial_\alpha z_1(\alpha)\beta/2 - V_1(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2 |V(\alpha,\alpha-\beta)|^2\beta/2},$$

to take $K_1 = L_1 + L_2$,

$$L_1 = \frac{\partial_\alpha z_2(\alpha)\partial_\alpha z_1(\alpha)((\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p - V_1(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2 |V(\alpha,\alpha-\beta)|^2},$$

$$L_2 = \frac{\partial_\alpha z_2(\alpha)\partial_\alpha z_1(\alpha)(\partial_\alpha z_1(\alpha)\beta/2 - (\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p)}{|\partial_\alpha z(\alpha)|^2 |V(\alpha,\alpha-\beta)|^2}.$$

We find

$$L_1 = \frac{\partial_\alpha z_2(\alpha)\partial_\alpha z_1(\alpha)}{|\partial_\alpha z(\alpha)|^2} \frac{1}{1+\frac{V_2^2(\alpha,\alpha-\beta)}{V_1^2(\alpha,\alpha-\beta)}} \frac{(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p - V_1(\alpha,\alpha-\beta)}{V_1^2(\alpha,\alpha-\beta)}$$

and using (11.4), we obtain $L_1 \leq C$.

Since

$$L_2 = \frac{\partial_\alpha z_2(\alpha)\partial_\alpha z_1(\alpha)}{|\partial_\alpha z(\alpha)|^2} \mathcal{F}(z)(\alpha,\beta) \frac{(\partial_\alpha z_1(\alpha)\beta/2 - (\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p)}{\beta^2/4}$$

we have $L_2 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$. Next let us write $K_2 = L_3 + L_4$, for

$$L_3 = \frac{\partial_\alpha z_2(\alpha) V_1(\alpha,\alpha-\beta)((\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p - V_1(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2 |V(\alpha,\alpha-\beta)|^2 \beta/2},$$

$$L_4 = \frac{\partial_\alpha z_2(\alpha) V_1(\alpha,\alpha-\beta)(\partial_\alpha z_1(\alpha)\beta/2 - (\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p)}{|\partial_\alpha z(\alpha)|^2 |V(\alpha,\alpha-\beta)|^2 \beta/2}.$$

In a similar way we find that

$$L_3 = \frac{\partial_\alpha z_2(\alpha)}{|\partial_\alpha z(\alpha)|^2} \frac{1}{1+\frac{V_2^2(\alpha,\alpha-\beta)}{V_1^2(\alpha,\alpha-\beta)}} \frac{(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p - V_1(\alpha,\alpha-\beta)}{V_1(\alpha,\alpha-\beta)} \frac{2}{\beta}.$$

By (11.3) one gets

$$L_3 \leq C \frac{|\partial_\alpha z_2(\alpha)|}{|\partial_\alpha z(\alpha)|^2} \frac{|(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p|}{|\beta|/2} \leq C.$$

As before we conclude that

$$L_4 \leq C\|\mathcal{F}(z)\|_{L^\infty} \frac{|\partial_\alpha z_1(\alpha)\beta/2 - (\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p|}{|\beta|^2} \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}.$$

We consider now $J_2 = K_3 + K_4$, where

$$K_3 = \frac{|\partial_\alpha z_2(\alpha)|^2}{|\partial_\alpha z(\alpha)|^2} \frac{\partial_\alpha z_2(\alpha)\beta/2 - V_2(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}$$

and

$$K_4 = \frac{\partial_\alpha z_2(\alpha)}{|\partial_\alpha z(\alpha)|^2} \frac{V_2(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} \frac{\partial_\alpha z_2(\alpha)\beta/2 - V_2(\alpha,\alpha-\beta)}{\beta/2}.$$

Using (11.1), we find

$$K_3 \leq C + \frac{|\partial_\alpha z_2(\alpha)|^2}{|\partial_\alpha z(\alpha)|^2} \frac{|\partial_\alpha z_2(\alpha)\beta/2 - (\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2})_p|}{|V(\alpha,\alpha-\beta)|^2} \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}.$$

and

$$\begin{aligned} K_4 &\leq \frac{|\partial_\alpha z_2(\alpha)|}{|\partial_\alpha z(\alpha)|^2}\Big(\frac{|V_2(\alpha,\alpha-\beta) - W_2(\alpha,\alpha-\beta)|}{(1+\frac{(V_1(\alpha,\alpha-\beta))^2}{(V_2(\alpha,\alpha-\beta))^2})|V_2(\alpha,\alpha-\beta)||\beta/2|} + \frac{|\partial_\alpha z_2(\alpha)\beta/2 - W_2(\alpha,\alpha-\beta)|}{|V(\alpha,\alpha-\beta)||\beta/2|}\Big) \\ &\leq C\|\mathcal{F}(z)\|_{L^\infty}^{1/2} \frac{|(\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2})_p|}{|\beta/2|} + \|\mathcal{F}(z)\|_{L^\infty} \frac{|\partial_\alpha z_2(\alpha)\beta/2 - (\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2})_p|}{(\beta/2)^2} \\ &\leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}, \end{aligned}$$

that is $K_3 + K_4 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$.

Putting all the previous estimates together we get $|A_1(\alpha,\alpha-\beta)| \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$.

Regarding

$$A_2(\alpha,\alpha-\beta)=\frac{V_1(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}-\frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_1(\alpha)}{\tan(\beta/2)}$$

we have the splitting $A_2=I_5+I_6+I_7+I_8$, where

$$I_5=\frac{V_1(\alpha,\alpha-\beta)-(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p}{|V(\alpha,\alpha-\beta)|^2},$$

$$I_6=\mathcal{F}(z)(\alpha,\beta)\frac{(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p-\partial_\alpha z_1(\alpha)\beta/2}{\beta^2/4},$$

$$I_7=\frac{\partial_\alpha z_1(\alpha)}{\beta/2}(\mathcal{F}(z)(\alpha,\beta)-\frac{1}{|\partial_\alpha z(\alpha)|^2}),$$

$$I_8=\frac{\partial_\alpha z_1(\alpha)}{|\partial_\alpha z(\alpha)|^2}(\frac{2}{\beta}-\frac{1}{\tan(\beta/2)}).$$

Then the same arguments used above allows us to obtain $|A_2|\leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$.

Lemma 11.2 *Let $B(\alpha,\beta)$ be defined by*

$$B(\alpha,\alpha-\beta)=V_1(\alpha,\alpha-\beta)\frac{V(\alpha,\alpha-\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}-\partial_\alpha z_1(\alpha)\frac{(\partial_\alpha^2 z(\alpha))^\perp\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4\tan(\beta/2)}$$

Then it satisfies the inequality

$$|B(\alpha,\alpha-\beta)|\leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^3_{C^{2,\delta}}|\beta|^{\delta-1}.$$

Proof: Let us decompose $B(\alpha,\beta)=I_1+I_2$ where

$$I_1=(V_1(\alpha,\alpha-\beta)-(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p)\frac{V(\alpha,\alpha-\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4},$$

and

$$I_2=(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p\frac{V(\alpha,\alpha-\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}-\partial_\alpha z_1(\alpha)\frac{(\partial_\alpha^2 z(\alpha))^\perp\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4\tan(\beta/2)}.$$

Using the identity (11.5), we can rewrite I_1 as follows:

$$I_1=k(z_1(\alpha)-z_1(\alpha-\beta))\frac{1}{(1+\frac{V_2^2(\alpha,\beta)}{V_1^2(\alpha,\beta)})^{3/2}}\frac{V(\alpha,\alpha-\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|}$$

to get $|I_1|\leq C\|z\|_{C^1}$.

Next we consider $I_2=J_1+J_2$, where

$$J_1=W_1(\alpha,\alpha-\beta)\frac{(V(\alpha,\alpha-\beta)-W(\alpha,\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4},$$

and

$$J_2 = W_1(\alpha, \alpha-\beta)\frac{W(\alpha, \alpha-\beta)^{\perp}\cdot\partial_\alpha z(\alpha)}{|V(\alpha, \alpha-\beta)|^4} - \partial_\alpha z_1(\alpha)\frac{(\partial^2_\alpha z(\alpha))^{\perp}\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4\tan(\beta/2)}.$$

Using (11.2), (11.5), and the fact that $(\beta/2)_p/\tan(\beta/2)$ is bounded, we obtain $|J_1| \leq C\|z\|_{C^1}$. To continue we can rewrite J_2 as follows:

$$J_2 = W_1(\alpha, \alpha-\beta)\frac{(W(\alpha, \alpha-\beta) - \partial_\alpha z(\alpha)\beta/2)^{\perp}\cdot\partial_\alpha z(\alpha)}{|V(\alpha, \alpha-\beta)|^4} - \partial_\alpha z_1(\alpha)\frac{(\partial^2_\alpha z(\alpha))^{\perp}\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4\tan(\beta/2)},$$

and $J_2 = K_1 + K_2 + K_3 + K_4$, where

$$K_1 = (W_1(\alpha, \alpha-\beta) - \partial_\alpha z_1(\alpha)\beta/2)\frac{(W(\alpha, \alpha-\beta) - \partial_\alpha z(\alpha)\beta/2)^{\perp}\cdot\partial_\alpha z(\alpha)}{|V(\alpha, \alpha-\beta)|^4},$$

$$K_2 = \partial_\alpha z_1(\alpha)\beta/2\frac{(W(\alpha, \alpha-\beta) - \partial_\alpha z(\alpha)\beta/2 - \partial^2_\alpha z(\alpha)\beta^2/4)^{\perp}\cdot\partial_\alpha z(\alpha)}{|V(\alpha, \alpha-\beta)|^4},$$

$$K_3 = 2\partial_\alpha z_1(\alpha)(\partial^2_\alpha z(\alpha)^{\perp}\cdot\partial_\alpha z(\alpha))(\mathcal{F}(z)(\alpha,\beta)^2 - \frac{1}{|\partial_\alpha z(\alpha)|^4})/\beta,$$

$$K_4 = \frac{\partial_\alpha z_1(\alpha)(\partial^2_\alpha z(\alpha))^{\perp}\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4}(\frac{2}{\beta} - \frac{1}{\tan(\beta/2)}).$$

Clearly we have $|K_4| \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$, and using (11.7) we obtain $|K_1| \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^3_{C^2}$. Furthermore the estimate (11.8) allows us to obtain $|K_2| \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^3_{C^{2,\delta}}|\beta|^{\delta-1}$. Next we consider in K_3 the factor $L(\alpha,\beta)$ given by

$$L(\alpha,\beta) = (\mathcal{F}(z)(\alpha,\beta)^2 - \frac{1}{|\partial_\alpha z(\alpha)|^4})/\beta.$$

We can write $L(\alpha,\beta)$ as follows:

$$\frac{(|\partial_\alpha z(\alpha)|^2\beta^2/4+|V(\alpha,\alpha-\beta)|^2)}{|\partial_\alpha z(\alpha)|^4|V(\alpha,\alpha-\beta)|^2}\frac{(\partial_\alpha z(\alpha)\beta/2+V(\alpha,\alpha-\beta))\cdot(\partial_\alpha z(\alpha)\beta/2-V(\alpha,\alpha-\beta))}{|V(\alpha,\alpha-\beta)|^2\beta}. \tag{11.9}$$

Then proceeding as in the previous lemma we get $|K_3| \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^3_{C^2}$ and this ends the proof. q.e.d.

Lemma 11.3 *Given $C(\alpha,\beta)$ by the following equality*

$$C(\alpha, \alpha-\beta) = \frac{V^{\perp}(\alpha, \alpha-\beta)\varpi(\alpha-\beta)\beta}{|V(\alpha, \alpha-\beta)|^4} - \frac{2\partial^{\perp}_\alpha z(\alpha)\varpi(\alpha)}{|\partial^2_\alpha z(\alpha)|^4\sin^2(\beta/2)},$$

we obtain

$$|C(\alpha, \alpha-\beta)| \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|_{C^2}\|\varpi\|_{C^1}\frac{1}{|\beta|}.$$

Proof: We decompose $C(\alpha,\alpha-\beta)=I_1+I_2+I_3+I_4+I_5$ where

$$I_1=\frac{(V(\alpha,\alpha-\beta)-W(\alpha,\alpha-\beta))^{\perp}\varpi(\alpha-\beta)\beta}{|V(\alpha,\alpha-\beta)|^4},$$

$$I_2=\frac{(W(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^{\perp}\varpi(\alpha-\beta)\beta}{|V(\alpha,\alpha-\beta)|^4},$$

$$I_3=\frac{\partial_\alpha^{\perp} z(\alpha)\beta^2(\varpi(\alpha-\beta)-\varpi(\alpha))}{2|V(\alpha,\alpha-\beta)|^4},$$

$$I_4=8\partial_\alpha^{\perp} z(\alpha)\varpi(\alpha)(\mathcal{F}(z)(\alpha,\beta)^2-\frac{1}{|\partial_\alpha z(\alpha)|^4})/\beta^2,$$

$$I_5=2\frac{\partial_\alpha^{\perp} z(\alpha)\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^4}(4/\beta^2-\sin^2(\beta/2)).$$

Using (11.2) and (11.5) we get $|I_1|\leq C\|\mathcal{F}(z)\|^{1/2}_{L^\infty}\|\varpi\|_{L^\infty}$. Using (11.7) clearly we obtain $|I_2|\leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|_{C^2}\|\varpi\|_{L^\infty}/|\beta|$. For the next term it holds

$$|I_3|\leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|_{C^1}\|\varpi\|_{C^1}/|\beta|.$$

The reference (11.6) gives $|I_5|\leq C\|\mathcal{F}(z)\|^{3/2}_{L^\infty}\|\varpi\|_{L^\infty}$. Finally, the estimate given in the previous lemma for the term

$$(\mathcal{F}(z)(\alpha,\beta)^2-\frac{1}{|\partial_\alpha z(\alpha)|^4})/\beta,$$

written in (11.9) allows us to conclude $|I_4|\leq\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|_{C^1}\|\varpi\|_{L^\infty}/|\beta|$.

Lemma 11.4 *Let $Q_1(\alpha,\beta)$ be given by*

$$Q_1(\alpha,\alpha-\beta)=-\frac{(\partial_\alpha z(\alpha))^{\perp}}{2}(\frac{1}{|V(\alpha,\alpha-\beta)|^2}-\frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}+\frac{2\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^4\beta}).$$

Then it satisfies the estimate $\|Q_1\|_{L^\infty}\leq\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^{2,\delta}}|\beta|^{\delta-1}$.

Proof: To simplify we will consider

$$C(\alpha,\alpha-\beta)=\frac{1}{|V(\alpha,\alpha-\beta)|^2}-\frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}+\frac{4\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^4\beta},$$

and we will show that $\|C\|_{L^\infty}\leq\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^{2,\delta}}|\beta|^{\delta-1}$. We can rewrite

$$C(\alpha,\alpha-\beta)=\frac{(\partial_\alpha z(\alpha)\beta+2V(\alpha,\alpha-\beta))\cdot(\partial_\alpha z(\alpha)\beta-2V(\alpha,\alpha-\beta))}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2|\beta|^2}+\frac{4\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^4\beta},$$

and then take $C(\alpha,\alpha-\beta)=I_1+I_2+I_3$ where

$$I_1 = -\frac{|2V(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta|^2}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2|\beta|^2},$$

$$I_2 = -\frac{2\partial_\alpha z(\alpha)\beta \cdot (2V(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta - \partial_\alpha^2 z(\alpha)\beta^2/2)}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2|\beta|^2},$$

$$I_3 = -\frac{\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)\beta}{|\partial_\alpha z(\alpha)|^2}\Big(\frac{1}{|V(\alpha,\alpha-\beta)|^2} - \frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}\Big).$$

Since

$$|I_1| \leq \frac{|2V(\alpha,\alpha-\beta) - 2W(\alpha,\alpha-\beta)|^2}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2|\beta|^2} + \frac{|2W(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta|^2}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2|\beta|^2},$$

using (11.1), (11.3) and the inequality (11.7) we control the term I_1. For I_2 it holds

$$|I_2| \leq \frac{4|V(\alpha,\alpha-\beta) - W(\alpha,\alpha-\beta)|}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)||\beta|} + \frac{4|W(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta - \partial_\alpha^2 z(\alpha)\beta^2/2|}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)||\beta|},$$

and using (11.1), (11.4), and (11.8) we get the appropriate inequality. For I_3 we write

$$I_3 = -\frac{4\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^2}\Big(\mathcal{F}(z)(\alpha,\beta) - \frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}\Big)/\beta,$$

and proceed as before.

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References

[1] D. Ambrose. Well-posedness of Two-phase Hele-Shaw Flow without Surface Tension. *Euro. Jnl. of Applied Mathematics* 15 597-607, 2004.

[2] J. Bear, Dynamics of Fluids in Porous Media, *American Elsevier*, New York, 1972.

[3] G. Baker, D. Meiron and S. Orszag. Generalized vortex methods for free-surface flow problems. *J. Fluid Mech.* 123 477-501, 1982.

[4] L.A. Caffarelli and A. Córdoba, Phase transitions: Uniform regularity of the intermediate layers., *J. Reine Angew. Math.,* 593, (2006), 209-235.

[5] P. Constantin and M. Pugh. Global solutions for small data to the Hele-Shaw problem. *Nonlinearity*, 6 (1993), 393 - 415.

[6] A. Córdoba and D. Córdoba, A pointwise estimate for fractionary derivatives with applications to P.D.E., *Proc. Natl. Acad. Sci.,* 100, 26, (2003), 15316-15317.

[7] A. Córdoba and D. Córdoba. A maximum principle applied to Quasi-geostrophic equations. *Comm. Math. Phys.* 249 (2004), no. 3, 511–528.

[8] D. Córdoba and F. Gancedo. Contour dynamics of incompressible 3-D fluids in a porous medium with different densities. *Comm. Math. Phys.* 273, 2, 445-471 (2007).

[9] D. Córdoba and F. Gancedo. A maximum principle for the Muskat problem for fluids with different densities. To appear in *Comm. Math. Phys.*, Preprint 2007.

[10] Dahlberg, Bjorn E. J. On the Poisson integral for Lipschitz and C^1-domains. Studia Math. 66 (1979), no. 1, 13–24.

[11] R. A. Fefferman, C. E. Kenig and J. Pipher. The theory of weights and the Dirichlet problem for elliptic equations. *Ann. of Math.* (2) 134 (1991), no. 1, 65–124.

[12] F. Gancedo. Existence for the α-patch model and the QG sharp front in Sobolev spaces. *Adv. Math.*, Vol 217/6: 2569-2598, 2008.

[13] Hele-Shaw. *Nature* 58, 34, 1898.

[14] T.Y. Hou, J.S. Lowengrub and M.J. Shelley. Removing the Stiffness from Interfacial Flows with Surface Tension. *J. Comput. Phys.*, 114: 312-338, 1994.

[15] M. Muskat. The flow of homogeneous fluids through porous media. *New York*, 1937.

[16] P.G. Saffman and Taylor. The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. *Proc. R. Soc. London, Ser. A* 245, 312-329, 1958.

[17] M. Siegel, R. Caflisch and S. Howison. Global Existence, Singular Solutions, and Ill-Posedness for the Muskat Problem. *Comm. Pure and Appl. Math.*, 57: 1374-1411, 2004.

[18] E. Stein. Harmonic Analysis. *Princeton University Press.* Princeton, NJ, 1993.

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