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ON TOTALLY GEODESIC INVARIANT SUBMANIFOLDS OF AN S-MANIFOLD

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In [1], S-manifolds, which reduce in a special case to Sasakian manifolds, were defined. In this note, a condition for an invariant submanifold of codimension greater than 2 in an S-manifold to be totally geodesic is obtained.

0. INTRODUCTION.- D. Blair, (Blair, [1]), has defined S-manifolds which reduce, in a special case, to Sasakian manifolds. On the other hand, many authors have studied invariant submanifolds of Sasakian manifolds, (see, e.g., Kon, [5] and Kon, [6]). Kobayashi and Tsuchiya, (Kobayashi and Tsuchiya, [4]), have investigated some topics in the geometry of invariant submanifolds of S-manifols. Specially, they have obtained a condition for an invariant submanifold of codimension 2 in an S-manifold of constant invariant f-sectional curvature to be totally geodesic.

The purpose of the present note is to study invariant submanifolds of codimension greater that 2 and with flat normal connection in S-manifolds whose invariant f-sectional curvature is constant and to obtain a condition for them to be totally geodesic. To this end, in section 1, we give a brief summary of notations and formulas for submanifolds and, in section 2, definitions and some properties of S-manifolds. In section 3 we get the main result.

1. PRELIMINARIES.- Let N^n be a Riemannian manifold of dimension n and M^m an m-dimensional submanifold of N^n. Let g be the metric tensor field on N^n as well as the induced metric on M^m. We denote by $\tilde{\nabla}$ the covariant differentiation in N^n and by ∇ the covariant differentiation in M^m determined by the induced metric. Let T(N) (resp. T(M)) be the Lie algebra of vector fields on N^n (resp. on M^m) and $T(M)^\perp$ the set of all vector fields normal to M^m.

The Gauss - Weingarten formulas are given by

$$(1.1) \qquad \tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y),$$
$$\tilde{\nabla}_X V = -A_V X + D_X V, \quad X,Y \in T(M), \; V \in T(M)^\perp,$$

where D is the connection in the normal bundle, σ is the second fundamental form of M^m, A_V is the Weingarten endomorphism associated with V and it satisfies:

$$g(A_V X, Y) = g(\sigma(X,Y), V).$$

We denote by $\tilde{R}$, R and R^D the curvature tensors associated with $\tilde{\nabla}$, ∇ and D respectively. If R^D vanishes identically the normal connection D is said to be flat. The Ricci equation is given by

$$(1.2) \qquad \tilde{R}(X,Y,U,V) = R^D(X,Y,U,V) - g([A_U,A_V]X,Y),$$
$$X,Y \in T(M), \; U,V \in T(M)^\perp,$$

where $[A_U,A_V]X = A_U A_V X - A_V A_U X$.

Finally, the submanifold M^m is said to be totally geodesic in N^n if its second fundamental form is identically zero.

2. S-MANIFOLDS.- Let N^{2n+s} be a $(2n+s)$-dimensional manifold with an f-structure f of rank 2n. If there exist on N^{2n+s} vector fields $\xi_1,\dots,\xi_s$, such that, if $\eta_1,\dots,\eta_s$ are dual 1-forms, then

$$(2.1)\qquad \eta_\alpha(\xi_\beta) = \delta_{\alpha\beta};\ f\xi_\alpha = 0;\ \eta_\alpha \circ f = 0;$$
$$f^2 = -I + \sum_\alpha \xi_\alpha \otimes \eta_\alpha,\ \alpha,\beta\in\{1,\dots,s\},$$

N^{2n+s} is said to have an f-structure with complemented frames. Further, the f-structure is said to be normal if

$$[f,f] + 2\sum_\alpha \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f,f]$ is the Nijenhuis torsion of f. Moreover, it is known that there exists a Riemannian metric g on N^{2n+s} satifying, (Yano, [7]):

$$(2.2)\qquad g(X,Y) = g(fX,fY) + \Phi(X,Y),\ X,Y\in T(N),$$

where $\Phi(X,Y) = \sum_\gamma \eta_\gamma(X)\eta_\gamma(Y)$. The fundamental 2-form F on N^{2n+s} is defined by

$$F(X,Y) = g(X,fY),\ X,Y\in T(N).$$

A normal f-structure with F closed is called a K-structure and N^{2n+s} is called a K-manifold. In such a manifold, the ξ_α are Killing vector fields, (Blair, [1]).

Let $\mathcal{L}$ denote the distribution determined by $-f^2$ and $\mathcal{M}$ the complementary distribution. $\mathcal{M}$ is determined by $f^2 + I$ and spanned by $\xi_1,\dots,\xi_s$. If $X\in\mathcal{L}$, then $\eta_\alpha(X) = 0$, for any α and if

$X \in \mathcal{M}$, then $fX = 0$.

A K-structure such that $F = d\eta_\alpha$, $\alpha = 1,\dots,s$, is called an S-structure and N^{2n+s} is called an S-manifold. These manifolds have been studied in (Blair, [1]). For the Riemannian connection $\tilde{\nabla}$ of g on an S-manifold N^{2n+s}, the following were also proved:

(2.3) $\tilde{\nabla}_X \xi_\alpha = -fX$, $X \in T(N)$, $\alpha \in \{1,\dots,s\}$,

(2.4) $(\tilde{\nabla}_X f)Y = \sum_\alpha \left[g(fX,fY)\xi_\alpha + \eta_\alpha(Y) f^2 X \right]$, $X,Y \in T(N)$.

A plane section π is called an invariant f-section if it is determined by a vector $X \in \mathcal{L}(p)$, $p \in N^{2n+s}$, such that $\{X,fX\}$ is an orthonormal pair spanning the section. The sectional curvature $K(X,fX)$, denoted by $H(X)$, is called an invariant f-sectional curvature. If N^{2n+s} is an S-manifold of constant invariant f-sectional curvature k, then its curvature tensor has the form, (Kobayashi and Tsuchiya, [4])

(2.5) $$\tilde{R}(X,Y,Z,W) = \sum_{\alpha,\beta} \{ g(fX,fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX,fZ)\eta_\alpha(Y)\eta_\beta(W) + g(fY,fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY,fW)\eta_\alpha(X)\eta_\beta(Z) \} + ((1/4)(k+3s)\{ g(X,W)g(fY,fZ) - g(X,Z)g(fY,fW) + g(fY,fW)\Phi(X,Z) - g(fY,fZ)\Phi(X,W) \} + (1/4)(k-s)\{ F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W) \},\ X,Y,Z,W \in T(N).$$

In the case $s = 1$, an S-manifold is a Sasakian manifold. For $s \geq 2$, examples of S-manifolds are given in (Blair, [1]), (Blair, [2]), (Blair, Ludden and Yano, [3]). Thus, the bundle space of a principal toroidal bundle over a Kaehler manifold with certain conditions is an S-manifold. In this way, a generalization of the Hopf fibration $\pi' : S^{2n+1} \longrightarrow \mathbb{P}\mathbb{C}^n$ is introduced as a canonical example of an S-manifold playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry.

Now, let M^m be an m-dimensional submanifold immersed in an S-manifold N^{2n+s}. For any $X \in T(M)$, we write

(2.6) $fX = TX + NX,$

where TX is the tangential component of fX and NX is the normal component of fX. Then, T is an endomorphism of the tangent bundle and N is a normal-bundle valued 1-form on the tangent bundle. It is easy to show that if T does not vanish, it defines an f-structure in the tangent bundle. On the other hand, if one of the ξ_α is normal to M^m, then $T \equiv 0$, because

$$g(X,fY) = F(X,Y) = d\eta_\alpha(X,Y) = 0, \quad X,Y \in T(M),$$

The submanifold M^m is said to be invariant if all of ξ_α ($\alpha = 1,\dots,s$) are always tangent to M^m and N is identically zero, i.e., $fX \in T(M)$, for any $X \in T(M)$. It is easy to show that an invariant submanifold of an S-manifold is such that $fV \in T(M)^\perp$, for any $V \in T(M)^\perp$. Moreover, it is an S-manifold too and, so, $m = 2p+s$. For later use, we prove the following

<u>Lemma</u> <u>2.1.-</u> *Let M^{2p+s} be an invariant submanifold of an S-manifold N^{2n+s}. Then, for any $X \in T(M)$, $V \in T(M)^\perp$, $\alpha \in \{1,\dots,s\}$:*

(2.7) $A_{fV}X = fA_VX = -A_VfX.$

Proof: By using the Weingarten formula (1.1), (2.4) and the fact that M^{m+s} is an invariant submanifold, it is easy to show that $A_{fV}X = fA_VX$. Now, if $Y \in T(M)$, we have

$$g(A_{fV}X,Y) = g(X,A_{fV}Y) = g(X,fA_VY) = -g(A_VfX,Y)$$

and (2.7) holds.

3. INVARIANT SUBMANIFOLDS WITH FLAT NORMAL CONNECTION.- In this section, let $N^{2n+s}(k)$ be an S-manifold whose invariant f-sectional curvature is a constant k. Let M^{2p+s} be an invariant submanifold of $N^{2n+s}(k)$ such that the normal connection of M^{2p+s} is flat, i. e., $R^D \equiv 0$. Then, by using the Ricci equation (1.2) and Lemma 2.1, we have

$$\tilde{R}(X,fY,V,fV) = 2g(A_V X, A_V Y),$$

for any vector field $X,Y \in T(M)$ and any unit vector field $V \in T(M)^{\perp}$. Now, from (2.5) we obtain

$$(3.1) \qquad (s-k)g(fX,fY) = 4g(A_V X, A_V Y).$$

Then, we get the following

<u>Proposition</u> <u>3.1.-</u> *Let M^{2p+s} be an invariant submanifold of an S-manifold $N^{2n+s}(k)$ with flat normal connection. Then, $k \leq s$ and the equality holds if and only if M^{2p+s} is totally geodesic.*

Now, we prove

<u>Theorem</u> <u>3.2.-</u> *Let M^{2p+s} be an invariant submanifold of an S-manifold $N^{2n+s}(k)$. If the codimension of M^{2p+s} is greater than 2, then the normal connection of M^{2p+s} is flat if and only if $k = s$ and M^{2p+s} is totally geodesic.*

Proof: From the Ricci equation (1.2) and (2.5), it is clear that if M^{2p+s} is totally geodesic, then its normal connection is flat. Now, we suppose that M^{2p+s} is not totally geodesic. We can choose a local field of orthonormal frames for vector fields in M^{2p+s} in the form

$$\{E_1,\ldots,E_p,E_{p+1} = fE_1,\ldots,E_{2p} = fE_p,\xi_1,\ldots,\xi_s\}.$$

If $A_V(E_i) = 0$, for some unit vector field $V\in T(M)^{\perp}$, then, from (3.1), we get that M^{2p+s} is totally geodesic, by virtue of Proposition 3.1. Thus, $A_V(E_i) \neq 0$, for any E_i and V, and so, $A_V(E_1),\ldots,A_V(E_{2p})$ are linearly independent.

On the other hand, it is easy to show, by using (3.1) again, that

$$A_V A_W + A_W A_V = 0, \tag{3.2}$$

for any orthonormal vector fields $V,W\in T(M)^{\perp}$. Thus, from (1.2) and (3.2), we get

$$\tilde{R}(X,Y,V,W) = 2g(A_V X,A_W Y),\ X,Y\in T(M).$$

Using (2.5), we obtain

$$(s-k)g(X,fY)g(V,fW) = 4g(A_V X,A_W Y). \tag{3.3}$$

If the codimension of M^{2p+s} is greater than 2, we can take a unit vector field W in $T(M)^{\perp}$ which is orthogonal to V and fV. Regarding to (3.3), it follows that $g(A_V X,A_W Y) = 0$, for any $X,Y\in T(M)$. Consequently, the vector fields

$$A_V(E_1),\ldots,A_V(E_{2p}),A_W(E_1),\ldots,A_W(E_{2p})$$

are linearly independent, which is a contradiction. Therefore, M^{2p+s} is totally geodesic and $k = s$.

Finally, for codimension 2, we have the following

<u>Theorem</u> <u>3.3.-</u> (Kobayashi and Tsuchiya, [4]) *Let M^{m+s} be an invariant submanifold of codimension 2 in an S-manifold $N^{2n+s}(k)$. Then, M^{m+s} is totally geodesic if and only if M^{m+s} is of constant invariant f-sectional curvature.*

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