Rev. Acad. Ciencias Zaragoza, **42** *(1987)*

A NOTE ON F-SECTIONAL CURVATURES OF S-MANIFOLDS

L.M. FERNÁNDEZ

Departamento de Algebra, Computación, Geometría y Topología.
Facultad de Matemáticas. Universidad de Sevilla.
Tarifa, s/n. 41012 SEVILLA (España).

This paper presents some characterizations of S-manifolds whose invariant f-sectional curvature is constant. The antiinvariant f-sectional curvature, the axiom of invariant f-planes and the axiom of antiinvariant f-planes are used in order to get the results.

0.- INTRODUCTION. For manifolds with an f-structure, David E. Blair (Blair, [1]) has introduced the analogue of Kaehler structure in the almost complex case and of quasi-Sasakian structure in the almost contact case, thus defining S-manifolds. He has also proved that the invariant f-sectional curvature determines the curvature of an S-manifold completely.

In this paper, we shall present some characterizations of S-manifolds whose invariant f-sectional curvature is constant. In section 1, we shall give a brief summary of basic formulas on S-manifolds. In section 2, we shall use the antiinvariant f-sectional curvature to characterize S-manifolds with constant invariant f-sectional curvature. In the last section, we shall prove that if an S-manifold satisfies the axiom of invariant f-planes, then it is of constant invariant f-sectional curvature. The same result is obtained, under certain restrictions on the dimension of the S-manifold, using the axiom of antiinvariant f-planes.

1.- PRELIMINARIES. Let M^{2n+s} be an S-manifold of dimension 2n+s, with structure tensors $(f,\xi_1,\dots,\xi_s,\eta_1,\dots,\eta_s,g)$. Let T(M) be the Lie Algebra of vector fields in M^{2n+s}. Then, the structure tensors satisfy the following equations (Blair, [1]):

$$(1.1)\qquad \eta_\alpha(\xi_\beta)=\delta_{\alpha\beta};\ f\xi_\alpha=0;\ \eta_\alpha(fX)=0;$$

$$f^2=-I+\sum_\alpha \xi_\alpha\times\eta_\alpha;$$

$$g(X,Y)=g(fX,fY)+\phi(X,Y),\ X,Y\in T(M),\ \alpha,\beta\in\{1,\dots,s\},$$

where $\phi(X,Y)=\sum_\alpha \eta_\alpha(X)\eta_\alpha(Y)$. Thus, the tensor f is an f-structure (Yano, [7]) of rank 2n and the metric g is compatible with f. Moreover, f is normal and so:

$$(1.2)\qquad [f,f]+2\sum_\alpha \xi_\alpha\times d\eta_\alpha=0$$

where [f,f] is the Nijenhuis torsion of f. The covariant differentiation ∇ of M^{2n+s} satisfies (Blair, [1]), if $X,Y\in T(M)$, $\alpha\in\{1,\dots,s\}$:

$$(1.3)\qquad \nabla_X\xi_\alpha=-fX.$$

$$(1.4)\qquad (\nabla_X f)Y=\sum_\alpha [g(fX,fY)\xi_\alpha+\eta_\alpha(Y)f^2X].$$

Furthermore, on an S-manifold we have $F=d\eta_\alpha$, $\alpha=1,\dots,s$, where F is the fundamental 2-form defined by $F(X,Y)=g(X,fY)$, $X,Y\in T(M)$.

Let $\mathcal{L}$ denote the distribution determined by $-f^2$ and $\mathcal{M}$ the complement distribution. $\mathcal{M}$ is determined by f^2+I and spanned by $\{\xi_1,\dots,\xi_s\}$. If $X\in\mathcal{L}$, then $\eta_\alpha(X)=0$ for any α and if $X\in\mathcal{M}$, then $fX=0$.

Examples of S-manifolds are given in (Blair, [1]), (Blair, [2]), (Blair, Ludden and Yano, [3]). Thus, the bundle space of a principal toroidal bundle over a Kaehler manifold with certain conditions is an S-manifold. In this way, a generalization of the Hopf Fibration $\pi': S^{2n+1}\longrightarrow \mathbb{PC}^n$ is introduced as a canonical example of an S-manifold (playing the role of complex space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry) as follows:

Let Δ denote the diagonal map. We define a principal toroidal bundle over $\mathbb{PC}^n$ by the following diagram:

$$
\begin{array}{ccc}
H^{2n+s} & \xrightarrow{\tilde{\Delta}} & S^{2n+1} \times \dots \times S^{2n+1} \\
\downarrow \pi & & \downarrow \pi' \times \dots \times \pi' \\
\mathbb{P}\mathbb{C}^n & \xrightarrow{\Delta} & \mathbb{P}\mathbb{C}^n \times \dots \times \mathbb{P}\mathbb{C}^n
\end{array}
$$

that is:

$$H^{2n+s} = \{(p_1,\dots,p_s) \in S^{2n+1} \times \dots \times S^{2n+1} \,/\, \pi'(p_1) = \dots = \pi'(p_s)\}.$$

By virtue of Theorem 3.1 in (Blair, [1]), H^{2n+s} is an S-manifold.

For later use, we recall the following (Blair, [1]):

<u>1.1.- Lemma</u>. On an S -manifold M^{2n+s}:

(1.5) $R(X,Y,fX,fY) = R(X,Y,X,Y) + sP(X,Y,X,fY)$;

(1.6) $R(X,fY,fX,Y) = -R(X,fY,X,fY) - sP(X,Y,X,fY)$;

(1.7) $R(X,fY,fZ,W) = R(X,Y,Z,W)$,

for any $X,Y,Z,W \in \mathcal{L}$, where:

$$P(X,Y,Z,W) = F(Y,Z)g(X,W) - F(X,Z)g(Y,W) - F(Y,W)g(X,Z) + F(X,W)g(Y,Z).$$

2.- INVARIANT AND ANTIINVARIANT f-SECTIONAL CURVATURES OF AN S-MANIFOLD. Let M^{2n+s} be an S-manifold. By a plane section we mean a 2-dimensional lineal subspace of a tangent space. A plane section π is called an invariant f-section (resp. an antiinvariant f-section) if $f\pi = \pi$ (resp. if $f\pi$ is perpendicular to π). The sectional curvature for an invariant (resp. antiinvariant) f-section is called an invariant (resp. antiinvariant) f-sectional curvature.

An invariant f-section is determined by a unit vector $X \in \mathcal{L}(p)$, $p \in M^{2n+s}$ such that $\{X, fX\}$ is an orthonormal pair spanning the section. On the other hand, it is easy to show that orthonormal vectors $X,Y \in \mathcal{L}(p)$, $p \in M^{2n+s}$, span an antiinvariant f-section if and only if X, Y and fX are orthonormal.

We denote by $K(X,Y)$ the sectional curvature of M^{2n+s} determined by orthonormal vectors $X,Y \in \mathcal{L}(p)$, $p \in M^{2n+s}$ and by $H(X)$ the invariant f-sectional curvature of an invariant f-section spanned by $\{X,fX\}$, that is, $H(X) = K(X,fX)$.

The fact that the invariant f-sectional curvature determines the curvature of an S-manifold completely is well known, (Blair, [1]). Moreover, in (Kobayashi and Tsuchiya, [5]) it is proved that if an S-manifold has constant invariant f-sectional curvature k, then its curvature tensor has the form:

(2.1) $$R(X,Y,Z,W) = \sum_{\alpha\beta}\sum [g(fX,fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX,fZ)\eta_\alpha(Y)\eta_\beta(W) + g(fY,fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY,fW)\eta_\alpha(X)\eta_\beta(Z)] + \tfrac{1}{4}(k+3s)[g(X,W)g(fY,fZ) - g(X,Z)g(fY,fW) + g(fY,fW)\phi(X,Z) - g(fY,fZ)\phi(X,W)] + \tfrac{1}{4}(k-s)[F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)],\ X,Y,Z,W\in T(M).$$

Now, we can prove:

2.1.- Theorem. Let M^{2n+s} be an S-manifold with $n\geq 2$. If the invariant f-sectional curvature at any point is independent of the choice of the invariant f-section at the point, then it is constant on the manifold and the curvature tensor is given by formula (2.1), where k is the constant invariant f-sectional curvature.

Proof: By virtue of Theorem 2.6 in (Blair, [1]), it is easy to see that the curvature tensor has the form of (2.1), with k a function on the manifold. Then, the Ricci tensor S and the scalar curvature ρ of M^{2n+s} are given by:

(2.2) $$S(X,Y) = \tfrac{1}{2}(n(k+3s) + k-s)g(fX,fY) + 2n\sum_{\alpha\beta}\sum \eta_\alpha(X)\eta_\beta(Y),\ X,Y\in T(M).$$

(2.3) $$\rho = \tfrac{1}{2}(n(2n+1)(k+3s) + n(k-s)).$$

Now, from the second Bianchi identity:

$$2\nabla_a S^a_j - \nabla_j\rho = 0,$$

where S^a_j are the components of the Ricci tensor of type (1,1). Making use of (2.2) and (2.3), we have:

$$(n+1)(n-1)\nabla_j k + (n+1)\sum_\alpha \eta^j_\alpha \xi^a_\alpha \nabla_a k = 0,$$

that is:

$$(n-1)dk + \sum_\alpha (\xi_\alpha k)\eta_\alpha = 0.$$

Applying this to ξ_β, $\beta = 1,\dots,s$, we get:

$$(n-1)(dk)\xi_\beta + \xi_\beta k = 0$$

and so, $\xi_\beta k = 0$, $\beta = 1,\dots,s$. Then, $dk = 0$, for $n \neq 1$ and the proof is complete.

As an example, it is well known (Blair, [1]), (Blair, [2]) and (Blair, Ludden and Yano, [3]) that H^{2n+s} has constant invariant f-sectional curvature $1 - 3s/4$. In general, if M^{2n+s}

is the bundle space of a principal toroidal bundle over a Kaehler manifold of constant holomorphic sectional curvature K, which is an S-manifold, then M^{2n+s} has constant invariant f-sectional curvature K - 3s/4.

With regard to the antiinvariant f-sectional curvature of an S-manifold, we have:

<u>2.2.- Proposition</u>. Let $M^{2n+s}(k)$ be an S-manifold of constant invariant f-sectional curvature k. Then, $M^{2n+s}(k)$ has constant antiinvariant f-sectional curvature equal to $\frac{1}{4}(k+3s)$.

<u>Proof:</u> By virtue of (2.1), if $X,Y\in\mathcal{L}$ span an antiinvariant f-section, then we have:

$$K(X,Y)=R(X,Y,Y,X) = \tfrac{1}{4}(k+3s)$$

as desired.

Now, we want to prove the converse. We need the following:

<u>2.3.- Lemma</u>. Let M^{2n+s} be an S-manifold. If $X,Y\in\mathcal{L}$ are orthonormal vectors, then:

(2.4) $K(X,Y) = K(fX,fY)$;

(2.5) $K(X,fY) = K(fX,Y)$.

Moreover, if X,Y span an antiinvariant f-section, then:

(2.6) $R(X,fX,fY,Y) = K(X,Y) + K(X,fY) - 2s$.

<u>Proof:</u> (2.4) and (2.5) follow from (1.7). Now, if $X,Y\in\mathcal{L}$ span an antiinvariant f-section, from the first Bianchi identity, we get:

(2.7) $R(X,fX,fY,Y) =-R(X,Y,fX,fY) + R(X,fY,fX,Y)$

and making use of (1.6), since $g(X,fY) = 0 = g(Y,fX) =$ $= g(X,fX) = g(Y,fY)$, we have:

$$R(X,Y,fX,fY) = R(fX,fY,X,Y) =-R(X,Y,Y,X) + s =$$
$$=-K(X,Y) + s$$

and

$$R(X,fY,fX,Y) =-R(X,fY,X,fY) - s = K(X,fY) - s.$$

Then, replacing these into (2.7), we obtain the result.

<u>2.4.- Theorem</u>. Let M^{2n+s} be an S-manifold with $n\geq 3$. If M^{2n+s} has constant antiinvariant f-sectional curvature c, then M^{2n+s} has constant invariant f-sectional curvature equal to 4c-3s.

<u>Proof:</u> Let X,Y be orthonormal vectors fields which span an antiinvariant f-section. Then, $(X+Y)/\sqrt{2}$ and $(fX-fY)/\sqrt{2}$ span an antiinvariant f-section too. Then, making use of Lemma 1.1

and Lemma 2.3, we get:

$$c = \tfrac{1}{4}K(X+Y,fX-fY) = \tfrac{1}{4}R(X+Y,fX-fY,fX-fY,X+Y) =$$
$$= \tfrac{1}{4}[H(X) + H(Y) - 2K(X,Y) - 2K(X,fY) + 6s].$$

Since $K(X,Y) = k(X,fY) = c$, we obtain:

$$H(X) + H(Y) = 8c - 6s.$$

Now, let p be an arbitrary point of M^{2n+s} and let X,Y be unit vectors in $\mathcal{L}(p)$. Since $n \geqslant 3$, we can choose a unit vector $Z \in \mathcal{L}(p)$ orthogonal to the plane sections spanned by $\{X,fX\}$ and $\{Y,fY\}$. It is easy to show that the plane sections spanned by $\{X,Z\}$ and $\{Y,Z\}$ are antiinvariant f-sections. Then we know that $H(X) + H(Z) = 8c-6s = H(Y) + H(Z)$. Thus, $H(X) = H(Y)$. Since X and Y are arbitrary vectors, the invariant f-sectional curvature does not depend on the choice of the invariant f-section at p. But p is an arbitrary point of M^{2n+s} too. Now, from Theorem 2.1, M^{2n+s} is of constant invariant f-sectional curvature equal to 4c-3s, by virtue of (2.8).

These results should be compared with the corresponding results for Kaehler manifolds (s = 0), (Chen and Ogiue, [4]).

3.- THE AXIOM OF INVARIANT (ANTIINVARIANT) f-PLANES. An S-manifold M^{2n+s} is said to satisfy tha axiom of invariant (resp. antiinvariant) f-planes if for each $p \in M^{2n+s}$ and each invariant (resp. antiinvariant) f-section π at p, there exists a 2-dimensional totally geodesic submanifold N of M^{2n+s} such that $p \in N$ and $T_p(N) = \pi$.

<u>3.1.- Theorem</u>. Let M^{2n+s} be an S-manifold. Then, M^{2n+s} satisfies the axiom of invariant f-planes if and only if M^{2n+s} is of constant invariant f-sectional curvature.

The proof is a very lengthy computation, but similar to that given by Ogiue (Ogiue, [6]), for Sasakian manifolds. Now, we shall prove:

<u>3.2.- Theorem</u>. Let M^{2n+s} be an S-manifold with $n \geqslant 3$ such that M^{2n+s} satisfies the axiom of antiinvariant f-planes. Then, M^{2n+s} is of constant invariant f-sectional curvature.

<u>Proof:</u> Let p be an arbitrary point of M^{2n+s} and let $X,Y \in \mathcal{L}(p)$ be orthonormal vectors spanning an antiinvariant f-section π. Let N be a 2-dimensional totally geodesic submanifold of M^{2n+s} such that $p \in N$ and $T_p(N) = \pi$. Since π is an antiinvariant f-section, fX is normal to N. Then, from

Weingarten's formula, we get:

$$R(X,Y,fX,X) = g(\nabla_X\nabla_Y fX,X) - g(\nabla_Y\nabla_X fX,X) -$$

$$- g(\nabla_{[X,Y]} fX,X) = g(\nabla_X D_Y fX,X) - g(\nabla_Y D_X fX,X) =$$

$$= g(D_X D_Y fX,X) - g(D_Y D_X fX,X) = 0,$$

where we have used the fact that N is totally geodesic and so, $A_V = 0$, for any vector field V normal to N.

Now, since X and Y span an antiinvariant f-section at p, then X+Y and fX-fY span an antiinvariant f-section too. Then:

$$R(X+Y,fX-fY,fX+fY,X+Y) = 0.$$

Using Lemma 1.1 and Lemma 2.3, a direct expansion gives:

(3.1) $H(X) = H(Y)$.

Now, let X and Y be unit arbitrary vectors in $\mathcal{L}(p)$. If the section {X,Y} is an invariant f-section, then H(X) = H(Y). If it is not an invariant f-section, since $n \geqslant 3$, we can choose a unit vector Z in $\mathcal{L}(p)$, orthogonal to the sections {X,fX} and {Y,fY}. Then, from (3.1), H(X) = H(Z) = H(Y). Since X and Y are arbitrary vectors, the invariant f-sectional curvature does not depend on the choice of the invariant f-section at p. But p is arbitrary too. So, from Theorem 2.1, we complete the proof.

This result should be compared with that in the case of s = 0, (Chen and Ogiue, [4]).

REFERENCES.

[1] Blair, D.E., 1.970. J.Diff. Geom., 4. (155 - 167).

[2] -----------, 1.971. Ann. Stiin. Univ. „Al. I. Cuza". Din Iasi (serie nova). T. XVII, Fasc. 1. (171 - 177).

[3] Blair, D.E., Ludden, G.D. and Yano, K., 1.973. Trans. Am. Math. Soc., 181. (175 - 184).

[4] Chen, B.Y. and Ogiue, K., 1.973. Duke Math. J., V. 40, n. 4. (797 - 799).

[5] Kobayashi, M. and Tsuchiya, S., 1.972. Kodai Math. Sem. Rep., 24. (430 - 450).

[6] Ogiue, K., 1.964. Kodai Math. Sem. Rep., 16. (223 - 232).

[7] Yano, K., 1.963. Tensor, 14. (99 - 109).