LOCALIZATION AT HYPERPLANE ARRANGEMENTS: COMBINATORICS AND D-MODULES

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ABSTRACT. We describe an algorithm deciding if the annihilating ideal of the meromorphic function $\frac{1}{f}$, where $f = 0$ defines an arrangement of hyperplanes, is generated by linear differential operators of order 1. The algorithm is based on the comparison of two characteristic cycles and uses a combinatorial description due to Àlvarez-Montaner, García–López and Zarzuela of the characteristic cycle of the $\mathcal{D}$-module of meromorphic functions with respect to f.

1. INTRODUCTION

Let $R = k[x_1, \ldots, x_n]$ be the ring of polynomials in n variables where k is a field of characteristic zero. Let $A_n := R\langle \partial_1, \ldots, \partial_n \rangle$ be the associated Weyl algebra, i.e. the ring extension generated by the partial derivatives $\partial_i = \frac{\partial}{\partial x_i}$, with the relations given by $\partial_i \partial_j = \partial_j \partial_i$ and $\partial_i r - r \partial_i = \frac{\partial r}{\partial x_i}$, where $r \in R$. For any unexplained terminology concerning the theory of rings of differential operators we will use [4]. For any polynomial $f \in R$, I. N. Bernstein [3] proved that the localization ring $R_f = \{ \frac{g}{f^m} \mid g \in R, m \in \mathbb{N} \}$ has a natural structure as a finitely generated left A_n-module. Moreover it has finite length so the ascending chain of A_n-submodules

$$A_n \cdot \frac{1}{f} \subseteq A_n \cdot \frac{1}{f^2} \subseteq \cdots \subseteq A_n \cdot \frac{1}{f^j} \subseteq \cdots \subseteq R_f$$

stabilizes. In particular we have

$$R_f \cong A_n \cdot \frac{1}{f^\ell} \cong \frac{A_n}{Ann_{A_n}(\frac{1}{f^\ell})}$$

where the integer $-\ell$ is the smallest integer root of the Bernstein-Sato polynomial of f (see [3]) and $R_f \not\cong A_n \cdot \frac{1}{f^j}$ for $j < \ell$ [25, Lem. 1.3]. T. Oaku [16], and later T. Oaku and N. Takayama [17], described an algorithm to compute this presentation of R_f using the theory of Gröbner bases over rings of differential operators. The algorithm has been implemented in the package D-modules [15] for Macaulay 2 [12] and in the system Kan/sm1 [21]. The computation of the Bernstein-Sato polynomial is very expensive due to the use of Gröbner bases so different short cuts have been considered to describe R_f. We will follow the approach given in [8], [9] where the authors use the module of logarithmic derivations introduced by K. Saito [20].

Let $Ann_{A_n}^{(j)}(\frac{1}{f^\ell})$ denote the ideal generated by the set of differential operators of order $\leq j$ that annihilate $\frac{1}{f^\ell}$. We have the ascending chain of ideals

$$Ann_{A_n}^{(1)}(\frac{1}{f^\ell}) \subseteq Ann_{A_n}^{(2)}(\frac{1}{f^\ell}) \subseteq \cdots \subseteq Ann_{A_n}(\frac{1}{f^\ell}).$$

Any of the terms $Ann_{A_n}^{(j)}(\frac{1}{f^\ell})$ of this chain can be described without using the Bernstein-Sato polynomial since its computation reduces to the one of commutative syzygies for a set of elements involving the partial derivatives of f of order less or equal to j.

For the case $j = 1$ the annihilator can be described in terms of logarithmic derivations. Namely, let $Der(-\log f)$ be the R-module of logarithmic derivations with respect to f, i.e. derivations $\delta = \sum_i a_i \partial_i \in Der_k(R)$ such that $\delta(f) = af$ for some $a \in R$. Then, $Ann_{A_n}^{(1)}(\frac{1}{f^k})$ is the ideal generated by the set

$$\{\delta + k\frac{\delta(f)}{f} \mid \delta \in Der(-\log f)\}.$$

Our motivation is the natural question: *When is $Ann_{A_n}(\frac{1}{f^\ell})$ generated by operators of order less or equal to j?* In this paper we will restrict ourselves to the case $j = 1$.

The analytic counterpart to this question has been treated in [10] following ideas introduced and developed in [6]. They got an affirmative answer for the case of locally quasi-homogeneous free divisors $D \subset \mathbb{C}^n$. Namely, let $\mathcal{O}_X$ be the sheaf of holomorphic functions on $X = \mathbb{C}^n$. Let $\mathcal{D}_X$ denote the corresponding sheaf of differential operators with coefficients in $\mathcal{O}_X$. If $Der_{\mathcal{O}_X}(-\log D)_x := Der_{\mathcal{O}_{X,x}}(-\log f)$, where $f \in \mathcal{O}_{X,x}$ is a local reduced equation of the germ $(D, x) \subset (\mathbb{C}^n, x)$, is a free $\mathcal{O}_{X,x}$-module at every point $x \in D$ and f is locally quasi-homogeneous then $Ann_{\mathcal{D}_{X,x}}(\frac{1}{f})$ is generated by operators of order 1 [10].

These results are closely related to the so-called *Logarithmic Comparison Theorem* (LCT) [7]. We say that LCT holds for a divisor $D \subset \mathbb{C}^n$ if the inclusion $i_D : \Omega(\log D) \hookrightarrow \Omega(\star D)$ is a quasi-isomorphism, where $\Omega(\star D)$ is the de Rham complex of differential meromorphic forms with poles along D and $\Omega(\log D)$ is its subcomplex of logarithmic forms. When LCT holds the complex $\Omega(\log D)$ computes the cohomology $H^p(X \setminus D, \mathbb{C})$ of the complementary of D by Grothendieck Comparison Theorem [14]. Locally quasi-homogeneous free divisors satisfy LCT [7].

T. Torrelli [24] gave some criteria to answer our main question and also conjectured the following link with LCT.

Conjecture 1.1. *Let $f \in \mathcal{O}_{X,x}$ is a local reduced equation of the germ $(D, x) \subset (\mathbb{C}^n, x)$. Then, the ideal $Ann_{\mathcal{D}_{X,x}}(\frac{1}{f})$ is generated by operators of order 1 if and only if LCT holds for (D, x).*

In the present work we turn our attention to the case of arrangement of hyperplanes $\mathcal{A} \subseteq \mathbb{A}_k^n$. It has been proved by A. Leykin (see [25]) that R_f is generated by $\frac{1}{f}$ so we have to study when $Ann_{A_n}^{(1)}(\frac{1}{f}) = Ann_{A_n}(\frac{1}{f})$. The equality holds for the union of a generic hyperplane arrangement with an hyperbolic arrangement as it has been proved by T. Torrelli [24] giving an explicit set of generators and solving a conjecture posed by U. Walther [25].

It has been conjectured in [22] that LCT holds for central arrangements of hyperplanes over any field k of characteristic zero. H. Terao observed that, using the methods in [7], LCT holds for tame arrangements over $\mathbb{C}$, i.e. arrangements such that the projective dimension of the module of logarithmic p-forms $\Omega^p(\log f)$ is less or equal than p. The conjecture is also proved for arbitrary

arrangements over $\mathbb{C}$ in dimension less or equal than 5 [23] and over any field k of characteristic zero for the case of tame arrangements and the case of all arrangements in dimension less or equal than 4 [26].

Our aim is to describe an indirect method to compare the ideals $Ann_{A_n}^{(1)}(\frac{1}{f})$ and $Ann_{A_n}(\frac{1}{f})$ using an invariant that we may attach to finitely generated A_n-modules, the characteristic cycle. We intend to use the interplay of A_n-modules and combinatorial techniques to deal with examples not covered in [24] and that can not be treated using the methods in [9]. One of our main examples will be a non-tame arrangement such that $Ann_{A_n}(\frac{1}{f})$ is generated by operators of order 1. As far as we know for these examples a direct approach using the methods in [16] or in [17] implemented in the available Computer Algebra systems is not possible due to crashing memory problems.

The scripts of the source codes we will use in this work as well as the output in full detail of the examples are available at the web page
`http://www.ma1.upc.edu/~jalvz/acu.html`.

2. USING THE CHARACTERISTIC CYCLE

Let $R = k[x_1, \ldots, x_n]$ be the ring of polynomials in n variables where k is a field of characteristic zero. The associated Weyl algebra $A_n := R\langle \partial_1, \ldots, \partial_n \rangle$ has a natural increasing filtration given by the order of differential operators, the order of a monomial $x^\alpha \partial^\beta$ being $|\beta| = \sum_i \beta_i$. The corresponding associated graded ring $gr(A_n)$ is isomorphic to the polynomial ring $R[\xi_1, \ldots, \xi_n]$ graded by the natural degree of polynomials on the variables ξ_i. A finitely generated A_n-module M has an increasing sequence of finitely generated R-submodules such that the associated graded module $gr(M)$ is a finitely generated $gr(A_n)$-module. All the A_n–modules considered here are left modules.

The *characteristic variety* of M is the closed algebraic set $Ch(M) \subseteq \mathbb{A}_k^{2n}$ defined by the *characteristic ideal* $J(M) := \operatorname{rad}(Ann_{gr(A_n)}(gr(M)))$, where $rad()$ stands for the radical ideal. A finitely generated A_n-module M is said to be *holonomic* if $M = 0$ or $\dim Ch(M) = n$. The *characteristic cycle* of a holonomic A_n-module M is defined as:

$$ChC(M) = \sum m_i \Lambda_i$$

where the sum is taken over all the irreducible components Λ_i of the characteristic variety $Ch(M)$, and m_i is the multiplicity of $\operatorname{gr}(M)$ at a generic point of Λ_i.

For each i, the irreducible variety Λ_i that appear in the characteristic cycle is the conormal space to the $X_i := \pi(\Lambda_i)$ in $X = \mathbb{A}_k^n$ where $\pi : \mathbb{A}_k^n \times \mathbb{A}_k^n \to \mathbb{A}_k^n$ is the first projection, see [19, §10]. So we have the following description

$$ChC(M) = \sum m_i \, T_{X_i}^* X.$$

Remark 2.1. Let $I_i \subseteq R$ be the defining ideal of a linear variety $X_i \subseteq X$ of codimension h then $T_{X_i}^* X$ is the characteristic cycle of the local cohomology module $H_{I_i}^h(R)$.

2.1. **Comparison using the characteristic cycle.** Let $f \in R$ be any polynomial and $-\ell$ be the smallest integer root of the Bernstein-Sato polynomial associated to f. Our aim is to use the characteristic cycle to decide whether $Ann_{A_n}(\frac{1}{f^\ell})$ is generated by differential operators of order 1.

Remark 2.2. We have to point out that if $Ann_{A_n}(\frac{1}{f^\ell})$ is generated by differential operators of order 1 for any given $\ell \in \mathbb{N}$ then the smallest integer root of the Bernstein-Sato polynomial of f is strictly greater than $-\ell - 1$, [24, Prop. 1.3].

To our purpose we consider the A_n-module

$$\widetilde{M}^{\log f^\ell} := \frac{A_n}{Ann_{A_n}^{(1)}(\frac{1}{f^\ell})}.$$

Notice that, although the localization module R_f is holonomic by [3] the module $\widetilde{M}^{\log f^\ell}$ may not be holonomic, e.g. $f = (xz + y)(x^4 + y^5 + xy^4)$.

Proposition 2.3. *Let $f \in k[x_1, \dots, x_n]$ be any polynomial and let $-\ell$ be the smallest integer root of the Bernstein-Sato polynomial associated to f. If $\widetilde{M}^{\log f^\ell}$ is holonomic, then $Ann_{A_n}(\frac{1}{f^\ell})$ is generated by differential operators of order 1 if and only if $ChC(R_f) = ChC(\widetilde{M}^{\log f^\ell})$. If $\widetilde{M}^{\log f^\ell}$ is not holonomic, then $Ann_{A_n}(\frac{1}{f^\ell})$ can not be generated by differential operators of order 1.*

Proof.- By [3], the localization module R_f is isomorphic to $\frac{A_n}{Ann_{A_n}(\frac{1}{f^\ell})}$ so we have the short exact sequence of A_n-modules

$$0 \longrightarrow K_{f^\ell} \longrightarrow \widetilde{M}^{\log f^\ell} \longrightarrow R_f \longrightarrow 0$$

where $K_{f^\ell} := \frac{Ann_{A_n}(\frac{1}{f^\ell})}{Ann_{A_n}^{(1)}(\frac{1}{f^\ell})}$.

Notice that the question about $Ann_{A_n}(\frac{1}{f^\ell})$ being generated by operators of order 1 reduces then to the vanishing of the kernel K_{f^ℓ}.

If $\widetilde{M}^{\log f^\ell}$ is not holonomic then $K_{f^\ell} \neq 0$. Otherwise, by the additivity of the characteristic cycle with respect to holonomic A_n-modules

$$K_{f^\ell} = 0 \iff ChC(\widetilde{M}^{\log f^\ell}) = ChC(R_f).$$

$\square$

Besides computing the Bernstein-Sato polynomial of f it may also be difficult to compute the characteristic cycles of the A_n-modules $\widetilde{M}^{\log f^\ell}$ and R_f. To compute the characteristic variety of these modules we need Gröbner basis over rings of differential operators. The software packages D-modules [15] for Macaulay 2 [12] and Kan/sm1 [21] can do the job for some examples but then, in order to get the irreducible components of the characteristic variety, we must perform primary decomposition so it is expensive to develop large examples. Algorithms for primary decomposition are defined over the field of rational numbers but this is not an issue for most examples due to the good behavior of these modules with respect to flat base change.

To compute the characteristic cycle of the localization module R_f we can also use a geometric formula given by J. Briançon, P. Maisonobe and M. Merle in [5, Thm. 3.4.2]). Their formula has been implemented in [2] but we have to point out that primary decomposition is also needed in the procedure.

2.2. **The case of hyperplane arrangements.** In this work we will restrict ourselves to the case of hyperplane arrangements since we can avoid the computation of Bernstein-Sato polynomials. Namely, as it was proved by A. Leykin, the smallest integer root of the Bernstein-Sato polynomial of any arrangement is -1 [25] so we have to study the A_n-module $\widetilde{M}^{\log f}$.

To obtain the characteristic cycle of $\widetilde{M}^{\log f}$ we can avoid the computation of primary decompositions since the singular support (see [4]) of this module is the divisor $\{f = 0\}$ so the components that appear in the characteristic cycle can be described in terms of the combinatorial data associated to the arrangement of hyperplanes.

Let $\mathcal{A} = \{H_1, \ldots, H_r\}$ be an arrangement of hyperplanes defined by a polynomial $f = f_1 \cdots f_r \in R$, f_i being a linear form in R. The arrangement $\mathcal{A}$ defines a partially ordered set $P(\mathcal{A})$ whose elements correspond to the intersections of irreducible components of $\mathcal{A}$ and where the order is given by inclusion, i.e. for $p, q \in P(\mathcal{A})$ we write $p < q$ if $q \subset p$.

Given $p \in P(\mathcal{A})$, we will denote by X_p the linear affine variety corresponding to p and by $I_p \subseteq R$ the radical ideal which defines X_p. Notice that the poset $P(\mathcal{A})$ is isomorphic to the poset of ideals $\{I_p\}_p$, ordered by reverse inclusion. We denote by $\text{ht}(p)$ the height of the prime ideal I_p.

We have
$$ChC(\widetilde{M}^{\log f}) = T_X^* X + \sum n_p\, T_{X_p}^* X,$$
where n_p is the multiplicity of $\text{gr}(\widetilde{M}^{\log f})$ at a generic point of $T_{X_p}^* X$.

On the other hand the characteristic cycle of the localization module R_f can be completely described using the combinatorial approach given in [1]. Let $K(> p)$ be the simplicial complex attached to the subposet
$$P(> p) := \{q \in P(X) \mid q > p\} \subseteq P(\mathcal{A}).$$

Namely, $K(> p)$ has as vertices the elements of $P(> p)$ and a set of vertices $p_0, \ldots, p_r$ determines a r-dimensional simplex if $p_0 < \cdots < p_r$. Then, the characteristic cycle of the localization module R_f can be described in terms of the elements of the poset $P(\mathcal{A})$ and the dimensions of the reduced simplicial homology groups of $K(> p)$. Namely, the result we will use in this work, conveniently reformulated states the following:

Proposition 2.4. ([1, Cor. 1.3]) *Let $\mathcal{A} \subseteq \mathbb{A}_k^n$ be an arrangement of hyperplanes defined by a polynomial $f \in R = k[x_1, \ldots, x_n]$. Then, the characteristic cycle of the localization module R_f is*
$$ChC(R_f) = T_X^* X + \sum m_p\, T_{X_p}^* X,$$
where $m_p = \dim_k \tilde{H}_{\text{ht}(p)-2}(K(> p); k)$.

Let $f \in R$ be the defining polynomial of an hyperplane arrangement $\mathcal{A} \subseteq \mathbb{A}_k^n$. Our strategy for this family of examples boils down to the following

Algorithm 2.5. INPUT: *The defining polynomial $f \in R$ of an hyperplane arrangement $\mathcal{A} \subseteq \mathbb{A}_k^n$.* OUTPUT: **True** *if $Ann_{A_n}(\frac{1}{f})$ is generated by operators of order 1 and* **False** *otherwise.*

(1) Compute the characteristic cycle of R_f.

(2) Compute the ideal $Ann_{A_n}^{(1)}(\frac{1}{f})$.

(3) Compute the characteristic cycle of $\widetilde{M}^{\log f}$.

(3.1) Compute the characteristic variety $Ch(\widetilde{M}^{\log f})$.
(3.2) Check out the holonomicity of $\widetilde{M}^{\log f}$.
(3.3) Compute the irreducible components of $Ch(\widetilde{M}^{\log f})$.
(3.4) Compute the multiplicity associated to each component.

(4) Compare both characteristic cycles.

Recall that if $\widetilde{M}^{\log f}$ is not holonomic we must stop the process at Step 3.2 and return **False**. The correctness of the algorithm is given by the following

Theorem 2.6. *The previous algorithm is correct, i.e. there is an algorithm based on the comparison of two characteristic cycles deciding if $Ann_{A_n}(\frac{1}{f})$ is generated by differential operators of order 1, f being the defining polynomial of an arrangement of hyperplanes $\mathcal{A} \subseteq \mathbb{A}_k^n$.*

Proof.- By Proposition 2.3 and the previous result by A. Leykin we have only to prove that each step can be achieved algorithmically. For simplicity we will denote $I = Ann_{A_n}^{(1)}(\frac{1}{f})$.

· *Step 1:* By Proposition 2.4 we only have to compute the simplicial homology groups associated to the poset $\mathcal{P}(\mathcal{A})$. It can be algorithmically achieved.

· *Step 2:* Since the ideal I is generated by the operators of the form $\delta + \frac{\delta(f)}{f}$ for $\delta \in Der(-\log f)$ it is enough to compute a finite generating system of the R-module $Der(-\log f)$. This last R-module is in fact isomorphic to the module of syzygies of $(f_1, \ldots, f_n, -f)$ where $f_i = \partial f/\partial x_i$. More precisely, if $\{\mathbf{s}_1, \ldots, \mathbf{s}_r\}$ is a generating systems for $Syz(f_1, \ldots, f_n, -f)$, where $\mathbf{s}_i = (s_{i1}, \ldots, s_{in}, s_{i(n+1)})$, then I is generated by the family $\{\sum_j s_{ij}\partial_j + s_{i(n+1)} \,|\, i = 1, \ldots, r\}$ of operators in A_n. Therefore this step is solved by using commutative Gröbner basis computations in R.

· *Step 3.1:* It is performed by a direct computation of a system of generators of the graded ideal $gr(I)$ which uses Gröbner basis computation in A_n.

· *Step 3.2:* We have to compute the dimension of our module using Gröbner basis.

· *Step 3.3:* The components for the case of arrangements of hyperplanes are described in terms of the associated poset $\mathcal{P}(\mathcal{A})$ (see Section 2.2) so this step is solved by using commutative Gröbner basis computations in R

· *Step 3.4:* We have to compute for each $p \in \mathcal{P}(\mathcal{A})$ the generic multiplicity of the component $T_{X_p}^* X$, i.e. the multiplicity of $gr(I)$ at a generic point of $T_{X_p}^* X$. A point $(a,b) \in T_{X_p}^* X$ is generic if $(a,b) \notin T_{X_q}^* X$ for $q \in \mathcal{P}(\mathcal{A})$ and $q \neq p$ since each $T_{X_p}^* X$ is an affine linear variety in $\mathbb{A}_k^{2n}$. Once a generic point (a,b) has been chosen we consider a n-dimensional linear affine variety Y through (a,b) and transversal to $T_{X_p}^* X$. The generic multiplicity we are looking for is just the dimension of the vector space $(\frac{R[\xi]}{J+gr(I)})_{(a,b)}$ where $R[\xi] = R[\xi_1, \ldots, \xi_n] = gr(A_n)$, J is the defining ideal of Y and $()_{(a,b)}$ stands for the localization at the maximal ideal of the point $(a,b) \in \mathbb{A}_k^{2n}$. $\square$

2.3. **Some tips on the implementation.** The algorithm we present uses Gröbner basis computations in both the commutative ring R and the non-commutative ring A_n so the complexity is very high. It does not prevent us from developing some relatively large examples that can not be treated by other known algorithms. In the sequel we will point out the major bottlenecks we may find in the process as well as the Computer Algebra systems we have used to develop our examples.

The general version of Proposition 2.4, namely the computation of the characteristic cycle of local cohomology modules supported on arrangements of linear varieties, has been implemented by J. Pfeifle in the software package `polymake` [11]. The script interfaces with the computer algebra system `Singular` to construct the poset associated to the arrangement. Since we have to compute the homology groups of simplicial complexes it becomes very difficult to develop arrangements having a lot of components in a large dimensional affine space.

Computing a set of generators of $I = Ann_{A_n}^{(1)}(\frac{1}{f})$ can be done in almost any Computer Algebra system such as `Macaulay 2` [12] or `Singular` [13] since we have to use Gröbner basis over R. We have done our examples using `Macaulay 2`. However, we should point out that we can speed up the computations using in an iterative way on the components of our arrangement $f = f_1 \cdots f_r \in R$ the following straightforward property.

Lemma 2.7. *Let $f, g \in R$ be polynomials with no common factors. Then*
$$Der(-\log(f \cdot g)) = Der(-\log f) \cap Der(-\log g)$$

To compute the characteristic variety of $\widetilde{M}^{\log f}$ and its dimension, in particular the computation of $\mathrm{gr}\,(I)$, we have to use the theory of Gröbner basis over A_n so it can be implemented in the software package `D-modules` [15] for `Macaulay 2` or `Kan/sm1` [21]. The components of the characteristic variety are described from the combinatorial data given by the poset associated to the arrangement. We recall that to construct the poset we must compute the sums of ideals in the minimal primary decomposition of the defining ideal of the arrangement so it can be done using Gröbner basis over R. For each irreducible component $T_{X_p}^* X$ of the characteristic variety $Ch(\widetilde{M}^{\log f})$ we produce a generic point and a linear variety Y containing this point that is transversal to $T_{X_p}^* X$ using `Maple`. After a suitable change of variables that translates the generic point to the origin we compute the multiplicity of $\mathrm{gr}\,(I)$ at the origin using `Maple` or `Singular`.

3. A FAMILY OF EXAMPLES

We include in this Section some examples that can be treated with our method and are not covered by Torrelli's result [24], i.e. they are not the union of a generic hyperplane arrangement with an hyperbolic arrangement. They live in dimension greater than 4 so we do not know weather LCT holds for these kind of examples but we are able to prove that the corresponding annihilator is generated by operators of order 1. We first describe with details the case of the arrangement $\mathcal{A} \subseteq \mathbb{A}_k^5$ defined by the polynomial in $R = k[x, y, z, t, u]$
$$f = xyztu(x + y + z + t + u)(x + y)(z + t).$$

· *Characteristic cycle of R_f:*   For the arrangement of hyperplanes $\mathcal{A}$ we are considering, the characteristic cycle of the localization module could be computed with the script developed by J. Pfeifle in 17.6 seconds in an Intel Pentium IV 3 GHz computer with 500 MB RAM. The components are the conormal bundles relative to the elements of the associated poset $P(\mathcal{A})$ that consists of 8 (resp. 24, 31, 15, 1) elements of height 1 (resp. 2, 3, 4, 5). We edited the output of the script where we get the multiplicities corresponding to each component as follows:

```
COMPONENTS AND MULTIPLICITIES

<x>: 1            <y>: 1           <z>: 1           <t>: 1           <u>: 1

<x+y+z+t+u>: 1    <x+y>:  1        <z+t>: 1

<y,x>: 2          <z,x>: 1         <t,x>: 1         <u,x>: 1         <y+z+t+u,x>: 1

<z+t,x>: 1        <z,y>: 1         <t,y>: 1         <u,y>: 1         <y,x+z+t+u>: 1
```

```
<z+t,y>: 1        <t,z>: 2         <u,z>: 1         <z,x+y+t+u>: 1   <z,x+y>: 1
<u,t>: 1          <t,x+y+z+u>: 1   <t,x+y>: 1       <u,x+y+z+t>: 1   <u,x+y>: 1
<u,z+t>: 1        <z+t+u,x+y>: 1   <z+t,x+y+u>: 1   <z+t,x+y>: 1

<z,y,x>: 2        <t,y,x>: 2       <u,y,x>: 2       <z+t+u,y,x>: 2   <z+t,y,x>: 2
<t,z,x>: 2        <u,z,x>: 1       <z,y+t+u,x>: 1   <u,t,x>: 1       <t,y+z+u,x>: 1
<u,y+z+t,x>: 1    <u,z+t,x>: 1     <z+t,y+u,x>: 1   <t,z,y>: 2       <u,z,y>: 1
<z,y,x+t+u>: 1    <u,t,y>: 1       <t,y,x+z+u>: 1   <u,y,x+z+t>: 1   <u,z+t,y>: 1
<z+t,y,x+u>: 1    <u,t,z>: 2       <t,z,x+y+u>: 2   <t,z,x+y>: 2     <u,z,x+y+t>: 1
<u,z,x+y>: 1      <t+u,z,x+y>: 1   <u,t,x+y+z>: 1   <u,t,x+y>: 2     <t,z+u,x+y>: 1
<u,z+t,x+y>: 3

<t,z,y,x>: 4      <u,z,y,x>: 2     <t+u,z,y,x>: 2   <u,t,y,x>: 2     <t,z+u,y,x>: 2
<u,z+t,y,x>: 6    <u,t,z,x>: 2     <t,z,y+u,x>: 2   <u,z,y+t,x>: 1   <u,t,y+z,x>: 1
<u,t,z,y>0): 2    <t,z,y,x+u>: 2   <u,z,y,x+t>: 1   <u,t,y,x+z>: 1   <u,t,z,x+y>: 6

<u,t,z,y,x>: 12
```

· *Characteristic cycle of* $\widetilde{M}^{\log f}$*:*   First we compute the ideal $Ann_{A_5}^{(1)}(\frac{1}{f})$ using the method in Section 2. We use the following script for `Macaulay 2`

```
i1 : load "D-modules.m2";

i2 : R=QQ[x,y,z,t,u]; W=makeWA R;

i3 : f=x*y*z*t*u*(x+y+z+t+u)*(x+y)*(z+t);

i5 : kernel matrix{{f,diff(x,f),diff(y,f),diff(z,f),diff(t,f),diff(u,f)}};

i6 : matrix{{-1,dx,dy,dz,dt,du}}*gens o5;

i7 : Ann1=ideal o6;
```

The characteristic ideal of $\widetilde{M}^{\log f}$ is:

```
i8 : charideal:= charIdeal Ann1

o8 = ideal(x*dx+y*dy+z*dz+t*dt+u*du,
   x*u*du+y*u*du+z*u*du+t*u*du+u^2*du,
   z*u*dz+t*u*dt-z*u*du-t*u*du,z*t*dz-z*t*dt,
   x*z*dz+y*z*dz+z^2*dz+x*t*dt+y*t*dt+2*z*t*dt+t^2*dt+z*u*du+t*u*du,
   x*y*dy+y^2*dy+y*z*dz+y*t*dt+y*u*du)
```

The A_5-module $\widetilde{M}^{\log f}$ is holonomic:

```
i9 : dim charideal

o9 = 5
```

For each component of the characteristic variety we have to compute the multiplicity at a generic point. Here we present a sample in a `Maple` session with the computation of the multiplicity at a generic point of the conormal space to the origin $T_0\mathbb{A}_k^5$. For the other components we proceed in an analogous way.

```
> with(Groebner):

> multg:=proc(a,b,c,d,e) local section,hil,mult;

section:=subs(dx=a,dy=b,dz=c,dt=d,du=e,charideal);

section:=`minus`(section,{0});
```

```
hil:=hilbertdim(section,tdeg(x,y,z,t,u));

if hil=0  then mult:=subs(s=1,hilbertseries(section,tdeg(x,y,z,t,u),s))

else error "the point is not generic" end if end;
```

Namely, for a point $(0,0,0,0,0,a,b,c,d,e) \in T_0\mathbb{A}_k^5$ we check out if the point is generic and if it is the case we compute the corresponding multiplicity.

```
> multg(1,2,3,4,5);
                              12
> multg(1,1,1,1,1);
Error, (in multg) the point is not generic

> multg(3,5,4,7,11);
                              12
```

Computing the multiplicities of each component we check the coincidence between the characteristic cycles of $\widetilde{M}^{\log f}$ and R_f so we conclude that $Ann_{A_n}(\frac{1}{f})$ is generated by operators of order 1 by Theorem 2.6. We should point out the following short cut to the method for this particular example

Claim.- In order to compare the characteristic cycles of $\widetilde{M}^{\log f}$ and R_f for the polynomial $f = xyztu(x+y+z+t+u)(x+y)(z+t)$ it is enough to compare the multiplicities of both A_5-modules at a generic point of the conormal space to the origin $T_0\mathbb{A}_k^5$.

Proof.- When considering a generic point in a component of the characteristic variety having the coordinate $x \neq 0$ (resp. $y \neq 0$, $z \neq 0$, $t \neq 0$, $u \neq 0$) we can restrict ourselves to study the arrangement $yztu(z+t)$ (resp. $xztu(z+t)$, $xytu(x+y)$, $xyzu(x+y)$, $xyzt(x+y)(z+t)$).

These arrangements are the union of a generic and an hyperbolic arrangement so, by the results in [24] and Theorem 2.6, the multiplicities of the characteristic cycle must coincide outside the origin. $\qquad\square$

To end this Section we want to remark that, when trying to test the capabilities of our method, we have been also able to treat (among others) the following examples:

In $\mathbb{A}_k^6$:

$f = xyztuv(x+y)(x+y+z+t+u+v)$
$f = xyztuv(x+y)(z+t+u)(x+y+z+t+u+v)$

In $\mathbb{A}_k^7$:

$f = xyztuvw(x+y)(x+y+z+t+u+v+w)$
$f = xyztuvw(x+y)(z+t+u+v)(x+y+z+t+u+v)$.

In $\mathbb{A}_k^8$:

$f = xyztuvws(x+y+z+t+u+v+w+s)(x+y)(z+t+u)$
$f = xyztuvws(x+y+z+t+u+v+w+s)(x+y+z)(t+u+v+w)$

4. Orlik-Terao's hyperplane arrangement

Our test example throughout this Section is going to be the hyperplane arrangement $\mathcal{A} \subseteq \mathbb{A}_k^4$ considered by P. Orlik and H. Terao in [18] defined by the polynomial in $R = k[x,y,z,t]$

$f = xyzt(x+y)(x+z)(x+t)(y+z)(y+t)(z+t)(x+y+z)(x+y+t)(x+z+t)(y+z+t)(x+y+z+t).$

This is the smallest known example of a non-tame arrangement however it satisfies Logarithmic Comparison Theorem since this is a central arrangement in dimension 4 [26]. We want to support Torrelli's conjecture proving that $Ann_{A_n}(\frac{1}{f})$ is generated by operators of order 1. Notice that this example is not a generic arrangement or the union of a generic and an hyperbolic arrangement so it does not fall into the set of examples considered in [24].

· *Characteristic cycle of R_f:* For the arrangement of hyperplanes $\mathcal{A}$ we are considering, the characteristic cycle of the localization module could be computed in 2 minutes and 32 seconds. The components are the conormal bundles relative to the elements of the associated poset $P(\mathcal{A})$ that consists of 15 (resp. 55, 45, 1) elements of height 1 (resp. 2, 3, 4). We edited the output of the script where we get the multiplicities corresponding to each component as follows:

```
COMPONENTS AND MULTIPLICITIES
<x>:1              <y>:1              <z>:1              <t>:1              <x+y>:1
<x+z>:1            <x+t>:1            <y+z>:1            <y+t>:1            <z+t>:1
<x+y+z>:1          <x+y+t>:1          <x+z+t>:1          <y+z+t>:1          <x+y+z+t>:1


<y,x>:2            <z,x>:2            <t,x>:2            <y+z,x>:2          <y+t,x>:2
<z+t,x>:2          <y+z+t,x>:2        <z,y>:2            <t,y>:2            <y,x+z>:2
<y,x+t>:2          <z+t,y>:2          <y,x+z+t>:2        <t,z>:2            <z,x+y>:2
<z,x+t>:2          <z,y+t>:2          <z,x+y+t>:2        <t,x+y>:2          <t,x+z>:2
<t,y+z>:2          <t,x+y+z>:2        <y-z,x+z>:1        <y-t,x+t>:1        <y+z,x-z>:1
<y+t,x-t>:1        <z+t,x+y>:2        <y-z-t,x+z+t>:1    <y+z+t,x-z-t>:1    <z-t,x+t>:1
<y+z,x+z>:1        <y+t,x+z>:2        <z+t,x-t>:1        <y-z+t,x+z>:1      <y+z+t,x+z>:1
<y+z,x+t>:2        <y+t,x+t>:1        <z+t,x+t>:1        <y+z-t,x+t>:1      <y+z+t,x+t>:1
<z-t,y+t>:1        <z+t,y-t>:1        <y+z,x-z+t>:1      <y+z,x+z+t>:1      <z+t,y+t>:1
<y+t,x+z-t>:1      <y+t,x+z+t>:1      <z+t,x+y-t>:1      <z+t,x+y+t>:1      <z-t,x+y+t>:1
<y-t,x+z+t>:1      <y+z+t,x-t>:1      <y-z,x+z+t>:1      <y+z+t,x-z>:1      <y+z+t,x+z+t>:1


<z,y,x>:9          <t,y,x>:9          <z+t,y,x>:9        <t,z,x>:9          <z,y+t,x>:9
<t,y+z,x>:9        <z-t,y+t,x>:4      <z+t,y-t,x>:4      <z+t,y+t,x>:4      <t,z,y>:9
<z,y,x+t>:9        <t,y,x+z>:9        <z-t,y,x+t>:4      <z+t,y,x-t>:4      <z+t,y,x+t>:4
<t,z,x+y>:9        <z,y-t,x+t>:4      <z,y+t,x-t>:4      <z,y+t,x+t>:4      <t,y-z,x+z>:4
<t,y+z,x-z>:4      <t,y+z,x+z>:4      <z-t,y-t,x+t>:1    <z+t,y+t,x-t>:4    <2z+t,2y+t,2x-t>:1
<z+t,y-t,x+t>:4    <z+2t,y-t,x+t>:1   <z-t,y+t,x-t>:1    <2z+t,2y-t,2x+t>:1 <z+2t,y+t,x-t>:1
<z-t,y+t,x+t>:4    <z-t,y+2t,x+t>:1   <z+t,y-t,x-t>:1    <2z-t,2y+t,2x+t>:1 <z+t,y+2t,x-t>:1
<z+t,y+t,x+t>:1    <z-2t,y+t,x+t>:1   <z+t,y-2t,x+t>:1   <z-t,y+t,x+2t>:1   <z+t,y-t,x+2t>:1
<z+t,y+t,x-2t>:1   <z-t,y-t,x+2t>:1   <z-t,y+2t,x-t>:1   <z+2t,y-t,x-t>:1   <2z+t,2y+t,2x+t>:1


<t,z,y,x>:104
```

· *Characteristic cycle of $\widetilde{M}^{\log f}$:* First we compute the ideal $Ann_{A_4}^{(1)}(\frac{1}{f})$ using the same script for `Macaulay 2` we used in the previous example.

The output of the script where we get the characteristic ideal of $Ann^{(1)}(\frac{1}{f})$ is too big to be edited here (the output file has a size of 7 KB) but we can check out that $\widetilde{M}^{\log f}$ is an holonomic module. As in the previous example, a tedious but straightforward application of our method allow us to compute the multiplicity of each component of the characteristic variety at a generic point. However it is worthwhile to point out the following short cut for this particular example

Claim.- In order to compare the characteristic cycles of $\widetilde{M}^{\log f}$ and R_f for the Orlik-Terao's polynomial it is enough to compare the multiplicities of both A_4-modules at a generic point of the conormal space to the origin $T_0 \mathbb{A}_k^4$.

Proof.- From the computation of $Der(-\log f)$ we deduce that the divisor $\{f = 0\}$ is free at any point different from the origin since there is a set of four logarithmic vector fields $\{\delta_1, \ldots, \delta_4\}$ of the form $\delta_i = \sum_j a_{ij} \partial_j$ such that the determinant of the matrix (a_{ij}) equals $(z+t)f$. By symmetry there is a set of four logarithmic vector fields such that the corresponding determinant equals xf. The same is also true for yf, zf and tf. By Saito's criterion [20] the divisor $\{f = 0\}$ is free outside the origin.

By [10] the equality $Ann^{(1)}(\frac{1}{h}) = Ann(\frac{1}{h})$ holds for any free arrangement $h = 0$ in $\mathbb{C}^n$. So, in our case, the surjective morphism

$$\widetilde{M}^{\log f} \longrightarrow R_f \longrightarrow 0$$

is an isomorphism when restricted to the neighborhood of any point $a \in \mathbb{C}^4$ different from the origin.

Notice that we have been using the field of complex numbers $\mathbb{C}$ to apply the results in [20] and [10]. Since Orlik-Terao's arrangement is in fact defined over the field of rational numbers $\mathbb{Q}$ we can extend this result to any field of characteristic zero by flat base change.   □

We applied the method explained in the previous example. The generic multiplicity of $T_0 \mathbb{C}^4$ in $Ch(\widetilde{M}^{\log f})$ is 104. The characteristic cycles of $\widetilde{M}^{\log f}$ and R_f coincide so we conclude that $Ann_{A_n}(\frac{1}{f})$ is generated by operators of order 1 by Theorem 2.6.

5. EXTENDING TO LARGE EXAMPLES

Let $f \in R = k[x_1, \ldots, x_n]$ be the defining polynomial of an arrangement of hyperplanes $\mathcal{A} \subseteq \mathbb{A}_k^n$. Assume that $Ann_{A_n}(\frac{1}{f})$ is generated by operators of order 1. Our aim in this section is to use our method to prove that the arrangement of hyperplanes $\mathcal{A}' \subseteq \mathbb{A}_k^{n+1}$ defined by the polynomial $f \cdot t \in R' := k[x_1, \ldots, x_n, t]$ satisfies the same property.

Let A_{n+1} and A_n be the rings of differential operators corresponding to R' and R respectively. We will also denote the corresponding affine spaces $X' = \mathbb{A}_k^{n+1}$ and $X = \mathbb{A}_k^n$.

· *Characteristic cycle of $R'_{f \cdot t}$:* The characteristic cycle of the A_{n+1}-module $R'_{f \cdot t}$ can be deduced from the characteristic cycle of A_n-module R_f using the formula given by Briançon-Maisonobe-Merle in [5]

Theorem 5.1. ([5, Thm. 3.4.2]) *Let M be a regular holonomic A_n-module with characteristic cycle $ChC(M) = \sum_i m_i \ T_{X_i}^* X$ and let $g \in R$ be a polynomial. Then*

$$ChC(M_g) = \sum_{g(X_i) \neq 0} m_i (\Gamma_i + T_{X_i}^* X)$$

with $\Gamma_i = \sum_j m_{ij} \Gamma_{ij}$, where Γ_{ij} are the irreducible components of multiplicity m_{ij} of the divisor defined by g in $T_{g|_{X_i}}^*$.

Corollary 5.2. *Assume that $ChC(R_f) = \sum_i m_i T_{X_i}^* X$. Then we have*

$$ChC(R'_{f \cdot t}) = \sum_i m_i \left(T_{X_i}^* X' + T_{X_i'}^* X' \right)$$

where we are considering X_i as a subvariety of X' and $X_i' = X_i \cap \{t = 0\}$.

Proof.- Notice that, in order to apply Briançon-Maisonobe-Merle's formula, we can reduce to the case where the characteristic cycle of the module M only has one component.

Let M be an A_n-module such that $ChC(M) = T^*_{X_i}X$ and $M' := M \otimes_R R'$ the corresponding A_{n+1}-module whose characteristic cycle is $ChC(M') = T^*_{X_i}X'$. Then we have $ChC(M'_t) = \Gamma_i + T^*_{X_i}X'$, where Γ_i is computed according to the formula above. Notice that $\Gamma_i = \sum_j m_{ij}\Gamma_{ij}$ is nothing but the characteristic cycle of the local cohomology module $H^1_{(t)}(M')$ so it may be computed using direct images.

Let M be a A_n-module. The direct image corresponding to the injection $i : \mathbb{A}^n_k \to \mathbb{A}^{n+1}_k$ is the A_{n+1}-module $i_+(M)$ defined as

$$i_+(M) = k[\partial_t]\widehat{\otimes}_k M = M[\partial_t].$$

The characteristic variety of $i_+(M)$ can be computed from the characteristic variety of M. Namely, we have:

$$Ch(i_+(M)) = \{(\mathbf{x}, 0, \xi, \tau) \mid (\mathbf{x}, \xi) \in Ch(M)\} \subseteq T^*X' := \mathrm{Spec}\,(R'[\xi_1, \ldots, \xi_n, \tau]),$$

where we have considered $Ch(M) \subseteq T^*X := \mathrm{Spec}\,(R[\xi_1, \ldots, \xi_n])$.

For the case of localizations we are considering we have a good behavior with respect to flat base change

$$H^1_{(t)}(M') = H^1_{(t)}(M\widehat{\otimes}_k k[t]) = H^1_{(t)}(k[t])\widehat{\otimes}_k M = (A_t/A_t \cdot (t))\widehat{\otimes}_k M = i_+(M)$$

where A_t is the ring of differential operators corresponding to the polynomial ring $k[t]$. Then

$$ChC(M'_t) = T^*_{X_i}X' + ChC(i_+(M)) = T^*_{X_i}X' + T^*_{X'_i}X',$$

where we are considering X_i as a subvariety of X' and $X'_i = X_i \cap \{t = 0\}$. $\square$

· *Characteristic cycle of $\widetilde{M'}^{\log f \cdot t}$:* The characteristic cycle of the A_{n+1}-module

$$\widetilde{M'}^{\log f \cdot t} := \frac{A_{n+1}}{Ann^{(1)}_{A_{n+1}}(\frac{1}{f \cdot t})}$$

can be deduced from the characteristic cycle of A_n-module $\widetilde{M}^{\log f} := \frac{A_n}{Ann^{(1)}_{A_n}(\frac{1}{f})}$ as follows

Proposition 5.3. *Assume that $ChC(\widetilde{M}^{\log f}) = \sum_i m_i\, T^*_{X_i}X$. Then we have*

$$ChC(\widetilde{M'}^{\log f \cdot t}) = \sum_i m_i\, (T^*_{X_i}X' + T^*_{X'_i}X')$$

where we are considering X_i as a subvariety of X' and $X'_i = X_i \cap \{t = 0\}$.

Proof.- From the equality $Der(-\log(f \cdot t)) = Der(-\log f) \cap Der(-\log t)$ we deduce the equality

$$Ann^{(1)}_{A_{n+1}}(\frac{1}{f \cdot t}) = A_{n+1} \cdot Ann^{(1)}_{A_n}(\frac{1}{f}) + A_{n+1}(t\partial_t + 1).$$

Since the variables t and ∂_t commute with x_i and ∂_i for $i = 1, \ldots, n$ we also have

$$\mathrm{gr}\,(Ann^{(1)}_{A_{n+1}}(\frac{1}{f \cdot t})) = \mathrm{gr}\,(A_{n+1}) \cdot \mathrm{gr}\,(Ann^{(1)}_{A_n}(\frac{1}{f})) + \mathrm{gr}\,(A_{n+1})(t\tau)$$

Then, from the above equality on graded ideals we get the desired result. $\square$

From the results above we get

Theorem 5.4. *Let $f \in k[x_1, \ldots, x_n]$ be the defining polynomial of an arrangement of hyperplanes $\mathcal{A} \subseteq \mathbb{A}_k^n$ such that $Ann_{A_n}(\frac{1}{f})$ is generated by operators of order 1. Then $Ann_{A_{n+m}}(\frac{1}{f \cdot t_1 \cdots t_m})$ is also generated by operators of order 1 for all $m \in \mathbb{N}$, where $f \cdot t_1 \cdots t_m \in k[x_1, \ldots, x_n, t_1, \ldots, t_m]$.*

5.1. **Extending Orlik-Terao's example.** Any arrangement of hyperplanes $\mathcal{A} \subseteq \mathbb{A}_k^n$ that contains Orlik-Terao's example as subarrangement is non-tame by [26, T2] so it is easy to find examples of non-tame arrangements in any dimension just adding new factors to the defining polynomial of Orlik-Terao's example. Even though it is not known wether LCT holds for this kind of examples in dimension greater than 4 (greater than 5 over $\mathbb{C}$) we are able to prove that the corresponding annihilator is generated by operators of order 1. By Theorem 5.4 we may extend the example given by Orlik-Terao's arrangement to higher dimension just adding new variables.

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