Finite-dimensional global attractors in Banach spaces

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Abstract

We provide bounds on the upper box-counting dimension of negatively invariant subsets of Banach spaces, a problem that is easily reduced to covering the image of the unit ball under a linear map by a collection of balls of smaller radius. As an application of the abstract theory we show that the global attractors of a very broad class of parabolic partial differential equations (semilinear equations in Banach spaces) are finite-dimensional.

Keywords: Global attractors, Negatively invariant sets, Box-counting dimension, Banach–Mazur distance, Auerbach basis

1. Introduction

There are now many techniques available for bounding the (box-counting) dimension of the global attractors that have been shown to exist for a variety of interesting models of mathematical physics (for many examples see Temam, 1988).

To our knowledge, the earliest such result is due to Mallet-Paret (1976), who showed that if K is a compact subset of a Hilbert space H, $f : H \to H$ is a continuously differentiable map, $f(K) \supset K$ ('K is negatively invariant'), and the derivative of f is everywhere equal to the sum of a compact map and a contraction, then the upper box-counting dimension of K is finite ($\dim_B(K) < \infty$). This method was subsequently generalised by Mañé (1981) to treat subsets of Banach spaces.

Both these methods rely on approximating the image of any ball B under f as the sum of a ball in a finite-dimensional subspace of X plus a 'small' error. Douady & Oesterlé (1980) developed a method for subsets of $\mathbb{R}^n$ that instead approximates $f(B)$ by an ellipse, plus an error. This approach, extended to subsets of Hilbert spaces by Constantin & Foias (1985), produces significantly improved bounds on the dimension and remains the most powerful to date.

We concentrate here on the case of subsets of Banach spaces, since the Hilbert space theory has been well developed. The problem reduces, as we shall see, to bounding the number of balls required to cover the image of the unit ball under a certain family of linear maps.

We choose to work with the upper box-counting dimension ($\dim_B(\cdot)$) for two main reasons. First, it provides an upper bound on the topological ($\dim_T(\cdot)$) and Hausdorff dimension ($\dim_H(\cdot)$) (see Hurewicz & Wallman (1941) for more on the topological dimension and a proof that $\dim_T(\cdot) \leq \dim_H(\cdot)$; and Falconer (2003) for more on the Hausdorff dimension and a proof that $\dim_H(X) \leq \dim_B(X)$ in general). Secondly, a set with finite upper box-counting dimension can be embedded into a finite-dimensional Euclidean space using a linear map (a result which fails for sets that only have finite Hausdorff dimension) – this was originally proved by Mañé in the same paper as the result which forms the main topic of our work here, but his result has been significantly improved in the Hilbert space case by Foias & Olson (1996) [the inverse is Hölder continuous] and Hunt & Kaloshin (1999) [explicit bounds on the Hölder exponent], and in the Banach space case by Robinson (2009) [a Banach space version of Hunt & Kaloshin's result].

We now give a formal definition of the box-counting dimension. Let X be a Banach space and K a compact subset of X. Define $N_X(K, \epsilon)$ as the minimum number of balls in X of radius ϵ needed to cover K. The (upper) box-counting dimension $\dim_B(K)$ of K is defined by:

$$\dim_B(K) = \limsup_{\epsilon \to 0} \frac{\log N_X(K, \epsilon)}{-\log \epsilon} \tag{1.1}$$

(the lim sup in the definition is necessary; there are sets for which the lim sup is not equal to the lim inf, see for example Mattila, 1995). Essentially this definition extracts the exponent d from the scaling law $N_X(K, \epsilon) \sim \epsilon^{-d}$. More rigorously, $\dim_B(K)$ is the smallest real number such that for any $d > \dim_B(K)$ there exists an $\epsilon_0 > 0$ such that $N_X(K, \epsilon) \leqslant \epsilon^{-d}$ for all $0 < \epsilon < \epsilon_0$.

The arguments that provide bounds on the dimension of attractors all follow similar lines, which we can formalise in the following lemma.

Lemma 1.1. *Let K be a compact subset of a Banach space X, let $f : X \to X$ be continuously differentiable in a neigbourhood of K, and let K be negatively invariant for f, i.e. $f(K) \supseteq K$. Suppose that there exist α, $0 < \alpha < 1$ and $M \geq 1$ such that for any $x \in K$,*

$$N_X(Df(x)[B_X(0,1)], \alpha) \leq M. \tag{1.2}$$

Then

$$\dim_B(K) \leq \frac{\log M}{-\log \alpha}. \tag{1.3}$$

Proof. First, we ensure that (1.2) is sufficient to provide bounds on the number of balls required to cover $f(B_X(x,r))$ when r is small enough. Since f is continuously differentiable and K is compact, for any $\eta > 0$ there exists an $r_0 = r_0(\eta)$ such that for any $0 < r < r_0$ and any $x \in K$,

$$f(B_X(x,r)) \subseteq f(x) + Df(x)[B_X(0,r)] + B_X(0, \eta r),$$

where $A + B$ is used to denote the set $\{a + b : \ a \in A, \ b \in B\}$. It follows that

$$N_X(f[B_X(x,r)], (\alpha + \eta)r) \leq M \tag{1.4}$$

for all $r \leq r_0(\eta)$.

Now fix η with $0 < \eta < 1 - \alpha$, and let $r_0 = r_0(\eta)$. Cover K with $N_X(K, r_0)$ balls of radius r_0. Apply f to every element of this cover. Since $f(K) \supseteq K$, this provides a new cover of K formed by sets all of the form $f(B_X(x, r_0))$, for some $x \in K$. It follows from (1.4) that each of these images can be covered by M balls of radius $(\alpha + \eta)r_0$, ensuring that $N_X(K, (\alpha + \eta)r_0) \leq M N_X(K, r_0)$. Applying this argument k times implies that

$$N_X(K, (\alpha + \eta)^k r_0) \leq M^k(K, r_0).$$

It follows from the definition of $\dim_B(K)$ that

$$\dim_B(K) \leq \frac{\log M}{-\log(\alpha + \eta)},$$

and since $\eta > 0$ we arbitrary we obtain (1.3). □

The key to applying this approach is to be able to prove (1.2), i.e. to find a way of estimating the number of balls of radius α required to cover

$Df(x)B(0,1)$. Our argument is based (as was that of Mallet-Paret and Mañé) on the fact that one can find a sharp estimate for coverings by balls in $(\mathbb{K}^n, \ell^\infty)$, and then use an isomophism T between $(\mathbb{K}^n, \ell^\infty)$ and an n-dimensional linear subspace U of X to relate coverings with respect to these two different norms. We observe here that only the product $\|T\|\|T^{-1}\|$ occurs in these estimates; this shows that covering results are related to the Banach–Mazur distance between $(U, \|\cdot\|)$ and $(\mathbb{K}^n, \ell^\infty)$, which is bounded by $\log n$ independently of U. Although this follows from the powerful general result that $d_{\rm BM}(X,Y) \le \log n$ for any two n-dimensional normed spaces, we give a simple proof of the particular result that we require here.

We use our covering results to prove that a negatively invariant compact set for a nonlinear map with a derivative that is a sum of a strong contraction with a compact map has finite box-counting dimension. Our proof is much simpler than that of Mañé and our bound on the dimension in a Banach space improves on his.

In Section 3 we derive interesting corollaries of the main theorem, and consider some applications. In particular, we show that the global attractors of a wide class of evolution equations are finite dimensional.

2. Coverings in Banach spaces

In this section we provide a bound of the form

$$N_X(T[B_X(0,1)], \alpha) \le M$$

for linear maps T that are the sum of a compact map and a contraction. When $f : X \to X$ is such that $Df(x)$ is of this form for all $x \in K$, and the contraction constant is uniformly bounded over K, we show that (1.2) holds uniformly over K.

2.1. The Banach–Mazur distance and coverings of balls in finite-dimensional subspaces

The key result concerns coverings of a ball in a finite-dimensional subspace of X by balls of smaller radius. We use ideas related to the Banach–Mazur distance to provide the estimate we need.

2.1.1. The Banach–Mazur distance

Let X and Y be normed spaces. If there exists $T \in \mathscr{L}(X,Y)$ that is bijective and has $T^{-1} \in \mathscr{L}(Y,X)$, we say that X and Y are isomorphic and that T is an isomorphism between X and Y. The Banach-Mazur distance between two isomorphic normed spaces X and Y, $d_{\mathrm{BM}}(X,Y)$, is defined as

$$\log\left(\inf\left\{\|T\|_{\mathscr{L}(X,Y)}\|T^{-1}\|_{\mathscr{L}(Y,X)}:\ T \in \mathscr{L}(X,Y),\ T^{-1} \in \mathscr{L}(Y,X)\right\}\right).$$

Clearly $d_{\mathrm{BM}}(X,Y) = 0$ if and only if X and Y are isometrically isomorphic; in particular this is the case for any two separable Hilbert spaces of the same cardinality.

It is a consequence of John's Theorem on bounding ellipses of minimal volume (John, 1948) that for any two n-dimensional real Banach spaces U and V, $d_{\mathrm{BM}}(U,V) \le \log n$, see Bollobás (1990, Theorem 4.15). Here we give a simple proof that if U is any n-dimensional normed space over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), $d_{\mathrm{BM}}(U, \mathbb{K}^n_\infty) \le \log n$, where $\mathbb{K}^n_\infty$ denotes $\mathbb{K}^n$ equipped with the ℓ^∞ norm: for $\underline{z} \in \mathbb{K}^n$ with $\underline{z} = (z_1, \dots, z_n)$, $z_j \in \mathbb{K}$, we define

$$\|\underline{z}\|_\infty = \max_{j=1,\dots,n} \|z_j\|_{\mathbb{K}}.$$

In order to show that $d_{\mathrm{BM}}(U, \mathbb{K}^n_\infty) \le \log n$ we will use an Auerbach basis for X. The proof of the existence of such a basis when X is real is standard (see Bollobás, 1990, Theorem 4.13, for example); we give a proof of the complex case in an appendix.

Lemma 2.1. *Let X be a n-dimensional normed vector space (which may be real or complex). Then, there exists a basis $B = \{x_1, \cdots, x_n\}$ for X and a basis $B^* = \{f_1, \cdots, f_n\}$ for X^* with $\|x_i\|_X = \|f_i\|_{X^*} = 1$ ($i = 1, \dots, n$) such that $f_i(x_j) = \delta_{ij}$, $i, j = 1, \dots, n$.*

Given this lemma the proof of the following proposition is straightforward.

Proposition 2.2. *Let U be an n-dimensional Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $d_{\mathrm{BM}}(U, \mathbb{K}^n_\infty) \le \log n$.*

Note that Mañé's paper contains a proof of the bound $d_{\mathrm{BM}}(U, \mathbb{R}^n_\infty) \le \log(n2^n)$ when U is real.

Proof. Let $\{x_1, \dots, x_n\}$ be an Auerbach basis for U, and $\{f_1, \dots, f_n\}$ the corresponding basis for U^*. Define a map $J : \mathbb{K}^n_\infty \to U$ by setting

$$J(\underline{z}) = \sum_{j=1}^n z_j x_j.$$

Then

$$\|J(\underline{z})\|_X = \left\| \sum_{j=1}^n z_j x_j \right\|_X \leq \sum_{j=1}^n |z_j| \leq n \|\underline{z}\|_\infty,$$

and so

$$\|J\|_{\mathscr{L}(\mathbb{K}^n_\infty, U)} \leq n.$$

On the other hand, if $x = \sum_{j=1}^n z_j x_j \in U$ with $\|x\|_X \leq 1$ then since $z_j = f_j(x)$,

$$\|J^{-1}(x)\|_\infty = \|\underline{z}\|_\infty = \max_{j=1,\dots,n} |z_j| = \max_{j=1,\dots,n} |f_j(x)| \leqslant \|x\|_X,$$

which implies that

$$\|J^{-1}\|_{\mathscr{L}(X, X_B^\infty)} \leqslant 1.$$

□

2.1.2. The covering lemma

We use this result to prove our covering lemma:

Lemma 2.3. *If U is an n-dimensional subspace of a real Banach space X, then*

$$N_X(B_U(0,r), \rho) \leq (n+1)^n \left(\frac{r}{\rho}\right)^n \qquad 0 < \rho \leq r, \tag{2.5}$$

where the balls in the cover can be taken to have centres in U. The same result holds in a complex Banach space if one replaces the right-hand side of (2.5) with its square.

Proof. Assume first that $\mathbb{K} = \mathbb{R}$. Since U and $\mathbb{R}^n_\infty$ are n-dimensional, $d_{\rm BM}(U, \mathbb{R}^n_\infty) \leq \log n$: in particular, there exists a linear isomorphism $T : \mathbb{R}^n_\infty \to U$ such that $\|T\|\|T\|^{-1} \leq n$. Since

$$B_U(0,r) = TT^{-1}(B_U(0,r)) \subseteq T(B_{\mathbb{R}^n_\infty}(0, \|T^{-1}\|r)),$$

and $B_{\mathbb{R}^n_\infty}(0, \|T^{-1}\|\, r)$ can be covered by

$$\left(1 + \frac{\|T^{-1}\|r}{\rho/\|T\|}\right)^n = \left(1 + \|T\|\|T^{-1}\|\,\frac{r}{\rho}\right)^n \leq \left(1 + n\,\frac{r}{\rho}\right)^n \leq (n+1)^n \left(\frac{r}{\rho}\right)^n$$

balls in $\mathbb{R}^n_\infty$ of radius $\rho/\|T\|$, it follows that $B_U(0,r)$ can be covered by the same number of U-balls of radius ρ. If X is complex one requires $(1+(a/b))^{2n}$ b-balls in $\mathbb{C}^n_\infty$ to cover a ball of radius a. □

2.2. Coverings of $T[B_X(0,1)]$ via finite-dimensional approximations

We now have good estimates for the coverings of balls in finite-dimensional linear subspaces, but we want to cover the images of balls under linear maps. In order to do this we show that, given a linear map T that is the sum of a compact map and a contraction, $T[B_X(0,1)]$ can be well-approximated by $T[B_Z(0,1)]$, where Z is a finite-dimensional subspace of X.

We denote by $\mathscr{L}(X)$ the space of bounded linear transformations from X into itself, by $\mathcal{K}(X)$ the closed subspace of $\mathscr{L}(X)$ consisting of all compact linear transformations from X into itself, and define

$$\mathscr{L}_\lambda(X) = \left\{T \in \mathscr{L}(X) : T = L + C, \text{ with } C \in \mathcal{K}(X) \text{ and } \|L\|_{\mathscr{L}(X)} < \lambda\right\}. \tag{2.6}$$

By $\operatorname{dist}(A, B)$ we denote the Hausdorff semi-distance between A and B,

$$\operatorname{dist}(A, B) = \sup_{a\in A}\left(\inf_{b\in B}\|a - b\|_X\right).$$

Lemma 2.4. *Let X be a Banach space and $T \in \mathscr{L}_{\lambda/2}(X)$. Then there exists a finite-dimensional subspace Z of X such that*

$$\operatorname{dist}(T[B_X(0,1)], T[B_Z(0,1)]) < \lambda. \tag{2.7}$$

We denote by $\nu_\lambda(T)$ the minimum $n \in \mathbb{N}$ such that (2.7) holds for some n-dimensional subspace of X.

Proof. Write $T = L + C$, where $C \in \mathcal{K}(X)$ and $L \in \mathscr{L}(X)$ with $\|L\|_{\mathscr{L}(X)} < \lambda/2$. We show first that for any $\epsilon > 0$ there is a finite-dimensional subspace Z such that

$$\operatorname{dist}(C[B_X(0,1)], C[B_Z(0,1)]) < \epsilon. \tag{2.8}$$

Suppose that this is not the case. Choose some $x_1 \in X$ with $\|x_1\|_X = 1$, and let $Z_1 = \operatorname{span}\{x_1\}$. Then

$$\operatorname{dist}(C[B_X(0,1)], C[B_{Z_1}(0,1)]) \geq \epsilon,$$

and so there exists an $x_2 \in X$ with $\|x_2\|_X = 1$ such that

$$\|Cx_2 - Cx_1\|_X \geqslant \epsilon.$$

With $Z_2 = \operatorname{span}\{x_1, x_2\}$, one can find an x_3 with $\|x_3\|_X = 1$ such that

$$\|Cx_3 - Cx_1\|_X \geqslant \epsilon \quad\text{and}\quad \|Cx_3 - Cx_2\|_X \geqslant \epsilon.$$

Continuing inductively one can construct in this way a sequence $\{x_j\}$ with $\|x_j\| = 1$ such that

$$\|Cx_i - Cx_j\|_X \geq \epsilon \qquad i \neq j,$$

contradicting the compactness of C.

Now let $\tilde{\lambda} < \lambda$ be such that $2\|L\|_{\mathscr{L}(X)} < \tilde{\lambda} < \lambda$, and choose Z using the above argument so that

$$\operatorname{dist}(C[B_X(0,1)], C[B_Z(0,1)]) < \lambda - \tilde{\lambda}.$$

If $x \in B_X(0,1)$ and $z \in B_Z(0,1)$, then

$$\|Tx - Tz\|_X \leq \|L(x-z)\|_X + \|Cx - Cz\|_X \leq \tilde{\lambda} + \|Cx - Cz\|_X$$

Hence,

$$\begin{aligned}\operatorname{dist}(T[B_X(0,1)], T[B_Z(0,1)]) &\leqslant \tilde{\lambda} + \operatorname{dist}(C[B_X(0,1)], C[B_Z(0,1)]) \\ &< \lambda.\end{aligned}$$

This completes the proof. ☐

2.3. Uniform estimates for $x \in K$

Theorem 2.5 (after Mañé, 1981)**.** *Let X be a Banach space, $U \subset X$ an open set, and $f : U \to X$ a continuously differentiable map. Suppose that K is a compact set and assume that for some λ with $0 < \lambda < \frac{1}{2}$,*

$$Df(x) \in \mathscr{L}_{\lambda/2}(X) \qquad \text{for all} \quad x \in K.$$

Then $n = \sup_{x \in K} \nu_\lambda(Df(x))$ and $D = \sup_{x \in K} \|Df(x)\|$ are finite, and

$$N(Df(x)[B_X(0,1)], 2\lambda) \leqslant \left[(n+1)\frac{D}{\lambda}\right]^{\alpha n} \qquad \text{for all} \quad x \in K, \tag{2.9}$$

where $\alpha = 1$ if X is real and $\alpha = 2$ if X is complex. It follows that

$$\dim_B(K) \leqslant \alpha n \left\{\frac{\log((n+1)D/\lambda)}{-\log(2\lambda)}\right\}, \tag{2.10}$$

Proof. First we show that $n = \sup_{x\in K} \nu_\lambda(Df(x))$ is finite. For each $x \in K$, there exists a finite-dimensional linear subspace Z_x such that

$$\operatorname{dist}(Df(x)[B_X(0,1)], Df(x)[B_{Z_x}(0,1)]) < \lambda.$$

Since $Df(\cdot)$ is continuous, it follows that there exists a $\delta_x > 0$ such that

$$\operatorname{dist}(Df(y)[B_X(0,1)], Df(y)[B_{Z_x}(0,1)]) < \lambda$$

for all $y \in B_X(x, \delta_x)$, i.e. $\nu_\lambda(y) \le \nu_\lambda(x)$ for all such y. The open cover of K formed by the union of $B_X(x, \delta_x)$ over x has a finite subcover, from whence it follows that $n < \infty$.

Now, since $n = \sup_{x\in K} \nu_\lambda(Df(x)) < \infty$, for each $x \in K$ there is a subspace Z_x of X with $\dim(Z_x) \leqslant n$ such that

$$\operatorname{dist}(Df(x)[B_X(0,1)], Df(x)[B_{Z_x}(0,1)]) < \lambda.$$

For ease of notation we now drop the x subscript on Z_x, and write $T = Df(x)$.

Noting that $T(Z)$ is also an n-dimensional subspace of X, one can use Lemma 2.3 to cover the ball $B_{T(Z)}(0, \|T\|)$ with balls $B_X(y_i, \lambda)$, $1 \leqslant i \leqslant k$, such that $y_i \in B_X(0, \|T\|)$ for each i and

$$k \leqslant \left[(n+1)\frac{\|T\|}{\lambda}\right]^{\alpha n}.$$

Thus

$$T[B_Z(0,1)] \subseteq B_{T(Z)}(0, \|T\|) = B_X(0, \|T\|) \cap T(Z) \subseteq \bigcup_{i=1}^{k} B_X(y_i, \lambda). \quad (2.11)$$

We complete the proof by showing that

$$\bigcup_{i=1}^{k} B_X(y_i, 2\lambda) \supseteq T[B_X(0,1)].$$

Indeed, if $x \in B_X(0,1)$ then it follows from (2.7) that there is a $y \in T[B_Z(0,1)]$ such that $\|Tx - y\|_X < \lambda$. Since $y \in T[B_Z(0,1)]$, it follows from (2.11) that $\|y - y_i\|_X \leqslant \lambda$ for some $i \in \{1, \dots, k\}$, and so

$$\|Tx - y_i\|_X \leqslant \|Tx - y\|_X + \|y - x_i\|_X < 2\lambda,$$

i.e. $x \in B_X(y_i, 2\lambda)$.

The result now follows as stated since n is uniform over $x \in K$. $\square$

Some immediate improvement of the above is possible if we work in a Hilbert space. A result due to Cheypzhov & Vishik (2002) (Lemma III.2.1) shows that in $\mathbb{R}^n$ we need no more than $4^n(r/\rho)^n$ balls of radius ρ to cover a ball of radius $r > \rho$. This implies that one can replace the factor $(n+1)^n$ in (2.5) by 7^n, and obtain a corresponding improvement in the bound of Theorem 2.5. We do not pursue this direction further here, since the use of ellipses in place of balls (an approach initiated by Douady & Oesterlé, 1980, and developed further by Constantin & Foias, 1985) leads to significant improvements on the possible dimension estimates and is now the standard approach.

3. Corollaries & Applications

3.1. When $Df \in \mathscr{L}_1(X)$

The following corollary can be found in Hale, Maghalaẽs, & Oliva (2002):

Corollary 3.1. *Suppose that X is a Banach space, $U \subset X$ an open set, and $f : U \to X$ a continuously differentiable map. Suppose that $K \subset U$ is a compact set such that $f(K) \supseteq K$, and that $Df(x) \in \mathscr{L}_1(X)$ for all $x \in K$. Then* $\dim_B(K) < \infty$.

Proof. It follows from an argument similar to that used in the proof of Theorem 2.5 to show that $n < \infty$ that in fact there exists an $\alpha < 1$ such that $Df(x) \in \mathscr{L}_\alpha(X)$ for all $x \in K$. Note that

$$D[f^p] = Df(f^{p-1}(x)) \circ \cdots \circ Df(x),$$

and that if $C_i \in \mathcal{K}(X)$ and $L_i \in \mathscr{L}(X)$, $i = 1, 2$, then

$$(C_1 + L_1) \circ (C_2 + L_2) = \underbrace{[C_1 \circ C_2 + C_1 \circ L_2 + L_1 \circ C_2]}_{\in\ \mathcal{K}(X)} + L_1 \circ L_2,$$

it follows that if $Df(x) \in \mathscr{L}_\alpha(X)$ with $\alpha < 1$ then $[D(f^p)](x) \in \mathscr{L}_{\alpha^p}(X)$. It follows that for p large enough, $D(f^p)(x) \in \mathscr{L}_\lambda$ for some $\lambda < 1/4$, for every $x \in K$. One can now apply Theorem 2.5 to f^p in place of f (noting that $f^p(K) \supseteq K$) to deduce that $d_f(K) < \infty$. $\square$

3.2. When D_xT has finite rank

Corollary 3.2. *Let X be a Banach space and assume that $T \in \mathcal{C}^1(X)$, K is a compact set such that $T(K) = K$, and D_xT has finite rank $\nu(x)$ with $\sup_{x\in K} \nu(x) := \nu < \infty$. Then,*

$$\dim_B(K) \leqslant \nu.$$

Proof. Clearly, for each $\lambda > 0$ and $x \in K$, $D_xT \in \mathscr{L}_{\frac{\lambda}{2}}(X)$ for all $\lambda > 0$. Consequently, for each $0 < \lambda < \frac{1}{2}$,

$$\dim_B(K) \leqslant \nu \frac{\log\left((\nu+1)\frac{D}{\lambda}\right)}{\log(1/2\lambda)}.$$

Taking the limit as $\lambda \to 0$ we have that $\dim_B(K) \leqslant \nu$. □

3.3. An ordinary differential equation

Corollary 3.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function. Assume that the semigroup $\{S(t) : t \geqslant 0\}$ in $\mathbb{R}^n$ associated to the ordinary differential equation*

$$\dot{x} = f(x) \qquad x(0) = x_0 \in \mathbb{R}^n.$$

has a global attractor $\mathcal{A}$. If $\operatorname{rank}(D_xf) \leqslant k \leqslant n$ for all $x \in \mathcal{A}$, then $\dim_B(\mathcal{A}) \leqslant k$.

In particular, if $f : \mathbb{R}^k \to \mathbb{R}^k$, $\beta > 0$ and there is a constant $M > 0$ such that $f(x) \cdot x < 0$ for $\|x\|_{\mathbb{R}^k} \geqslant M$, then the semigroup $\{S(t) : t \geqslant 0\}$ associated to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & -\beta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$
$$\begin{pmatrix} x \\ y \end{pmatrix}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

has a global attractor $\mathcal{A}$ in $\mathbb{R}^k \times \mathbb{R}^k$ with $\dim_B(\mathcal{A}) \leqslant k$.

Corollary 3.4. *Let $A : D(A) \subset X \to X$ be a sectorial operator with* $\mathrm{Re}\sigma(A) > 0$. *If $f : X^\alpha \to X$ is continuously differentiable and Lipschitz continuous in bounded subsets of X^α and the semigroup $\{S(t) : t \geqslant 0\}$ in X^α associated to the abstract parabolic problem*

$$\dot{x} + Ax = f(x) \qquad \text{with} \qquad x(0) = x_0 \in X^\alpha$$

has a global attractor $\mathcal{A}$ and either e^{-At} is compact for each $t > 0$ or $f_x \in \mathcal{K}(X^\alpha, X)$ is compact for each $x \in \mathcal{A}$, then $\dim_B(\mathcal{A}) < \infty$.

Proof. For $x \in \mathcal{A}$, let

$$S(t)x = \mathrm{e}^{-At}x + \int_0^t \mathrm{e}^{-A(t-s)} f(S(s)x)ds$$

hence, the derivative $S_x(t) \in \mathscr{L}(X^\alpha)$ with respect to x of $S(t)$ at x satisfies

$$S_x(t) = \mathrm{e}^{-At} + \int_0^t \mathrm{e}^{-A(t-s)} f'(S(s)x)S_x(s)ds.$$

Hence, for t suitably large, the hypothesis of Corollary 3.1 are satisfied and the result follows.

□

Now we show how a rough estimate on the dimension of the attractor can be obtained. First note that, if $A : D(A) \subset X \to X$ is a sectorial operator with compact resolvent, X^β, $\beta \geqslant 0$ denotes the fractional power spaces associated to A, $\alpha > 0$, there is a sequence of finite rank projections $\{P_n\}_{n\in\mathbb{N}}$ and sequences of positive real numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{M_n\}_{n\in\mathbb{N}}$ such that

$$\|\mathrm{e}^{-At}(I - P_n)\|_{\mathscr{L}(X^\gamma, X^\beta)} \leqslant M_n t^{-(\beta-\gamma)} \mathrm{e}^{-\lambda_n t},\ t \geqslant 0,\ 0 \leqslant \gamma \leqslant \beta \leqslant \alpha. \quad (3.12)$$

We say that A is an admissible sectorial operator if it is sectorial and there is a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ and $M > 0$ such that (3.12) with $M_n = M$ for all $n \in \mathbb{N}$. It is not difficult to see that, if A is an admissible sectorial operator, then A has compact resolvent.

$$\|S_x(t)\|_{\mathscr{L}(X^\alpha)} \leqslant M + MN \int_0^t (t-s)^{-\alpha} \|S_x(s)\|_{\mathscr{L}(X^\alpha)} ds,$$

where

$$N = \sup\left\{\|f'(x)\|_{\mathscr{L}(X^\alpha,X)} : x\in\mathcal{A}\right\}$$

It follows from the generalized Gronwall inequality that

$$\|S_x(t)\|_{\mathscr{L}(X^\alpha)} \leqslant \frac{\bar{M}}{1-\alpha}\mathrm{e}^{(MN\Gamma(1-\alpha))^{1/(1-\alpha)}\,t}.$$

Now, if $Q_n = (I - P_n)$,

$$\|Q_nT_x(t)\|_{\mathscr{L}(X^\alpha)} \leqslant M\mathrm{e}^{-\lambda_n t} + MN\int_0^t (t-s)^{-\alpha}\mathrm{e}^{-\lambda_n(t-s)}\|S_x(s)\|_{\mathscr{L}(X^\alpha)}ds$$

and

$$\begin{aligned}
&\|Q_nT_x(t)\|_{\mathscr{L}(X^\alpha)}\\
&\leqslant M\mathrm{e}^{-\lambda_n t} + \frac{\bar{M}MN}{1-\alpha}\mathrm{e}^{(MN\Gamma(1-\alpha))^{1/(1-\alpha)}}\int_0^t (t-s)^{-\alpha}\mathrm{e}^{-(\lambda_n+(MN\Gamma(1-\alpha))^{1/(1-\alpha)})(t-s)}ds\\
&\leqslant M\mathrm{e}^{-\lambda_n t} + \frac{\bar{M}MN}{1-\alpha}\mathrm{e}^{(MN\Gamma(1-\alpha))^{1/(1-\alpha)}t}\int_0^t u^{-\alpha}\mathrm{e}^{-(\lambda_n+(MN\Gamma(1-\alpha))^{1/(1-\alpha)})u}du\\
&\leqslant M\mathrm{e}^{-\lambda_n t} + \frac{\bar{M}MN}{1-\alpha}\;\frac{\mathrm{e}^{(MN\Gamma(1-\alpha))^{1/(1-\alpha)}t}\Gamma(1-\alpha)}{\lambda_n+(MN\Gamma(1-\alpha))^{1/(1-\alpha)}} = \Lambda_n(t),\ t\geqslant 0.
\end{aligned}$$

From the admissibility of A and (3.12), $\|Q_n\|_{\mathscr{L}(X^\alpha)} \leqslant M$, for all $n\in\mathbb{N}$. Hence,

$$\|Q_nT_x(t)Q_n\|_{\mathscr{L}(X^\alpha)} \leqslant M\Lambda(t),\ t\geqslant 0.$$

Choose $t = 1$ and $n_0 \in \mathbb{N}$ such that $\Lambda(1) < \lambda < \frac{1}{4}$. If $F = S(1)$, $L = Q_{n_0}S(1)$ and $C = P_{n_0}S(1)$, then $\mathcal{A}$ is invariant under F. Furthermore, $F_x = L_x + C_x$ with $L_x = Q_{n_0}S_x(1)$ and $C_x = P_{n_0}S_x(1)$ and if $Z_x = R(C_x)$ and W_x is a subspace of X such that $C_x : W_x \to Z_x$ is an isomorphism, $F_x \in \mathscr{L}_{\lambda/2}(X)$ for all $x\in\mathcal{A}$ and for some $\lambda < \frac{1}{2}$. In addition,

$$\nu = \sup_{x\in\mathcal{A}}\dim(Z_x) \leqslant \dim(R(P)).$$

This proves that all the assumptions of Theorem 2.5 are satisfied and consequently

$$\dim_B(\mathcal{A}) \leqslant \nu\frac{\log\left((\nu+1)\,\frac{D}{\lambda}\right)}{\log(1/2\lambda)} < \infty. \tag{3.13}$$

4. Conclusion

As specific examples of the result in Section 3 (a) we mention the attractor for the damped wave equation with critical exponent (see Arrieta et al (1992)) and as an application of the results in Section 3 (d) we mention the attractors of dissipative parabolic equations in $L^p(\Omega)$, $W^{1,p}(\Omega)$ as in Arrieta et al (2000) or the Navier-Stokes Equation in space dimension 2 as in Temam (1988).

As an example of a problem which does not define a semigroup in a Hilbert space we mention that of Arrieta et al (submitted for publication), for Section 3 (d). For Section 3 (a), the examples are the attractors for functional differential equations for which the natural phase space is not a Hilbert space (see Hale et al., 2002, for example).

Appendix: Auerbach bases in a complex Banach spaces

In this appendix we give a proof of the existence of an Auerbach basis in a complex n-dimensional Banach space. The result is standard in a real Banach space, and we use the real version in our proof.

Lemma 4.1. *Let X be a n-dimensional complex normed vector space. Then, there exists a basis $B = \{x_1, \cdots, x_n\}$ for X and a basis $B^* = \{f_1, \cdots, f_n\}$ for X^* with $\|x_i\|_X = \|f_i\|_{X^*} = 1$ ($i = 1, \ldots, n$) such that $f_i(x_j) = \delta_{ij}$, $i, j = 1, \ldots, n$.*

Proof. Given a basis $B_0 = \{y_1, \cdots, y_n\}$ of X we consider the real space X_2 given as the linear span of over $\mathbb{R}$ of

$$B_2 = \{y_1, \ldots, y_n, \mathrm{i}y_1, \ldots, \mathrm{i}y_n\},$$

equipped with the norm $\|z\|_{X_2} = \|z\|_X$. We now apply the real version of the result to X_2, to produce a basis $\{x_1, \ldots, x_{2n}\}$ for X_2 and $\{\varphi_1, \ldots, \varphi_{2n}\}$ for X_2^* such that $\varphi_j(x_k) = \delta_{jk}$ and $\|\varphi_j\| = \|x_j\| = 1$.

Since $x_j \in X$ and the $\{x_j\}$ must span X over $\mathbb{C}$, we can reorder and relabel the $\{x_j\}$ (and the corresponding $\{\varphi_j\}$) so that $\{x_1, \ldots, x_n\}$ span X over $\mathbb{C}$. It follows that

$$\{x_1, \ldots, x_n, \mathrm{i}x_1, \ldots, \mathrm{i}x_n\}$$

span X over $\mathbb{R}$. Furthermore, each $\mathrm{i}x_j$ must be a linear combination of $\{x_{n+1}, \ldots, x_{2n}\}$, since it follows from the fact that $\{x_1, \ldots, x_n\}$ form a basis for X over $\mathbb{C}$ that $\{\mathrm{i}x_1, x_2, x_n\}$ are linearly independent (over $\mathbb{C}$, so certainly over $\mathbb{R}$).

For $k = 1, \ldots, n$ we define an element $f_k \in X^*$ via

$$f_k(z) = \varphi_k(z) - \mathrm{i}\varphi_k(\mathrm{i}z) \qquad z \in X,$$

where in order to interpret $\varphi_k(z)$ we consider z as an element of X_2 (expand in terms of the basis $\{x_1, \ldots, x_n\}$ and split the real and imaginary parts of the coefficients). We now show that the $\{f_k\}$ have the properties we require.

First, note that for $1 \leq k, j \leq n$,

$$f_k(x_j) = \varphi_k(x_j) - \mathrm{i}\varphi_k(\mathrm{i}x_j) = \delta_{kj},$$

since $\mathrm{i}x_j$ is a linear combination of $\{x_{n+1}, \ldots, x_{2n}\}$, and thus $\varphi_k(\mathrm{i}x_j) = 0$. All that remains is to show that $\|x_k^*\| = 1$. To do this we follow the standard argument (e.g. Yosida, 1980), writing $f_k(z) = r\mathrm{e}^{-\mathrm{i}\theta}$. Then

$$|f_k(z)| = \mathrm{e}^{\mathrm{i}\theta} f_k(z) = f_k(\mathrm{e}^{\mathrm{i}\theta} z),$$

so that $f_k(\mathrm{e}^{\mathrm{i}\theta} z)$ is real and positive. It follows that

$$|f_k(z)| = f_k(\mathrm{e}^{\mathrm{i}\theta} z) = \varphi_k(\mathrm{e}^{\mathrm{i}\theta} z) \leq \|\mathrm{e}^{\mathrm{i}\theta} z\|_{X_2} \leq \|\mathrm{e}^{\mathrm{i}\theta} z\|_X = \|x\|_X,$$

and the lemma is proved. □

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