

# Gröbner $\delta$ -bases and Gröbner bases for differential operators\*

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October 30, 2001

2000 *Mathematics Subject Classification*: PRIMARY 13N10; SECONDARY 13P10,16S32,32C38,68W30.

## Abstract

This paper deals with the notion of Gröbner  $\delta$ -base for some rings of linear differential operators by adapting the works of W. Trinks, A. Assi, M. Insa and F. Pauer. We compare this notion with the one of Gröbner base for such rings. As an application, and following a previous work of A. Assi, we give some results on finiteness and on flatness of finitely generated left modules over these rings.

## 1 Introduction.

We will study Gröbner  $\delta$ -bases for some rings of linear differential operators.

We have adapted to the differential case some notions and some results obtained by W. Trinks in [TRI] and A. Assi (in [ASS-1] and [ASS-2]) for the case of a commutative polynomial ring with coefficients in a commutative unitary ring.

The notion of Gröbner  $\delta$ -base we introduce here is equivalent to the one of Gröbner base defined by M. Insa and F. Pauer in [IN-PA]. Nevertheless, we reserve the name Gröbner base for the classical notion introduced in [CAS-1] (see also [CAS-2]). Besides the  $\mathbf{k}$ -algebras appearing in [IN-PA], the cases  $\mathcal{H} = \mathbf{k}[[X]][X^{-1}]$  and  $\mathcal{H} = \mathbf{k}\{X\}[X^{-1}]$  (when  $\mathbf{k} = \mathbf{R}, \mathbf{C}$ ) will be especially interesting in order to extend the results of [ACG-1] and [ACG-2] to the rings of linear differential operators with coefficients in  $\mathcal{H}$ .

Section 2 is devoted to the definition of the class of rings of linear differential operators we will study and to the theory of Gröbner  $\delta$ -bases. We have, in these rings, a reduction algorithm which allows the effective construction of a Gröbner  $\delta$ -base for a given ideal, defined by a finite system of generators. This is the aim of the sections 3,4 and 5.

In section 6 we compare the notions of Gröbner  $\delta$ -base and Gröbner base in the case of the Weyl algebras. We prove that any Gröbner base (in the sense of [CAS-1] (see also [CAS-2])) of a left ideal of a Weyl algebra is a Gröbner  $\delta$ -base with respect to an appropriate well-ordering. We also prove that the converse is not true.

We can deduce adapted algorithms for membership problem, elimination problem and syzygies problem by using Gröbner  $\delta$ -bases (instead of Gröbner bases) that could be in some cases with better complexity.

In section 7 we apply previous results to the study of flatness of some modules in a (local) relative situation. We also give a finiteness results for some modules. These flatness results could be compared to those of [SAB] for Rees modules over Rees rings.

It is a pleasure to thank Professor A. Assi for his help and useful suggestions.

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\*Research supported by DGESIC BFM2001-3164, A.I.H.F.2000-0044 and FQM-218.

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## 2 Gröbner $\delta$ -bases.

Here,  $\mathbf{k}$  is a field of zero characteristic. Let us denote by  $\mathbf{k}[[X]] = \mathbf{k}[[x_1, \dots, x_n]]$  the ring of formal power series and by  $\mathbf{k}((X))$  its quotient field.

Let us denote by  $\mathbf{k}((X))[\partial] = \mathbf{k}((X))[\partial_1, \dots, \partial_n]$  the ring of linear differential operators with coefficients in  $\mathbf{k}((X))$ , where  $\partial_i$  stands for the partial derivative with respect to the variable  $x_i$ .

Let us consider a noetherian sub- $\mathbf{k}$ -algebra  $\mathcal{H} \subset \mathbf{k}((X))$  stable under the action of the partial derivatives  $\partial_1, \dots, \partial_n$ . Let us denote by  $\mathcal{D}$  (or  $\mathcal{H}[\partial]$ ) the sub- $\mathbf{k}$ -algebra (of  $\mathbf{k}((X))[\partial]$ ) of linear differential operators generated by  $\mathcal{H}$  and  $\{\partial_1, \dots, \partial_n\}$ .

More generally, we will consider differential rings as  $\mathcal{D} = \mathcal{H}[\partial_1, \dots, \partial_n]$  for any noetherian sub- $\mathbf{k}$ -algebra  $\mathcal{H}$  of  $\mathbf{k}((X)) = \mathbf{k}((x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}))$ , stable under the action of  $\partial_i$  for  $i = 1, \dots, n$ .

The ring  $\mathcal{D}$  is the set of formal finite sums

$$\sum_{\alpha \in \mathbf{N}^n} p_\alpha \partial^\alpha,$$

where  $p_\alpha \in \mathcal{H}$ .

Let  $<$  be a well-ordering, compatible with the sum in  $\mathbf{N}^n$  (i.e. a well-ordering such that, for all  $\gamma \in \mathbf{N}^n$ , we have  $\alpha + \gamma < \beta + \gamma$  if and only if  $\alpha < \beta$ ).

**Definition 1** Let  $P = \sum_{\alpha \in \mathbf{N}^n} p_\alpha \partial^\alpha$  be a non-zero element of  $\mathcal{D}$ . The Newton  $\delta$ -diagram of  $P$  is the set

$$\mathcal{N}^\delta(P) = \{\alpha \in \mathbf{N}^n : p_\alpha \neq 0\}.$$

**Definition 2** Let  $P$  be a non-zero element of  $\mathcal{D}$ . We call the element of  $\mathbf{N}^n$ ,  $\max_{<} \{\mathcal{N}^\delta(P)\}$ , the  $\delta$ -exponent of  $P$  with respect to  $<$ . It will be denote by  $\exp_{<}^\delta(P)$  or by  $\exp^\delta(P)$  when no confusion is possible.

**Definition 3** Let  $P$  be a non-zero element of  $\mathcal{D}$ . We call the element  $p_\alpha \in \mathcal{H}$ , where  $\alpha = \exp^\delta(P)$ , the  $\delta$ -coefficient of  $P$  with respect to  $<$ . It will be denote by  $c_{<}^\delta(P)$  or by  $c^\delta(P)$  when no confusion is possible.

With these notations we have the following (see [MOR, page 106-108]):

**Lemma 4** Given two non-zero elements  $P, Q$  in  $\mathcal{D}$ . Then the following properties hold:

1.  $\exp^\delta(PQ) = \exp^\delta(P) + \exp^\delta(Q)$  and  $\exp^\delta([P, Q]) < \exp^\delta(PQ)$ .
2. If  $\exp^\delta(P) \neq \exp^\delta(Q)$  then  $\exp^\delta(P + Q) = \max_{<} \{\exp^\delta(P), \exp^\delta(Q)\}$ .
3. If  $\exp^\delta(P) = \exp^\delta(Q)$  and  $c^\delta(P) + c^\delta(Q) \neq 0$  then  $\exp^\delta(P + Q) = \exp^\delta(P) = \exp^\delta(Q)$  and  $c^\delta(P + Q) = c^\delta(P) + c^\delta(Q)$ .
4. If  $\exp^\delta(P) = \exp^\delta(Q)$  and  $c^\delta(P) + c^\delta(Q) = 0$  then  $\exp^\delta(P + Q) < \exp^\delta(P)$ .

All the ideals we will consider in  $\mathcal{D}$  will be left ideals.

Let  $I$  be a non-zero ideal of  $\mathcal{D}$ . We denote by

$$\text{Exp}_{<}^\delta(I) = \left\{ \exp^\delta(P) : P \in I \setminus \{0\} \right\} \subseteq \mathbf{N}^n.$$

We write  $\text{Exp}^\delta(I)$  when no confusion is possible.

**Remark 5** By Lemma 4 we have  $\text{Exp}^\delta(I) + \mathbf{N}^n = \text{Exp}^\delta(I)$ . So, by Dickson's Lemma (see for example [CLO]), there is a finite generating subset  $F$  of  $\text{Exp}^\delta(I)$ , i.e.

$$\text{Exp}^\delta(I) = \cup_{\alpha \in F} (\alpha + \mathbf{N}^n).$$

Any of the subset  $F$  is called a  $\delta$ -stair of  $I$ .

We denote by  $\mathcal{H}[\zeta] = \mathcal{H}[\zeta_1, \dots, \zeta_n]$  the (commutative) polynomial ring with coefficients in  $\mathcal{H}$  and with variables  $\zeta_1, \dots, \zeta_n$ .

**Definition 6** Let  $P = \sum_{\alpha} p_\alpha \partial^\alpha$  be a non-zero element of  $\mathcal{D}$ . The  $\delta$ -initial form of  $P$ , with respect to  $<$ , is  $\text{in}_{<}^\delta(P) = c^\delta(P) \zeta^{\exp^\delta(P)} \in \mathcal{H}[\zeta]$ . We write  $\text{in}^\delta(P)$  when no confusion is possible.

**Definition 7** Let  $I$  be a non-zero ideal of  $\mathcal{D}$ . The ideal generated by

$$\left\{ \text{in}^\delta(P) : P \in I \setminus \{0\} \right\}$$

is called the  $\delta$ -initial ideal of  $I$  with respect to  $<$  (and it is denoted by  $\text{in}_<^\delta(I)$ ). We write  $\text{in}^\delta(I)$  when no confusion is possible.

**Remark 8** Note that  $\text{in}^\delta(I)$  is a  $\zeta$ -monomial ideal in  $\mathcal{H}[\zeta]$ . If  $I$  is generated by  $\{P_1, \dots, P_m\}$  (i.e.  $I = \mathcal{D}(P_1, \dots, P_m)$ ), then the ideals  $\mathcal{H}[\zeta](\text{in}^\delta(P_1), \dots, \text{in}^\delta(P_m))$  and  $\text{in}^\delta(I)$  may be different.

**Definition 9** Let  $I$  be a non-zero ideal of  $\mathcal{D}$ . A finite family  $\{P_1, \dots, P_m\} \subset I$  is called a Gröbner  $\delta$ -base of  $I$ , with respect to the well ordering  $<$ , if

$$\text{in}^\delta(I) = \mathcal{H}[\zeta](\text{in}^\delta(P_1), \dots, \text{in}^\delta(P_m)).$$

**Remark 10** When  $\mathcal{H} = \mathbf{k}$  (i.e. in the ring  $\mathbf{k}[\partial] = \mathbf{k}[\partial_1, \dots, \partial_n]$ ), the notion of Gröbner  $\delta$ -base and the one of Gröbner base coincide. Here  $\mathbf{k}[\partial]$  is the ring of linear differential operators with constant coefficients which is a commutative polynomial ring.

### 3 Reduction in $\mathcal{D}$ .

Let  $F$  be a non-empty set in  $\mathcal{D}$  and  $\alpha \in \mathbf{N}^n$ . Here we will use some notations of [ASS-1]. Let

$$K(\alpha; F) = \left\{ c^\delta(P) : P \in F, \alpha \in \exp^\delta(P) + \mathbf{N}^n \right\}.$$

We denote by  $C(\alpha; F)$  the ideal, in  $\mathcal{H}$ , generated by  $K(\alpha; F)$ , i.e.

$$C(\alpha; F) = \mathcal{H}K(\alpha; F).$$

If  $K(\alpha; F) = \emptyset$  then  $C(\alpha; F) = \{0\}$ .

**Example 1** Let consider  $F = \{P_1, P_2\} \subseteq \mathbf{k}[x_1, x_2][\partial_1, \partial_2]$ , where  $P_1 = a(x_1)\partial_1^2 + b(x_2)\partial_1$  and  $P_2 = c(x_2)\partial_2^2 + e(x_1)\partial_2$  with  $a(x_1), e(x_1) \in \mathbf{k}[x_1]$ ,  $a(x_1) \neq 0$  and  $b(x_2), c(x_2) \in \mathbf{k}[x_2]$  with  $c(x_2) \neq 0$ . We consider the lexicographic order (c.f. [CLO]) in  $\mathbf{N}^2$  with  $\partial_1 > \partial_2$ .

Then  $\exp^\delta(P_1) = (2, 0)$  and  $\exp^\delta(P_2) = (0, 2)$ . Since  $K((1, 1); F) = \emptyset$  we have  $C((1, 1); F) = \{0\}$ . We also have

$$C((2, 2); F) = \langle a(x_1), c(x_2) \rangle, \quad C((2, 0); F) = \langle a(x_1) \rangle, \quad C((0, 2); F) = \langle c(x_2) \rangle,$$

where  $\langle N \rangle$  stands for the ideal (in  $\mathbf{k}[x_1, x_2]$ ) generated by  $N$ .

It is easy to check that

1. If  $\beta \in \alpha + \mathbf{N}^n$  then  $K(\alpha; F) \subseteq K(\beta; F)$  and  $C(\alpha; F) \subseteq C(\beta; F)$ .
2. If  $F_1 \subseteq F_2$  then  $K(\alpha; F_1) \subseteq K(\alpha; F_2)$  and  $C(\alpha; F_1) \subseteq C(\alpha; F_2)$  for all  $\alpha \in \mathbf{N}^n$ .

**Remark 11** If  $I$  is a non-zero ideal of  $\mathcal{D}$  then:

- a)  $K(\alpha; I) = \{c^\delta(P) : P \in I, \exp^\delta(P) = \alpha\}$ . Moreover,  $K(\alpha; I) \cup \{0\} = C(\alpha; I)$ .
- b)  $C(0; I) = I \cap \mathcal{H}$ .

**Remark 12** From now on, we suppose  $\mathcal{H}$  verifying two additional conditions:

- 1) For any subset  $\{f_1, \dots, f_r\} \subset \mathcal{H}$  and for any  $f \in \mathcal{H}$  we can decide if  $f \in \mathcal{H}(f_1, \dots, f_r)$ , and in this case, it is possible to find  $q_1, \dots, q_r \in \mathcal{H}$  such that  $f = \sum_{i=1}^r q_i f_i$ .
- 2) For any subset  $\{f_1, \dots, f_r\} \subset \mathcal{H}$  it is possible to find a system of generators of the  $\mathcal{H}$ -module of syzygies of  $\{f_1, \dots, f_r\}$ .

The algebras

$$\mathcal{H} = \mathbf{k}[X], \quad \mathbf{k}(X), \quad \mathbf{k}[[X]], \quad \mathbf{k}((X)), \quad \mathbf{k}[[X]][x_1^{-1}, \dots, x_n^{-1}]$$

and the algebra  $\mathbf{k}\{X\}[x_1^{-1}, \dots, x_n^{-1}]$  with  $\mathbf{k} = \mathbf{R}$  or  $\mathbf{C}$  verify conditions 1) and 2).

**Definition 13** Let  $F = \{P_1, \dots, P_m\} \subseteq \mathcal{D} \setminus \{0\}$  with  $P_i \neq 0$ ,  $1 \leq i \leq m$  and  $P \in \mathcal{D}$ . We will say that  $P$  is reduced with respect to  $F$ , if  $\exp^\delta(P) \notin \bigcup_{i=1}^m (\exp^\delta(P_i) + \mathbf{N}^n)$  or if  $\exp^\delta(P) \in \bigcup_{i=1}^m (\exp^\delta(P_i) + \mathbf{N}^n)$  then  $c^\delta(P) \notin C(\exp^\delta(P); F)$ .

Let  $F$  be a non-empty subset of  $\mathcal{D} \setminus \{0\}$ . We denote

$$R(F) = \{R \in \mathcal{D} : R \text{ is reduced to respect } F\}.$$

**Remark 14**  $R(F)$  is not necessarily a vector space over  $\mathbf{k}$ .

**Theorem 15 (Reduction algorithm)** Let  $F = \{P_1, \dots, P_m\} \subseteq \mathcal{D}$ , with  $P_i \neq 0$ ,  $i = 1, \dots, m$  and  $P \in \mathcal{D}$ . Then there exist  $Q_1, \dots, Q_m, R \in \mathcal{D}$  such that

1.  $P = \sum_{i=1}^m Q_i P_i + R$ .
2.  $R \in R(F)$ .
3.  $\max_{1 \leq i \leq m} \{\exp^\delta(Q_i P_i), \exp^\delta(R)\} = \exp^\delta(P)$ .

*Proof* We proceed by induction on  $\exp^\delta(P) = \alpha$ .

If  $\alpha = 0$ , then  $P \in \mathcal{H}$ . So, we can consider two cases:

1. If  $0 \notin \bigcup_{i=1}^m (\exp^\delta(P_i) + \mathbf{N}^n)$ , then

$$P = \sum_{i=1}^m 0 P_i + P, \quad \text{with} \quad P \in R(F).$$

2. If  $0 \in \bigcup_{i=1}^m (\exp^\delta(P_i) + \mathbf{N}^n)$ , then we consider the set

$$\Lambda = \left\{ i : 0 \in \exp^\delta(P_i) + \mathbf{N}^n \right\} = \{i : \exp^\delta(P_i) = 0\}.$$

Thus for  $i \in \Lambda$  we have  $P_i \in \mathcal{H}$  and we can consider two cases:

- (a) If  $P \in \mathcal{H}(P_i : i \in \Lambda)$  then  $P = \sum_{i \in \Lambda} q_i P_i$  with  $q_i \in \mathcal{H}$  (according our assumption on  $\mathcal{H}$  we can calculate such elements  $q_i$ ). In this case we have  $P = \sum Q_i P_i + R$  where  $Q_i = q_i$ , for  $i \in \Lambda$ ;  $Q_i = 0$  for  $i \notin \Lambda$  and  $R = 0$ .
- (b) If  $P \notin \mathcal{H}(P_i : i \in \Lambda)$  then  $P \in R(F)$ .

Suppose  $\alpha > 0$  and the theorem proved for  $\exp^\delta(P) < \alpha$ .

Let  $P \in \mathcal{D}$  be such that  $\exp^\delta(P) = \alpha$ . We have two possible cases:

1. If  $\alpha \notin \bigcup_{i=1}^m (\exp^\delta(P_i) + \mathbf{N}^n)$ , then  $P = \sum_{i=1}^m 0 P_i + P$  and  $P \in R(F)$ .
2. If  $\alpha \in \bigcup_{i=1}^m (\exp^\delta(P_i) + \mathbf{N}^n)$  then we consider the set

$$\Lambda = \left\{ i : \alpha \in \exp^\delta(P_i) + \mathbf{N}^n \right\}$$

and the following two cases are possible:

- (a) If  $c^\delta(P) \in C(\alpha; F)$ , then there exists  $(q_i)_{i \in \Lambda} \in \mathcal{H}$  such that

$$c^\delta(P) = \sum_{i \in \Lambda} q_i c^\delta(P_i).$$

We may write,

$$P^{(1)} = P - \sum_{i \in \Lambda} q_i \partial^{\gamma^i} P_i, \quad \text{with} \quad \gamma^i + \exp^\delta(P_i) = \alpha.$$

By construction,  $\exp^\delta(P^{(1)}) < \exp^\delta(P)$ . Hence, by induction, we may write  $P^{(1)} = \sum_{i=1}^m Q'_i P_i + R'$ , with  $R' \in R(F)$  and finally  $P = \sum_{i \notin \Lambda} Q'_i P_i + \sum_{i \in \Lambda} (Q'_i + q_i \partial^{\gamma^i}) P_i + R'$ .

(b) If  $c^\delta(P) \notin C(\alpha; F)$  then  $P \in R(F)$ .

So, we have proved the existence of  $Q_1, \dots, Q_m, R$  verifying conditions 1. and 2. of the statement. The condition 3. is easy to verify. That ends the proof.  $\square$

**Remark 16** We call  $R \in \mathcal{D}$  a remainder of the reduction of  $P$  by  $(P_1, \dots, P_m) \subseteq \mathcal{D}^m$ . We denote by  $\tilde{R}(P; P_1, \dots, P_m)$ , the set of remainders of the reduction of  $P$  by  $\{P_1, \dots, P_m\}$ .

**Remark 17** The proof of Theorem 15 provides an algorithm to reduce an operator  $P \in \mathcal{D}$  to respect a subset  $F$  of  $\mathcal{D}$ .

**Theorem 18** Let  $I$  be a non-zero ideal of  $\mathcal{D}$  and  $\{P_1, \dots, P_r\} \subset I$ . Then the following statements are equivalent:

1.  $\{P_1, \dots, P_r\}$  is a Gröbner  $\delta$ -base of  $I$ .
2. For  $\alpha \in \mathbf{N}^n$ , we have  $C(\alpha; I) = C(\alpha; P_1, \dots, P_r)$ .
3. For  $P \in I$  we have  $\tilde{R}(P; P_1, \dots, P_r) = \{0\}$ .

*Proof.* 1.  $\implies$  2.:  $C(\alpha; P_1, \dots, P_r)$  is clearly contained in  $C(\alpha; I)$ . Conversely, let  $p(x) \in C(\alpha; I)$  then there exists  $P \in I \setminus \{0\}$  such that  $in^\delta(P) = p(x)\zeta^\alpha$  and  $in^\delta(P) \in in^\delta(I)$ . But by hypothesis, we have

$$in^\delta(I) = \mathcal{H}[\zeta](in^\delta(P_1), \dots, in^\delta(P_r)).$$

Let us denote

$$in^\delta(P_i) = p_i(x)\zeta^{\alpha_i} \quad \text{with} \quad 1 \leq i \leq r,$$

then

$$p(x)\zeta^\alpha = \sum_{i=1}^r q_i(x, \zeta) p_i(x)\zeta^{\alpha_i},$$

where

$$q_i(x, \zeta) = \sum_{\beta} q_{i\beta}(x)\zeta^\beta \in \mathcal{H}[\zeta] \quad \text{with} \quad q_{i\beta}(x) \in \mathcal{H}.$$

Thus,

$$p(x)\zeta^\alpha = \sum_{i,\beta} q_{i\beta}(x) p_i(x)\zeta^{\beta+\alpha_i},$$

hence,

$$p(x)\zeta^\alpha \in \mathcal{H}[\zeta](p_i(x)\zeta^{\alpha_i} : \alpha \in \alpha_i + \mathbf{N}^n)$$

and so,

$$p(x) \in C(\alpha; P_1, \dots, P_r).$$

Therefore  $C(\alpha; I) \subseteq C(\alpha; P_1, \dots, P_r)$  and it follows that

$$C(\alpha; I) = C(\alpha; P_1, \dots, P_r).$$

2.  $\implies$  3.: Let  $P \in I \setminus \{0\}$ , then by Theorem 15, there exists  $Q_1, \dots, Q_r, R \in \mathcal{D}$  such that

$$P = \sum_{i=1}^r Q_i P_i + R,$$

where  $R \in \tilde{R}(P; P_1, \dots, P_r)$ .

Suppose  $R \neq 0$ . Since  $R = P - \sum_{i=1}^r Q_i P_i \in I$ , we can consider two cases:

- i) If  $\exp^\delta(R) \notin \bigcup_{i=1}^r (\exp^\delta(P_i) + \mathbf{N}^n)$ , then  $C(\exp^\delta(R); P_1, \dots, P_r) = (0)$ . Therefore  $c^\delta(R) \notin C(\exp^\delta(R); P_1, \dots, P_r)$  and by hypothesis 2,  $c^\delta(R) \notin C(\exp^\delta(R); I)$ . But this is impossible since  $R \in I$ .
- ii) If  $\exp^\delta(R) \in \bigcup_{i=1}^r (\exp^\delta(P_i) + \mathbf{N}^n)$ , then

$$c^\delta(R) \notin C(\exp^\delta(R); P_1, \dots, P_r) = C(\exp^\delta(R); I)$$

because  $R$  is reduced with respect to  $\{P_1, \dots, P_r\}$ , and this contradicts that  $R \in I$ .

Therefore  $R = 0$ .

3.  $\implies$  1.: We must show that  $\text{in}^\delta(I) = \mathcal{H}[\zeta](\text{in}^\delta(P_1), \dots, \text{in}^\delta(P_r))$ . Clearly  $\langle \text{in}^\delta(P_1), \dots, \text{in}^\delta(P_r) \rangle \subseteq \text{in}^\delta(I)$ . Let  $P \in I$  be a non-zero operator. Then we can write

$$P = p_{\alpha_0} \partial^{\alpha_0} + \widehat{P}$$

where  $p_{\alpha_0} \in \mathcal{H} \setminus \{0\}$  and  $\exp^\delta(\widehat{P}) < \alpha_0$ .

Then, by hypothesis and by Theorem 15, we have:

$$\alpha_0 \in \bigcup_{i=1}^r (\exp^\delta(P_i) + \mathbf{N}^n) \quad \text{and} \quad c^\delta(P) \in C(\alpha_0; P_1, \dots, P_r).$$

We consider the set  $\Lambda = \{i : \alpha_0 \in \exp^\delta(P_i) + \mathbf{N}^n\}$ . Then  $c^\delta(P) = \sum_{i \in \Lambda} q_i^{(1)} c^\delta(P_i)$ .

Let

$$P^{(1)} = P - \sum_{i \in \Lambda} q_i^{(1)} \partial^{\gamma^i} P_i, \quad \text{with} \quad \gamma^i + \exp^\delta(P_i) = \alpha_0,$$

then  $P^{(1)} \in I$  and  $\exp^\delta(P^{(1)}) < \alpha_0$ .

Now we can consider two cases:

i) If  $P^{(1)} = 0$ , then  $P = \sum_{i \in \Lambda} q_i^{(1)} \partial^{\gamma^i} P_i$  and it can be checked that

$$\text{in}^\delta(P) = \sum_{i \in \Lambda} q_i^{(1)} \zeta^{\gamma^i} \text{in}^\delta(P_i).$$

ii) If  $P^{(1)} \neq 0$ , then by repeating the same procedure, we can obtain a family  $P^{(k)} \in I$  with  $\exp^\delta(P^{(k)}) < \exp^\delta(P^{(k-1)})$ . So, as  $<$  is a well-ordering in  $\mathbf{N}^n$ , there exists  $l$ , such that  $P^{(l)} = 0$ .

This completes the proof.  $\square$

As a straightforward consequence of Theorem 18 we get the following result:

**Corollary 19** *Any Gröbner  $\delta$ -base of an ideal  $I \subseteq \mathcal{D}$  is a system of generators of  $I$ . Moreover, if  $\{P_1, \dots, P_r\}$  is a Gröbner  $\delta$ -base of  $I$  then*

$$\text{Exp}^\delta(I) = \bigcup_{i=1}^r (\exp^\delta(P_i) + \mathbf{N}^n).$$

## 4 $S^\delta$ -operators.

Let  $F = \{P_1, \dots, P_r\} \subseteq \mathcal{D} \setminus \{0\}$ . Let

$$K(F) = \left\{ \alpha \in \mathbf{N}^n : \exists N \subseteq F, \alpha = \text{lcm}\{\exp^\delta(P); P \in N\} \right\},$$

where *lcm* stands for *less common multiple*, and

$$F_\alpha = \left\{ (\lambda_1, \dots, \lambda_r) \in \mathcal{H}^r : \sum_{k=1}^r \lambda_k c^\delta(P_k) = 0 \text{ where } \lambda_k = 0 \text{ if } \alpha \notin \exp^\delta(P_k) + \mathbf{N}^n \right\} \subseteq \mathcal{H}^r.$$

$F_\alpha$  is isomorphic to the  $\mathcal{H}$ -module of syzygies of

$$\left\{ c^\delta(P_k) : \alpha \in \exp^\delta(P_k) + \mathbf{N}^n, 1 \leq k \leq r \right\}.$$

Since  $\mathcal{H}$  is a noetherian algebra then  $F_\alpha$  is finitely generated (as a  $\mathcal{H}$ -module). Let  $\{(\lambda_1^\tau, \dots, \lambda_r^\tau)\}$ ,  $1 \leq \tau \leq r_\alpha$ , be a system of generators of  $F_\alpha$ .

**Definition 20** *With the notations as above, for  $\tau = 1, \dots, r_\alpha$ , the element*

$$S_{\alpha, \tau}^\delta = \sum_{k=1}^r \lambda_k^\tau \partial^{\alpha - \exp^\delta(P_k)} P_k$$

*will be called a  $S^\delta$ -operator of the set  $F_\alpha$ .*

**Proposition 21** *With the notations as above, we have*

$$\exp^\delta(S_{\alpha, \tau}^\delta) < \alpha.$$

*Proof.* We can write

$$\begin{aligned} S_{\alpha, \tau}^{\delta} &= \sum_{k=1}^r \lambda_k^{\tau} \partial^{\alpha - \exp^{\delta}(P_k)} P_k = \\ &= \sum_{k=1}^r \lambda_k^{\tau} \partial^{\alpha - \exp^{\delta}(P_k)} c^{\delta}(P_k) \partial^{\exp^{\delta}(P_k)} + \sum_{k=1}^r \sum_{\beta < \exp^{\delta}(P_k)} \lambda_k^{\tau} \partial^{\alpha - \exp^{\delta}(P_k)} p_{\beta, k} \partial^{\beta}. \end{aligned}$$

Since

$$\begin{aligned} \partial^{\alpha - \exp^{\delta}(P_k)} c^{\delta}(P_k) &= c^{\delta}(P_k) \partial^{\alpha - \exp^{\delta}(P_k)} + A_k \quad \text{with} \quad \exp^{\delta}(A_k) < \alpha - \exp^{\delta}(P_k), \\ \partial^{\alpha - \exp^{\delta}(P_k)} p_{\beta, k} &= p_{\beta, k} \partial^{\alpha - \exp^{\delta}(P_k)} + B_k \quad \text{with} \quad \exp^{\delta}(B_k) < \alpha - \exp^{\delta}(P_k) \end{aligned}$$

and  $\sum_{k=1}^r \lambda_k^{\tau} c^{\delta}(P_k) = 0$ , finally

$$S_{\alpha, \tau}^{\delta} = \sum_{k=1}^r \lambda_k^{\tau} A_k \partial^{\exp^{\delta}(P_k)} + \sum_{k=1}^r \sum_{\beta < \exp^{\delta}(P_k)} \left( \lambda_k^{\tau} p_{\beta, k} \partial^{\alpha - \exp^{\delta}(P_k) + \beta} + B_k \partial^{\beta} \right).$$

□

**Proposition 22** *Let  $I$  be a non-zero ideal of  $\mathcal{D}$  and  $\{P_1, \dots, P_r\}$  be a system of generators of  $I$ . Then the following are equivalent:*

1.  $\{P_1, \dots, P_r\}$  is a  $\delta$ -Gröbner base of  $I$ .
2. For all  $P \in I$ , we have  $\tilde{R}(P; P_1, \dots, P_r) = \{0\}$ .
3. For all  $S^{\delta}$ -operator,  $S_{\alpha, \tau}^{\delta}$ , of  $\{P_1, \dots, P_r\}$  we have  $0 \in \tilde{R}(S_{\alpha, \tau}^{\delta}; P_1, \dots, P_r)$ .

*Proof.* 1.  $\implies$  2.: See Theorem 18.

2.  $\implies$  3.: Since  $S_{\alpha, \tau}^{\delta} \in I$ , then, by assumption,  $0 \in \tilde{R}(S_{\alpha, \tau}^{\delta}; P_1, \dots, P_r)$ .

3.  $\implies$  1.: Let  $P \in I$  be a non-zero operator. We must show that  $\text{in}^{\delta}(P) \in \mathcal{H}[\zeta](\text{in}^{\delta}(P_1), \dots, \text{in}^{\delta}(P_r))$ . We may write  $P = \sum_{i=1}^r H_i P_i$ , with  $H_i \in \mathcal{H}[\partial]$ .

Suppose

$$\alpha_0 = \max_i \left\{ \exp^{\delta}(H_i P_i) \right\} \quad \text{and} \quad \exp^{\delta}(H_{i_k} P_{i_k}) = \alpha_0, \quad k = 0, \dots, t.$$

Hence, by Lemma 4,

$$\exp^{\delta}(H_{i_k}) + \exp^{\delta}(P_{i_k}) = \alpha_0, \quad k = 0, \dots, t.$$

We can consider two cases:

- a) If  $\sum_{k=0}^t c^{\delta}(H_{i_k}) c^{\delta}(P_{i_k}) \neq 0$  then

$$\text{in}^{\delta}(P) = c^{\delta}(P) \zeta^{\alpha_0}, \quad \text{with} \quad c^{\delta}(P) = \sum_{k=0}^t c^{\delta}(H_{i_k}) c^{\delta}(P_{i_k}).$$

Therefore,

$$\text{in}^{\delta}(P) = \sum_{k=0}^t c^{\delta}(H_{i_k}) c^{\delta}(P_{i_k}) \zeta^{\alpha_0} = \sum_{k=0}^t c^{\delta}(H_{i_k}) \zeta^{\alpha_0 - \exp^{\delta}(P_{i_k})} \text{in}^{\delta}(P_{i_k})$$

and so,  $\text{in}^{\delta}(P) \in \mathcal{H}[\zeta](\text{in}^{\delta}(P_1), \dots, \text{in}^{\delta}(P_r))$ .

- b) Suppose now  $\sum_{k=0}^t c^{\delta}(H_{i_k}) c^{\delta}(P_{i_k}) = 0$ . Let us denote  $\alpha^i = \exp^{\delta}(P_i)$ ,  $i = 1, \dots, r$ ; we consider the set  $\Lambda = \{i : \exp^{\delta}(H_i) + \alpha^i = \alpha_0\}$ , and we suppose  $\gamma = \text{lcm}\{\exp^{\delta}(P_i) : i \in \Lambda\}$ .

We may write

$$P = \sum_{i \notin \Lambda} H_i P_i + \sum_{i \in \Lambda} c^{\delta}(H_i) \partial^{\exp^{\delta}(H_i)} P_i + \sum_{i \in \Lambda} (H_i - c^{\delta}(H_i) \partial^{\exp^{\delta}(H_i)}) P_i.$$

We can identify  $(c^{\delta}(H_i))_{i \in \Lambda}$  with an element of  $F_{\gamma}$ . Let  $\underline{\lambda}^1, \dots, \underline{\lambda}^p$  be a family of generators of  $F_{\gamma}$  where

$$\underline{\lambda}^{\tau} = (\lambda_1^{\tau}, \dots, \lambda_r^{\tau}) \quad \text{with} \quad \lambda_j^{\tau} = 0 \quad \text{if} \quad \gamma \notin \exp^{\delta}(P_j) + \mathbf{N}^n.$$

Now for each  $i \in \{1, \dots, r\}$  we define  $s_i$  as follows:

$$s_i = \begin{cases} c^\delta(H_i) & \text{if } i \in \Lambda \\ 0 & \text{if } i \notin \Lambda. \end{cases}$$

Hence  $\underline{s} = (s_1, \dots, s_r) \in F_\gamma$ , and then there exist  $u_1, \dots, u_p \in \mathcal{H}$  such that  $\underline{s} = \sum_{\tau=1}^p u_\tau \underline{\lambda}^\tau$ . Let us denote  $\beta^i = \exp^\delta(H_i)$ , for  $i \in \Lambda$ , then

$$\sum_{i \in \Lambda} c^\delta(H_i) \partial^{\beta^i} P_i = \sum_{\tau=1}^p u_\tau \left( \sum_{i=1}^r \lambda_i^\tau \partial^{\beta^i} P_i \right).$$

The element  $\alpha_0$  is, by definition, a common multiple of the elements  $\{\exp^\delta(P_i) : i \in \Lambda\}$  then there exists  $\epsilon \in \mathbf{N}^n$  such that  $\alpha_0 = \gamma + \epsilon$  and so  $\beta^i = \gamma - \alpha^i + \epsilon$ .

If  $j \notin \Lambda$  and  $\gamma \in \exp^\delta(P_j) + \mathbf{N}^n$  we denote  $\beta^j = \gamma - \alpha^j + \epsilon$ . Therefore,

$$\sum_{i \in \Lambda} c^\delta(H_i) \partial^{\beta^i} P_i = \sum_{\tau=1}^p u_\tau \left( \sum_{i=1}^r \partial^\epsilon \lambda_i^\tau \partial^{\gamma - \alpha^i} P_i \right) + \sum_{\tau=1}^p u_\tau \left( \sum_{i | \gamma - \alpha^i > 0} B_i^\tau \partial^{\gamma - \alpha^i} P_i \right)$$

where  $\exp^\delta(B_i^\tau) < \epsilon$ .

Therefore, by Definition 20,

$$\sum_{i \in \Lambda} c^\delta(H_i) \partial^{\beta^i} P_i = \sum_{\tau=1}^p u_\tau \partial^\epsilon S_{\gamma, \tau}^\delta + \sum_{i | \gamma - \alpha^i > 0} \left( \sum_{\tau=1}^p u_\tau B_i^\tau \right) \partial^{\gamma - \alpha^i} P_i.$$

But by hypothesis, we have

$$S_{\gamma, \tau}^\delta = \sum_{j=1}^r Q_j^{\gamma, \tau} P_j,$$

with  $\gamma > \exp^\delta(S_{\gamma, \tau}^\delta) = \max_{1 \leq j \leq r} \{\exp^\delta(Q_j^{\gamma, \tau} P_j)\}$ . Hence,

$$\sum_{i \in \Lambda} c^\delta(H_i) \partial^{\beta^i} P_i = \sum_{j=1}^r \left( \sum_{\tau=1}^p u_\tau \partial^\epsilon Q_j^{\gamma, \tau} \right) P_j + \sum_{j | \gamma - \alpha^j > 0} \left( \sum_{\tau=1}^p u_\tau B_j^\tau \right) \partial^{\gamma - \alpha^j} P_j.$$

Therefore

$$P = \sum_{i=1}^r H'_i P_i$$

where

- If  $i \in \Lambda$ ,

$$H'_i = H_i - c^\delta(H_i) \partial^{\beta^i} + \sum_{\tau=1}^p u_\tau \partial^\epsilon Q_i^{\gamma, \tau} + \sum_{\tau=1}^p u_\tau B_i^\tau \partial^{\gamma - \alpha^i}.$$

- If  $i \notin \Lambda$  and  $\gamma - \alpha^i > 0$ ,

$$H'_i = H_i + \sum_{\tau=1}^p u_\tau \partial^\epsilon Q_i^{\gamma, \tau} + \sum_{\tau=1}^p u_\tau B_i^\tau \partial^{\gamma - \alpha^i}.$$

- If  $i \notin \Lambda$  and  $\gamma - \alpha^i$  is not greater than 0,

$$H'_i = H_i + \sum_{\tau=1}^p u_\tau \partial^\epsilon Q_i^{\gamma, \tau}.$$

Hence, we have obtained an expression for  $P$  as a combination of the  $P_i$  where  $\exp^\delta(H'_i P_i) < \alpha_0$ , then  $\max_i \{\exp^\delta(H'_i P_i)\} < \alpha_0$ . But this process stops because  $<$  is a well-ordering in  $\mathbf{N}^n$ . So, there exists an expression of  $P$  with the conditions of the case a).

□



## 5 Construction of a Gröbner $\delta$ -base.

Let  $I$  be a non-zero ideal of  $\mathcal{D}$  and let  $F = \{P_1, \dots, P_r\}$  be a system of generators of  $I$ . We will show here how to build a Gröbner  $\delta$ -base of the ideal  $I$  (with respect to a ordering  $<$ ). We will follow the main lines of Buchberger's algorithm, adapted to our case (see [BUCH], [TRI] and [ASS-1]).

Let  $K(F) = \{\alpha^1, \dots, \alpha^s\}$  (see Section 4). Let  $\{S_{\alpha_j, \tau}^\delta\}, 1 \leq j \leq s, 1 \leq \tau \leq r_j$  the family of  $S^\delta$ -operators associated to  $F$ .

We suppose that  $\{P_1, \dots, P_r\}$  is not a Gröbner  $\delta$ -base for  $I$ , then (by Proposition 22) there exists  $S_{\alpha_0, \tau}^\delta$  such that  $0 \notin \tilde{R}(S_{\alpha_0, \tau}^\delta; P_1, \dots, P_r)$ , then let

$$P_{r+1} \in \tilde{R}(S_{\alpha_0, \tau}^\delta; P_1, \dots, P_r)$$

and we repeat this process with  $\{P_1, \dots, P_r, P_{r+1}\}$ .

**Remark 23** *If a  $S^\delta$ -operator,  $S$ , of  $F$  verify that  $0 \in \tilde{R}(S; P_1, \dots, P_r)$  then  $0 \in \tilde{R}(S; P_1, \dots, P_r, P_{r+1})$ .*

The following Proposition assures that this procedure terminates

**Proposition 24** *With the notations as above, there exists  $\rho \in \mathbf{N}$  such that for all  $S^\delta$ -operator  $S$  of  $\{P_1, \dots, P_{r+\rho}\}$  we have  $0 \in \tilde{R}(S; P_1, \dots, P_{r+\rho})$ .*

*Proof.* See [MOR, pages 131-133].  $\square$

## 6 Gröbner bases and the Gröbner $\delta$ -bases.

In this section we will work on the Weyl algebra  $A_n(\mathbf{k}) = \mathbf{k}[X][\partial]$ , so we suppose here  $\mathcal{H} = \mathbf{k}[X] = \mathbf{k}[x_1, \dots, x_n]$ .

Let  $<_x, <_\partial$  be monomial orderings in  $\mathbf{N}^n$ .

We denote by  $X^\alpha \partial^\beta$  the monomial

$$x_1^{\alpha_1} \dots x_n^{\alpha_n} \partial_1^{\beta_1} \dots \partial_n^{\beta_n}$$

Let us define on  $\mathbf{N}^n \times \mathbf{N}^n$  the total ordering (denoted  $<$ ) by

$$(\alpha(1), \beta(1)) < (\alpha(2), \beta(2)) \iff \begin{cases} \beta(1) <_\partial \beta(2) \\ \text{or} \\ \beta(1) = \beta(2) \quad \text{and} \quad \alpha(1) <_x \alpha(2). \end{cases}$$

**Remark 25** *The relation  $<$ , defined in  $\mathbf{N}^n \times \mathbf{N}^n$ , is a monomial ordering. This well-ordering is called an elimination order (see for example [CLO]).*

For the notion of Gröbner base on  $A_n(\mathbf{k})$  and some related results we follow here [CAS-1] (see also [CAS-2]).

**Theorem 26** *Let  $G = \{P_1, \dots, P_r\}$  be a system of generators for a non-zero ideal  $I \subset A_n(\mathbf{k})$ . Then if  $G$  is a Gröbner base for  $I$ , with respect to  $<$ , then  $G$  is a Gröbner  $\delta$ -base for  $I$  with respect to  $<_\partial$ .*

*Proof.* Let  $P \in I$  be a non-zero operator. We must show that

$$\text{in}^\delta(P) \in \mathcal{H}[\zeta](\text{in}^\delta(P_1), \dots, \text{in}^\delta(P_r)).$$

For  $i = 1, \dots, r$ , we may write  $P_i = a_i \partial^{\alpha_i} + \widehat{P}_i$  where  $\exp^\delta(P_i) = \alpha_i$ ,  $\exp^\delta(\widehat{P}_i) < \alpha_i$  and  $a_i \in \mathcal{H}$ . Thus  $\text{in}^\delta(P_i) = a_i \zeta^{\alpha_i}$ . By the division algorithm in  $A_n(\mathbf{k})$ , (see [CAS-1] and [CAS-2]) there exists  $Q_{i_1}, \dots, Q_{i_N} \in A_n(\mathbf{k})$ ,  $1 \leq i_j \leq r$ , satisfying  $P = Q_{i_1} P_{i_1} + \dots + Q_{i_N} P_{i_N}$  where  $\exp_{<}(Q_i P_i) \neq \exp_{<}(Q_j P_j)$  for  $i \neq j$ .

We can suppose

$$\exp_{<}(Q_{i_N} P_{i_N}) < \exp_{<}(Q_{i_{N-1}} P_{i_{N-1}}) < \dots < \exp_{<}(Q_{i_1} P_{i_1}).$$

We can write

$$Q_{i_j} = c_{i_j} \partial^{\beta_{i_j}} + \widehat{Q}_{i_j}$$

where  $\exp^\delta(Q_{i_j}) = \beta_{i_j}$ ,  $\exp^\delta(\widehat{Q}_{i_j}) < \beta_{i_j}$ ,  $c_{i_j} \in \mathcal{H}$ . Thus  $\exp_{<}(Q_{i_j}) = (\exp_{<_x}(c_{i_j}), \beta_{i_j})$ .

Therefore,

$$P = \sum_{j=1}^N c_{i_j} a_{i_j} \partial^{\beta_{i_j} + \alpha_{i_j}} + \sum_{j=1}^N c_{i_j} A_{i_j} \partial^{\alpha_{i_j}} + \sum_{j=1}^N c_{i_j} \partial^{\beta_{i_j}} \widehat{P}_{i_j} + \sum_{j=1}^N \widehat{Q}_{i_j} a_{i_j} \partial^{\alpha_{i_j}} + \sum_{j=1}^N \widehat{Q}_{i_j} \widehat{P}_{i_j}$$

where,  $\exp^\delta \left( \sum_{j=1}^N c_{i_j} a_{i_j} \partial^{\beta_{i_j} + \alpha_{i_j}} \right) \leq \max_{1 \leq j \leq N} \{\beta_{i_j} + \alpha_{i_j}\}$ ,

$$\exp^\delta \left( \sum_{j=1}^N c_{i_j} A_{i_j} \partial^{\alpha_{i_j}} \right) < \max_{1 \leq j \leq N} \{\beta_{i_j} + \alpha_{i_j}\},$$

$$\exp^\delta \left( \sum_{j=1}^N c_{i_j} \partial^{\beta_{i_j}} \widehat{P}_{i_j} \right) < \max_{1 \leq j \leq N} \{\beta_{i_j} + \alpha_{i_j}\},$$

$$\exp^\delta \left( \sum_{j=1}^N \widehat{Q}_{i_j} a_{i_j} \partial^{\alpha_{i_j}} \right) < \max_{1 \leq j \leq N} \{\beta_{i_j} + \alpha_{i_j}\} \text{ and}$$

$$\exp^\delta \left( \sum_{j=1}^N \widehat{Q}_{i_j} \widehat{P}_{i_j} \right) < \max_{1 \leq j \leq N} \{\beta_{i_j} + \alpha_{i_j}\}.$$

Let  $j_0$  be such that

$$\beta_{i_{j_0+1}} + \alpha_{i_{j_0+1}} < \beta_{i_{j_0}} + \alpha_{i_{j_0}} = \beta_{i_{j_0-1}} + \alpha_{i_{j_0-1}} = \cdots = \beta_{i_1} + \alpha_{i_1}.$$

Since  $\sum_{j=1}^{j_0} c_{i_j} a_{i_j} \neq 0$ , we have  $\text{in}^\delta(P) = \left( \sum_{j=1}^{j_0} c_{i_j} a_{i_j} \right) \zeta^{\beta_{i_1} + \alpha_{i_1}}$ . Therefore,

$$\text{in}^\delta(P) = \sum_{j=1}^{j_0} c_{i_j} \text{in}^\delta(P_{i_j}) \zeta^{\beta_{i_j}}$$

and so  $\text{in}^\delta(P) \in \mathcal{H}[\zeta](\text{in}^\delta(P_1), \dots, \text{in}^\delta(P_r))$ . This completes the proof.  $\square$

The converse result is not true as we show in the following example:

**Example 2** Let  $I \subset A_2(\mathbf{C}) = \mathbf{C}[x_1, x_2][\partial_1, \partial_2]$  be the left ideal generated by the operators

$$P_1 = x_1 \partial_1 + a \partial_2 + b, \quad P_2 = (x_2 - x_1) \partial_2 - d$$

with  $a, b, d \in \mathbf{C}[x_1, x_2]$ .

We will prove<sup>1</sup> that  $\{P_1, P_2\}$  is a Gröbner  $\delta$ -base which is not a Gröbner base of  $I$ , for a particular choice of the polynomials  $a, b, d$ .

We have

$$\exp_{<}(P_1) = (1, 0, 1, 0), \quad \exp_{<}(P_2) = (1, 0, 0, 1).$$

Then

$$S(P_1, P_2) = \partial_2 P_1 + \partial_1 P_2 = x_2 \partial_1 \partial_2 + \partial_2 a \partial_2 + \partial_2 b - \partial_1 d - \partial_2,$$

and then

$$\exp_{<}(S(P_1, P_2)) = (0, 1, 1, 1) \notin \langle (1, 0, 1, 0), (1, 0, 0, 1) \rangle = \langle \exp_{<}(P_1), \exp_{<}(P_2) \rangle.$$

So,  $G = \{P_1, P_2\}$  is not a Gröbner base of the ideal  $I$ , for any  $a, b, d \in \mathbf{C}[x_1, x_2]$ .

We will prove that, for some  $a, b, d \in \mathbf{C}[x_1, x_2]$ , the set  $G = \{P_1, P_2\}$  is a Gröbner  $\delta$ -base of  $I$ .

We have

$$\exp^\delta(P_1) = (1, 0), \quad c^\delta(P_1) = x_1$$

and

$$\exp^\delta(P_2) = (0, 1), \quad c^\delta(P_2) = x_2 - x_1.$$

We will compute the associated  $S^\delta$ -operators (see Definition 20).

As

$$\alpha = \text{lcm}((1, 0), (0, 1)) = (1, 1)$$

we must first compute a system of generators of

$$F_{(1,1)}(P_1, P_2) = \left\{ (\lambda_1, \lambda_2) \in \mathbf{C}[x_1, x_2] : \lambda_1 c^\delta(P_1) + \lambda_2 c^\delta(P_2) = 0 \right\}.$$

In fact we have

$$\text{Syzy}(c^\delta(P_1), c^\delta(P_2)) = F_{(1,1)}(P_1, P_2) = \langle (x_2 - x_1, -x_1) \rangle$$

<sup>1</sup>By using the degree lexicographical order with  $\partial_2 <_{\partial} \partial_1$  and  $x_2 <_x x_1$

and then

$$\begin{aligned} S_{(1,1),(x_2-x_1,-x_1)}^\delta &= (x_2 - x_1)\partial^{(1,1)-(1,0)} P_1 - x_1\partial^{(1,1)-(0,1)} P_2 = \\ &= (x_2 - x_1)a\partial_2^2 + (x_2 - x_1)\partial_2(a)\partial_2 + (x_2 - x_1)b\partial_2 + \\ &\quad + (x_2 - x_1)\partial_2(b) + x_1\partial_2 + x_1d\partial_1 + x_1\partial_1(d). \end{aligned}$$

Now we reduce  $S_{(1,1),(x_2-x_1,-x_1)}^\delta$  by  $(P_1, P_2)$ , say

$$\begin{aligned} S_{(1,1),(x_2-x_1,-x_1)}^\delta - dP_1 - \partial_2aP_2 - bP_2 &= \\ = (x_2 - x_1)\partial_2(b) + x_1\partial_2 + x_1\partial_1(d) - a\partial_2 + a\partial_2(d) + \partial_2(a)d &= \\ = (x_2 - x_1)\partial_2(b) + (x_1 - a)\partial_2 + x_1\partial_1(d) + \partial_2(ad). \end{aligned}$$

Then  $\{P_1, P_2\}$  is a Gröbner  $\delta$ -base of  $I$  if  $a = x_1$ ,  $b \in \mathbf{C}[x_1]$  and  $d \in \mathbf{C}$ .

**Remark 27** The example before proves a little more. Let us consider  $\mathcal{H} = \mathbf{C}[x_1, \dots, x_n]$  (for  $n \geq 3$ ) and the ring of differential operators  $\mathcal{D} = \mathcal{H}[\partial_1, \partial_2]$  (which is a sub-algebra of the Weyl algebra  $A_n(\mathbf{C})$ ).

Let  $I \subset \mathcal{D}$  be the left ideal generated by the operators

$$P_1 = x_1\partial_1 + a\partial_2 + b, \quad P_2 = (x_2 - x_1)\partial_2 - d$$

with  $a, b, d \in \mathbf{C}[x_1, \dots, x_n]$ . An analogous computation to the one of example 2 proves that  $\{P_1, P_2\}$  is a Gröbner  $\delta$ -base of  $I$  if  $a = x_1$ ,  $b \in \mathbf{C}[x_1, x_3, \dots, x_n]$  and  $d \in \mathbf{C}[x_3, \dots, x_n]$ .

## 7 Applications: Flatness and finiteness.

As elementary applications of Gröbner  $\delta$ -bases, we have the effective solution for the ideal membership problem, variable elimination problem and effective intersection of ideals. We also can calculate a generating system of the  $\mathcal{H}[\partial]$ -module of syzygies of a finite subset  $\{P_1, \dots, P_r\}$  of  $\mathcal{H}[\partial]$ , as well as a free resolution of a finitely generated (left)  $\mathcal{H}[\partial]$ -module. Calculating free resolutions of a  $\mathcal{H}[\partial]$ -module, we have found examples where the use of Gröbner  $\delta$ -bases is, in some sense, more efficient than the one of Gröbner bases (see [MOR, page 189-190]).

In this section the ring  $\mathcal{H}$  is a noetherian sub- $\mathbf{k}$ -algebra of

$$\mathbf{k}((\tilde{X})) = \mathbf{k}((x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})),$$

stable under the action of  $\partial_i$  for  $i = 1, \dots, n$  and satisfying the two additional conditions of Remark 12. We denote as before  $\mathcal{D} = \mathcal{H}[\partial] = \mathcal{H}[\partial_1, \dots, \partial_n]$ .

The aim of this section is to characterize flatness and finiteness of a  $\mathcal{D}$ -module by using the notion of Gröbner  $\delta$ -bases, following the work of A. Assi [ASS-2] in the commutative case.

We can see the quotient  $\mathcal{D}/I$  as a family of  $A_n(\mathbf{k})$ -modules, the space of parameters being  $\mathbf{C}^m$ . In this section we will see when this family is flat.

Let  $S$  be a multiplicatively closed subset of  $\mathcal{H}$ . The ring  $S^{-1}\mathcal{H}$  is a noetherian sub- $\mathbf{k}$ -algebra of  $\mathbf{k}((\tilde{X}))$ , stable under the action of the derivations  $\partial_1, \dots, \partial_n$  and satisfying the two additional conditions of Remark 12. So, we can consider the sub- $\mathbf{k}$ -algebra  $S^{-1}\mathcal{D}$  of  $\mathbf{k}((\tilde{X}))[\partial]$ , generated by  $S^{-1}\mathcal{H}$  and  $\partial_1, \dots, \partial_n$ .

One can define in  $S^{-1}\mathcal{D}$  the notions of section 2.

Let  $I \subset \mathcal{D}$  a left ideal. We denote by  $S^{-1}I$  the ideal of  $S^{-1}\mathcal{D}$  generated by  $I$  and by  $\text{in}^\delta(S^{-1}I)$  the ideal (of  $S^{-1}\mathcal{H}[\zeta]$ ) generated by  $\{\text{in}^\delta(P) : P \in (S^{-1}I) \setminus \{0\}\}$ . Here  $S^{-1}\mathcal{H}[\zeta]$  denotes the polynomial ring in the variables  $\zeta = (\zeta_1, \dots, \zeta_n)$  and coefficients in  $S^{-1}\mathcal{H}$ . We have:

**Proposition 28** Suppose  $\{P_1, \dots, P_r\}$  is a Gröbner  $\delta$ -base of  $I$ . If  $S$  be a multiplicatively closed subset of  $\mathcal{H}$  then  $\text{in}^\delta(S^{-1}I)$  is generated by  $\{\text{in}^\delta(P_1), \dots, \text{in}^\delta(P_r)\}$  in  $S^{-1}\mathcal{H}[\zeta]$ . In particular,  $\text{in}^\delta(S^{-1}I) = S^{-1}(\text{in}^\delta(I))$  and  $\{P_1/1, \dots, P_r/1\}$  is a Gröbner  $\delta$ -base of  $S^{-1}I$ .

Let  $\mathfrak{P}$  be a prime ideal of  $\mathcal{H}$ . Then  $S = \mathcal{H} \setminus \mathfrak{P}$  is a multiplicatively closed subset of  $\mathcal{H}$ . We denote  $\mathcal{H}_{\mathfrak{P}} = S^{-1}\mathcal{H}$ ,  $\mathcal{D}_{\mathfrak{P}} = S^{-1}\mathcal{D}$  and  $I_{\mathfrak{P}} = S^{-1}I$  and .

For each ideal  $\mathcal{K}$  in  $\mathcal{H}$ , we denote  $V(\mathcal{K}) = \{\mathfrak{P} \in \text{Spec}(\mathcal{H}) : \mathcal{K} \subseteq \mathfrak{P}\}$ , which is a Zariski closed subset of  $\text{Spec}(\mathcal{H})$ . Here we endowed the set  $\text{Spec}(\mathcal{H})$  of prime ideals of  $\mathcal{H}$  with its Zariski topology.

Let consider  $J = \prod_{i=1}^s C(\alpha(i); I)$  as an ideal in  $\mathcal{H}$ , where  $\{\alpha(1), \dots, \alpha(s)\}$  is a  $\delta$ -stair of the ideal  $I$  (see Remark 5). Let us denote  $U = \text{Spec}(\mathcal{H}) \setminus V(J)$ . We have:

**Theorem 29** With the notations as above, let  $\mathfrak{P} \in U$ . Then  $\mathcal{D}_{\mathfrak{P}}/I_{\mathfrak{P}}$  is a free (and then a flat)  $\mathcal{H}_{\mathfrak{P}}$ -module.

*Proof.* Let  $M$  be the free  $\mathcal{H}_{\mathfrak{P}}$ -module generated by  $\{\partial^\alpha : \alpha \in \mathbf{N}^n \setminus \text{Exp}^\delta(I)\}$ . Obviously we have  $\text{Exp}^\delta(I) = \text{Exp}^\delta(I_{\mathfrak{P}})$ . Let us consider a Gröbner  $\delta$ -base  $\{P_1, \dots, P_r\}$  of  $I$ . By Proposition 28,  $\{P_1/1, \dots, P_r/1\}$  is a Gröbner  $\delta$ -base of  $I_{\mathfrak{P}}$ . Now, applying the reduction algorithm with respect to  $\{P_1/1, \dots, P_r/1\}$  (see Theorem 15), each  $P \in \mathcal{D}_{\mathfrak{P}}$  can be written as a sum

$$P = P' + P''$$

with  $P' \in I_{\mathfrak{P}}$  and  $P'' \in M$ . Here we have used the equality  $C(\alpha(i); I_{\mathfrak{P}}) = \mathcal{H}_{\mathfrak{P}}$  for each  $i = 1, \dots, s$ .

So, we have proved that  $\mathcal{D}_{\mathfrak{P}} = I_{\mathfrak{P}} + M$  and it is obvious that  $I_{\mathfrak{P}} \cap M = (0)$ , so the  $\mathcal{H}_{\mathfrak{P}}$ -modules  $\mathcal{D}_{\mathfrak{P}}/I_{\mathfrak{P}}$  and  $M$  are isomorphic. Then  $M$  is a free  $\mathcal{H}_{\mathfrak{P}}$ -module.  $\square$

**Proposition 30** *With the notations as above, we have*

1. *If  $C(0; I) = I \cap \mathcal{H} \neq (0)$ , then  $U = \text{Spec}(\mathcal{H}) \setminus V(C(0; I))$  is the maximal open set of flatness.*
2. *If  $C(\alpha(k); I) = \mathcal{H}$  for each  $k \in \{1, \dots, s\}$ , then  $\mathcal{D}/I$  is a flat  $\mathcal{H}$ -module.*

*Proof.*

1. We have  $C(0; I) = \mathcal{H} \cap I$  (see Remark 11). Suppose  $U$  is not maximal, then there exists  $\mathfrak{P} \in \text{Spec}(\mathcal{H}) \setminus U$  such that  $\mathcal{D}_{\mathfrak{P}}/I_{\mathfrak{P}}$  is  $\mathcal{H}_{\mathfrak{P}}$ -flat. If  $C(0; I) \neq (0)$  then  $\mathcal{H}_{\mathfrak{P}} \cap I_{\mathfrak{P}} \neq (0)$ , which is impossible by flatness of  $\mathcal{H}_{\mathfrak{P}}[\partial]/I_{\mathfrak{P}}$  over  $\mathcal{H}_{\mathfrak{P}}$ .
2. We have  $C(\alpha_k; I) = \mathcal{H}$  for each  $\alpha_k$  in a  $\delta$ -stair of  $I$ . So, we have  $U = \text{Spec}(\mathcal{H})$  and then  $\mathcal{H}[\partial]/I$  is  $\mathcal{H}$ -flat.

$\square$

**Example 3** *Let us denote  $\mathbf{C}[X] = \mathbf{C}[x_1, \dots, x_n]$  and consider the ideal of Example 2, i.e.  $I = \{P_1, P_2\} \subset \mathcal{D} = \mathbf{C}[X][\partial_1, \partial_2]$  where*

$$P_1 = x_1\partial_1 + x_1\partial_2 + b, \quad P_2 = (x_2 - x_1)\partial_2 - d$$

*with  $b \in \mathbf{C}[x_1, x_3, \dots, x_n]$  and  $d \in \mathbf{C}[x_3, \dots, x_n]$ . We will suppose  $b$  is a multiple of  $x_1$ . In particular,  $\mathcal{D}/I$  is not a flat  $\mathbf{C}[x_1, \dots, x_n]$ -module, because the class of  $\partial_1 + \partial_2 + b/x_1$  mod.  $I$  has  $x_1$ -torsion.*

*We know by Example 2 and Remark 27 that  $\{P_1, P_2\}$  is a Gröbner  $\delta$ -base of  $I$ . Then a  $\delta$ -stair of  $I$  is  $\{\text{exp}^\delta(P_1), \text{exp}^\delta(P_2)\}$ , i.e.  $\{(1, 0), (0, 1)\}$ .*

*Moreover, by Theorem 18, we have*

$$C((1, 0); I) = C((1, 0); P_1, P_2) = \langle x_1 \rangle, \quad C((0, 1); I) = C((0, 1); P_1, P_2) = \langle x_2 - x_1 \rangle.$$

*Let us consider  $J = C((1, 0); I)C((0, 1); I)$ , i.e.  $J = \langle x_1(x_2 - x_1) \rangle$ . By Theorem 29,  $\mathbf{C}[X]_{\mathfrak{P}}[\partial_1, \partial_2]/I_{\mathfrak{P}}$  is a flat  $\mathbf{C}[X]_{\mathfrak{P}}$ -module for  $\mathfrak{P} \in U = \text{Spec}(\mathbf{C}[X]) \setminus V(J)$ .*

**Theorem 31** *Let  $I$  be an ideal of  $\mathcal{D}$ . The following are equivalent:*

1.  *$\mathcal{D}/I$  is a finitely generated  $\mathcal{H}$ -module.*
2. *For each  $i = 1, \dots, n$  there exists  $a_i \in \mathbf{N}$  such that*

$$\alpha(i) = a_i \epsilon_i \in \text{Exp}^\delta(I)$$

*and*

$$C(\alpha(i); I) = \mathcal{H},$$

*here  $\epsilon_i$  is the  $i$ -th element of the canonical base of  $\mathbf{N}^n$ .*

*Proof.*  $1 \implies 2$ .: For each  $i \in \{1, \dots, n\}$  we consider the sub- $\mathcal{H}$ -module  $M \subset \mathcal{D}/I$  generated by the set

$$\left\{ 1 + I, \partial_i + I, \dots, \partial_i^k + I, \dots \right\}.$$

So, by the finiteness of  $M$  over  $\mathcal{H}$ , there exists  $\alpha(i) = a_i \epsilon_i \in \text{Exp}^\delta(I)$  such that  $C(\alpha(i); I) = \mathcal{H}$ , for some  $a_i \in \mathbf{N}$ .

$2 \implies 1$ .: Let us write  $\overline{\Delta} = \mathbf{N}^n \setminus \bigcup_{i=1}^n (\alpha(i) + \mathbf{N}^n)$ . Let us consider  $M$  as the  $\mathcal{H}$ -module generated by the finite set  $\{\partial^\alpha; \alpha \in \overline{\Delta}\}$ . We have  $\mathcal{D} = I + M$  and then  $\mathcal{D}/I$  is a quotient of  $M$ . Thus  $\mathcal{D}/I$  is finitely generated as  $\mathcal{H}$ -module.  $\square$

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