Rate of growth of hypercyclic and frequently hypercyclic functions for the Dunkl operator

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Abstract

For the Dunkl operator Λ_α ($\alpha > -1/2$) on the space of entire functions on the complex space $\mathbb{C}$, the critical rate of growth for the integral means $M_p(f, r)$ of their hypercyclic functions f is obtained. The rate of growth of the corresponding frequently hypercyclic functions is also analyzed.

1 Introduction. In this paper, we consider the vector space $H(\mathbb{C})$ of all entire functions $\mathbb{C} \to \mathbb{C}$, endowed with the topology of local uniform convergence. Under this topology, $H(\mathbb{C})$ becomes an F-space, that is, a completely metrizable topological vector space. Recall that a (continuous and linear) operator T on a (Hausdorff) topological vector space X is said to be *hypercyclic* if there exists a vector $x \in X$, also called *hypercyclic*, whose orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in X. We refer the reader to the books [6] and [26] for rather complete accounts and further information on hypercyclic operators.

In 1952, MacLane [32] proved that the differentiation operator $D : H(\mathbb{C}) \to H(\mathbb{C})$, given by

$$Df(z) = f'(z),$$

is hypercyclic. Moreover, he showed that D-hypercyclic entire functions can be of exponential type 1. In 1984, Duyos-Ruiz [19] they cannot be of exponential type less than 1. An optimal result on the possible rates of growth for the differentiation operator was subsequently obtained by Grosse-Erdmann [22] and, independently, by Shkarin [37]. Specifically, they obtained that there is no D-hypercyclic entire function f for which there exists a constant $C > 0$ satisfying $|f(z)| \leq C \frac{e^r}{\sqrt{r}}$ for $|z| = r$ large enough but, given a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(r) \to \infty$ as $r \to \infty$, there exists a D-hypercyclic entire function such that $|f(z)| \leq \varphi(r) \frac{e^r}{\sqrt{r}}$ for $|z| = r$ large enough.

The notion of frequent hypercyclicity that was recently introduced by Bayart and Grivaux [4], [5]. We recall that the lower density of a subset A of $\mathbb{N}$ is defined as

$$\underline{\mathrm{dens}}\,(A) = \liminf_{N \to \infty} \frac{\#\{n \in A : n \leq N\}}{N},$$

where $\#$ denotes the cardinality of a set.

A vector $x \in X$ is called *frequently hypercyclic* for T if, for every non-empty open subset U of X,

$$\underline{\mathrm{dens}}\,\{n \in \mathbb{N} : T^n x \in U\} > 0.$$

The operator T is called *frequently hypercyclic* if it possesses a frequently hypercyclic vector.

The problem of determining possible rates of growth of frequently hypercyclic entire functions for differentiation operator was studied in [10], [11] and [17]. More generally, the problem of isolating the possible rates of growth of hypercyclic or frequently hypercyclic entire functions for convolution operators on $H(\mathbb{C})$ (operators that conmute with the differentiation operator) has been considered in [8], [13] and [14]. Finally, the rates of growth of hypercyclic entire functions for weighted backward shifts B_w on $H(\mathbb{C})$ –considered as a special sequence space– have been established in [25]. More classes of hypercyclic non-convolution operators on $H(\mathbb{C})$ –different from composition operators– have been analyzed by a number of authors, see e.g. [2, 21, 28, 29, 31, 33, 34, 35].

One remarkable example is the *Dunkl operator*

$$\Lambda_\alpha : H(\mathbb{C}) \to H(\mathbb{C}) \quad (\alpha > -\frac{1}{2}),$$

which is a differential-difference operator given by

$$\Lambda_\alpha f(z) = \frac{d}{dz} f(z) + \frac{2\alpha + 1}{z} \Big(\frac{f(z) - f(-z)}{2} \Big).$$

Note that for $\alpha = -\frac{1}{2}$ we get $\Lambda_\alpha = D$. The operator Λ_α was introduced in 1989 by Dunkl [18]. It is connected to the theory of sampling signals, and there are in the literature a lot of papers dealing with the Dunkl operator, see for instance [16] and the references given in it.

Here we will concentrate on the dynamical aspects of Λ_α. The hypercyclicity of this operator was established by Betancor, Sifi and Trimeche in [9] (see also [30]and [15]). The purpose of this note is to study the rate of growth of hypercyclic and frequently hypercyclic entire functions for the Dunkl operator. In fact, the critical rate of growth for the p-integral means $M_p(f, r)$ of their hypercyclic functions f is obtained. Moreover, we get permissible and non-permissible rates of growth for the means of the corresponding frequently hypercyclic functions.

2 Preliminaries and notation. As usual, throughout this paper constants $C > 0$ can take different values at different occurrences. We write $a_n \sim b_n$ for positive sequences (a_n) and (b_n) if a_n/b_n and b_n/a_n are bounded.

For an entire function f and $1 \le p < \infty$ we consider the p-integral means

$$M_p(f, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p} \quad (r > 0)$$

and

$$M_\infty(f, r) = \sup_{|z|=r} |f(z)| \quad (r > 0).$$

These means have been considered by Blasco, Bonet, Bonilla and Grosse-Erdmann (see [10, 11]) in order to establish the possible rates of growth of hypercyclic or frequently hypercyclic entire functions for the operator D. Some of the techniques from [10, 11, 22] will be used and generalized in this paper in order to analyze the corresponding problem for Λ_α.

With this aim, some properties of the Dunkl operators are needed. Let us define

$$d_n(\alpha) = \frac{2^n \left([\frac{n}{2}]! \right)}{\Gamma(\alpha + 1)} \Gamma \left([\frac{n+1}{2}] + \alpha + 1 \right) \quad \text{for } n \ge 0 \text{ and } \alpha > -\frac{1}{2},$$

and $d_n(\alpha) = 0$ when $n < 0$, where $[x]$ denotes the integer part of x. Notice that $d_n(\alpha)$ tends rapidly to $+\infty$ as $n \to \infty$. Then, for every $k \in \mathbb{N}$, we have (see e.g. [9, pag. 106]) that

$$\Lambda_\alpha^k(z^n) = \frac{d_n(\alpha)}{d_{n-k}(\alpha)} z^{n-k} \quad \text{for all } z \in \mathbb{C}.$$

Moreover, given $f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} z^n \in H(\mathbb{C})$, the following holds:

$$\Lambda_\alpha^n f(0) = \frac{f^{(n)}(0)}{n!} d_n(\alpha).$$

By using Stirling's asymptotic formula (see e.g. [1]) $\Gamma(x) = (2\pi)^{1/2} x^{x-\frac{1}{2}} e^{-x} \Psi(x)$ for $x > 0$ (where $\Psi(x) \to 1$ as $x \to +\infty$) and the facts $\Gamma(n+1) = n!$, $[\frac{n}{2}] + [\frac{n+1}{2}] = n$ $(n \ge 0)$, one can easily obtain the following lemma, which will be used in the forthcoming sections.

Lemma 1. *For each $\alpha > -1/2$, we have the equivalence*

$$d_n(\alpha) \sim \frac{(n + \alpha + 1)^{n+\alpha+1}}{e^{n+\alpha+1}}.$$

An important tool in the setting of linear dynamics is the Universality Criterion. There are several versions of it, see for instance [23] or [26]. The one given in Theorem 1 is sufficient for our goals. Prior to this, it is easy to extend the notion of hypercyclicity. A sequence of continuous linear mappings

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$T_n : X \to Y \ (n \geq 1)$ between two topological vector spaces X, Y is said to be *universal* whenever there is a vector $x_0 \in X$ –called universal for (T_n)– such that the set $\{T_n x_0 : n \in \mathbb{N}\}$ is dense in Y. Of course, an operator $T : X \to X$ is hypercyclic if and only if the sequence (T^n) of its iterates is universal.

Theorem 1. *Assume that X and Y are topological vector spaces, such that X is a Baire space and Y is separable and metrizable. Let $T_n : X \to Y \ (n \geq 1)$ be a sequence of continuous linear mappings. Suppose also that there subsets $X_0 \subset X$ and $Y_0 \subset Y$ that are respectively dense in X and Y satisfying that, for any pair $(x_0, y_0) \in X_0 \times Y_0$, there are sequences $\{n_1 < n_2 < \cdots < n_k < \cdots\} \subset \mathbb{N}$ and $(x_k) \subset X$ such that*

$$x_k \to 0, \ T_{n_k} x_0 \to 0 \quad and \quad T_{n_k} x_k \to y_0 \quad as \quad n \to \infty.$$

Then (T_n) is universal. In fact, the set of universal vectors for (T_n) is residual (hence dense) in X.

Corresponding notion and criterium for frequent universal sequences will be given in Section 4. In that section the well-known Hausdorff–Young inequality (see for instance [27] or [36]) will be used several times. We state a version of it in the next theorem for the sake of completeness. Recall that if $p \in [1, \infty]$ then the conjugate exponent of p is the unique $q \in [1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 2. *Consider the spaces $L^p([0, 2\pi])$ of Lebesgue integrable functions $[0, 2\pi] \to \mathbb{C}$ of order p. For each $F \in L^p([0, 2\pi])$, let $\hat{F}(n) = \frac{1}{2\pi} \int_0^{2\pi} F(t) e^{-int} \, dt$ $(n \geq 0)$ be the sequence of its Fourier coefficients. If $1 < p \leq 2$ and q is the conjugate exponent of p then*

$$\left(\sum_{n=0}^{\infty} |\hat{F}(n)|^q \right)^{1/q} \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |F(t)|^p \, dt \right)^{1/p}.$$

3 Hypercyclic entire functions for the Dunkl operator.

For hypercyclicity with respect to the Dunkl operator, the rate of growth $e^r/r^{\alpha+1}$ turns out to be critical. We denote $\mathbb{R}^+ := (0, +\infty)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Theorem 3. *Let $1 \leq p \leq \infty$. Then the following holds:*

(a) *For any function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(r) \to \infty$ as $r \to \infty$ there is a Λ_α-hypercyclic entire function f with*

$$M_p(f, r) \leq \varphi(r) \frac{e^r}{r^{\alpha+1}} \quad for \ r > 0 \ sufficiently \ large.$$

(b) *There is no Λ_α-hypercyclic entire function f satisfying, for some $C > 0$, that*

$$M_p(f, r) \leq C \frac{e^r}{r^{\alpha+1}} \quad for \ all \ r > 0.$$

Proof. Thanks to the elementary inequality

$$M_p(f,r) \leq M_q(f,r) \ \text{ for all } r > 0 \ \ (1 \leq p \leq q)$$

it is enough to prove part (a) for the case $p = \infty$ and part (b) for the case $p = 1$.

Let us prove (a). We can assume without loss of generality that $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(r) \to \infty$ as $r \to \infty$ is monotonic and continuous on $[0, +\infty)$ with $\varphi(0) > 0$. Consider the following space

$$\widetilde{X} = \left\{ f \in H(\mathbb{C}) : \sup_{r>0} \frac{M_\infty(f,r)\, r^{\alpha+1}}{\varphi(r)\, e^r} < \infty \right\}.$$

It is not difficult to see that $\widetilde{X}$ is a Banach space under the norm $\|f\| := \sup_{r>0} \frac{M_\infty(f,r)\, r^{\alpha+1}}{\varphi(r)\, e^r}$ that is continuously embedded in $H(\mathbb{C})$.

In addition, we define the set X as the closure of the polynomials in $\widetilde{X}$, and the set $X_0 = Y_0 \subset H(\mathbb{C})$ as the collection of polynomials in z. Note that, trivially, X_0 is dense in X and Y_0 is dense in $Y := H(\mathbb{C})$. Now, for $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, we consider the mappings

$$T_n : X \to Y, \ T_n = \Lambda_\alpha^n|_X,$$

which are continuous, and

$$S_n : Y_0 \to X, \ S_n = S^n \ \text{ with } \ S(z^k) = \frac{d_k(\alpha)}{d_{k+1}(\alpha)} z^{k+1},$$

extended linearly to Y_0.

Now, fix a pair of polynomials P, Q, that is, $(P, Q) \in X_0 \times Y_0$. Then $P(z) = \sum_{k=0}^p a_k z^k$ and $Q(z) = \sum_{k=0}^q b_k z^k$ for certain $a_0, \dots, a_p, b_0, \dots, b_q \in \mathbb{C}$. Take as (n_k) the full sequence $\{1, 2, 3, \dots\}$ of natural numbers and $f_n(z) := S^n Q$ (with $S^n = S \circ S \circ \cdots \circ S$, n times) for all $n \in \mathbb{N}$. It follows from the properties of Λ_α given in Section 2 that $T_n f_n = Q$ (so, trivially, $T_n f_n \to Q$ as $n \to \infty$) and $T_n P \to 0$ in $H(\mathbb{C})$ as $n \to \infty$ (because $\Lambda_\alpha^n(z^k) = 0$ if $n \geq k$).

Thus the conditions of the Universality Criterion (Theorem 1) are satisfied if we can show that $\{f_n\}_{n \geq 1}$ converges to 0 in X. By linearity, one can assume that $Q(z) = z^k$, a monomial ($k \in \mathbb{N}_0$), in which case

$$\lim_{n\to\infty} S_n Q(z) = \lim_{n\to\infty} \frac{d_k(\alpha)}{d_{k+n}(\alpha)} z^{k+n}.$$

Therefore, all we need to show is that

$$\lim_{n\to\infty} \frac{z^n}{d_n(\alpha)}$$

converges to 0 in X.

To this end, fix $\varepsilon > 0$ as well as an $n \in \mathbb{N}$. We choose $R > 0$ such that $\varphi(r) \geq 1/\varepsilon$ for $r \geq R$. Then we have that

$$\sup_{r \leq R} \frac{r^{\alpha+1}}{\varphi(r)e^r} \frac{r^n}{d_n(\alpha)} \leq \frac{R^{\alpha+1}}{\inf_{r>0} \varphi(r)} \frac{R^n}{d_n(\alpha)} \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover a simple calculation involving the derivative of $r^{n+\alpha+1}e^{-r}$ shows that

$$\sup_{r \geq R} \frac{1}{\varphi(r)} \frac{r^{\alpha+1}}{e^r} \frac{r^n}{d_n(\alpha)} \leq \varepsilon \frac{(n+\alpha+1)^{n+\alpha+1}}{e^{n+\alpha+1} d_n(\alpha)},$$

and Lemma 1 implies that the last term is bounded by $C\varepsilon$ for any $n \in \mathbb{N}$. This shows that $\lim_{n\to\infty} \frac{z^n}{d_n(\alpha)}$ converges to 0 in X.

In order to prove part (b), the use of the Cauchy estimates leads us to

$$|\Lambda_\alpha^n f(0)| = \left| \frac{f^{(n)}(0)}{n!} \right| d_n(\alpha) \leq \frac{d_n(\alpha)}{r^n} M_1(f,r).$$

Assume, by way of contradiction, that there is $C > 0$ such that $M_1(f,r) \leq C \frac{e^r}{r^{\alpha+1}}$ for all $r > 0$. Hence we find that

$$|\Lambda_\alpha^n f(0)| \leq C \frac{d_n(\alpha)}{r^{n+\alpha+1}} e^r \leq C \frac{d_n(\alpha)}{(n+\alpha+1)^{n+\alpha+1}} e^{n+\alpha+1} \quad \text{for all } n \geq 1.$$

Now Lemma 1 implies that the sequence $\{\Lambda_\alpha^n f(0)\}_{n \geq 1}$ is bounded, so that f cannot be hypercyclic for Λ_α. This is the desired contradiction. $\qquad\square$

Remark 1. An operator $T : H(\mathbb{C}) \to H(\mathbb{C})$ is called a *weighted backward shift* if there is a sequence $\{a_n\}_{n \geq 1} \subset \mathbb{C} \setminus \{0\}$ such that

$$(T f)(z) = \sum_{n=0}^{\infty} a_{n+1} c_{n+1} z^n \quad (z \in \mathbb{C})$$

provided that $f(z) = \sum_{n=0}^{\infty} c_n z^n$. We denote this operator by $T = B_{(a_n)}$. It is not difficult to see that $B_{(a_n)}$ is well defined and continuous if and only if $\sup_{n \in \mathbb{N}} |a_n|^{1/n} < \infty$. In [24] (see also [7]) is was proved that, under the assumption $\sup_{n \in \mathbb{N}} |a_n|^{1/n} < \infty$, the operator $B_{(a_n)}$ is hypercyclic if and only if $\limsup_{n \to \infty} \left| \prod_{k=1}^{n} a_k \right|^{1/n} = \infty$. This result contains MacLane's theorem as a special case because the differentiation operator D is the weighted backward shift $B_{(a_n)}$ with $a_n = n$ $(n \in \mathbb{N})$. Now, observe that, for $\alpha > -1/2$, the Dunkl operator is $B_{(a_n)}$ with $a_n = \frac{d_n(\alpha)}{d_{n-1}(\alpha)}$. In this case, from Lemma 1 one easily derives that $\sup_{n \in \mathbb{N}} |a_n|^{\frac{1}{n}} < \infty$ and $\limsup_{n \to \infty} \left| \prod_{k=1}^{n} a_k \right|^{1/n} = \limsup_{n \to \infty} d_n(\alpha)^{\frac{1}{n}} = C \lim_{n \to \infty} (n+\alpha+1)^{\frac{n+\alpha+1}{n}} = \infty$. Then the Dunkl operator is hypercyclic (note that we get an alternative proof of this result given in [9]; as a matter of fact, a much more general result is shown in [9, Theorem 4.1]).

Moreover, Grosse-Erdmann proved in [24, Theorems 1–2] that, under the assumption that $(|a_n|)$ is nondecreasing, the critical rate of growth of $M_\infty(f, \cdot)$ allowed for a $B_{(a_n)}$-hypercyclic entire function f is $\mu(r) = \max_{n \geq 0} |r^n / \prod_{k=1}^n a_k|$. If $B_{(a_n)} = \Lambda_\alpha$ then $(|a_n|)$ is nondecreasing and $\mu(r) = \max_{n \geq 0} |r^n / d_n(\alpha)|$, so the cited theorems of [24] are at our disposal. Nevertheless, we have opted for a more direct proof, which in turn yields critical rates for all $M_p(f, \cdot)$ $(1 \leq p \leq \infty)$.

4 Frequently hypercyclic entire functions for the Dunkl operator.

First of all, it is easy to extend the notion of frequent hypercyclicity to a sequence of mappings, see [12].

Definition 1. Let X and Y be topological spaces, and let $T_n : X \to Y$ $(n \in \mathbb{N})$ be a sequence of continuous linear mappings. Then an element $x \in X$ is called *frequently universal* for the sequence (T_n) if, for every non-empty open subset U of Y,

$$\underline{\mathrm{dens}}\{n \in \mathbb{N} : T_n x \in U\} > 0.$$

The sequence (T_n) is called *frequently universal* if it possesses a frequently universal element.

Of course, an operator $T : X \to X$ on a topological vector space X is frequently hypercyclic if and only if the sequences of its iterates (T^n) is frequently universal. In [12], a *Frequent Universality Criterion* was obtained that generalizes the Frequent Hypercyclicity Criterion of Bayart and Grivaux [5]. We state it here (see Theorem 4 below) only for Fréchet spaces. Recall that a collection of series $\sum_{k=1}^\infty x_{k,j}$ $(j \in I)$ in a Fréchet space X is said to be *unconditionally convergent, uniformly in $j \in I$,* if for every continuous seminorm p on X and every $\varepsilon > 0$ there is some $N \geq 1$ such that for every finite set $F \subset \mathbb{N}$ with $F \cap \{1, 2, \ldots, N\} = \varnothing$ and every $j \in I$ we have that $p(\sum_{k \in F} x_{k,j}) < \varepsilon$.

Theorem 4. *Let X be a Fréchet space, Y a separable Fréchet space, and $T_n : X \to Y$ $(n \in \mathbb{N})$ a sequence of operators. Suppose that there are a dense subset Y_0 of Y and mappings $S_n : Y_0 \to X$ $(n \in \mathbb{N})$ such that, for all $y \in Y_0$,*

(i) $\sum_{n=1}^k T_k S_{k-n} y$ *converges unconditionally in Y, uniformly in $k \in \mathbb{N}$,*

(ii) $\sum_{n=1}^\infty T_k S_{k+n} y$ *converges unconditionally in Y, uniformly in $k \in \mathbb{N}$,*

(iii) $\sum_{n=1}^\infty S_n y$ *converges unconditionally in X,*

(iv) $T_n S_n y \to y$ *as $n \to \infty$.*

Then the sequence (T_n) is frequently universal.

We note that the sums in (i) can be understood as infinite series by adding zero terms.

As an auxiliary result we shall need the following estimate. In the following, we shall adopt the convention $\frac{1}{2p} = 0$ for $p = \infty$.

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Lemma 2. *Let* $1 \leq q \leq 2$ *and* p *be the conjugate exponent of* q. *Then there is some* $C > 0$ *such that, for all* $r > 0$, *we have*

$$\sum_{n=0}^{\infty} \frac{r^{qn}}{(d_n(\alpha))^q} \leq C \left(\frac{e^r}{r^{\alpha + \frac{1}{2} + \frac{1}{2p}}} \right)^q.$$

Proof. Consider the entire functions

$$E_\alpha(z; \theta, \beta) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\theta)^\beta \Gamma(\alpha n + 1)}$$

for $\alpha, \theta > 0, \beta \in \mathbb{R}$. In the special case of $\beta = 0$ these are the Mittag-Leffler functions, see for instance [20, Section 18.1]. Barnes [3, pp. 289–292] studied the functions $E_\alpha(z; \theta, \beta)$ and, by using Stirling's formula, he derived asymptotical expansions from which one deduces that, for $0 < \alpha \leq 2$,

(1) $\qquad E_\alpha(r; \theta, \beta) = \alpha^{\beta-1} r^{-\beta/\alpha} e^{r^{1/\alpha}} \left(1 + O(r^{-1/\alpha}) \right)$ as $r \to +\infty$.

Now, by the definition of $d_n(\alpha)$, we get

$$\sum_{n=0}^{\infty} \frac{r^{qn}}{(d_n(\alpha))^q} \leq \sum_{n=0}^{\infty} \frac{(\Gamma(\alpha+1))^q r^{2qn}}{(2^{2n} \, n! \, \Gamma(n+\alpha+1))^q} + \sum_{n=0}^{\infty} \frac{(\Gamma(\alpha+1))^q r^{(2n+1)q}}{(2^{2n+2} \, n! \, \Gamma(n+\alpha+2))^q}$$

$$= (\Gamma(\alpha+1))^q \left(\sum_{n=0}^{\infty} \frac{r^{2qn}}{2^{2qn} \, n!^q \, (\Gamma(n+\alpha+1))^q} + \frac{r^q}{2^{2q}} \sum_{n=0}^{\infty} \frac{r^{2nq}}{(2^{2qn} \, n!^q \, (\Gamma(n+\alpha+2))^q} \right)$$

Thanks to (1), we obtain for $r > 0$ that

$$\sum_{n=0}^{\infty} \frac{r^{2qn}}{2^{2qn} \, n!^q \, (\Gamma(n+\alpha+1))^q} \leq C \sum_{n=0}^{\infty} \frac{((qr)^{2q})^n}{(n+1)^{q(\alpha+1)-1/2} \Gamma(2qn+1)}$$

$$\leq C \left((qr)^{2q} \right)^{-\frac{q(\alpha+1)-\frac{1}{2}}{2q}} e^{\left((qr)^{2q} \right)^{\frac{1}{2q}}}$$

$$\leq C \left(\frac{e^r}{r^{\alpha + \frac{1}{2} + \frac{1}{2p}}} \right)^q,$$

where the constant C is not necessarily the same in each occurrence.

Analogously, we have for all $r > 0$ that

$$\frac{r^q}{2^{2q}} \sum_{n=0}^{\infty} \frac{r^{2nq}}{(2^{2qn} \, n!^q \, (\Gamma(n+\alpha+2))^q} \leq C r^q \left(\frac{e^r}{r^{\alpha+1+\frac{1}{2}+\frac{1}{2p}}} \right)^q \leq C \left(\frac{e^r}{r^{\alpha+\frac{1}{2}+\frac{1}{2p}}} \right)^q.$$

Finally, it is enough to add up both inequalities to get the desired result. $\qquad\square$

Our first main result in this section gives growth rates for which Λ_α-frequently hypercyclic functions exist.

Theorem 5. *Let* $1 \le p \le \infty$, *and put* $a = \alpha + \frac{1}{2} + \frac{1}{2\max\{2,p\}}$. *Then, for any function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\varphi(r) \to \infty$ *as* $r \to \infty$, *there is an entire function* f *with*

$$M_p(f,r) \le \varphi(r)\frac{e^r}{r^a} \quad \text{for } r > 0 \text{ sufficiently large}$$

that is frequently hypercyclic for the Dunkl operator.

Proof. Since

$$M_p(f,r) \le M_2(f,r) \quad \text{for } 1 \le p < 2,$$

we need only prove the result for $p \ge 2$.

Thus let $2 \le p \le \infty$. We shall make use of the Frequent Universality Criterion (Theorem 4). Assuming without loss of generality that $\inf_{r>0} \varphi(r) > 0$, we consider the vector space

$$X := \left\{ f \in H(\mathbb{C}) : \sup_{r>0} \frac{M_p(f,r)\,r^{\alpha+\frac{1}{2}+\frac{1}{2p}}}{\varphi(r)\,e^r} < \infty \right\}.$$

If we endow X with the norm $\|f\| := \sup_{r>0} \frac{M_p(f,r)r^{\alpha+\frac{1}{2}+\frac{1}{2p}}}{\varphi(r)\,e^r}$ then it is not difficult to see that $(X, \|\cdot\|)$ is a Banach space that is continuously embedded in $H(\mathbb{C})$.

Let $Y_0 \subset H(\mathbb{C})$ be the set of polynomials, and we consider the mappings

$$T_n = \Lambda_\alpha^n|_X : X \to H(\mathbb{C}),$$

which are continuous, and

$$S_n : Y_0 \to X, \ S_n = S^n \quad \text{with } S(z^k) = \frac{d_k(\alpha)}{d_{k+1}(\alpha)}z^{k+1}.$$

Then we have for any polynomial f and any $k \in \mathbb{N}$ that

$$\sum_{n=1}^{k} T_k S_{k-n}f = \sum_{n=1}^{k} \Lambda_\alpha^n f,$$

that converges unconditionally convergent in $H(\mathbb{C})$, uniformly for $k \in \mathbb{N}$, because $\sum_{n=1}^{\infty} \Lambda_\alpha^n f$ is a finite series. Moreover, we have according to the properties given in Section 2 that

$$T_n S_n f = f \quad \text{for any } n \in \mathbb{N},$$

and

$$\sum_{n=1}^{\infty} T_k S_{k+n}f = \sum_{n=1}^{\infty} S_n f.$$

Thus the conditions (i)–(iv) in Theorem 4 are satisfied if we can show that $\sum_{n=1}^{\infty} S_n f$ converges unconditionally in X, for any polynomial $f(z) = z^k$ ($k \in \mathbb{N}_0$). In which case

$$\sum_{n=1}^{\infty} S_n f(z) = \sum_{n=1}^{\infty} \frac{d_k(\alpha)}{d_{k+n}(\alpha)} z^{k+n}.$$

Therefore, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{z^n}{d_n(\alpha)}$$

converges unconditionally in X. To this end, let $\varepsilon > 0$ and $N \in \mathbb{N}$. By Theorem 2 we obtain for any finite set $F \subset \mathbb{N}$ that

$$M_p\Big(\sum_{n \in F} \frac{z^n}{d_n(\alpha)}, r\Big) \leq \Big(\sum_{n \in F} \frac{r^{qn}}{(d_n(\alpha))^q}\Big)^{1/q},$$

where q is the conjugate exponent of p. Hence, if $F \cap \{0, 1, \ldots, N\} = \varnothing$, then

$$\Big\| \sum_{n \in F} \frac{z^n}{d_n(\alpha)} \Big\| \leq \Big(\sup_{r>0} \frac{r^{q((\alpha+\frac{1}{2})+\frac{1}{2p})}}{\varphi(r)^q e^{qr}} \sum_{n>N} \frac{r^{qn}}{(d_n(\alpha))^q}\Big)^{1/q}.$$

We choose $R > 0$ such that $\varphi(r)^q \geq 1/\varepsilon$ for $r \geq R$. Then we have that

$$\sup_{r \leq R} \frac{r^{q((\alpha+\frac{1}{2})+\frac{1}{2p})}}{\varphi(r)^q\, e^{qr}} \sum_{n>N} \frac{r^{qn}}{(d_n(\alpha))^q} \leq \frac{R^{q((\alpha+\frac{1}{2})+\frac{1}{2p})}}{\inf_{r>0}\varphi(r)^q} \sum_{n>N} \frac{R^{qn}}{(d_n(\alpha))^q} \longrightarrow 0$$

as $N \to \infty$. Moreover, Lemma 2 implies that

$$\sup_{r \geq R} \frac{1}{\varphi(r)^q} \frac{r^{q((\alpha+\frac{1}{2})+\frac{1}{2p})}}{e^{qr}} \sum_{n>N} \frac{r^{qn}}{(d_n(\alpha))^q} \leq C\varepsilon \quad \text{for any } N \in \mathbb{N},$$

where C is a constant only depending on q.

This shows that

$$\Big\| \sum_{n \in F} \frac{z^n}{d_n(\alpha)} \Big\|^q \leq (1 + C)\, \varepsilon,$$

if $\min F > N$ and N is sufficiently large, so that $\sum_{n=1}^{\infty} \frac{z^n}{d_n(\alpha)}$ converges unconditionally in X, as required. $\square$

The following result, that concludes the paper, gives lower estimates on the possible growth rates.

Theorem 6. *Let* $1 \leq p \leq \infty$, *and put* $a = \alpha + \frac{1}{2} + \frac{1}{2\min\{2,p\}}$. *Assume that* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a function with* $\psi(r) \to 0$ *as* $r \to \infty$. *Then there is no* Λ_α-*frequently hypercyclic entire function* f *that satisfies*

(2) $$M_p(f, r) \leq \psi(r)\frac{e^r}{r^a} \quad \text{for } r > 0 \text{ sufficiently large.}$$

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Proof. First, for $p = 1$ the result follows immediately from Theorem 3(b) (notice that one may even take $\psi(r) \equiv C$ here). Moreover, since

$$M_2(f, r) \leq M_p(f, r) \quad \text{for } 2 < p \leq \infty,$$

it suffices to prove the result for $p \leq 2$.

Thus let $1 < p \leq 2$ and q is the conjugate exponent of p. We obviously may assume that ψ is decreasing. Suppose that f satisfy (2). With the help of Theorem 2 we get that

$$\Big(\sum_{n=0}^{\infty} \Big(\frac{|f^{(n)}(0)|}{n!} r^n \Big)^q \Big)^{1/q} \leq M_p(f, r) \leq \psi(r) \frac{e^r}{r^{\alpha + \frac{1}{2} + \frac{1}{2p}}}$$

for $r > 0$ sufficiently large. Thus we have that, for large r,

$$(3) \qquad \sum_{n=0}^{\infty} \Big(\frac{|f^{(n)}(0)|}{n!} d_n(\alpha) \Big)^q \cdot \frac{r^{qn + \frac{q}{2p} + (\alpha + \frac{1}{2})q} e^{-qr}}{\psi(r)^q (d_n(\alpha))^q} \leq 1.$$

Using Stirling's formula we see that the function

$$g(r) := \frac{r^{qn + \frac{q}{2p} + (\alpha + \frac{1}{2})q} e^{-qr}}{(d_n(\alpha))^q}$$

has its maximum at $a_n := n + \frac{1}{2p} + \alpha + \frac{1}{2}$ of order $1/\sqrt{n}$ and an inflection point at $b_n := a_n + \sqrt{\frac{a_n}{q}}$. On $I_n := [a_n, b_n]$, g therefore dominates the linear function h that satisfies $h(a_n) = g(a_n)$, $h(b_n) = 0$.

Now let $m \in \mathbb{N}$. If m is sufficiently large and $m < n \leq 2m$ then $I_n \subset [m, 3m]$. Hence we have for these n that

$$\int_m^{3m} \frac{r^{qn + \frac{q}{2p} + (\alpha + \frac{1}{2})q} e^{-qr}}{\psi(r)^q (d_n(\alpha))^q} \, dr \geq \int_{I_n} \frac{h(r)}{\psi(r)^q} \, dr$$

$$\geq C \frac{1}{\psi(m)^q} \frac{1}{\sqrt{n}} \sqrt{\frac{n}{q} + \frac{\frac{1}{2p} + \alpha + \frac{1}{2}}{q}}$$

$$\geq C \frac{1}{\psi(m)^q}.$$

Now, integrating (3) over $[m, 3m]$ we obtain that for m sufficiently large

$$\frac{1}{m} \sum_{n=m+1}^{2m} \Big(\frac{|f^{(n)}(0)|}{n!} d_n(\alpha) \Big)^q \leq C \, \psi(m)^q.$$

Hence

$$\frac{1}{m} \sum_{n=0}^{m} \Big(\frac{|f^{(n)}(0)|}{n!} d_n(\alpha) \Big)^q \longrightarrow 0 \quad \text{as } m \to \infty.$$

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Therefore we have

$$\underline{\text{dens}}\{n \in \mathbb{N} : |\Lambda_\alpha^n f(0)| > 1\} = \liminf_{m \to \infty} \frac{1}{m} \#\{n \le m : |\Lambda_\alpha^n f(0)| > 1\}$$
$$\le \liminf_{m \to \infty} \frac{1}{m} \sum_{n=0}^{m} \Big(\frac{|f^{(n)}(0)|}{n!} d_n(\alpha) \Big)^q = 0.$$

If we take $U := (\delta_0)^{-1}(\{z \in \mathbb{C} : |z| > 1\})$ (where δ_0 represents the 0-evaluation functional $g \in H(\mathbb{C}) \mapsto g(0) \in \mathbb{C}$, that is continuous) and $T := \Lambda_\alpha$, then U is a nonempty open subset of $H(\mathbb{C})$ and the last display shows that $\underline{\text{dens}}\{n \in \mathbb{N} : T_n f \in U\} = 0$, which prevents f to be frequently hypercyclic for the Dunkl operator. This proves the theorem. $\square$

References

[1] L. V. Ahlfors, *Complex Analysis*, 3rd edition, McGraw-Hill, London, 1979.

[2] R.M. Aron and D. Markose, On universal functions, *J. Korean Math. Soc.* **41** (2004), 65–76.

[3] E. W. Barnes, The asymptotic expansion of integral functions defined by Taylor's series, *Phil. Trans. Royal Soc. London* A **206** (1906), 249–297.

[4] F. Bayart and S. Grivaux, Hypercyclicité: le rôle du spectre ponctuel unimodulaire, *C. R. Math. Acad. Sci. Paris* **338** (2004), 703–708.

[5] F. Bayart and S. Grivaux, Frequently hypercyclic operators, *Trans. Amer. Math. Soc.* **358** (2006), 5083–5117.

[6] F. Bayart and E. Matheron, *Dynamics of Linear Operators*, Cambridge Tracts in Mathematics, Cambridge University Press, 2009.

[7] L. Bernal-González, Universal functions for Taylor shifts, *Complex Variables* **31** (1996), 121–129.

[8] L. Bernal-González and A. Bonilla, Exponential type of hypercyclic entire functions, *Arch. Math. (Basel)* **78** (2002), 283–290.

[9] J.J. Betancor, M. Sifi and K. Trimeche, Hypercyclic and chaotic convolution operators associated with the Dunkl operator on $\mathbb{C}$, *Acta Math. Hungar.* **106** (1-2) (2005), 101–116.

[10] O. Blasco, A. Bonilla and K.-G. Grosse-Erdmann, Rate of growth of frequently hypercyclic functions, *Proc. Edinburgh Math. Soc.* **53**, (2010), 39–59.

[11] J. Bonet and A. Bonilla, Chaos of the differentiation operator on weighted Banach spaces of entire functions, *Complex Analysis Oper. Theory* **7** (2013), 33–42.

[12] A. Bonilla and K.-G. Grosse-Erdmann, Frequently hypercyclic operators and vectors, *Ergodic Theory Dynam. Systems* **27** (2007), 383-404.

[13] A. Bonilla and K.-G. Grosse-Erdmann, On a theorem of Godefroy and Shapiro, *Integral Equations Operator Theory* **56** (2006), 151–162.

[14] K.C. Chan and J.H. Shapiro, The cyclic behavior of translation operators on Hilbert spaces of entire functions, *Indiana Univ. Math. J.* **40** (1991), 1421–1449.

[15] F. Chouchene, H. Mejjaoli, M. Mili, and K. Trimeche, Hypercyclic and chaotic convolution associated with the Jacobi-Dunkl operator. *Mediterr. J. Math.* **11** (2014), 577–600.

[16] O. Ciaurri and J.L. Varona, Transformadas de Dunkl y teoremas de muestreo, *Bol. Soc. Esp. Mat. Apl.* **42** (2008), 103–115.

[17] D. Drasin and E. Saksman, Optimal growth of frequently hypercyclic entire functions, *J. Funct. Anal.* **263** (2012), 3674–3688.

[18] C.F. Dunkl, Differential-difference operators associated with reflections groups, *Trans. Amer. Math. Soc.* **311** (1989), 167–183.

[19] S.M. Duyos-Ruiz, Universal functions and the structure of the space of entire functions, *Soviet Math. Dokl.* **30** (1984), 713–716.

[20] A. Erdélyi, *Higher transcendental functions*, Volume III, McGraw-Hill Book Company, New York, 1955.

[21] G. Fernández and A.A. Hallack, Remarks on a result about hypercyclic non-convolution operators, *J. Math. Anal. Appl.* **309** (2005), 52–55.

[22] K.-G. Grosse-Erdmann, On the universal functions of G. R. MacLane, *Complex Variables Theory Appl.* 15 (1990), 193–196.

[23] K.-G. Grosse-Erdmann, Universal families and hypercyclic operators, *Bull. Amer. Math. Soc. (N.S.)* **36** (1999), 345–381.

[24] K.-G. Grosse-Erdmann, Hypercyclic and chaotic weighted shifts, *Studia Math.* **139** (2000), 47-68.

[25] K.-G. Grosse-Erdmann, Rate of growth of hypercyclic entire functions, *Indag. Math. (N.S.)* **11** (2000), 561–571.

[26] K.-G. Grosse-Erdmann and A. Peris, *Linear Chaos*, Springer, London, 2011.

[27] Y. Katznelson, *An introduction to harmonic analysis*, second edition, Dover Publications, New York, 1976.

[28] V.E. Kim, Hypercyclicity and chaotic character of generalized convolution operators generated by Gel'fond-Leont'ev Operators, *Math. Notes* **85** (6)(2009), 807–813.

[29] V.E. Kim, Commutation relations and hypercyclic operators, *Arch. Math.* **99** (2012), 247–253.

[30] V.E. Kim and V.V. Napalkov, Generalization of the Dunkl operator on the space of entire functions, *Doklady Math.* **77** (3) (2008), 365–367.

[31] F. León-Saavedra and P. Romero de la Rosa, Fixed points and orbits of non-convolution operators, *Fixed Point Theory Appl.* **2014**, 2014:221.

[32] G.R. MacLane, Sequences of derivatives and normal families, *J. Analyse Math.* **2** (1952/53), 72–87.

[33] H. Petersson, Supercyclic and hypercyclic non-convolution operators on Hilbert-Schmidt entire functions, *Math. Proc. R. Irish Acad.* **105 A** (2005), 87–101.

[34] H. Petersson, Supercyclic and hypercyclic non-convolution operators on Hilbert-Schmidt entire functions, *Math. Proc. Royal Irish Acad.* **105-A** (2) (2005), 87–101.

[35] H. Petersson, Supercyclic and hypercyclic non-convolution operators, *J. Operator Theory* **55** (2006), 135–151.

[36] W. Rudin, *Real and complex analysis*, third edition, McGraw-Hill Book Co., New York, 1987.

[37] S.A. Shkarin, On the growth of D-universal functions, *Moscow Univ. Math. Bull.* **48** (1993), 49–51.

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