On the "Favard theorem" and its extensions [1]

Francisco Marcellán [a,2] and Renato Álvarez-Nodarse [b,3]

[a] *Departamento de Matemáticas. Universidad Carlos III de Madrid. Ave. de la Universidad 30, E-28911 Leganés, Madrid, SPAIN*

[b] *Departamento de Análisis Matemático, Universidad de Sevilla, Apdo. 1160, E-41080 Sevilla, and Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, E-18071, Granada, SPAIN*

Abstract

In this paper we present a survey on the "Favard theorem" and its extensions.

Key words: Favard Theorem, recurrence relations

1 Introduction.

Given a sequence $\{P_n\}_{n=0}^{\infty}$ of monic polynomials satisfying a certain recurrence relation, we are interested in finding a general inner product, if one exists, such that the sequence $\{P_n\}_{n=0}^{\infty}$ is orthogonal with respect to it.

The original "classical" result in this direction is due to J. Favard [10] even though his result seems to be known by different mathematicians. The first who obtained a similar result was Stieltjes in 1894 [23]. In fact, from the point of view of $J-$continued fractions obtained from the contraction of an $S-$continued fraction with positive coefficients, Stieltjes proved the existence of a positive linear functional such that the denominators of the approximants are orthogonal with respect to it [23, §11]. Later on, Stone gave another approach using the spectral resolution of a self-adjoint operator associated to a Jacobi matrix [24, Th. 10.23]. In his paper [21, page 454] Shohat claims "We

[1] This work has been partially supported by the European project INTAS-93-219-ext and by the Spanish Dirección General de Enseñanza Superior (DGES) grant PB-96-0120-C03-01.

[2] Phone: +34 91 624 94 42, Fax: +34 91 624 91 51. E-mail: pacomarc@ing.uc3m.es

[3] Phone: +34 954 55 79 97, Fax: +34 95 455 79 72. E-mail: ran@cica.es

have been in possession of this proof for several years. Recently J. Favard published an identical proof in the *Comptes Rendus*". Also I. P. Natanson in his book [17, page 167] said "This theorem was also discovered (independent of Favard) by the author (Natanson) in the year 1935 and was presented by him in a seminar led by S. N. Bernstein. He then did not publish the result since the work of Favard appeared in the meantime". The "same" theorem was also obtained by Perron [19], Wintner [28] and Sherman [20], among others.

The Favard's result essencially means that if a sequence of monic polynomials $\{P_n\}_{n=0}^{\infty}$ satisfies a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x), \tag{1.1}$$

with a_n, $b_n \in \mathbb{R}$, $b_n > 0$, then there exists a positive Borel measure μ such that $\{P_n\}_{n=0}^{\infty}$ is orthogonal with respect to the inner product

$$\langle p, q \rangle = \int_{\mathbb{R}} p\, q\, d\mu. \tag{1.2}$$

This formulation is equivalent to the following: Given the linear operator t : $\mathbb{P} \to \mathbb{P}$, $p(t) \to tp(t)$, characterize an inner product such that the operator t is Hermitian with respect to the inner product.

A first extension of this problem is due to Chihara [5]. If $\{P_n\}_{n=0}^{\infty}$ satisfies a three-term recurrence relation like (1.1) with a_n, $b_n \in \mathbb{C}$, $b_n \neq 0$, find a linear functional $\mathfrak{L}$ defined on $\mathbb{P}$, the linear space of polynomials with complex coefficients, such that $\{P_n\}_{n=0}^{\infty}$ is orthogonal with respect to the general inner product $\langle p, q \rangle = \mathfrak{L}[pq]$, where $p, q \in \mathbb{P}$. Notice that in the case analyzed by Favard [10] the linear functional has an integral representation

$$\mathfrak{L}[p] = \int_{\mathbb{R}} p\, d\mu.$$

Favard's Theorem is an inverse problem in the sense that from information about polynomials we can deduce what kind of inner product induces orthogonality for such polynomials. The aim of this contribution is to survey some extensions of the Favard Theorem when a sequence of monic polynomials $\{P_n\}_{n=0}^{\infty}$ satisfies recurrence relations of a different form than (1.1).

In the first place, in [8] a similar problem is studied relating to polynomials orthogonal with respect to a positive Borel measure ν supported on the unit circle, which satisfy a recurrence relation

$$\Phi_n(z) = z\, \Phi_{n-1}(z) + \Phi_n(0)\Phi_{n-1}^*(z), \quad |\Phi_n(0)| < 1, \tag{1.3}$$

where $\Phi_n^*(z) = z^n \overline{\Phi_n(1/\overline{z})}$.

Thus, a Favard Theorem means, in this case, if we can identify an inner product in $\mathbb{P}$ such that $\{\Phi_n\}_{n=0}^{\infty}$ satisfying (1.3) is the corresponding sequence of orthogonal polynomials.

The structure of the paper is as follows. In Section 2 we present a survey of results surrounding the Favard Theorem when a sequence of polynomials satisfies a linear relation like (1.1). In particular, we show that the interlacing property for the zeros of two consecutive polynomials gives basic information about the preceding ones in the sequence of polynomials.

In Section 3, an analogous approach is presented in the case of the unit circle in a more general situation when $|\Phi_n(0)| \neq 1$. Furthermore, an integral representation for the corresponding inner product is given. The connection with the trigonometric moment problem is stated when we assume that the nth polynomial Φ_n is coprime with Φ_n^*.

In Section 4, we present some recent results about a natural extension of the above Favard theorems taking into account their interpretation in terms of operator theory. Indeed, the multiplication by t is a Hermitian operator with respect to (1.2) and a unitary operator with respect to the inner product

$$\langle p, q \rangle = \int_{\mathbb{R}} p(e^{i\theta}) \overline{q(e^{i\theta})} \, d\nu(\theta). \tag{1.4}$$

Thus we are interested in characterizing inner products such that the multiplication by a fixed polynomial is a Hermitian or a unitary operator. The connection with matrix orthogonal polynomials is stated, and some examples relating to Sobolev inner products are given.

2 The Favard theorem on the real line.

2.1 Preliminaries.

In this subsection we summarize some definitions and preliminary results that will be useful throughout the work. Most of them can be found in [5].

Definition 2.1 *Let $\{\mu_n\}_{n=0}^{\infty}$ be a sequence of complex numbers (moment sequence) and $\mathfrak{L}$ a functional acting on the linear space of polynomials $\mathbb{P}$ with complex coefficients. We say that $\mathfrak{L}$ is a moment functional associated with $\{\mu_n\}_{n=0}^{\infty}$ if $\mathfrak{L}$ is linear, i.e., for all polynomials π_1 and π_2 and any complex* *numbers* α_1 *and* α_2

$$\mathfrak{L}[\alpha_1\,\pi_1+\alpha_2\,\pi_2]=\alpha_1\mathfrak{L}[\pi_1]+\alpha_2\mathfrak{L}[\pi_2], \quad \textit{and} \quad \mathfrak{L}[x^n]=\mu_n, \quad n=0,1,2,\ldots \ .$$

Definition 2.2 *Given a sequence of polynomials* $\{P_n\}_{n=0}^{\infty}$*, we say that* $\{P_n\}_{n=0}^{\infty}$ *is a sequence of orthogonal polynomials (SOP) with respect to a moment functional* $\mathfrak{L}$ *if for all nonnegative integers* n *and* m *the following conditions hold:*

(1) P_n *is a polynomial of exact degree* n*,*
(2) $\mathfrak{L}[P_nP_m]=0, \quad m\neq n$*,*
(3) $\mathfrak{L}[P_n^2]\neq 0$*.*

Usually, the last two conditions are replaced by

$$\mathfrak{L}[x^mP_n(x)]=K_n\delta_{nm}, \quad K_n\neq 0\,, \quad 0\leq m\leq n\,,$$

where δ_{nm} *is the Kronecker symbol.*

The next theorems are direct consequences of the above definition [5, Chapter I, §2,3, pages 8-17].

Theorem 2.1 *Let* $\mathfrak{L}$ *be a moment functional and* $\{P_n\}_{n=0}^{\infty}$ *a sequence of polynomials. Then the following are equivalent:*

(1) $\{P_n\}_{n=0}^{\infty}$ *is an SOP with respect to* $\mathfrak{L}$*.*
(2) $\mathfrak{L}[\pi P_n]=0$ *for all polynomials* π *of degree* $m<n$*, while* $\mathfrak{L}[\pi P_n]\neq 0$ *if the degree of* π *is* n *.*
(3) $\mathfrak{L}[x^mP_n(x)]=K_n\delta_{nm}$*, where* $K_n\neq 0$*, for* $m=0,1,...,n$*.*

Theorem 2.2 *Let* $\{P_n\}_{n=0}^{\infty}$ *be an SOP with respect to* $\mathfrak{L}$*. Then, for every polynomial* π *of degree* n

$$\pi(x)=\sum_{k=0}^{n}d_kP_k(x), \quad \textit{where} \quad d_k=\frac{\mathfrak{L}[\pi P_k]}{\mathfrak{L}[P_k^2]}, \quad k=0,1,...,n. \tag{2.1}$$

A simple consequence of the above theorem is that an SOP is uniquely determined if we impose an additional condition that fixes the leading coefficient k_n of the polynomials ($P_n(x)=k_nx^n+$ lower order terms). When $k_n=1$ for all $n=0,1,2,\ldots$ the corresponding SOP is called a monic SOP (MSOP). If we choose $k_n=(\mathfrak{L}[P_n^2])^{-\frac{1}{2}}$, the SOP is called an orthonormal SOP (SONP).

The next question which obviously arises is the existence of an SOP. To answer this question, it is necessary to introduce the Hankel determinants Δ_n,

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}.$$

Theorem 2.3 *Let $\mathfrak{L}$ be a moment functional associated with the sequence of moments $\{\mu_n\}_{n=0}^{\infty}$. Then, the sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ is an SOP with respect to $\mathfrak{L}$ if and only if $\Delta_n \neq 0$ for all nonnegative n. Moreover, the leading coefficient k_n of the polynomial P_n is given by $k_n = \dfrac{K_n \Delta_{n-1}}{\Delta_n}$.*

Definition 2.3 *A moment functional $\mathfrak{L}$ is called positive definite if for every nonzero and nonnegative real polynomial π, $\mathfrak{L}[\pi] > 0$.*

The following theorem characterizes the positive definite functionals in terms of the moment sequences $\{\mu_n\}_{n=0}^{\infty}$. The proof is straightforward.

Theorem 2.4 *A moment functional $\mathfrak{L}$ is positive definite if and only if their moments are real and $\Delta_n > 0$ for all $n \geq 0$.*

Using the above theorem, we can define a positive definite moment functional $\mathfrak{L}$ entirely in terms of the determinants Δ_n. In other words, a moment functional $\mathfrak{L}$ is called positive definite if all its moments are real and $\Delta_n > 0$ for all $n \geq 0$. Notice also that for a MSOP, it is equivalent to say that $K_n > 0$ for all $n \geq 0$. This, and the fact that an SOP exists if and only if $\Delta_n \neq 0$, leads us to define more general moment functionals: the so-called *quasi-definite* moment functionals.

Definition 2.4 *A moment functional $\mathfrak{L}$ is said to be quasi-definite if and only if $\Delta_n \neq 0$ for all $n \geq 0$.*

We can write the explicit expression of the MOP in terms of the moments of the corresponding functional:

$$P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}, \quad \Delta_{-1} \equiv 1, \quad n = 0, 1, 2, \dots . \tag{2.2}$$

One of the simplest characteristics of orthogonal polynomials is the so-called three-term recurrence relation (TTRR) that connects every three consecutive polynomials of the SOP.

Theorem 2.5 *If $\{P_n\}_{n=0}^{\infty}$ is a MSOP with respect to a quasi-definite moment functional, then the polynomials P_n satisfy a three-term recurrence relation*

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n = 1, 2, 3, \dots , \tag{2.3}$$

where $\{c_n\}_{n=0}^{\infty}$ and $\{\lambda_n\}_{n=0}^{\infty}$ are given by

$$c_n = \frac{\mathfrak{L}[x\, P_{n-1}^2]}{\mathfrak{L}[P_{n-1}^2]}, \quad n \geq 1, \quad \text{and} \quad \lambda_n = \frac{\mathfrak{L}[xP_{n-1}P_{n-2}]}{\mathfrak{L}[P_{n-2}^2]} = \frac{\mathfrak{L}[P_{n-1}^2]}{\mathfrak{L}[P_{n-2}^2]}\,, \quad n \geq 2\ ,$$

respectively, and $P_{-1}(x) \equiv 0$, $P_0(x) \equiv 1$.

The proof of the above theorem is a simple consequence of the orthogonality of the polynomials and Theorem 2.2. A straightforward calculation shows that ($\lambda_1 = \mathfrak{L}[1]$)

$$\lambda_{n+1} = \frac{K_n}{K_{n-1}} = \frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}, \qquad n = 1, 2, 3, \dots,$$

and $\Delta_{-1} \equiv 1$. From Theorem 2.4 and Definition 2.4 it follows that, if $\lambda_n \neq 0$, then $\mathfrak{L}$ is quasi-definite whereas, if $\lambda_n > 0$, then $\mathfrak{L}$ is positive definite. Notice also that from the above expression we can obtain the square norm $K_n \equiv \mathfrak{L}[P_n^2]$ of the polynomial P_n as

$$K_n \equiv \mathfrak{L}[P_n^2] = \lambda_1\lambda_2 \cdots \lambda_{n+1}. \tag{2.4}$$

A useful consequence of Theorem 2.5 are the Christoffel-Darboux identities.

Theorem 2.6 *Let $\{P_n\}_{n=0}^{\infty}$ be a MSOP which satisfies (2.3) with $\lambda_n \neq 0$ for all nonnegative n. Then*

$$\sum_{m=0}^{n} \frac{P_m(x)P_m(y)}{K_m} = \frac{1}{K_n}\frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{x - y}, \qquad n \geq 0\,, \tag{2.5}$$

and

$$\sum_{m=0}^{n} \frac{P_m^2(x)}{K_m} = \frac{1}{K_n}[P_{n+1}'(x)P_n(x) - P_{n+1}(x)P_n'(x)], \quad n \geq 0\,. \tag{2.6}$$

For an arbitrary normalization (not necessarily the monic one) of the polynomials P_n, the three-term recurrence relation becomes

$$xP_{n-1}(x) = \alpha_n\, P_n(x) + \beta_n P_{n-1}(x) + \gamma_n P_{n-2}(x). \tag{2.7}$$

In this case, the coefficients α_n and β_n can be obtained comparing the coefficients of x^n and x^{n-1}, respectively, in both sides of (2.7) and γ_n is given by $\frac{\mathfrak{L}[xP_{n-1}P_{n-2}]}{\mathfrak{L}[P_{n-2}^2]}$. This leads to

$$\alpha_n = \frac{k_{n-1}}{k_n}, \qquad \beta_n = \frac{b_{n-1}}{k_{n-1}} - \frac{b_n}{k_n}, \qquad \gamma_n = \frac{k_{n-2}}{k_{n-1}}\frac{K_{n-1}}{K_{n-2}}, \tag{2.8}$$

where k_n is the leading coefficient of P_n and b_n denotes the coefficient of x^{n-1} in P_n, i.e., $P_n(x) = k_n x^n + b_n x^{n-1} + \cdots$. Notice also that knowing two of the coefficients α_n, β_n, and γ_n, one can find the third one using (2.7) provided, for example, that $P_n(x_0) \neq 0$ for some x_0 (usually $x_0 = 0$) and for all $n = 1, 2, 3, \ldots$.

The above TTRR (2.7) can be written in matrix form,

$$x\mathbf{P}_{n-1} = J_n\mathbf{P}_{n-1} + \alpha_n P_n(x)\mathbf{e}_n, \tag{2.9}$$

where

$$\mathbf{P}_{n-1} = \begin{bmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \\ P_{n-2}(x) \\ P_{n-1}(x) \end{bmatrix}, \quad J_n = \begin{bmatrix} \beta_1 & \alpha_1 & 0 & \ldots & 0 & 0 \\ \gamma_2 & \beta_2 & \alpha_2 & \ldots & 0 & 0 \\ 0 & \gamma_3 & \beta_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \beta_{n-1} & \alpha_{n-1} \\ 0 & 0 & 0 & \ldots & \gamma_n & \beta_n \end{bmatrix}, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{2.10}$$

Denoting by $\{x_{n,j}\}_{1\leq j\leq n}$ the zeros of the polynomial P_n, we see from (2.9) that each $x_{n,j}$ is an eigenvalue of the corresponding tridiagonal matrix of order n and $[P_0(x_{n,j}), ..., P_{n-1}(x_{n,j})]^T$ is the associated eigenvector. From the above representation many useful properties of zeros of orthogonal polynomials can be found.

2.2 The zeros of orthogonal polynomials.

Definition 2.5 *Let $\mathfrak{L}$ be a moment functional. The support of the functional $\mathfrak{L}$ is the largest interval $(a, b) \subset \mathbb{R}$ where $\mathfrak{L}$ is positive definite.*

The following theorem holds.

Theorem 2.7 *Let (a, b) be the support of the positive definite functional $\mathfrak{L}$, and let $\{P_n\}_{n=0}^{\infty}$ be the MSOP associated with $\mathfrak{L}$. Then,*

(1) All zeros of P_n are real, simple, and located inside (a, b).
(2) Two consecutive polynomials P_n and P_{n+1} have no common zeros.
(3) Let $\{x_{n,j}\}_{j=1}^{n}$ denote the zeros of the polynomial P_n, with $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$. Then,

$$x_{n+1,j} < x_{n,j} < x_{n+1,j+1}, \qquad j = 1, 2, 3, ..., n \ .$$

The last property is usually called the interlacing property*.*

Proof: Notice that, in the case when the SOP is an SNOP, i.e, $K_n = 1$ for all n, then the matrix J_n is a symmetric real matrix ($J_n = J_n^T$, where J_n^T denotes the transposed matrix of J_n). So its eigenvalues, and thus, the zeros of the orthogonal polynomials are real. To prove that all zeros are simple, we can use the Christoffel-Darboux identity (2.6). Let x_k be a multiple zero of P_n, i.e., $P_n(x_k) = P_n'(x_k) = 0$. Then (2.6) gives

$$0 < \sum_{m=0}^{n} \frac{P_m^2(x_k)}{K_m} = \frac{1}{K_n}[P_{n+1}'(x_k)P_n(x_k) - P_{n+1}(x_k)P_n'(x_k)] = 0 \,.$$

This contradiction proves the statement. Let $\{x_k\}_{k=1}^{p}$ be the zeros of P_n inside (a, b). Then, $P_n(x)\prod_{k=1}^{p}(x - x_k)$ does not change sign in (a, b) and $\mathfrak{L}[P_n(x)\prod_{k=1}^{p}(x - x_k)] \neq 0$, so $p = n$, i.e., all the zeros of P_n are inside (a, b). Thus, the statement 1 is proved. To prove 2, we use the TTRR. In fact, if x_k is a zero of P_n and P_{n+1}, then it must be a zero of P_{n-1}. Continuing this process by induction, we get that x_k must be a zero of $P_0(x) \equiv 1$, which is a contradiction. Before proving the interlacing property 3 we will prove a theorem due to Cauchy [22, page 197].

Theorem 2.8 *Let B be a principal $(n-1) \times (n-1)$ submatrix of a real symmetric $n \times n$ matrix A, with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$. Then, if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of A,*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n \ .$$

Proof: Let A be the $n \times n$ matrix

$$A = \begin{pmatrix} B & a \\ a^T & b \end{pmatrix},$$

and assume that the theorem is not true, i.e., $\mu_i > \lambda_i$ or $\lambda_{i+1} > \mu_i$ (since the matrix A is real symmetric, all its eigenvalues are real). Let i be the first such index. If $\mu_i > \lambda_i$ (the other case is similar), there exists a real number τ such that $\mu_i > \tau > \lambda_i$. Then, $B - \tau I_{n-1}$, where I_k denotes the identity $k \times k$ matrix, is nonsingular ($\det(B - \tau I_{n-1}) \neq 0$), and the matrix

$$H = \begin{pmatrix} B - \tau I_{n-1} & 0 \\ 0 & b - \tau - a^T (B - \tau I_{n-1})^{-1} a \end{pmatrix}$$

$$= \begin{pmatrix} I_{n-1} & 0 \\ -a^T (B - \tau I_{n-1})^{-1} & 1 \end{pmatrix} \begin{pmatrix} B - \tau I_{n-1} & a \\ a^T & b - \tau \end{pmatrix} \begin{pmatrix} I & -(B - \tau I_{n-1})^{-1} a \\ 0 & 1 \end{pmatrix},$$

is congruent to $A - \tau I_n$. Then, by the inertia theorem, the matrix H has the same number of positive eigenvalues as $A - \tau I_n$, i.e., $i - 1$. But H has at least as many positive eigenvalues as $B - \tau I_{n-1}$, i.e., i. The contradiction proves the theorem. ■

Obviously, the interlacing property 3 can be obtained as a simple corollary of the Cauchy Theorem, since the matrix J_n associated with the SONP is a real symmetric matrix and we can choose as A the matrix J_{n+1} whose zeros are the zeros of the polynomial P_{n+1} and then, the principal submatrix B is the matrix J_n whose eigenvalues coincide with the zeros of P_n. This completes the proof of Theorem 2.7. ■

2.3 The Favard Theorem and some applications.

In this subsection we will prove the so-called Favard Theorem.

Theorem 2.9 *Let $\{c_n\}_{n=0}^{\infty}$ and $\{\lambda_n\}_{n=0}^{\infty}$ be two arbitrary sequences of complex numbers, and let $\{P_n\}_{n=0}^{\infty}$ be a sequence of polynomials defined by the relation*

$$P_n(x) = (x - c_n) P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n = 1, 2, 3, \dots\,, \tag{2.11}$$

where $P_{-1}(x) = 0$ and $P_0(x) = 1$. Then, there exists a unique moment functional $\mathcal{L}$ such that

$$\mathcal{L}[1] = \lambda_1, \qquad \mathcal{L}[P_n \, P_m] = 0 \quad \text{if} \quad n \neq m.$$

Moreover, $\mathcal{L}$ is quasi-definite and $\{P_n\}_{n=0}^{\infty}$ is the corresponding MSOP if and only if $\lambda_n \neq 0$, and $\mathcal{L}$ is positive definite if and only if c_n are real numbers and $\lambda_n > 0$ for all $n = 1, 2, 3, \dots$.

Proof: To prove the theorem, we will define the functional $\mathcal{L}$ by induction on $\mathbb{P}_n$, the linear subspace of polynomials with degree at most n. We put

$$\mathcal{L}[1] = \mu_0 = \lambda_1, \qquad \mathcal{L}[P_n] = 0, \quad n = 1, 2, 3, \dots\ . \tag{2.12}$$

So, using the three-term recurrence relation (2.11), we can find all the moments in the following way: Since $\mathfrak{L}[P_n] = 0$, the TTRR gives

$$0 = \mathfrak{L}[P_1] = \mathfrak{L}[x - c_1] = \mu_1 - c_1\lambda_1, \quad \text{then} \quad \mu_1 = c_1\lambda_1,$$

$$0 = \mathfrak{L}[P_2] = \mathfrak{L}[(x - c_2)P_1 - \lambda_2 P_0] = \mu_2 - (c_1 + c_2)\mu_1 + (c_1c_2 - \lambda_2)\lambda_1,$$

then we can find μ_2, etc. Continuing this process, we can find, recursively, μ_{n+1} by using the TTRR, and they are uniquely determined. Next, using (2.11) and (2.12), we deduce that

$$x^k P_n(x) = \sum_{i=n-k}^{n+k} d_{n,i} P_i(x)\,.$$

Then, $\mathfrak{L}[x^k\, P_n] = 0$ for all $k = 0, 1, 2, ..., n - 1$. Finally,

$$\mathfrak{L}[x^n P_n] = \mathfrak{L}[x^{n-1}(P_{n+1} + c_{n+1}P_n + \lambda_{n+1}P_{n-1})] = \lambda_{n+1}\mathfrak{L}[x^{n-1}P_{n-1}],$$

so, $\mathfrak{L}[x^n P_n] = \lambda_{n+1}\lambda_n \cdots \lambda_1$.

Moreover, $\mathfrak{L}$ is quasi-definite and $\{P_n\}_{n=0}^{\infty}$ is the corresponding MSOP if and only if for all $n \geq 1$, $\lambda_n \neq 0$, while $\mathfrak{L}$ is positive definite and $\{P_n\}_{n=0}^{\infty}$ is the corresponding MSOP if and only if for all $n \geq 1$, $c_n \in \mathbb{R}$ and $\lambda_n > 0$. ■

Next, we will discuss some results dealing with the zeros of orthogonal polynomials.

The following theorem is due to Wendroff [27] (for a different point of view using the Bézoutian matrix see [2]).

Theorem 2.10 *(Wendroff [27])*
Let P_n and P_{n-1} be two monic polynomials of degree n and $n-1$, respectively. If $a < x_1 < x_2 < \cdots < x_n < b$ are the real zeros of P_n and $y_1 < y_2 < \cdots < y_{n-1}$ are the real zeros of P_{n-1}, and they satisfy the interlacing property, i.e.,

$$x_i < y_i < x_{i+1}, \qquad i = 1, 2, 3, ..., n - 1,$$

then there exists a family of polynomials $\{P_k\}_{k=0}^{n}$ orthogonal on $[a, b]$ such that the above polynomials P_n and P_{n-1} belong to it.

Proof: Let $c_n = x_1 + x_2 + \cdots + x_n - y_1 - y_2 - \cdots - y_{n-1}$. Then, the polynomial $P_n(x) - (x - c_n)P_{n-1}(x)$ is a polynomial of degree at most $n - 2$, i.e.,

$$P_n(x) - (x - c_n)P_{n-1}(x) \equiv -\lambda_n R(x),$$

where R is a monic polynomial of degree r at most $n - 2$. Since

$$x_1 - c_n = (y_1 - x_2) + \cdots + (y_{n-1} - x_n) < 0,$$

and $P_{n-1}(x_1) \neq 0$ (this is a consequence of the interlacing property), then $\lambda_n \neq 0$ and $R(x_1) \neq 0$. Moreover, $P_n(y_i) = -\lambda_n R(y_i)$. Now, using the fact that $P_n(y_i)P_n(y_{i+1}) < 0$ (again this is a consequence of the interlacing property), we conclude that also $R(y_i)R(y_{i+1}) < 0$, and this immediately implies that R has exactly $n-2$ real zeros and they satisfy $y_i < z_i < y_{i+1}$ for $i = 1, 2, ..., n-2$.

If we now define the polynomial P_{n-2} of degree exactly $n-2$, $P_{n-2} \equiv R$, whose zeros interlace with the zeros of P_{n-1}, we can construct, just repeating the above procedure, a polynomial of degree $n-3$ whose zeros interlace with the ones of P_{n-2}, etc. So we can find all polynomials P_k for $k = 1, 2, ..., n$.

Notice also that, by construction,

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x),$$

so

$$\lambda_n = \frac{(x_1 - c_n)P_{n-1}(x_1)}{P_{n-2}(x_1)} > 0,$$

because $\text{sign}P_{n-1}(x_1) = (-1)^{n-1}$ and $\text{sign}P_{n-2}(x_1) = (-1)^{n-2}$, which is a consequence of the interlacing property $x_1 < y_1 < z_1$. ■

We point out here that it is possible to complete the family $\{P_k\}_{k=0}^{n}$ to obtain a MSOP. To do this, we can define the polynomials P_k for $k = n+1, n+2, ...$ recursively by the expression

$$P_{n+j}(x) = (x - c_{n+j})P_{n+j-1}(x) - \lambda_{n+j}P_{n+j-2}(x), \qquad j = 1, 2, 3, \dots ,$$

where c_{n+j} and λ_{n+j} are real numbers chosen such that $\lambda_{n+j} > 0$ and the zeros of P_{n+j} lie on (a, b). Notice also that, in such a way, we have defined, from two given polynomials P_{n-1} and P_n, a sequence of polynomials satisfying a three-term recurrence relation of the form (2.11). So Theorem 2.9 states that the corresponding sequence is an orthogonal polynomial sequence with respect to a quasi-definite functional. Moreover, since the coefficients in (2.11) are real and $\lambda_{n+j} > 0$, the corresponding functional is positive definite.

Theorem 2.11 *(Vinuesa & Guadalupe [26], Nevai & Totik [18])*
Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences of real numbers such that

$$\cdots < x_3 < x_2 < x_1 = y_1 < y_2 < y_3 < \cdots .$$

Then there exists a unique system of monic polynomials $\{P_n\}_{n=0}^{\infty}$ orthogonal with respect to a positive definite functional on the real line such that $P_n(x_n) = P_n(y_n) = 0$ and $P_n(t) \neq 0$ for $t \notin [x_n, y_n]$, $n = 1, 2, \dots$.

Proof: Set $P_0 = 1$, $\lambda_0 = 0$ and $c_0 = x_1$. Define $\{P_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ by

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \qquad n \geq 1,$$

$$\lambda_n = (x_n - y_n)\left[\frac{P_{n-2}(x_n)}{P_{n-1}(x_n)} - \frac{P_{n-2}(y_n)}{P_{n-1}(y_n)}\right]^{-1}, \qquad c_n = x_n - \lambda_n \frac{P_{n-2}(x_n)}{P_{n-1}(x_n)}. \tag{2.13}$$

The above two formulas come from the TTRR and from the requirement $P_n(x_n) = P_n(y_n) = 0$. By induction one can show that $P_n(x) \neq 0$ if $x \notin [x_n, y_n]$, $P_n(x_n) = P_n(y_n) = 0$ and $\lambda_{n+1} > 0$ for $n = 0, 1, 2, ...$ Then, from Theorem 2.9 $\{P_n\}_{n=0}^{\infty}$ is a MSOP with respect to a positive definite moment functional. ■

Notice that, in the case $x_n = -y_n$, for $n = 1, 2, 3, ...$, the expression (2.13) for λ_n and c_n reduces to

$$\lambda_n = x_n \frac{P_{n-2}(x_n)}{P_{n-1}(x_n)}, \qquad c_n = 0.$$

3 The Favard Theorem on the unit circle.

3.1 Preliminaries.

In this subsection we will summarize some definitions and results relating to orthogonal polynomials on the unit circle $\mathbb{T} = \{|z| = 1, \ z \in \mathbb{C}\}$.

Definition 3.1 *Let $\{\mu_n\}_{n\in\mathbb{Z}}$ be a bisequence of complex numbers (moment sequence) such that $\mu_{-n} = \overline{\mu}_n$ and $\mathfrak{L}$ be a functional on the linear space of Laurent polynomials $\Lambda = Span\{z^k\}_{k\in\mathbb{Z}}$. We say that $\mathfrak{L}$ is a moment functional associated with $\{\mu_n\}$ if $\mathfrak{L}$ is linear and $\mathfrak{L}(x^n) = \mu_n$, $n \in \mathbb{Z}$.*

Definition 3.2 *Given a sequence of polynomials $\{\Phi_n\}_{n=0}^{\infty}$ we say that $\{\Phi_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials (SOP) with respect to a moment functional $\mathfrak{L}$ if*

(i) Φ_n *is a polynomial of exact degree* n,
(ii) $\mathfrak{L}(\Phi_n(z) \cdot z^{-m}) = 0$, *if* $0 \leq m \leq n-1$,
$\mathfrak{L}(\Phi_n(z) \cdot z^{-n}) = S_n \neq 0$, *for every* $n = 0, 1, 2, ...$.

For such a linear functional $\mathfrak{L}$ we can define a Hermitian bilinear form in $\mathbb{P}$ (the linear space of polynomials with complex coefficients) as follows:

$$\langle p(z), q(z) \rangle = \mathfrak{L}(p(z) \cdot \overline{q(1/\overline{z})}), \tag{3.1}$$

where $\overline{q(z)}$ denotes the complex conjugate of the polynomial $q(z)$.

Notice that Definition 3.2 means that $\{\Phi_n\}_{n=0}^{\infty}$ is an SOP with respect to the above bilinear form, and thus the idea of orthogonality appears, as usual, in the framework of Hermitian bilinear forms. Furthermore,

$$\langle zp(z), zq(z) \rangle = \langle p(z), q(z) \rangle \,, \tag{3.2}$$

i.e., the shift operator is unitary with respect to the bilinear form (3.1). In particular, the Gram matrix for the canonical basis $\{z^n\}_{n=0}^{\infty}$ is a structured matrix of Toeplitz type, i.e.,

$$\langle z^m, z^n \rangle = \left\langle z^{m-n}, 1 \right\rangle = \left\langle 1, z^{n-m} \right\rangle = \mu_{m-n}, \quad m, n \in \mathbb{N}.$$

In this case the entries (m, n) of the Gram matrix depend of the difference $m - n$.

In the following we will denote $T_n = [\mu_{k-j}]_{k,j=0}^{n}$.

Now we will deduce some recurrence relations for the respective sequence of monic orthogonal polynomials.

Theorem 3.1 *Let $\mathfrak{L}$ be a moment functional associated with the bisequence $\{\mu_n\}_{n\in\mathbb{Z}}$. The sequence of polynomials $\{\Phi_n\}_{n=0}^{\infty}$ is an SOP with respect to $\mathfrak{L}$ if and only if* $\det T_n \neq 0$ *for every $n = 0, 1, 2, \ldots$. Furthermore, the leading coefficient of Φ_n is $s_n = \frac{\det T_{n-1}}{\det T_n}$.*

Definition 3.3 *$\mathfrak{L}$ is said to be a positive definite moment functional if for every Laurent polynomial $q(z) = p(z)\overline{p(1/\overline{z})}$, $\mathfrak{L}(q) > 0$.*

Theorem 3.2 *$\mathfrak{L}$ is a positive definite functional if and only if* $\det T_n > 0$ *for every $n = 0, 1, 2, \ldots$.*

Definition 3.4 *$\mathfrak{L}$ is said to be a quasi-definite moment functional if* $\det T_n \neq 0$ *for every $n = 0, 1, 2, \ldots$.*

Remark: Compare the above definitions with those of Subsection 2.1.

In the following we will assume that the SOP $\{\Phi_n\}_{n=0}^{\infty}$ is normalized using the fact that the leading coefficient is one, i.e., we have a sequence of monic orthogonal polynomials (MSOP) given by $(n = 0, 1, 2, ...)$

$$\Phi_n(x) = \frac{1}{\det T_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_{-1} & \mu_0 & \cdots & \mu_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_1 \\ 1 & z & \cdots & z^n \end{vmatrix}, \quad \det T_{-1} \equiv 1 \ . \tag{3.3}$$

Unless stated otherwise, we will suppose the linear functional $\mathfrak{L}$ is quasi-definite.

Theorem 3.3 *(Geronimus [11]) If $\{\Phi_n\}_{n=0}^{\infty}$ is an MSOP with respect to a quasi-definite moment functional, it satisfies two recurrence relations:*

(i) $\Phi_n(z) = z\,\Phi_{n-1}(z) + \Phi_n(0)\Phi^*_{n-1}(z)$, $\quad \Phi_0(z) = 1$ *(forward recurrence relation),*

(ii) $\Phi_n(z) = (1 - |\Phi_n(0)|^2)z\,\Phi_{n-1}(z) + \Phi_n(0)\Phi^*_n(z)$, $\quad \Phi_0(z) = 1$ *(backward recurrence relation),*

where $\Phi^*_n(z) = z^n\overline{\Phi_n(1/\overline{z})}$ is called the reciprocal polynomial of Φ_n.

Proof:
(i) Let $R_{n-1}(z) = \Phi_n(z) - z\,\Phi_{n-1}(z)$. Thus, from orthogonality and (3.2)

$$\left\langle R_{n-1}(z), z^k \right\rangle = \mathfrak{L}(z^k \cdot \overline{R_{n-1}(1/\overline{z})}) = \mathfrak{L}(z^{k-n+1} \cdot z^{n-1}\overline{R_{n-1}(1/\overline{z})}) = 0,$$

for $k = 1, 2, ..., n-1$, and $\mathfrak{L}(z^{-j} \cdot z^{n-1}\overline{R_{n-1}(1/\overline{z})}) = 0$, $j = 0, 1, ..., n-2$.

This means that the polynomial of degree at most $n-1$, $z^{n-1}\overline{R_{n-1}(1/\overline{z})}$, with leading coefficient $\overline{\Phi_n(0)}$, is orthogonal to $\mathbb{P}_{n-2}$, i.e.,

$$z^{n-1}\overline{R_{n-1}(1/\overline{z})} = \overline{\Phi_n(0)}\Phi_{n-1}(z).$$

Thus, $R_{n-1}(z) = \Phi_n(0)\Phi^*_{n-1}(z)$.

(ii) From (i) we deduce

$$\Phi^*_n(z) = \Phi^*_{n-1}(z) + \overline{\Phi_n(0)}z\Phi_{n-1}(z).$$

Then, the substitution of $\Phi^*_{n-1}(z)$ in (i), using the above expression, leads to (ii). ■

Remark: Notice that, if we multiply both sides of (ii) by $1/z^n$, use the orthogonality of Φ_n as well as the explicit expression (3.3), we get the following identity:

$$\frac{\det T_n}{\det T_{n-1}} = (1 - |\Phi_n(0)|^2)\,\frac{\det T_{n-1}}{\det T_{n-2}}. \tag{3.4}$$

The values $\{\Phi_n(0)\}_{n=1}^{\infty}$ are called reflection coefficients or Schur parameters for the MSOP. Notice that the main difference with the recurrence relation analyzed in Section 2 is that here only two consecutive polynomials are involved and the reciprocal polynomial is needed. On the other hand, the basic parameters which appear in these recurrence relations are the value at zero of the orthogonal polynomial.

Theorem 3.4 *$\mathfrak{L}$ is a quasi-definite moment functional if and only if $|\Phi_n(0)| \neq 1$ for every $n = 1, 2, 3, \ldots$.*

Proof: If $\mathfrak{L}$ is quasi-definite the corresponding MSOP satisfies both (i) and (ii). If for some $n \in \mathbb{N}$, $|\Phi_n(0)| = 1$, then from (ii), $\Phi_n(z) = \Phi_n(0)\Phi_n^*(z)$. Thus,

$$\begin{aligned}\langle \Phi_n(z), z^n\rangle &= \Phi_n(0)\,\langle \Phi_n^*(z), z^n\rangle = \Phi_n(0)\left\langle z^n\overline{\Phi_n(1/\overline{z})}, z^n\right\rangle \\ &= \Phi_n(0)\left\langle \overline{\Phi_n(1/\overline{z})}, 1\right\rangle = \Phi_n(0)\,\langle 1, \Phi_n(z)\rangle = 0,\end{aligned}$$

which is a contradiction with the fact that $\{\Phi_n\}_{n=1}^{\infty}$ is a MSOP.

Assume now that a sequence of polynomials is defined by (i) with $|\Phi_n(0)| \neq 1$. We will prove by induction that there exists a moment functional $\mathfrak{L}$ which is quasi-definite and such that $\{\Phi_n\}_{n=1}^{\infty}$ is the corresponding sequence of MOP.

Let $\Phi_1(z) = z + \Phi_1(0)$. We define $\mu_1 = \mathfrak{L}(z) = -\Phi_1(0)\mu_0$. Thus $T_1 = \begin{pmatrix} \mu_0 & \mu_1 \\ \overline{\mu_1} & \mu_0 \end{pmatrix}$ is such that $\det T_1 = \mu_0^2(1 - |\Phi_1(0)|^2) \neq 0$.

Furthermore,

$$\langle \Phi_1(z), z\rangle = \mathfrak{L}(\Phi_1(z)\cdot 1/z) = \mu_0 + \Phi_1(0)\overline{\mu_1} = \mu_0(1 - |\Phi_1(0)|^2) \neq 0,$$

i.e., Φ_1 is a monic polynomial of degree 1 such that $\langle \Phi_1(z), 1\rangle = \mu_1 + \Phi_1(0)\mu_0 = 0$, i.e., is orthogonal to $\mathbb{P}_0$.

Assume $\{\Phi_0, \Phi_1, \ldots, \Phi_{n-1}\}$ are monic and orthogonal. Let $a_n = \Phi_n(0)$, $|a_n| \neq 1$, and construct a polynomial Φ_n of degree n such that

$$\Phi_n(z) = z\,\Phi_{n-1}(z) + \underbrace{\Phi_n(0)}_{a_n}\,\Phi_{n-1}^*(z).$$

If $\Phi_n(z) = z^n + c_{n,1}z^{n-1} + \cdots + c_{n,n-1}z + a_n$, we define $\mu_n = -c_{n,1}\mu_{n-1} - \cdots - c_{n,n-1}\mu_1 - a_n\mu_0$. Notice that this means that $\langle \Phi_1(z), 1\rangle = 0$.

On the other hand, for $1 \le k \le n-1$, using the recurrence relation (i)

$$\left\langle \Phi_n(z), z^k \right\rangle = \left\langle \Phi_{n-1}(z), z^{k-1} \right\rangle + a_n \left\langle \Phi^*_{n-1}(z), z^k \right\rangle = 0,$$

where the last term in the above sum vanishes since

$$\left\langle \Phi^*_{n-1}(z), z^k \right\rangle = \left\langle z^{n-k-1}, \Phi_{n-1}(z) \right\rangle.$$

Finally, using (3.4), we have

$$\langle \Phi_n(z), z^n \rangle = \frac{\det T_n}{\det T_{n-1}} = (1 - |\Phi_n(0)|^2)\, \frac{\det T_{n-1}}{\det T_{n-2}},$$

and thus, because of the induction hypothesis, $\langle \Phi_n(z), z^n \rangle \neq 0$. ■

Corollary 3.1 *The functional $\mathfrak{L}$ is positive definite if and only if $|\Phi_n(0)| < 1$, for $n = 1, 2, \ldots$.*

3.2 The zeros of the orthogonal polynomials.

In the following we will analyze the existence of an integral representation for a moment functional.

First, we will consider the case of positive definiteness.

Proposition 3.1 *[12] If α is a zero of $\Phi_n(z)$, then $|\alpha| < 1$.*

Proof: Let $\Phi_n(z) = (z-\alpha)q_{n-1}(z)$, where q_{n-1} is a polynomial of degree $n-1$. Then,

$$\begin{aligned}
&0 < \langle \Phi_n(z), \Phi_n(z)\rangle = \langle (z-\alpha)q_{n-1}(z), \Phi_n(z)\rangle = \langle z\, q_{n-1}(z), \Phi_n(z)\rangle \\
&= \langle z\, q_{n-1}(z), z\, q_{n-1}(z) - \alpha q_{n-1}(z)\rangle = \langle q_{n-1}(z), q_{n-1}(z)\rangle - \overline{\alpha}\, \langle z\, q_{n-1}(z), q_{n-1}(z)\rangle \\
&= \langle q_{n-1}(z), q_{n-1}(z)\rangle - \overline{\alpha}\, [\langle \Phi_n(z), q_{n-1}(z)\rangle + \alpha \langle q_{n-1}(z), q_{n-1}(z)\rangle] \\
&= (1 - |\alpha|^2)\, \langle q_{n-1}(z), q_{n-1}(z)\rangle,
\end{aligned}$$

and the result follows. ■

Corollary 3.2 *(Montaner & Alfaro [16]) If β is a zero of $\Phi^*_n(z)$, then $|\beta| > 1$.*

Remark: Notice that, in the quasi-definite case, we only can guarantee that $|\alpha| \neq 1$.

Next, we will define an absolutely continuous measure such that the induced inner product in $\mathbb{P}_n$ agrees with the restriction to $\mathbb{P}_n$ of our inner product associated with the positive definite linear functional. In order to do this, we need some preliminary result.

Lemma 3.1 *[8] Let ϕ_n be the nth orthonormal polynomial with respect to a positive definite linear functional. Then,*

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_k(e^{i\theta})\overline{\phi_j(e^{i\theta})}\frac{d\theta}{|\phi_n(e^{i\theta})|^2} = \delta_{j,k}, \quad 0 \leq j \leq k \leq n < \infty.$$

Proof: Notice that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_n(e^{i\theta})\overline{\phi_n(e^{i\theta})}\frac{d\theta}{|\phi_n(e^{i\theta})|^2} = 1, \tag{3.5}$$

and, for $j < n$,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \phi_n(e^{i\theta})\overline{\phi_j(e^{i\theta})}\frac{d\theta}{|\phi_n(e^{i\theta})|^2} &= \frac{1}{2\pi}\int_0^{2\pi} \overline{\left[\frac{\phi_j(e^{i\theta})}{\phi_n(e^{i\theta})}\right]} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i(n-j)\theta}\phi_j^*(e^{i\theta})}{\phi_n^*(e^{i\theta})} d\theta \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{z^{n-j-1}\phi_j^*(z)}{\phi_n^*(z)} d\,z = 0,\end{aligned} \tag{3.6}$$

because of the analyticity of the function in the last integral (see Corollary 3.2). Then, $\phi_n(z)$ is the nth orthonormal polynomial with respect to both, a positive linear functional and the absolutely continuous measure $d\nu_n = \frac{d\theta}{|\phi_n(e^{i\theta})|^2}$. Using the backward recurrence relation (Theorem 3.3, (ii)) for the orthonormal case, the polynomials $\{\phi_j\}_{j=0}^{n-1}$, which are uniquely defined by this recurrence relation, are orthogonal with respect to both, the linear functional and the measure $d\,\nu_n$. Thus, the result follows. ∎

Remark: In [8], an induction argument is used in order to prove the previous result. Indeed, assuming that for a fixed $k \leq n$,

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_k(e^{i\theta})\overline{\phi_j(e^{i\theta})}\frac{d\theta}{|\phi_n(e^{i\theta})|^2} = \delta_{jk}, \quad 0 \leq j \leq k,$$

they proved that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_{k-1}(e^{i\theta})\overline{\phi_l(e^{i\theta})}\frac{d\theta}{|\phi_n(e^{i\theta})|^2} = \delta_{k-1\,l}, \quad 0 \leq l \leq k-1.$$

Notice that the nth orthogonal polynomial defines in a unique way the previous ones; thus, the proof of the second statement (the induction) is not necessary. Of course, here we need not do this since we are using the backward recurrence relation for the orthogonal polynomials ϕ_n.

Notice also that the measure $d\nu_n = \frac{d\theta}{|\phi_n(e^{i\theta})|^2}$ defines a MSOP $\{\Psi_n\}_{n=0}^{\infty}$ such that $\Psi_m(z) = z^{m-n}\Phi_n(z)$, for $m \geq n$, where Φ_n is the monic polynomial corresponding to ϕ_n. Moreover, the sequence of reflection coefficients corresponding to this MSOP $\{\Psi_n\}_{n=0}^{\infty}$ is $\{\Phi_1(0), ..., \Phi_n(0), 0, 0, ...\}$. Usually, in the literature of orthogonal polynomials, this measure $d\nu_n$ is called a Bernstein-Szegö measure (see [25]).

In Section 2, Theorem 2.10, we proved that the interlacing property for the zeros of two polynomials P_{n-1} and P_n of degree $n-1$ and n, respectively, means that they are the $(n-1)$st and nth orthogonal polynomials of a MSOP. Indeed, the three-term recurrence relation for a MSOP plays a central role in the proof. In the case of the unit circle, we have an analogous result, which is known in the literature as the Schur-Cohn-Jury criterion [4].

Theorem 3.5 *A monic polynomial p of degree n has its n zeros inside the unit circle if and only if the family of parameters $\{a_k\}_{k=0}^{n}$ defined by the following backward algorithm*

$$q_n(z) = p(z), \quad q_n(0) = a_n,$$

$$q_k(z) = \frac{q_{k+1}(z) - a_{k+1}q_{k+1}^*(z)}{z(1-|a_{k+1}|^2)}, \quad a_k = q_k(0), \qquad k = n-1, n-2, ..., 0,$$

satisfies $|a_k| < 1$, $k = 1, 2, ..., n$.

Proof: Notice that the polynomials $\{q_k\}_{k=1}^{n}$, $q_0 = 1$, satisfy a backward recurrence relation like the polynomials orthogonal on the unit circle with truncated Schur parameters $\{a_k\}_{k=1}^{\infty}$. Because $\{a_1, a_2, ..., a_n, 0, 0, ...\}$ is induced by the measure $d\,\nu_n = \frac{d\theta}{|q_n(e^{i\theta})|^2} = \frac{d\theta}{|p(e^{i\theta})|^2}$, up to a constant factor, then $p = q_n(z)$ is the nth monic orthogonal polynomial with respect to the measure $d\,\nu_n$. According to Proposition 3.1 its zeros are located inside the unit disk.

Conversely, if the polynomial p has its zeros inside the unit disk, then $|a_n| = |q_n(0)| < 1$. On the other hand, since

$$q_{n-1}(z) = \frac{q_n(z) - a_n q_n^*(z)}{z(1-|a_n|^2)},$$

if α is a zero of q_{n-1} with $|\alpha| \geq 1$, then $q_n(\alpha) = a_n q_n^*(\alpha)$, and $0 < |q_n(\alpha)| < |q_n^*(\alpha)|$. This means that $\left|\frac{q_n(\alpha)}{q_n^*(\alpha)}\right| < 1$, but this is in contradiction with the fact that the zeros of $q_n(z)$ are inside the unit disk and thus, by the maximum modulus principle, $\left|\frac{q_n(z)}{q_n^*(z)}\right| \leq 1$ if $|z| < 1$, which is equivalent to $\left|\frac{q_n(z)}{q_n^*(z)}\right| \geq 1$ for $|z| \geq 1$. The same procedure applied to all $1 \leq k \leq n-2$ leads to the result. ■

Remark: The above criterion is a very useful qualitative result in the stability theory for discrete linear systems [4]. In fact, given the characteristic polynomial of the matrix of a linear system, we do not need to calculate its zeros (the eigenvalues of the matrix) in order to prove that they are located inside the unit disk, and then to prove the stability of the system.

3.3 The trigonometric moment problem revisited.

Next, we can state our main result.

Theorem 3.6 *[8] Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $|a_n| < 1$, $n = 1, 2, \ldots$. Let*

$$\Phi_0(z) = 1, \qquad \Phi_n(z) = z\,\Phi_{n-1}(z) + a_n \Phi_{n-1}^*(z), \quad n \geq 1.$$

Then, there exists a unique positive and finite Borel measure ν supported on $\mathbb{T}$ such that $\{\Phi_n\}_{n=0}^{\infty}$ is the corresponding MSOP. In other words, the positive definite linear functional associated with the reflection coefficients $\{a_n\}_{n=0}^{\infty}$ can be represented as

$$\mathcal{L}[p(z)] = \int_0^{2\pi} p(e^{i\theta})\, d\nu(\theta).$$

Proof: Let

$$\nu_n(\theta) = \int_0^\theta d\nu_n(t) = \int_0^\theta \frac{dt}{|\phi_n(e^{it})|^2},$$

where ϕ_n denotes the nth orthonormal polynomial with respect to $\mathfrak{L}$. The function ν_n is monotonic increasing in $[0,2\pi]$ and according to Lemma 3.1,

$$|\nu_n(\theta)| \leq \int_0^{2\pi} \frac{d\theta}{|\phi_n(e^{i\theta})|^2} \leq 2\pi d_0 < +\infty \qquad \forall n \in \mathbb{N}, \quad \theta \in [0,2\pi].$$

From Helly's selection principle (see e.g. [5]) there exists a subsequence $\{\nu_{n_k}\}_{n_k=0}^{\infty}$ and a monotonic increasing function ν such that $\lim\limits_{n_k\to\infty} \nu_{n_k}(\theta) = \nu(\theta)$. Furthermore, for every continuous function f on $\mathbb{T}$,

$$\lim_{n_k\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\nu_{n_k}(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\nu(\theta).$$

Finally,

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_k(e^{i\theta})\overline{\phi_j(e^{i\theta})}d\nu(\theta) = \lim_{n_l\to\infty}\frac{1}{2\pi}\int_0^{2\pi} \phi_k(e^{i\theta})\overline{\phi_j(e^{i\theta})}d\nu_{n_l}(\theta) = \delta_{j,k},$$

taking $n_l > \max\{k,j\}$. ■

To conclude the study of the positive definite case, we will show an analog of Theorem 2.11 of Section 2 in the following sense.

Theorem 3.7 *[1] Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $|z_n| < 1$. Then, there exists a unique sequence of monic polynomials Φ_n orthogonal with respect to a positive definite moment functional such that $\Phi_n(z_n) = 0$.*

Proof: Since $\Phi_1(z) = z + \Phi_1(0) = z - z_1$, then $\Phi_1(0) = -z_1$, and $|\Phi_1(0)| < 1$. Using induction, assume that z_{n-1} is a zero of Φ_{n-1} and $|\Phi_{n-1}(0)| < 1$. Let $\Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\Phi^*_{n-1}(z)$, for $n > 1$, and z_n be a zero of Φ_n. Then, substituting z_n in the above expression, we deduce

$$z_n\Phi_{n-1}(z_n) = -\Phi_n(0)\Phi^*_{n-1}(z_n).$$

But $\Phi^*_{n-1}(z_n) \neq 0$ (otherwise z_n would be a zero of Φ_{n-1}, which is a contradiction). Thus,

$$\Phi_n(0) = -z_n\frac{\Phi_{n-1}(z_n)}{\Phi^*_{n-1}(z_n)}, \quad \text{but then} \quad |\Phi_n(0)| = |z_n|\left|\frac{\Phi_{n-1}(z_n)}{\Phi^*_{n-1}(z_n)}\right| < |z_n| < 1,$$

since $\left|\frac{\Phi_{n-1}(z_n)}{\Phi^*_{n-1}(z_n)}\right| < 1$ by the maximum modulus principle (see the proof of Theorem 3.5). Then, the sequence $\{z_n\}_{n=1}^{\infty}$ defines uniquely a sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$, with $a_n = \Phi_n(0)$, and this sequence, according to Theorem 3.6, uniquely defines a sequence of orthogonal polynomials $\{\Phi_n\}_{n=0}^{\infty}$ with reflection parameters a_n such that $\Phi_n(z_n) = 0$. ■

In the quasi-definite case, as we already pointed out after Proposition 3.1, if ϕ_n is the nth orthonormal polynomial with respect to a quasi-definite moment functional $\mathcal{L}$, then the polynomials $z\phi_n(z)$ and $\phi_n^*(z)$ have no zeros in common. They are coprime, and by the Bézout identity [4], there exist polynomials $r(z)$ and $s(z)$ such that

$$z\, r(z)\phi_n(z) + s(z)\phi_n^*(z) = 1,$$

or, equivalently, if $u(z) = z\, r(z)$, i.e., $u(0) = 0$,

$$u(z)\phi_n(z) + s(z)\phi_n^*(z) = 1.$$

The next result is analogous to that stated in Lemma 3.1.

Theorem 3.8 *[3] There exists a unique real trigonometric polynomial $f(\theta)$ of degree at most n, such that*

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_n(e^{i\theta})e^{-ik\theta} f(\theta)d\theta = 0, \quad 0 \le k \le n-1, \tag{3.7}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\phi_n(e^{i\theta})\right|^2 f(\theta)d\theta = 1, \tag{3.8}$$

if and only if there exist $u,\ v \in \mathbb{P}_n$, with $u(0) = 0$, such that $u(z)\phi_n(z) + v(z)\phi_n^*(z) = 1$. Furthermore,

$$f(\theta) = |u(e^{i\theta})|^2 - |v(e^{i\theta})|^2.$$

Proof: If f satisfies (3.7) and (3.8), consider the function $g(\theta) = f(\theta)\phi_n(e^{i\theta})$, which is a trigonometric polynomial of degree at most $2n$. The conditions mean that the Fourier coefficients $\hat{g}(k)$ of $g(\theta)$ are $\hat{g}(j) = 0$, $j = 0, 1, ..., n-1$, and $\hat{g}(n)\phi_n^*(0) = 1$. Then, there exist polynomials $u,\ v \in \mathbb{P}_n$, such that $u(0) = 0$, $v(0))\phi_n^*(0) = 1$ and $g(\theta) = e^{in\theta}v(e^{i\theta}) - \overline{u(e^{i\theta})}$. In fact,

$$u(z) = -\sum_{j=1}^{n} \overline{\hat{g}(-j)} z^j \quad \text{and} \quad v(z) = \sum_{j=0}^{n} \hat{g}(j+n) z^j.$$

Now we introduce the trigonometric polynomial of degree at most $3n$, $h(\theta) = \phi_n(e^{i\theta})f(\theta)\overline{\phi_n(e^{i\theta})}$. Notice that

$$h(\theta) = \phi_n^*(e^{i\theta})v(e^{i\theta}) - \overline{u(e^{i\theta})\phi_n(e^{i\theta})},$$

and h is a real-valued function. Then,

$$\phi_n^*(e^{i\theta})v(e^{i\theta}) - \overline{u(e^{i\theta})\phi_n(e^{i\theta})} = \overline{\phi_n^*(e^{i\theta})v(e^{i\theta})} - u(e^{i\theta})\phi_n(e^{i\theta}),$$

or, equivalently,

$$s(\theta) = u(e^{i\theta})\phi_n(e^{i\theta}) + v(e^{i\theta})\phi_n^*(e^{i\theta}) \in \mathbb{R}.$$

This means that the algebraic polynomial of degree at most $2n$,

$$q(z) = u(z)\phi_n(z) + v(z)\phi_n^*(z),$$

is real-valued on the unit circle, and thus $\hat{q}(j) = \overline{\hat{q}(-j)} = 0$, i.e.,

$$q(z) = q(0) = u(0)\phi_n(0) + v(0)\phi_n^*(0) = 1.$$

This yields our result.

Conversely, assume there exist polynomials u, $v \in \mathbb{P}_n$ with $u(0) = 0$, such that

$$u(z)\phi_n(z) + v(z)\phi_n^*(z) = 1. \tag{3.9}$$

Let $f(\theta) = v(e^{i\theta})\overline{v(e^{i\theta})} - u(e^{i\theta})\overline{u(e^{i\theta})}$, a trigonometric polynomial of degree at most n. We will prove that the orthogonality conditions (3.7) and (3.8) hold.

Indeed, let $g(\theta) = f(\theta)\phi_n(e^{i\theta})$. Taking into account (3.9), we have

$$\overline{u(e^{i\theta})}\,\overline{\phi_n(e^{i\theta})} + \overline{v(e^{i\theta})}e^{-in\theta}\phi_n(e^{i\theta}) = 1, \quad \text{i.e.,} \quad e^{in\theta} = \overline{u(e^{i\theta})}\phi_n^*(e^{i\theta}) + \overline{v(e^{i\theta})}\phi_n(e^{i\theta}).$$

Then, using (3.9) as well as the last expression, we obtain

$$g(\theta) = \phi_n(e^{i\theta})\left[v(e^{i\theta})\overline{v(e^{i\theta})} - u(e^{i\theta})\overline{u(e^{i\theta})}\right] = e^{in\theta}v(e^{i\theta}) - \overline{u(e^{i\theta})}, \tag{3.10}$$

which yields our orthogonality conditions

$$\hat{g}(j) = 0, \quad j = 0, 1, ..., n-1, \quad \text{and} \quad \hat{g}(n)\phi_n^*(0) = 1.$$

In order to prove uniqueness of f, notice that if u, $v \in \mathbb{P}_n$, satisfy (3.9) together with $u(0) = 0$, then $f(\theta) = u(e^{i\theta})\phi_n(e^{i\theta})f(\theta) + v(e^{i\theta})\phi_n^*(e^{i\theta})f(\theta)$. By (3.10), we get

$$f(\theta)\phi_n(e^{i\theta}) = e^{in\theta}v(e^{i\theta}) - \overline{u(e^{i\theta})},$$

and

$$f(\theta)\phi_n^*(e^{i\theta}) = \overline{v(e^{i\theta})} - e^{in\theta}u(e^{i\theta}).$$

Thus, $f(\theta) = |v(e^{i\theta})|^2 - |u(e^{i\theta})|^2$. The uniqueness of f follows from the uniqueness of u, v. ■

To conclude this section, we will show with two simple examples how to find the function f explicitly.

Example 3.1 Let $\phi_3(z) = 2z^3 + 1$. Notice that because the zeros are inside the unit circle, we are in a positive definite case. Moreover, $\phi_3^*(z) = z^3 + 2$. Using the Euclidean algorithm for $z\phi_3(z)$ and $\phi_3^*(z)$, we find

$$2z^4 + z = 2z(z^3 + 2) - 3z, \quad \text{and} \quad z^3 + 2 = -3z(-\tfrac{1}{3}z^2) + 2.$$

Thus,

$$\tfrac{1}{6}z^2(2z^4 + z) + (z^3 + 2)(\tfrac{1}{2} - \tfrac{1}{3}z^3) = 1, \quad \text{and} \quad u(z) = \tfrac{1}{6}z^3, \quad v(z) = \tfrac{1}{2} - \tfrac{1}{3}z^3.$$

Then

$$f(\theta) = \left|\tfrac{1}{2} - \tfrac{1}{3}e^{3i\theta}\right|^2 - \tfrac{1}{36} = \tfrac{1}{3}(1 - \cos 3\theta) = \tfrac{1}{6}\left|e^{3i\theta} - 1\right|^2 \geq 0.$$

Example 3.2 Let $\phi_3(z) = z(z^2 + 4)$. Notice that now there are two zeros outside the unit circle. In this case, $\phi_3^*(z) = 4z^2 + 1$. An analogous procedure leads to

$$z\phi_3(z) = z^4 + 4z^2 = \tfrac{1}{4}z^2(4z^2 + 1) + \tfrac{15}{4}z^2, \quad \phi_3^*(z) = \tfrac{16}{15}(\tfrac{15}{4}z^2) + 1.$$

Thus

$$-\tfrac{16}{15}z^2(z^2 + 4) + (\tfrac{4}{15}z^2 + 1)(4z^2 + 1) = 1, \quad u(z) = -\tfrac{16}{15}z^2, \quad v(z) = \tfrac{4}{15}z^2 + 1,$$

so

$$f(\theta) = \left|\tfrac{4}{15}e^{2i\theta} - 1\right|^2 - \tfrac{256}{225} = -\tfrac{1}{15}(1 + 8\cos 2\theta),$$

which gives rise to a nonpositive case, i.e., to a signed measure on $[-\pi, \pi]$.

4 The Favard Theorem for nonstandard inner products.

To conclude this work, we will survey some very recent results concerning the Favard theorem for Sobolev-type orthogonal polynomials.

First of all, we want to point out that the Favard Theorem on the real line can be be considered in a functional-analytic framework as follows.

Theorem 4.1 *(Duran [6]) Let $\mathbb{P}$ be the linear space of real polynomials and B an inner product on $\mathbb{P}$. Then, the following conditions are equivalent:*

(1) The multiplication operator t, i.e., the operator $t : \mathbb{P} \to \mathbb{P}$, $p(t) \to t\,p(t)$, is Hermitian for B, that is, $B(t\,f, g) = B(f, t\,g)$ for every polynomial f, g.

(2) *There exists a nondiscrete positive measure μ such that $B(f,g) = \int f(t)g(t)d\mu(t)$.*
(3) *For any set of orthonormal polynomials (q_n) with respect to B the following three-term recurrence holds:*

$$tq_n(t) = a_{n+1}q_{n+1}(t) + b_nq_n(t) + a_nq_{n-1}(t), \quad n \geq 0, \tag{4.1}$$

with $q_{-1}(t) = 0$, $q_0(t) = 1$ and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ real sequences such that $a_n > 0$ for all n.

Notice that from the three-term recurrence relation (4.1) we get

$$\begin{aligned} t^2q_n(x) &= a_{n+2}a_{n+1}q_{n+2}(t) + (b_{n+1}a_{n+1} + b_na_{n+1})q_{n+1}(t) \\ &\quad +(a_{n+1}^2 + a_n^2 + b_n^2)q_n(t) + (a_nb_n + a_nb_{n-1})q_{n-1}(t) + a_na_{n-1}q_{n-2}(t), \end{aligned}$$

i.e., the sequence $\{q_n\}_{n=0}^{\infty}$ satisfies a five-term recurrence relation, which is a simple consequence of the symmetry of the operator $t^2 \equiv t \cdot t$.

Here we are interested in the converse problem, which is a natural extension of the Favard Theorem: To characterize the real symmetric bilinear forms such that the operator t^2 is a Hermitian operator. A nonstandard example of such an inner products is

$$B(f,g) = \int f(t)g(t)d\mu(t) + Mf'(0)g'(0), \qquad f,g \in \mathbb{P},$$

for which t^2 is Hermitian, i.e., $B(t^2f,g) = B(f,t^2g)$.

Theorem 4.2 *Let B be a real symmetric bilinear form on the linear space $\mathbb{P}$. Then the following conditions are equivalent:*

(1) *The operator t^2 is Hermitian for B, that is, $B(t^2f,g) = B(f,t^2g)$ for every polynomial f, g.*
(2) *There exist two functions μ and ν such that*

$$B(f,g) = \int f(t)g(t)d\mu(t) + 4\int f_0(t)g_0(t)d\nu(t), \tag{4.2}$$

where f_0 and g_0 denote the odd components of f and g, respectively, i.e.,

$$f_0(t) = \frac{f(t)-f(-t)}{2}, \qquad g_0(t) = \frac{g(t)-g(-t)}{2}.$$

Moreover, if we put $\alpha_n = \int t^n d\mu(t)$ and $\beta_n = 4\int t^n d\nu(t)$, then the matrix

$$a_{n,k} = \begin{cases} \alpha_{n+k} & \text{if } n \text{ or } k \text{ are even}, \\ \alpha_{n+k} + \beta_{n+k} & \text{otherwise}, \end{cases}$$

is positive definite if and only if B is an inner product. In this case the set of orthonormal polynomials with respect to an inner product of the form (4.2) satisfies a five-term recurrence relation

$$\begin{aligned} t^2 q_n(x) = A_{n+2} q_{n+2}(t) + B_{n+1} q_{n+1}(t) + C_n q_n(t) \\ + B_n q_{n-1}(t) + A_n q_{n-2}(t), \quad n \geq 0, \end{aligned} \tag{4.3}$$

where $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$, and $\{C_n\}_{n=0}^{\infty}$ are real sequences such that $A_n \neq 0$ for all n.

Also we get a generalization of the Favard Theorem.

Theorem 4.3 *Let $\{q_n\}_{n=0}^{\infty}$ be a set of polynomials satisfying the initial conditions $q_{-1}(t) = q_{-2}(t) = 0$, $q_0(t) = 1$ and the five-term recurrence relation (4.3). Then, there exist two functions μ and ν such that the bilinear form (4.2) is an inner product and the polynomials $\{q_n\}_{n=0}^{\infty}$ are orthonormal with respect to B.*

Remark: The above theorem does not guarantee the positivity of the measures μ and ν. In fact in [6] some examples of inner products of type (4.2) where both measures cannot be chosen to be positive, or μ is positive and ν cannot be chosen to be positive, are shown.

All the previous results can be extended to real symmetric bilinear forms such that the operator "multiplication by $h(t)$", where h is a fixed polynomial, is Hermitian for B, i.e., $B(h\,f, g) = B(f, h\,g)$.

The basic idea consists in the choice of an adequate basis of $\mathbb{P}$ which is associated with the polynomial h. Assume that $\deg h = N$, and let $E_h = \mathrm{span}[1, h, h^2, ...]$; then

$$\mathbb{P} = E_h \oplus t\,E_h \oplus \cdots \oplus t^{N-1} E_h\,.$$

If π_k denotes the projector operator in $t^k E_h$, then $\pi_k(p) = t^k q[h(t)]$. We introduce a new operator $\tilde{\pi}_k : \mathbb{P} \to \mathbb{P}$, $p \to q$, where q denotes a polynomial such that $\pi_k(p) = t^k q[h(t)]$. Then we obtain the following extension of Theorem 4.2:

Theorem 4.4 *Let B be a real symmetric bilinear form in $\mathbb{P}$. Then the following statements are equivalent:*

(1) The operator "multiplication by h" is Hermitian for B, i.e., $B(h\,f, g) = B(f, h\,g)$ for every polynomial f, g, where h is a polynomial of degree N.

(2) *There exist functions $\mu_{m,m'}$ for $0 \leq m \leq m' \leq N-1$ such that B is defined as follows:*

$$B(f,g) = \int (\pi_0(f), ..., \pi_{N-1}(f)) \begin{pmatrix} d\mu_{0,0} & \cdots & d\mu_{0,N-1} \\ \vdots & \ddots & \vdots \\ d\mu_{N-1,0} & \cdots & d\mu_{N-1,N-1} \end{pmatrix} \begin{pmatrix} \pi_0(g) \\ \vdots \\ \pi_{N-1}(g) \end{pmatrix}.$$

(3) *There exist functions μ_0 and $\mu_{m,m'}$ for $1 \leq m \leq m' \leq N-1$ such that B is defined as follows:*

$$B(f,g) = \int f\, g d\mu_0 + \int (\pi_1(f), ..., \pi_{N-1}(f)) \begin{pmatrix} d\mu_{1,1} & \cdots & d\mu_{1,N-1} \\ \vdots & \ddots & \vdots \\ d\mu_{N-1,1} & \cdots & d\mu_{N-1,N-1} \end{pmatrix} \begin{pmatrix} \pi_1(g) \\ \vdots \\ \pi_{N-1}(g) \end{pmatrix}.$$

(4) *There exist functions $\tilde{\mu}_{m,m'}$ for $0 \leq m \leq m' \leq N-1$ such that B is defined as follows:*

$$B(f,g) = \int (\tilde{\pi}_0(f), ..., \tilde{\pi}_{N-1}(f)) \begin{pmatrix} d\tilde{\mu}_{0,0} & \cdots & d\tilde{\mu}_{0,N-1} \\ \vdots & \ddots & \vdots \\ d\tilde{\mu}_{N-1,0} & \cdots & d\tilde{\mu}_{N-1,N-1} \end{pmatrix} \begin{pmatrix} \tilde{\pi}_0(g) \\ \vdots \\ \tilde{\pi}_{N-1}(g) \end{pmatrix}.$$

(5) *There exist functions $\tilde{\mu}_0$ and $\tilde{\mu}_{m,m'}$ for $1 \leq m \leq m' \leq N-1$ such that B is defined as follows:*

$$B(f,g) = \int f\, g d\tilde{\mu}_0 + \int (\tilde{\pi}_1(f), ..., \tilde{\pi}_{N-1}(f)) \begin{pmatrix} d\tilde{\mu}_{1,1} & \cdots & d\tilde{\mu}_{1,N-1} \\ \vdots & \ddots & \vdots \\ d\tilde{\mu}_{N-1,1} & \cdots & d\tilde{\mu}_{N-1,N-1} \end{pmatrix} \begin{pmatrix} \tilde{\pi}_1(g) \\ \vdots \\ \tilde{\pi}_{N-1}(g) \end{pmatrix}.$$

Proof: The equivalence $1 \iff 2 \iff 3$ was proved in [6]. 4 and 5 are a straightforward reformulation of the above statements 2 and 3, respectively. ■

In a natural way, matrix measures appear in connection with this extension of the Favard Theorem. This fact was pointed out in [7, Section 2]. Even more, if B is an inner product of Sobolev type,

$$B(f,g) = \int f(t)\, g(t) d\mu(t) + \sum_{i=1}^{N} \int f^{(i)}(t) g^{(i)}(t) d\mu_i(t), \tag{4.4}$$

where $\{\mu_i\}_{i=1}^N$ are atomic measures, it is straightforward to prove that there exists a polynomial h of degree depending on N and mass points such that h induces a Hermitian operator with respect to B. As an immediate consequence we get a higher-order recurrence relation of type

$$h(t)q_n(t) = c_{n,0}q_n(t) + \sum_{k=1}^{M} \left[c_{n,k}q_{n-k}(t) + c_{n+k,k}q_{n+k}(t)\right], \tag{4.5}$$

where M is the degree of h and $\{q_n\}_{n=0}^\infty$ is the sequence of orthogonal polynomials relative to B.

Furthermore, extra information about the measures $\{\mu_i\}_{i=1}^N$ in (4.4) is obtained in [9] when the corresponding sequence of orthonormal polynomials satisfies a recurrence relation like (4.5).

Theorem 4.5 *Assume that there exists a polynomial h of $\deg h \geq 1$ such that $B(h\,f,g) = B(f,h\,g)$, where B is defined by (4.4). Then the measures $\{\mu_i\}_{i=1}^N$ are necessarily of the form*

$$\mu_i(t) = \sum_{k=1}^{j(i)} \alpha_{i,k}\delta(t - t_{i,k}),$$

for some positive integers $j(i)$, where

(1) $\alpha_{i,k} \geq 0$, $k = 1, 2, ..., j(i)$, $i = 1, 2, ..., N$.
(2) $R_i = \{t_{i,k}\}_{k=1}^{j(i)} \neq \emptyset$ *are the distinct real zeros of* $h^{(i)}$, $i = 1, 2, ..., N$.
(3) $\mathrm{supp}\mu_i \subset \bigcap_{k=1}^{i} R_k$, $k = 1, 2, ..., N$.
(4) The degree of h is at least $N+1$ and there exists a unique polynomial H of minimal degree $m(H)$ satisfying $H(0) = 0$ and $B(H\,f,g) = B(f,H\,g)$.

The above situation corresponds to the so-called diagonal case for Sobolev-type orthogonal polynomials.

Finally, we state a more general result, which was obtained in [6].

Theorem 4.6 *Let $\mathbb{P}$ be the space of real polynomials and B a real symmetric bilinear form defined on $\mathbb{P}$. If $h(t) = (t-t_1)^{n_1}\cdots(t-t_k)^{n_k}$ and $N = \deg h$, then the following statements are equivalent:*

(1) The operator "multiplication by h" is Hermitian for B and $B(h\,f,t\,g) = B(t\,f,h\,g)$, i.e., the operators "multiplication by h" and "multiplication by t" commute with respect to B.
(2) There exist a function μ and constant real numbers $M_{i,j,l,l'}$ with $0 \leq i \leq$

$n_l - 1$, $0 \leq j \leq n_{l'} - 1$, $1 \leq l, l' \leq k$ *and* $M_{i,j,l,l'} = M_{j,i,l',l}$, *such that*

$$B(f,g) = \int f(t)\, g(t) d\mu(t) + \sum_{l,l'=1}^{k} \sum_{i=0}^{n_l - 1} \sum_{j=0}^{n_{l'}-1} M_{i,j,l,l'} f^{(i)}(t_l) g^{(i)}(t_{l'}).$$

To conclude, in view of the fact that the operator "multiplication by h" is Hermitian with respect to the complex inner product

$$\langle f, g \rangle = \int_{\Gamma} f(z) \overline{g(z)} d\mu(z), \tag{4.6}$$

where Γ is a harmonic algebraic curve defined by $\Im h(z) = 0$ and h a complex polynomial (see [15]), it seems natural to ask:

Problem 1 *To characterize the sesquilinear forms $B : \mathbb{P} \times \mathbb{P} \to \mathbb{C}$ such that the operator "multiplication by h" satisfies $B(h\, f, g) = B(f, h\, g)$ for every polynomial f, $g \in \mathbb{P}$, the linear space of polynomials with complex coefficients.*

In the same way (see [14]), given an inner product like (4.6), if Γ is an equipotential curve $|h(z)| = 1$, where h is a complex polynomial, then the operator "multiplication by h" is isometric with respect to (4.6). Thus, it is natural to formulate

Problem 2 *To characterize the sesquilinear forms $B : \mathbb{P} \times \mathbb{P} \to \mathbb{C}$ such that the operator "multiplication by h" satisfies $B(h\, f, h\, g) = B(f, g)$ for every polynomial f, $g \in \mathbb{P}$, the linear space of polynomials with complex coefficients.*

The connection between these problems and matrix polynomials orthogonal with respect to matrix measures supported on the real line and on the unit circle, respectively, has been shown in [15] and [14].

Acknowledgements

We thank Walter Gautschi for his helpful remarks which help us to improve the paper and also to the unknown referee for the historical remarks and suggestions, specially for pointed out the work by Natanson [17].

References

[1] M. P. Alfaro and L. Vigil, Solution of a problem of P. Turán on zeros of orthogonal polynomials on the unit circle. *J. Approx. Theory* **53** (1988) 195-197.

[2] M. Álvarez and G. Sansigre, On polynomials with interlacing zeros. In: *Polynômes Orthogonaux et Applications. C. Brezinski et al. Eds. Proceedings, Bar-le-Duc 1984.* Springer Verlag, Berlin, 1985, 255-258.

[3] A. Atzmon, n-Orthonormal operator polynomials. In: *Orthogonal Matrix-valued Polynomials and Applications.* I. Gohberg Ed. Birkhäuser, Basel, 1998, 47-63.

[4] S. Barnett, *Polynomials and Linear Control Systems.* Marcel Dekker, New York, 1983.

[5] T.S. Chihara, *An Introduction to Orthogonal Polynomials.* Gordon and Breach, New York, 1978.

[6] A. J. Duran, A generalization of Favard's Theorem for polynomials satisfying a recurrence relation. *J. Approx. Theory* **74** (1993) 83-109.

[7] A. J. Duran and W. Van Assche, Orthogonal matrix polynomials and higher-order recurrence relations. *Linear Algebra Appl.* **219** (1995) 261-280.

[8] T. Erdélyi, P. Nevai, J. Zhang, and J. S, Geronimo. A simple proof of "Favard's theorem" on the unit circle. *Atti. Sem. Mat. Fis. Univ. Modena* **39** (1991) 551-556.

[9] W.D. Evans, L.L. Littlejohn, F. Marcellán, C. Markett, and A. Ronveaux, On recurrence relations for Sobolev orthogonal polynomials. *SIAM J. Math. Anal.* **26** (1995), 446-467.

[10] J. Favard, Sur les polynômes de Tchebicheff. *C. R. Acad. Sci. Paris* **200** (1935) 2052–2053.

[11] Ya. L. Geronimus, *Polynomials orthogonal on a circle and their applications.* Amer. Math. Soc. Translations 1954.

[12] H. J. Landau, *Maximum entropy and the moment problem.* Bull. Amer. Math. Soc. (N.S.) **16** (1987) 47-77.

[13] F. Marcellán, Orthogonal polynomials and Toeplitz matrices: some applications In: *Rational Approximation and Orthogonal Polynomials.* M. Alfaro Ed. *Sem. Mat. García de Galdeano.* Univ. Zaragoza, 1989, 31-57.

[14] F. Marcellán and I. Rodríguez, A class of matrix orthogonal polynomials on the unit circle. *Linear Algebra Appl.* **121** (1989) 233-241.

[15] F. Marcellán and G. Sansigre, On a class of matrix orthogonal polynomials on the real line. *Linear Algebra Appl.* **181** (1993) 97-109.

[16] J. Montaner and M. Alfaro, On zeros of polynomials orthogonal with respect to a quasi-definite inner product on the unit circle. *Rend. Cir. Mat. Palermo (2)*, **44** (1995) 301-314.

[17] I. P. Natanson, *Constructive Function Theory* Vol. II, Approximation in Mean. Frederick Ungar Publ. Co., New York, 1965.

[18] P. Nevai and V. Totik, Orthogonal polynomials and their zeros. *Acta Sci. Math. (Szeged)* **53** (1989) 99-104.

[19] O. Perron, *Die Lehre von den Kettendrüchen.* Teubner, Leipzig, 1929. 2nd Edition.

[20] J. Sherman, On the numerators of the Stieltjes continued fractions. *Trans. Amer. Math. Soc.* **35** (1933) 64-87.

[21] J. Shohat, The relation of the classical orthogonal polynomials to the polynomials of Appell. *Amer. J. Math.* **58** (1936) 453-464.

[22] G. W. Stewart and J. G. Sun, *Matrix Perturbation Theory.* Academic Press, Boston, MA, 1990.

[23] T. J. Stieltjes, Recherches sur les fractions continues. *Ann. Fac. Sci Toulouse* **8** (1894) J1-122, **9** (1895) A1-10.

[24] M. H. Stone, *Linear Transformations in Hilbert Spaces.* Colloquium Publications Vol. 15, Amer. Math. Soc., Providence, RI, 1932.

[25] G. Szegö, *Orthogonal Polynomials.* Colloquium Publications Vol. 23, Amer. Math. Soc., Providence, RI, 1975. 4th Edition.

[26] J. Vinuesa and R. Guadalupe, Zéros extrémaux de polynômes orthogonaux. In: *Polynômes Orthogonaux et Applications.* C. Brezinski et al. Eds. Proceedings, Bar-le-Duc 1984. Springer Verlag, Berlin, 1985, 291-295.

[27] B. Wendroff, On orthogonal polynomials. *Proc. Amer. Math. Soc.* **12** (1961) 554–555.

[28] A. Wintner, *Spektraltheorie der unendlichen Matrizen, Einführung in den analytischen Apparat der Quantenmechanik.* Hitzel, Leipzig, 1929.