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An Approach to the Isotheory by Means of Extended Pseudoisotopisms

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Abstract. Based on the traditional concept of isotopism, extended isotopisms were introduced by the authors in 2006 in order to provide a fundamental basis to the isotheory of Santilli. Since that first attempt, distinct studies on extended isotopisms have focused on the construction of partial Latin squares having a Santilli autotopism in their autotopism group. In order to deal with new structures, we introduce in this paper the concept of extended pseudoisotopism. This is based on the use of onto linear transformations that are not necessarily injective. Some examples are exposed throughout the paper.

Keywords: Isotopism, isotheory
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INTRODUCTION

In 1942, Albert [1] introduced the concept of *isotopism of algebras*: Two algebras $(A_1,\cdot)$ and $(A_2,\circ)$ are *isotopic* if there exist three regular linear transformations α, β and γ from A_1 to A_2 such that

$$\alpha(u)\circ\beta(v)=\gamma(u\cdot v),\ \text{for all } u,v\in A_1. \tag{1}$$

The algebra A_2 is then said to be *isotopic* to A_1 and the triple $\Theta=(\alpha,\beta,\gamma)$ is said to be an *isotopism* between both algebras A_1 and A_2. If $\alpha=\beta=\gamma$, then this is an *isomorphism*.

Since the original paper of Albert, a wide amount of papers have appeared in the literature that deal with isotopisms of distinct types of algebras as division [2, 3, 4, 5], Jordan [6, 7, 8], alternative [9, 10], absolute valued [11, 12], structural [13] and real two-dimensional commutative [14] algebras. Isotopism of Lie algebras were already considered by Albert himself [1] and, shortly after, in 1944, by Bruck [15]. More recently, in 1978, Santilli [16] took up the concept in the frame of a dynamical system based on an unitary Lie algebra L endowed with an inner product $\cdot$, whose state space is determined by a set S of parameters as coordinates, velocity, time, temperature or density, among others. He generalized the associative product $u\cdot v$ between Hermitian generators of the corresponding universal enveloping associative algebra by considering the new product

$$u\hat{\cdot}_s v=u\cdot\hat{T}(s)\cdot v,\ \text{for all } u,v\in L \text{ and } s\in S, \tag{2}$$

where $\hat{T}:S\to L\setminus\{0\}$ is called *isotopic element*[1]. The *commutator product* $[u,v]=u\hat{\cdot}_s v-v\hat{\cdot}_s u$ preserves the Lie axioms and is called the *Lie-isotopic product*. The application to Lie's theory (enveloping algebras, Lie algebras and Lie groups) that emerges from this new product is the so-called *Lie-Santilli isotheory* [16, 17, 18, 19, 20, 21, 22].

For each state $s\in S$, there are also defined the elements

$$\hat{u}_s=u\cdot\hat{I}(s),\ \text{for all } u\in L, \tag{3}$$

where $\hat{I}:S\to L\setminus\{0\}$ satisfies that

i. $\hat{T}(s)\cdot\hat{I}(s)$ and $\hat{I}(s)\cdot\hat{T}(s)$ coincide with the identity element in L, for all $s\in S$.
ii. The set $\hat{L}_s=L\cdot\hat{I}(s)=\{\hat{u}_s:\ u\in L\}$ coincides with L, for all $s\in S$.

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[1] Note that, in order to clarify the relation that exists between the isotheory and the classical theory of isotopisms, the notation that is used throughout the current paper may differ from that used in [16] and its subsequent papers.

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iii. $u \cdot \hat{I}(s) \neq v \cdot \hat{I}(s)$, for all $s \in S$ and $u, v \in L$ such that $u \neq v$.

This map $\hat{I}$ is called *isounit*, due to the fact that it generalizes the unit of the ground field of the original algebra. In particular, since the product $\cdot$ is associative, the next equality holds

$$\hat{u}_s \hat{\cdot}_s \hat{I}(s) = \hat{u}_s \cdot \hat{T}(s) \cdot \hat{I}(s) = \hat{u}_s = \hat{I}(s) \cdot \hat{T}(s) \cdot \hat{u}_s = \hat{I}(s) \hat{\cdot}_s \hat{u}_s, \text{ for all } u \in L \text{ and } s \in S. \quad (4)$$

Further,

$$\hat{u}_s \hat{\cdot}_s \hat{v}_s = (u \cdot \hat{I}(s)) \cdot \hat{T}(s) \cdot (v \cdot \hat{I}(s)) = (u \cdot v) \cdot \hat{I}(s), \text{ for all } u, v \in L \text{ and } s \in S. \quad (5)$$

The set $\hat{L}_s = L$ constitutes, therefore, a Lie algebra with the commutator product applied to the product $\hat{\cdot}_s$. It involves the definition of the *isoproduct*

$$\hat{[}\hat{u}_s, \hat{v}_s\hat{]}_s = \hat{u}_s \hat{\cdot}_s \hat{v}_s - \hat{v}_s \hat{\cdot}_s \hat{u}_s = [u, v] \hat{\cdot}_s \hat{I}(s), \text{ for all } u, v \in L. \quad (6)$$

For each state $s \in S$, let α_s be the linear transformation from L to itself, which is defined so that $\alpha_s(u) = \hat{u}_s$, for all $u \in L$. The identity (6) is then equivalent to

$$\hat{[}\alpha_s(u), \alpha_s(v)\hat{]}_s = \alpha_s([u, v]), \text{ for all } u, v \in L. \quad (7)$$

The family of triples $\mathscr{F}_S = \{(\alpha_s, \alpha_s, \alpha_s) \colon s \in S\}$ constitutes, therefore, a local state isomorphism of Lie algebras on the underlying dynamical system. Nevertheless, to the best knowledge of the authors, even if distinct papers and monographs have dealt with the foundations of the Lie-Santilli isotheory and its extension to other algebraic structures as groups, rings or vector spaces, among others [18, 19, 23, 24], there does not exist at this time any comprehensive study that reinterprets the Lie-Santilli isotheory by means of the theory of local state isomorphisms. A further study in this regard is, therefore, necessary.

In 2006, in order to generalize the isotheory to a non-isomorphic frame and bring it closer to the classical theory of isotopisms, the authors used the concept of extended isotopism introduced in [25] as a way to reinterpret the dependence of the isounit on the state space of the underlying dynamical system as a family of classical isotopisms. At the time, distinct papers on extended isotopisms have focused on the construction of partial Latin squares having a Santilli isotopism in their autotopism group [26, 27, 28]. In order to deal with new algebraic structures, we generalize in this paper the concept of extended isotopism by reducing the condition of being injective. We introduce in this way the concepts of pseudoisotopism and extended pseudoisotopism, which constitute a new approach to lay the foundation for the isotheory.

SANTILLI EXTENDED ISOTOPISMS

The main strength of the isotheory consists of the dependence of the isounit on the state space of the underlying dynamical system. In the context of the Lie-Santilli isotheory, let us review how the authors reinterpreted this dependence in [26] in order to relate it with the classical theory of isotopisms.

Let $(L, \cdot)$ be a Lie algebra, not necessarily unitary, associated to a dynamical system, whose state space is determined by a set S of parameters. As a first step, we include a new parameter in the set S with three possible states 1, 2 and 3. The new set of parameters is denoted as $\overline{S} = S \times \{1, 2, 3\}$. Let us consider a map $\hat{I} \colon \overline{S} \to L \setminus \{0\}$ such that

i. The set $\hat{L}_{(s,t)} = L \cdot \hat{I}(s,t) = \{u \cdot \hat{I}(s,t) \colon u \in L\}$ coincides with L, for all $(s,t) \in \overline{S}$.

ii. $u \cdot \hat{I}(s,t) \neq v \cdot \hat{I}(s,t)$, for all $(s,t) \in \overline{S}$ and $u, v \in L$ such that $u \neq v$.

For each $s \in S$, let us consider the three linear transformations α_s, β_s and γ_s from L to itself so that $\alpha_s(u) = u \cdot \hat{I}_{(s,1)}$, $\beta_s(u) = u \cdot \hat{I}_{(s,2)}$ and $\gamma_s(u) = u \cdot \hat{I}_{(s,3)}$, for all $u \in L$. The conditions (i) and (ii) imposed to $\hat{I}$ involve these three transformations to be onto and injective. As a consequence, the triple $\Theta_s = (\alpha_s, \beta_s, \gamma_s)$ constitutes an isotopism between the Lie algebra $(L, \cdot)$ and the Lie algebra $(L, \hat{\cdot}_s)$, where $\hat{\cdot}_s$ is the product defined so that

$$u \hat{\cdot}_s v = \gamma_s(\alpha_s^{-1}(u) \cdot \beta_s^{-1}(v)), \text{ for all } u, v \in L. \quad (8)$$

If $\alpha_s = \beta_s$, then the triple Θ_s constitutes indeed an isotopism between the Lie algebra L endowed with the commutator product $[u,v] = u\cdot v - v\cdot u$ and the Lie algebra L endowed with the commutator product

$$\hat{[u,v]}_s = u\hat{\cdot}_s v - v\hat{\cdot}_s u = \gamma_s(\alpha_s^{-1}(u)\cdot\alpha_s^{-1}(v)) - \gamma_s(\alpha_s^{-1}(v)\cdot\alpha_s^{-1}(u)) = \gamma_s([\alpha_s^{-1}(u),\alpha_s^{-1}(v)]), \text{ for all } u,v\in L. \quad (9)$$

The family of triples $\mathscr{F}_S = \{(\alpha_s,\beta_s,\gamma_s):\ s\in S\}$ is called an *extended Santilli isotopism* of Lie-algebras.

EXTENDED PSEUDOISOTOPISMS

Extended isotopisms make possible to generalize the state isomorphism approach of the isotheory to a state isotopism approach. In the current section we generalize the latter by removing the injectivity in the conditions of the isounit $\hat{I}$. A first attempt in this regard was already exposed by the authors in [26] for the construction of non-injective isoalgebras by means of Santilli isotopisms. We formalize the ideas exposed in that paper by introducing here the notions of pseudoisotopism and extended pseudoisotopism. The original idea on which both concepts are based was introduced in the Ph. D. Thesis of the first author [29].

Let $(G_1,\cdot)$ and $(G_2,\circ)$ be two *groupoids*, that is, a pair of sets G_1 and G_2 endowed with two respective binary operations $\cdot: G_1\times G_1\to G_1$ and $\circ: G_2\times G_2\to G_2$. A triple $\Theta = (\alpha,\beta,\gamma)$ of onto maps from G_1 to G_2 is called a *pseudoisotopism* from $(G_1,\cdot)$ to $(G_2,\circ)$ if the following two conditions are satisfied

i. $\gamma(u\cdot v) = \gamma(u'\cdot v')$, for all $u,v,u',v'\in G_1$ such that $\alpha(u) = \alpha'(u)$ and $\beta(v) = \beta'(v)$.
ii. $\alpha(u)\circ\beta(v) = \gamma(u\cdot v)$, for all $u,v\in G_1$.

Observe that the second condition is consistent because of the first condition. If the three maps α, β and γ are injective, then the triple Θ is a an isotopism. If $\alpha=\beta=\gamma$, then Θ is called a *pseudoisomorphism*. If there exist three elements u_α, u_β and u_γ in G_1 such that $\delta(u) = u\cdot u_\delta$, for all $u\in G_1$ and $\delta\in\{\alpha,\beta,\gamma\}$, then the triple Θ is called a *Santilli pseudoisotopism*. Finally, if I is a set of indices, then every family $\mathscr{F} = \{(\alpha_i,\beta_i,\gamma_i)\}_{i\in I}$ formed by pseudoisotopisms from $(G_1,\cdot)$ to $(G_2,\circ)$ is called an *extended pseudoisotopism*. It is a *Santilli extended pseudoisotopism* if each triple of the family $\mathscr{F}$ is a Santilli pseudoisotopism. Let us finish the paper with some examples on this point.

Example 1 *In the field of complex numbers $\mathbb{C}$, let G be a group generated by $i\in\mathbb{C}$, endowed with the usual product $\cdot$ in $\mathbb{C}$, that is, $G=\{1,-1,i,-i\}$. Let us consider the subgroup $H=\{1,-1\}$ of G and let $\alpha: G\to H$ be such that $\alpha(1)=\alpha(-1)=1$ and $\alpha(i)=\alpha(-i)=-1$. The triple (α,α,α) is a pseudoisomorphism from $(G,\cdot)$ to $(H,\cdot)$ that preserves the structure of group.* ◁

Example 2 *Let $(\mathbb{R},+,\times)$ be the field of real numbers and let U be the set of differentiable real functions of one real variable. This set constitutes an $\mathbb{R}$-vector space with the natural operators*

$$(f+g)(x) = f(x)+g(x), \text{ for all } f,g\in U \text{ and } x\in\mathbb{R}. \quad (10)$$

$$(\lambda\cdot f)(x) = \lambda\times f(x), \text{ for all } f\in U \text{ and } x\in\mathbb{R}. \quad (11)$$

Let us consider the map $\alpha: U\to\mathbb{R}$ so that $\alpha(f) = \frac{\partial f}{\partial x}(2)$, for all $f\in U$. The latter is onto because, given $a\in\mathbb{R}$, the function $f(x)=\frac{a}{4}x^2$ satisfies that $\alpha(f)=a$. However, α is not injective. To see it, it is enough to consider the functions $f_1(x)=15x^2$ and $f_2(x)=5x^3$, for which $\alpha(f_1)=\alpha(f_2)=60$. The triple $\Theta_1=(\alpha,\alpha,\alpha)$ is a pseudoisomorphism from $(U,+)$ to $(\mathbb{R},+)$ because

$$a+b = \bigcup_{f,g\in U}\left\{\alpha(f+g): \frac{\partial f}{\partial x}(2)=a, \frac{\partial g}{\partial x}(2)=b\right\}.$$

The triple $\Theta_2=(\mathrm{Id},\alpha,\alpha)$ is a pseudoisomorphism from $(U,\cdot)$ to $(\mathbb{R},\times)$ where

$$a\times b = \bigcup_{f\in U}\left\{\alpha(a\cdot f): \frac{\partial f}{\partial x}(2)=b\right\}.$$

The pair (Θ_1,Θ_2) constitutes therefore a pseudoisotopism from $(U,+,\cdot)$ to $(\mathrm{R},+,\times)$. ◁

Example 3 *Let us consider the polynomial ring* $\mathbb{F}_2[x]$ *over the finite field* $\mathbb{F}_2$*. Let us define the maps* α_0 *and* α_1 *from* $\mathbb{F}_2[x]$ *to itself, so that* $\alpha_t(p) = p + t$*, for all polynomial* $p \in \mathbb{F}_2[x]$ *and* $t \in \{0,1\}$*. The family* $\mathscr{F} = \{(\alpha_{t_1}, \alpha_{t_2}, \alpha_{(t_1+t_2) \pmod 2})\}_{t_1,t_2 \in \{0,1\}}$ *is a Santilli extended isotopism from* $(\mathbb{F}_2[x], +)$ *to itself. Specifically,*

$$p + q = \alpha_{t_1}(p - t_1) + \alpha_{t_2}(q - t_2) = \alpha_{(t_1+t_2) \pmod 2}(p + q - ((t_1 + t_2) \pmod 2)), \text{ for all } t_1, t_2 \in \{0,1\}.$$

◁

CONCLUSIONS AND FURTHER STUDIES

Extended (pseudo)isotopisms are introduced here to extend the state isomorphism frame in which isotheory is comprehended to a more general state (pseudo)isotopism frame. A further study on both concepts is necessary. Specifically, a study that focuses on Santilli (pseudo)isotopisms of partial Latin rectangles can be an interesting starting point in this regard.

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