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GROTHENDIECK LOCALLY CONVEX SPACES OF CONTINUOUS VECTOR VALUED FUNCTIONS

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Let $\mathscr{C}(X, E)$ be the space of continuous functions from the completely regular Hausdorff space X into the Hausdorff locally convex space E, endowed with the compact-open topology. Our aim is to characterize the $\mathscr{C}(X, E)$ spaces which have the following property: weak-star and weak sequential convergences coincide in the equicontinuous subsets of $\mathscr{C}(X, E)'$. These spaces are here called Grothendieck spaces. It is shown that in the equicontinuous subsets of E' the $\sigma(E', E)$- and $\beta(E', E)$-sequential convergences coincide, if $\mathscr{C}(X, E)$ is a Grothendieck space and X contains an infinite compact subset. Conversely, if X is a G-space and E is a strict inductive limit of Fréchet-Montel spaces $\mathscr{C}(X, E)$ is a Grothendieck space. Therefore, it is proved that if E is a separable Fréchet space, then E is a Montel space if and only if there is an infinite compact Hausdorff X such that $\mathscr{C}(X, E)$ is a Grothendieck space.

1. Introduction. In this paper X will always denote a completely regular Hausdorff topological space, E a Hausdorff locally convex space, and $\mathscr{C}(X, E)$ the space of continuous functions from X into E, endowed with the compact-open topology. When E is the scalar field of reals or complex numbers, we write $\mathscr{C}(X)$ instead $\mathscr{C}(X, E)$.

It is well known that $\mathscr{C}(X, E)$ is a Montel space whenever $\mathscr{C}(X)$ and E so are, hence, if and only if X is discrete and E is a Montel space (see **[5]**, **[16]**).

We study what happens when X has the following weaker property: the compact subsets of X are G-spaces (see below for definitions).

We obtain in Theorem 4.4 that if E is a Fréchet-Montel space and X has that property, then $\mathscr{C}(X, E)$ is a Grothendieck locally convex space. The key in the proof is the following fact: every countable equicontinuous subset of $\mathscr{C}(X, E)'$ lies, via a Radon-Nikodým theorem, in a suitable $L^1(\tau, E'_\beta)$. As a consequence of a theorem of Mújica **[10]**, the same result is true when E is a strict inductive limit of Fréchet-Montel spaces.

In §3 we study the converse of 4.4. In Corollary 3.3 it is proved that if X contains an infinite compact subset, E is a Fréchet separable space and $\mathscr{C}(X, E)$ is a Grothendieck space, then E is a Montel space. This property characterizes the Montel spaces among the Fréchet separable spaces.

Finally, in §5 we study the Grothendieck property in $\mathscr{B}(\Sigma, E)$, the space of Σ-totally measurable functions, by using the results for $\mathscr{C}(X, E)$.

2. Generalities. A compact Hausdorff topological space K is called a G-space whenever $\mathscr{C}(K)$ is a Grothendieck Banach space, i.e. the weak-star and weak sequential convergences coincide in $\mathscr{C}(K)'$ [**6**].

We extend here this concept to completely regular spaces.

2.1. Definition. X is a G-space if every compact subset K of X is a G-space.

If X is compact, both definitions coincide [**6**]. Let us remark that there exist non-compact non-discrete G-spaces. Indeed, the topological subspace of the Stone-Čech compactification of a countable discrete set obtained removing a cluster point, is such a space.

We introduce a new definition of Grothendieck locally convex space, so that $\mathscr{C}(X)$ is a Grothendieck space if and only if X is a G-space.

2.2. Definition. E is a Grothendieck space whenever the $\sigma(E', E)$- and $\sigma(E', E'')$-sequential convergences coincide in the equicontinuous subsets of E'.

In [**17**] the TG-spaces are defined as those spaces E in which the $\sigma(E', E)$- and $\sigma(E', E'')$-sequential convergences coincide. When one deals with $\mathscr{C}(X)$ spaces, our definition seems to be more reasonable than that of [**17**] (see 2.4 and 2.5).

The following permanence properties of the class of Grothendieck locally convex spaces are easy to see, thus we state them without proof.

2.3. Proposition. (a) *E is a Grothendieck space if and only if every, or some, dense subspace of E so is.*

(b) *Let $T: E \to F$ be a linear continuous operator such that for every bounded subset B of F there is a bounded subset C of E so that B is contained in the closure of $T(C)$. Then F is a Grothendieck space if E so is.*

(c) *If E is the inductive limit of the sequence (E_n) of Grothendieck spaces, and if every bounded subset of E is contained in some E_n, then E is a Grothendieck space.*

2.4. Theorem. *$\mathscr{C}(X, E)$ is a Grothendieck space if and only if $\mathscr{C}(K, E)$ so is for every compact subset K of X. In particular, X is a G-space if and only if $\mathscr{C}(X)$ is a Grothendieck space.*

Proof. Let us recall that, if K is a compact subset of X, the restriction map T is a continuous linear operator from $\mathscr{C}(X, E)$ into $\mathscr{C}(K, E)$.

If $B \subset \mathscr{C}(K, E)$ is bounded, then the bounded subset C of $\mathscr{C}(X, E)$, whose elements g can be written $g = \sum_{n \le m} f_n(\cdot)e_n$ with $f_n \in \mathscr{C}(X)$, $0 \le f_n \le 1$, $\sum_{n \le m} f_n \le 1$, and $e_n \in \bigcup\{h(K)\colon h \in B\}$, satisfies $\overline{T(C)} \supset B$ (see [**14**, I.5.3]).

If $\mathscr{C}(X, E)$ is a Grothendieck space, $\mathscr{C}(K, E)$ so is by 2.3(b).

Conversely, let (g_n') be an equicontinuous and $\sigma(\mathscr{C}(X, E)', \mathscr{C}(X, E))$-null sequence. By [**14**, III.3 and III.4], there exist a compact subset K of X and an equicontinuous sequence (h_n') in $\mathscr{C}(K, E)'$ such that $g_n' = h_n' \circ T$ for all $n \in \mathbf{N}$. Since (h_n') is $\sigma(\mathscr{C}(K, E)', T(\mathscr{C}(K, E)))$-null and equicontinuous, it is also $\sigma(\mathscr{C}(K, E)', \mathscr{C}(K, E)'')$-null if $\mathscr{C}(K, E)$ is a Grothendieck space. It follows that (g_n') is $\sigma(\mathscr{C}(X, E)', \mathscr{C}(X, E)'')$-null.

2.5. Remark. We use an example of Haydon [**4**] to show that, while in the class of barrelled spaces the TG-spaces and the Grothendieck spaces do coincide, this is not true in general.

Choose, for each infinite sequence in $\mathbf{N}$, a cluster point in the Stone-Čech compactification of $\mathbf{N}$, and let X be the topological subspace of that compactification, formed by $\mathbf{N}$ and these cluster points. Then every compact subset of X is finite, $\mathscr{C}(X)$ is infrabarrelled and every $f \in \mathscr{C}(X)$ is bounded. By Theorem 2.4, X is a G-space. Let $f_n'(f) = n^{-1}f(n)$ for all $f \in \mathscr{C}(X)$ and $n \in \mathbf{N}$. Then (f_n') is a $\sigma(\mathscr{C}(X)', \mathscr{C}(X))$-null sequence in $\mathscr{C}(X)'$, that is not $\sigma(\mathscr{C}(X)', \mathscr{C}(X)'')$-null because it is not equicontinuous.

3. Necessary conditions for $\mathscr{C}(X, E)$ to be a Grothendieck space. It is well known, and easy to see, that $\mathscr{C}(X)$ and E are topologically isomorphic to complemented subspaces of $\mathscr{C}(X, E)$. By 2.3(b), $\mathscr{C}(X)$ and E must be Grothendieck spaces if $\mathscr{C}(X, E)$ is such a space.

However, unless X is pseudofinite, i.e. their compact subsets are finite (hence $\mathscr{C}(X, E)$ is a Grothendieck space if and only if E so is, by Theorem 2.4), E has a stronger property if $\mathscr{C}(X, E)$ is a Grothendieck space, as we prove in the next theorem. To prove it we recall the following result of [**2**]:

Theorem A. *Let E and F be Hausdorff locally convex spaces, and suppose that F contains a subspace topologically isomorphic to the subspace of c_0 whose elements have only finitely many non-zero coordinates.*

If the injective tensor product $F \otimes_\varepsilon E$ is a Grothendieck space, then the $\sigma(E', E)$- and $\beta(E', E)$-sequential convergences coincide in the equicontinuous subsets of E'.

As was noted in [**2**], if X is not pseudofinite, then $\mathscr{C}(X)$ contains a subspace topologically isomorphic to the above mentioned subspace of c_0. Moreover, the injective tensor product $\mathscr{C}(X) \otimes_\varepsilon E$ can be linear and topologically identified with a dense subspace of $\mathscr{C}(X, E)$, namely, the subspace of all finite dimensional valued elements of $\mathscr{C}(X, E)$. Thus we obtain from Theorem A and Proposition 2.3 (a):

3.1. THEOREM. *If $\mathscr{C}(X, E)$ is a Grothendieck space and X contains an infinite compact subset, then the $\sigma(E', E)$- and $\beta(E', E)$-sequential convergences coincide in the equicontinuous subsets of E'.*

3.2. REMARK. By Theorem 2.4, if X is pseudofinite and E is a Grothendieck Banach space, $\mathscr{C}(X, E)$ is a Grothendieck space. However, if E is infinite dimensional, the conclusion of Theorem 3.1 does not hold [**11**].

Using Theorem 3.1 and [**7**, 11.6.2], we obtain the following corollary, converse of Theorem 4.4:

3.3. COROLLARY. *If E is a Fréchet separable space, X is not pseudofinite and $\mathscr{C}(X, E)$ is a Grothendieck space, then E is a Montel space.*

3.4. REMARK. It is unknown for us if Corollary 3.3 is true without the separability assumption on E. This is related with the following question raised in [**7**, pg. 247]: is a Fréchet space E already a Montel space if every $\sigma(E', E)$-convergent sequence in E' converges for $\beta(E', E)$?

4. Sufficient conditions for $\mathscr{C}(X, E)$ to be a Grothendieck space. We shall need some facts about vector integration, many of those can be found in [**1**] and [**15**].

Let (X, Σ, τ) be a complete measure space with $\tau(X) \leq 1$. We denote by $\mathscr{S}(\Sigma, E)$ (resp. $\mathscr{B}(\Sigma, E)$, $L^1(\tau, E)$, $L^\infty(\tau, E)$) the vector space of Σ-simple (resp. Σ-totally measurable, τ-integrable, τ-essentially bounded) E-valued (classes of) functions. Recall that $\mathscr{S}(\Sigma, E)$ and $\mathscr{B}(\Sigma, E)$ are endowed with the uniform convergence topology, and that the topology of $L^1(\tau, E)$ is defined by the seminorms $u \to \int p(u(x))\, d\tau(x)$, where p runs over the set of all continuous seminorms in E (unless contrary specification, all integrals will be extended to X).

The following Radon-Nikodým theorem is proved in [**1**]:

THEOREM B. *If E is a quasi-complete* (CM)*-space, $\mu\colon \Sigma \to E$ is a countably additive vector measure, of bounded variation and τ-absolutely continuous, then there exists $u \in L^1(\tau, E)$ such that $\mu(A) = \int_A u(x)\, d\tau(x)$ for every $A \in \Sigma$.*

Let us recall that E is a quasi-complete (CM)-space, if, for instance, it is either a Fréchet-Montel space or a (DF)-Montel space [**1**].

Firstly we extend the classical duality theorem $L^1 - L^\infty$ to $L^1(\tau, E'_\beta)$, where E is a Fréchet-Montel space.

The following lemma can be easily proved. As usual, p_L will denote the gauge of the absolutely convex set L in its linear span.

4.1. LEMMA. *If $u \in \mathscr{S}(\Sigma, E')$, namely, $u = \sum_{i \le m} \chi_{A_i} e'_i$ with $(A_i)_{i \le m}$ disjoint in Σ, then*

$$\int p_{B^0}(u(x))\, d\tau(x) \le \tau\Big(\bigcup_{i \le m} A_i\Big) \sup_{i \le m} p_{B^0}(e'_i)$$

for every bounded subset B of E.

4.2. THEOREM. *Let E be a Fréchet-Montel space. The relation*

$$(1) \qquad u'(u) = \int u(x)(v(x))\, d\tau(x) \quad \textit{for all } u \in L^1(\tau, E'_\beta)$$

defined for $u' \in L^1(\tau, E'_\beta)'$ and $v \in L^\infty(\tau, E)$, is an algebraic isomorphism between $L^1(\tau, E'_\beta)'$ and $L^\infty(\tau, E)$.

Proof. Let $v \in L^\infty(\tau, E)$. The map $x \to u(x)(v(x))$ is measurable for every $u \in L^1(\tau, E'_\beta)$, because v is strongly measurable and the assertion is clearly true when $v \in \mathscr{S}(\Sigma, E)$.

Furthermore, if $Z \in \Sigma$ is a τ-null set such that $B = v(S \setminus Z)$ is bounded, then we have

$$(2) \qquad |u(x)(v(x))| \le p_{B^0}(u(x))$$

for every $x \in X \setminus Z$.

Hence $x \to u(x)(v(x))$ is τ-integrable, and we can define a linear form u' on $L^1(\tau, E'_\beta)$ by (1). Moreover, it follows from (2) that u' is continuous.

Conversely, fix $u' \in L^1(\tau, E'_\beta)'$. There exists a bounded subset B of E such that

$$(3) \qquad \int p_{B^0}(u(x))\, d\tau(x) \le 1 \quad \text{implies } |u'(u)| \le 1$$

for every $u \in L^1(\tau, E'_\beta)$.

We define a map $\mu\colon \Sigma \to E''$ by

$$(4) \qquad \mu(A)(e') = u'(\chi_A e')$$

for every $A \in \Sigma$ and $e' \in E'$ (it follows easily from Lemma 4.1 and (3) that $\mu(A) \in E''$). Since E is reflexive we can suppose that $\mu(A) \in E$.

Clearly, $\mu\colon \Sigma \to E$ is a finitely additive vector measure. We shall show that μ is countably additive: let A be the union of the disjoint sequence (A_n) in Σ. Given an absolutely convex zero-neighborhood U in E and $\varepsilon > 0$, we choose λ with $0 < \lambda < \infty$ such that $B \subset \lambda U$, and $m_0 \in \mathbf{N}$ such that $\lambda\tau(\bigcup_{n>m} A_n) \le \varepsilon$ for every $m \ge m_0$. Since

$$e'(\mu(A)) - \sum_{n\le m} e'(\mu(A_n)) = u'\left(\chi_{\bigcup_{n>m}A_n} e'\right)$$

it follows from Lemma 4.1 and (3) that

$$\left|e'(\mu(A)) - \sum_{n\le m} e'(\mu(A_n))\right| \le \varepsilon$$

for every $m \ge m_0$ and $e' \in U^0$, as desired.

Furthermore, if $A = \bigcup_{n\le m} A_n$ where $(A_n)_{n\le m}$ is disjoint in Σ, and if $\varepsilon > 0$, there exists $(e'_n)_{n\le m}$ in U^0 such that

$$\sum_{n\le m} p_U(\mu(A_n)) \le \sum_{n\le m} e'_n(\mu(A_n)) + \varepsilon = u'\left(\sum_{n\le m} \chi_{A_n} e'_n\right) + \varepsilon.$$

Hence the p_U-variation of μ satisfies the inequality $V_{p_U}\mu(A) \le \lambda\tau(A)$, from Lemma 4.1 and (3) again.

Thus μ is τ-absolutely continuous and has bounded variation. By Theorem B, there exists $v \in L^1(\tau, E)$ such that

$$\mu(A) = \int_A v(x)\, d\tau(x) \quad \text{for every } A \in \Sigma. \tag{5}$$

We claim that v is τ-essentially bounded and satisfies (1). Indeed, let $(U_j)_j$ be a countable basis in E of absolutely convex zero-neighborhoods. Choose, for each $j \in \mathbf{N}$, λ_j such that $0 < \lambda_j < \infty$ and $B \subset \lambda_j U_j$.

By Lemma 4.1, (3), (4) and (5), we have

$$\left|\int_A e'(v(x))\, d\tau(x)\right| \le \lambda_j \tau(A) \tag{6}$$

for all $e' \in U_j^0$, $A \in \Sigma$ and $j \in \mathbf{N}$.

Let $(e'_{j,k})_k$ be a sequence in U_j^0 such that $p_{U_j}(e) = \sup_k |e'_{j,k}(e)|$ for every $e \in E$.

By (6), there exists $Z \in \Sigma$ with $\tau(Z) = 0$ such that $|e'_{j,k}(v(x))| \le \lambda_j$ for all $x \in X \setminus Z$ and all $j, k \in \mathbf{N}$. Hence $v(X \setminus Z)$ is bounded in E.

Finally, it follows from (4) that (1) is true for all $u \in \mathcal{S}(\Sigma, E')$, and, by density, for every $u \in L^1(\tau, E'_\beta)$. This concludes the proof.

Assume that X is compact Hausdorff and Σ contains the Borel subsets of X. For each $u \in L^1(\tau, E'_\beta)$, denote by ν_u the vector measure of density u with respect to τ. If p is a continuous seminorm in E, the subset

F of $L^1(\tau, E'_\beta)$ defined by the condition $V_p\nu_u(X) < \infty$, is a linear subspace. If $u \in F$ then ν_u has bounded semivariation, thus it defines a continuous linear form on $\mathcal{S}(\Sigma, E)$, which extends by continuity to the whole space $\mathcal{B}(\Sigma, E)$ [**15**]. Let $Tu \in \mathcal{C}(X, E)'$ be the restriction to $\mathcal{C}(X, E)$ of this linear form, i.e.

$$(7) \qquad (Tu)(g) = \int g(x)\, d\nu_u(x)$$

for every $g \in \mathcal{C}(X, E)$.

4.3. LEMMA. *The map T: $F \to \mathcal{C}(X, E)'$ defined by* (7) *is a linear continuous operator, when $\mathcal{C}(X, E)'$ is endowed with the strong topology with respect to $\mathcal{C}(X, E)$.*

Proof. We have, for each $u \in F$,

$$(8) \qquad (Tu)(g) = \int u(x)(g(x))\, d\tau(x)$$

for every $g \in \mathcal{C}(X, E)$. Indeed, the dominated convergence theorem and a standard density argument show that it suffices to see (8) when g belongs to $\mathcal{S}(\Sigma, E)$, that is trivially true.

Let H be a bounded subset of $\mathcal{C}(X, E)$. Then $B = \bigcup\{g(X): g \in H\}$ is a bounded subset of E. Hence, by (8), $|(Tu)(g)| \leq \int p_{B^0}(u(x))\, d\tau(x)$ and the lemma follows.

We are now ready to prove the sufficient condition:

4.4. THEOREM. *Let X be a completely regular Hausdorff G-space and E a Fréchet-Montel space. Then $\mathcal{C}(X, E)$ is a Grothendieck space.*

Proof. By 2.4 we can suppose, without loss of generality, that X is compact.

Let $(g'_n)_n$ be an equicontinuous sequence in $\mathcal{C}(X, E)'$. By [**14**, III.4.5] there exists a continuous seminorm p in E such that $V_p\mu_n(X) \leq 1$, for every $n \in \mathbf{N}$, where μ_n is the representing measure of g'_n [**14**, III].

Let $\tau = \sum_n 2^{-n} V_p\mu_n$. τ is a countably additive $[0,1]$-valued Borel measure, by [**14**, III.2.5]. Let Σ be the completed σ-field of the Borel field of X with respect to τ. We shall denote also by τ and μ_n the natural extensions of the earlier measures to Σ.

Since E is a Montel space, the measure μ_n: $\Sigma \to E'_\beta$ is countably additive. Clearly $V_p\mu_n \leq 2^n$, thus μ_n has bounded variation and is τ-absolutely continuous (when it is considered as an E'_β-valued measure).

We apply Theorem B, obtaining, for each $n \in \mathbf{N}$, a function $u_n \in L^1(\tau, E'_\beta)$ such that μ_n is the vector measure of density u_n with respect to τ.

Clearly $u_n \in F$ and $Tu_n = g'_n$, for every $n \in \mathbf{N}$.

Fix $g'' \in \mathscr{C}(X, E)''$. By Lemma 4.3 and Theorem 4.2, there exists $v \in L^\infty(\tau, E)$ such that $g''(g'_n) = \int u_n(x)(v(x))\, d\tau(x)$ for every $n \in \mathbf{N}$.

Let Z be a set in Σ with $\tau(Z) = 0$ and $v(X \setminus Z)$ bounded. The function $v_1 = \chi_{X\setminus Z} v$ is totally measurable, because E is Montel and metrizable.

Given $\varepsilon > 0$, we can choose $v_2 \in \mathscr{S}(\Sigma, E)$ such that $p(v_3(x)) \leq \varepsilon/2$, for every $x \in X$, if $v_3 = v_1 - v_2$. Hence,

$$\left|\int u_n(x)(v_3(x))\, d\tau(x)\right| = \left|\int v_3(x)\, d\mu_n(x)\right| \leq \varepsilon/2 \tag{9}$$

for every $n \in \mathbf{N}$, because $V_p\mu_n(X) \leq 1$.

On the other hand, if (g'_n) is $\sigma(\mathscr{C}(X, E)', \mathscr{C}(X, E))$-null, then $(\mu_n(A)(e))$ is a null sequence, for every $e \in E$ and $A \in \Sigma$. Indeed, since X is a G-space, for each $e \in E$, the weak-star null sequence $(\mu_n(\cdot)(e))$ in $\mathscr{C}(X)'$, is also weak null, hence $(\mu_n(A)(e))$ is null for every Borel subset A of X, and so for every $A \in \Sigma$.

Since v_2 is simple, it follows that

$$\lim_{n\to\infty} \int u_n(x)(v_2(x))\, d\tau(x) = 0. \tag{10}$$

By (9) and (10), $(g''(g'_n))$ is a null sequence, and we have shown that (g'_n) is $\sigma(\mathscr{C}(X, E)', \mathscr{C}(X, E)'')$-null.

4.5. COROLLARY. *Let X be a completely regular Hausdorff G-space and E the inductive limit of the sequence (E_n) of Fréchet-Montel spaces, such that every bounded subset of E is localized in some E_n. Then $\mathscr{C}(X, E)$ is a Grothendieck space.*

Proof. We can again suppose X compact. By **[10]**, the inductive limit of the sequence $(\mathscr{C}(X, E_n))$ is a dense topological subspace of $\mathscr{C}(X, E)$. By Proposition 2.3 (a) and (c), and Theorem 4.4, it follows that $\mathscr{C}(X, E)$ is a Grothendieck space.

4.6. COROLLARY. *Let E be a Fréchet separable space. The following conditions are equivalent*:

(a) *E is a Montel space.*

(b) *There exists a non-pseudofinite completely regular Hausdorff space X such that $\mathscr{C}(X, E)$ is a Grothendieck space.*

(c) *For every completely regular Hausdorff G-space X, $\mathscr{C}(X, E)$ is a Grothendieck space.*

Proof. Use 4.4 and 3.3.

5. Application to spaces of totally measurable functions. Let X be a nonempty set and Σ a field of subsets of X. We will say that a subset B of X is open if for every $x \in B$ there is $A \in \Sigma$ with $x \in A$ and $A \subset B$. Endowed X with this topology, let X^* be the Hausdorff space associated to X, $\pi: X \to X^*$ the quotient map, and $\Sigma^* = \{\pi(A): A \in \Sigma\}$.

The following lemma is easily established:

5.1. LEMMA (a) *X^* is a completely regular Hausdorff zero-dimensional topological space.*

(b) *The map $A \in \Sigma \to \pi(A) \in \Sigma^*$ is a Boolean isomorphism.*

(c) *The map $g \in \mathscr{B}(\Sigma^*, E) \to g \circ \pi \in \mathscr{B}(\Sigma, E)$ is a topological isomorphism, and its restriction to $\mathscr{S}(\Sigma^*, E)$ so is onto $\mathscr{S}(\Sigma, E)$.*

(d) *The map $x^* \in X^* \to \{B^* \in \Sigma^*: x^* \in B^*\} \in \mathscr{P}(\Sigma^*)$ is one-to-one.*

By using 5.1, when one studies the linear topological properties of $\mathscr{B}(\Sigma, E)$, it can be supposed that X is a dense subspace of a Hausdorff compact zero-dimensional topological space K (namely, the Stone space of the Boolean algebra Σ), and Σ is the trace in X of the Boolean algebra of open and closed subsets of K. In this context we have the following theorem:

5.2. THEOREM. *There exists a subspace of $\mathscr{B}(\Sigma, E)$, containing $\mathscr{S}(\Sigma, E)$, that is topologically isomorphic to $\mathscr{C}(K, E)$.*

Proof. It is easy to check that the set of restrictions to X of all elements of $\mathscr{C}(K, E)$ is such a subspace.

By Proposition 2.3 (a), it follows that $\mathscr{B}(\Sigma, E)$ is a Grothendieck space if and only if $\mathscr{C}(K, E)$ so is. Hence we can apply to $\mathscr{B}(\Sigma, E)$ the results of §§3 and 4.

5.3. REMARK. The question of when $\mathcal{B}(\Sigma)$ (equivalently, $\mathcal{C}(K)$) is a Grothendieck space is related to the validity of the Vitali-Hahn-Saks theorem for finitely additive scalar measures on Σ, of bounded variation. For instance, if Σ is σ-complete, or more generally, Σ has the subsequential interpolation property, then $\mathcal{B}(\Sigma)$ is a Grothendieck space (see **[13]** and **[3]**).

Finally, we show that the following result of Mendoza **[8]**, can be easily deduced from their earlier results in **[9]** and our Theorem 5.2.

5.4. THEOREM. *Suppose Σ infinite. Then $\mathcal{B}(\Sigma, E)$ is infrabarrelled (resp. barrelled) if and only if E'_β has property* (B) *of Pietsch* [**12**, 1.5.8], *and E is infrabarrelled (resp. barrelled).*

Proof. Let us observe that $\mathcal{S}(\Sigma, E)$ is a large dense subspace of $\mathcal{B}(\Sigma, E)$. Indeed, if H is a bounded subset of $\mathcal{B}(\Sigma, E)$, then the set of all g in $\mathcal{S}(\Sigma, E)$ for which there exists $h \in H$ with $g(X) \subset h(X)$, is a bounded subset of $\mathcal{S}(\Sigma, E)$ whose closure in $\mathcal{B}(\Sigma, E)$ contains H.

Thus Theorem 5.2 implies that $\mathcal{B}(\Sigma, E)$ is infrabarrelled whenever $\mathcal{C}(K, E)$ so is, hence we have the first equivalence of the theorem, by **[9]**.

If $\mathcal{B}(\Sigma, E)$ is barrelled, then E is barrelled and $\mathcal{B}(\Sigma, E)$ is infrabarrelled, so E'_β has property (B). The converse follows easily because $\mathcal{C}(K, E)$ is topologically isomorphic to a dense subspace of $\mathcal{B}(\Sigma, E)$, by 5.2.

5.5. REMARK. We have also shown in 5.4 that, if Σ is infinite, $\mathcal{S}(\Sigma, E)$ is infrabarrelled if and only if E'_β has property (B) and E is infrabarrelled, a result of Mendoza **[8]**. In **[2]** we prove that $\mathcal{S}(\Sigma, E)$ is barrelled if and only if $\mathcal{S}(\Sigma)$ and E so are, and E is nuclear.

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