On the long time behaviour of non-autonomous Lotka-Volterra models with diffusion via the sub-super trajectory method

José A. Langa[a,1], Aníbal Rodríguez-Bernal[b,2], Antonio Suárez[*,c,3]

[a]*Dpto. Ecuaciones Diferenciales y Análisis Numérico. C/ Tarfia s/n, 41012. Sevilla. Spain*

[b]*Departamento de Matemática Aplicada, Universidad Complutense de Madrid, Madrid 28040, Spain and Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM.*

[c]*Dpto. Ecuaciones Diferenciales y Análisis Numérico. C/ Tarfia s/n, 41012. Sevilla. Spain*

Abstract

In this paper we study in detail the geometrical structure of global pullback and forwards attractors associated to non-autonomous Lotka-Volterra systems in all the three cases of competition, symbiosis or prey-predator. In particular, under some conditions on the parameters, we prove the existence of a unique non-degenerate global solution for these models, which attracts any other complete bounded trajectory. Thus, we generalize the existence of a unique strictly positive stable (stationary) solution from the autonomous case and we extend to Lotka–Volterra systems the result for scalar logistic equations. To this end we present the sub-supertrajectory tool as a generalization of the now classical sub-supersolution method. In particular, we also conclude pullback and forwards permanence for the above models.

Key words: Sub-supertrajectory method, Lotka-Volterra competition, symbiosis and prey-predator systems, attracting complete trajectories

[*]Corresponding author

Email addresses: langa@us.es (José A. Langa), arober@mat.ucm.es (Aníbal Rodríguez-Bernal), suarez@us.es (Antonio Suárez)

[1]Partly supported by grants MTM2008-0088, HF2008-0039 and PHB2006-003PC.

[2]Partly supported by grants MTM2006-08262, CCG07-UCM/ESP-2393 UCM-CAM Grupo de Investigación CADEDIF and PHB2006-003PC.

[3]Partly supported by grant MTM2006-07932.

1. Introduction

When phenomena from different areas of Science as Physics, Chemistry or Biology can be modeled by a system of partial differential equations, one of the most important questions is to determine the asymptotic regimes (or future stable configurations) to which solutions evolves in time. In this paper we will analyze the asymptotic dynamics of the following non-autonomous model for two species (u and v) within a habitat Ω, a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$,

$$\begin{cases} u_t - d_1 \Delta u = f(t,x,u,v) & x \in \Omega,\ t > s \\ v_t - d_2 \Delta v = g(t,x,u,v) & x \in \Omega,\ t > s \\ \mathcal{B}_1 u = 0,\ \mathcal{B}_2 v = 0 & x \in \partial\Omega,\ t > s \\ u(s) = u_s,\ v(s) = v_s, & \end{cases} \tag{1}$$

where $d_1, d_2 > 0$ and $\mathcal{B}_i$ denotes either one of the boundary operators

$$\mathcal{B}u = \begin{cases} u, & \text{Dirichlet case, or} \\ d\dfrac{\partial u}{\partial \vec{n}} + \sigma(x)u, & \text{Robin case,} \end{cases} \tag{2}$$

where $\vec{n}$ is the outward normal vector-field to $\partial\Omega$, $\sigma(x)$ a C^1 function and f and g are regular functions. Observe that the Neumann boundary condition is included in the Robin case taking $\sigma \equiv 0$, while Dirichlet boundary conditions can be understood as the limit case $\sigma(x) = \infty$. Finally note that no sign assumption is made on $\sigma(x)$.

We will denote the solutions of (1) as

$$u(t,s;u_s,v_s), \quad v(t,s;u_s,v_s), \quad \text{for } t > s.$$

As a particular class of models of the form (1) are the non-autonomous Lotka-Volterra models:

$$\begin{cases} u_t - d_1 \Delta u = u(\lambda(t,x) - a(t,x)u - b(t,x)v) & x \in \Omega,\ t > s \\ v_t - d_2 \Delta v = v(\mu(t,x) - c(t,x)u - d(t,x)v) & x \in \Omega,\ t > s \\ \mathcal{B}_1 u = 0,\ \mathcal{B}_2 v = 0 & x \in \partial\Omega,\ t > s \\ u(s) = u_s,\ v(s) = v_s. & \end{cases} \tag{3}$$

We refer for example to [2, 3, 4, 8] for the biological meaning of the parameters involved in (3).

In line with the ecological interpretation of these models we will only consider positive solutions, and in the light of this we note here that $u_s, v_s \geq 0$ implies that the solution of (1) satisfies $u, v \geq 0$.

We will cover the now classical three main population dynamics: competition if $b, c > 0$, symbiosis if $b, c < 0$ and prey-predator if $b > 0$ and $c < 0$. However we do not allow sign changes in the coefficients. We also make no assumptions on the time behavior of the coefficients (e.g. periodicity, or almost periodicity).

The asymptotic behavior, both forwards and in the pullback sense, for systems of the form (1)–(3) have been recently studied in [10].

Note that the dynamics of (3) is very much influenced by the stability properties of semitrivial solutions, i.e. solutions with a null component. Loosely speaking, if some semitrivial solution is stable for (3) then one expects that some solutions of (3) are driven to extinction. On the other hand, if semitrivial solutions are unstable for (3) then one expects that no semitrivial solution of (3) can be driven to extinction. Such situation is denoted permanence. Observe that as semitrivial solutions of (3) satisfy a nonautonomous logistic equation, the informal discussion above about stability or instability of semitrivial solutions of (3) can be addressed both in the forwards and in the pullback senses. Also, as we are dealing with nonautonomous problems there is no an immediate linearized eigenvalue problem to derive instability from, as there is in the autonomous case.

In this direction, in [10], we were able to prove some results on the permanence and asymptotic behaviour for these kind of systems, i.e., for any positive initial data u_s and v_s, within a finite time, the values of the solution $(u(t, s, x; u_s, v_s), v(t, s, x; u_s, v_s))$, for $x \in \Omega$, enter and remain within a compact set in $\mathbb{R}^2$ that is strictly bounded away from zero in each component.

Moreover, under some conditions on the parameters, it is also proved in [10] that all non–semitrivial solutions of (3) have the same asymptotic behavior as $t \to \infty$. These conditions include a smallness conditions for the coupling parameters:

$$\limsup_{t\to\infty} \|b\|_{L^\infty(\Omega)} \limsup_{t\to\infty} \|c\|_{L^\infty(\Omega)} < \rho_0$$

for some suitable constant $\rho_0 > 0$, and imply the forwards instability of semitrivial solutions.

We also showed that, under similar conditions, which now guarantee the pullback instability of semitrivial solutions, and a similar smallness condition on the coupling coefficients, now as $t \to -\infty$, if one of the bounded complete trajectories of (3) (which exists, as we showed, from the existence of the non–autonomous attractor) is non-degenerate at $-\infty$ (see Definition 5), then it is the unique such trajectory, and it also describes the unique pullback asymptotic behavior of all non–semitrivial bounded solutions of (3).

Thus, the main left open problem in [10], which we are now able to solve in this work, is proving that such complete solution, nondegenerate at $-\infty$, actually exists.

Note that when both results in [10] can be applied together, we obtain that there exists a unique bounded complete trajectory $(u^*(t), v^*(t))$, $t \in \mathbb{R}$, that is both forwards and pullback attracting for (3), i.e. (u^*, v^*) is a bounded trajectory such that, for any $s \in \mathbb{R}$ and any choice of nonnegative, nonzero initial data u_s, v_s the corresponding solution of (3) defined for $t > s$, satisfies

$$(u(t,s;u_s,v_s) - u^*(t), v(t,s;u_s,v_s) - v^*(t)) \to (0,0) \quad \text{as } t \to \infty, \text{ or } s \to -\infty. \tag{4}$$

Note that, in general, pullback and forwards asymptotic behaviour are unrelated (see [11, 9] for cases of pullback but not forwards permanence or attraction in non-autonomous reaction-diffusion equations). Moreover, a proper concept of forwards non-autonomous attractor is also unclear (see, for instance, [6]). However, our results leads to define this bounded complete and non-degenerate solution (u^*, v^*) as the right candidate for the forwards attractor, which is also the pullback one. In particular, we can conclude that this is just the *"stationary solution"* for the non-autonomous systems which generalizes the strictly positive stationary solution known in the autonomous models. This situation also occurs, under suitable conditions for scalar nonautonomous equations, see [18]. Therefore, our results here extend to Lotka–Volterra systems (3), the case of scalar autonomous and nonautonomous equations.

On the other hand, there exists a close relation between the asymptotic dynamics of a model and the one observed inside the global pullback attractor. Nevertheless, the former is often difficult to interpret unless we have additional information about the structures within the attractor which allow in some cases a complex dynamics. Therefore, the analysis of the geometrical structure of attractors is a fundamental problem. In our situation, the existence of a unique non-degenerate complete trajectory (u^*, v^*) leads us to an important consequence on the shape of the pullback attractor for (3) (in the cone of positive solutions). Indeed, we are able to show that the pullback attractor is just the intermediate bounded complete trajectories $(\tilde{u}(\cdot), \tilde{v}(\cdot))$ between the zero solution and (u^*, v^*), and that all of them are degenerate at $-\infty$, i.e., either $\tilde{u}(\cdot)$ or $\tilde{v}(\cdot)$ are degenerate at $-\infty$.

As mentioned before, our main goal in this paper is then showing that there exists a complete, bounded and nondegenerate (at $t = -\infty$) solution of (3).

To this end we introduce the sub-supertrajectory method as a tool to get existence of intermediate complete trajectories associated to the nonlinear process for (1). Thus, if for (1) we prove the existence of ordered positive non-degenerate subtrajectories and bounded supertrajectories, see Definition 2.4, we are able to conclude the existence of non-degenerate bounded complete trajectories; see Theorem 2.5. Note that our construction is independent of whether or not (3) has monotonicity properties. In the former case, our results lead to more precise results, see Corollaries 2.7 and 2.8. Then, Section 3 is devoted to give some further results on logistic nonautonomous equations, which we use for the Lotka–Volterra system and which are of independent interest. With these tools in Section 4, we apply the techniques in Section 2 for the Lotka–Volterra model (3) by constructing such sub–super trajectory pairs. Then we impose asymptotic conditions in the coefficients of (3) that imply the that the complete subtrajectory is non degenerate at $t = -\infty$. This implies the existence of complete nondegenerate solutions for (3). See Theorem 4.4 for the case of competition, Theorem 4.5 for the case of symbiosis and Theorem 4.6 for the prey–predator case. It is important to remark that the asymptotic conditions we impose on the coefficients in (3) are the same we had in [10] to guarantee the pullback instability of semitrivial solutions, which in turn imply that the system is pullback permanent. Finally note that Theorems 4.4, 4.5 and 4.6 also include some results on asymptotic trivial or semitrivial behavior for solutions of (3).

Note that the usual way in previous works ([10], [9], [17]) to get existence of complete trajectories associated to a particular system is by means of the pullback attractor. The sub-supertrajectory method adopts a different and, in this case, more fruitful strategy. For instance, thanks to the monotonicity in the competition and symbiosis cases, we get some results on the periodicity of the complete bounded trajectories if the non-linear terms are also periodic in time, as well as the existence of equilibria in the autonomous case. Moreover, we also get the existence of minimal and maximal global bounded trajectories associated to these systems; see Corollaries 2.7 and 2.8.

2. The sub-supertrajectory method for complete solutions

In this section we will develop the main general results in this paper. The use of sub-supertrajectory pairs to construct complete solutions can be found in Chueshov [7] or Langa and Suárez [12]. Both references use monotonicity properties of the equations, see Corollaries 2.7 and 2.8 below. In particular this applies to scalar equations, a property that will be used below (see Corollary 2.10). Here we use similar ideas to construct bounded complete trajectories, without such monotonicity assumptions.

2.1. Nonautonomous processes and nondegenerate solutions

We consider classical solutions (u, v) of (1) in the sense that $u, v \in C^{1,2}_{t,x}((s, \infty) \times \overline{\Omega})$.

For this we will assume that f, g are bounded on bounded sets of $\mathbb{R}\times\overline{\Omega}\times\mathbb{R}^2$ and are locally Hölder continuous in time.

Definition 2.1. *A pair of functions $(u, v) \in C^{1,2}_{t,x}(\mathbb{R} \times \overline{\Omega})$ is a complete trajectory of (1), if for all $s < t$ in $\mathbb{R}$, $(u(t), v(t))$ is the solution of (1) with initial data $u_s = u(s)$, $v_s = v(s)$.*

Note that if the solutions of (1) are globally defined, then we can define a non-autonomous *process* in some Banach space X appropriate for the solutions, i.e. a family of mappings $\{S(t, s)\}_{t\geq s} : X \to X$, $t, s \in \mathbb{R}$ satisfying:

a) $S(t, s)S(s, \tau)z = S(t, \tau)z$, for all $\tau \leq s \leq t$, $z \in X$,
b) $S(t, \tau)z$ is continuous in $t > \tau$ and z, and
c) $S(t, t)$ is the identity in X for all $t \in \mathbb{R}$.

$S(t, \tau)z$ arises as the value of the solution of the non-autonomous system (1) at time t with initial condition $z \in X$ at initial time τ. For an autonomous equation the solutions only depend on $t - \tau$, and we can write $S(t, \tau) = S(t - \tau, 0)$.

With this definition we can restate the definition of a complete trajectory as follows:

Definition 2.2. *Let S be a process. We call the continuous map $(u, v) : \mathbb{R} \to (C^2(\overline{\Omega}))^2$ a complete trajectory of (1) if, for all $s \in \mathbb{R}$,*

$$S(t, s)(u(s), v(s)) = (u(t), v(t)) \qquad \textit{for all} \qquad t \geq s.$$

In what follows we assume that (1) defines a process S.

Definition 2.3. *1. A positive function $u(t,x)$ is non–degenerate at ∞ (respectively $-\infty$) if there exists $t_0 \in \mathbb{R}$ such that u is defined in $[t_0,\infty)$ (respectively $(-\infty,t_0]$) and there exists a $C^1(\overline{\Omega})$ function $\varphi_0(x) > 0$ in Ω, (vanishing on $\partial\Omega$ in case of Dirichlet boundary conditions), such that for all $x \in \overline{\Omega}$,*

$$u(t,x) \geq \varphi_0(x) \qquad \text{for all} \quad t \geq t_0 \tag{5}$$

(respectively for all $t \leq t_0$).
2. A function $u(t,x)$ is bounded at ∞ (respectively $-\infty$) if there exists $t_0 \in \mathbb{R}$ such that u is defined in $[t_0,\infty)$ (respectively $(-\infty,t_0]$) and there exists a constant $C > 0$ such that $|u(t,x)| \leq C$ for all $x \in \overline{\Omega}$ and $t \geq t_0$ (respectively $t \leq t_0$.)

2.2. The sub-supertrajectory method for systems. Main result

Given $T_0 \leq \infty$ and two functions $w, z \in C((-\infty,T_0)\times\overline{\Omega})$ with $w \leq z$ we denote

$$[w,z] := \{u \in C((-\infty,T_0)\times\overline{\Omega}) : w \leq u \leq z\}.$$

Now we introduce the concept of complete sub-supertrajectory pair.

Definition 2.4. *Let $T_0 \leq \infty$ and $(\underline{u},\underline{v}),(\overline{u},\overline{v}) \in \mathcal{X} = C^{1,2}_{t,x}((-\infty,T_0)\times\overline{\Omega})$. We say that $(\underline{u},\underline{v}) - (\overline{u},\overline{v})$ is a complete sub-supertrajectory pair of (1) if*

1. *$\underline{u}(t) \leq \overline{u}(t)$ and $\underline{v}(t) \leq \overline{v}(t)$, in Ω, for all $t < T_0$.*
2. *$\mathcal{B}_1(\underline{u}) \leq 0 \leq \mathcal{B}_1(\overline{u})$ and $\mathcal{B}_2(\underline{v}) \leq 0 \leq \mathcal{B}_2(\overline{v})$ on $\partial\Omega$, for all $t < T_0$.*
3. *For all $x \in \Omega$, $t < T_0$*

$$\begin{aligned} &\underline{u}_t - d_1\Delta\underline{u} - f(t,x,\underline{u},v) \leq 0 \leq \overline{u}_t - d_1\Delta\overline{u} - f(t,x,\overline{u},v), \quad \forall v \in [\underline{v},\overline{v}], \\ &\underline{v}_t - d_2\Delta\underline{v} - g(t,x,u,\underline{v}) \leq 0 \leq \overline{v}_t - d_2\Delta\overline{v} - g(t,x,u,\overline{v}), \quad \forall u \in [\underline{u},\overline{u}]. \end{aligned}$$

Note that the concept of a sub-supersolution pair, defined for $t > s$, has been widely used and developed, see e.g. Pao [15], to construct solutions for the initial value problem (1).

Theorem 2.5. *Assume that there exists a complete sub-supertrajectory pair of (1), $(\underline{u},\underline{v}) - (\overline{u},\overline{v})$, in the sense of Definition 2.4. Moreover, assume $\underline{u}$, $\underline{v}$, $\overline{u}$ and $\overline{v}$ are bounded at $-\infty$.*

For any $s < T_0$, consider initial data u_s, v_s in (1) such that

$$\underline{u}(s) \le u_s \le \overline{u}(s) \quad \text{and} \quad \underline{v}(s) \le v_s \le \overline{v}(s). \tag{6}$$

Then there exists some $t_1 < T_0$ such that for any sequence $s_n \to -\infty$ there is a subsequence of

$$\{(u(\cdot, s_n; u_{s_n}, v_{s_n}), v(\cdot, s_n; u_{s_n}, v_{s_n})) = S(\cdot, s_n)(u_{s_n}, v_{s_n})\}$$

that we denote the same, converging uniformly in compact sets of $(-\infty, t_1]$ to a complete solution of (1) as in Definition 2.1.

In particular, there exists a complete trajectory $(u^*, v^*) \in \mathcal{X}$ of (1) such that

$$(u^*, v^*) \in \mathcal{I} := [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}].$$

Proof. For initial data satisfying (6), it is easy to show that $(\underline{u}, \underline{v}) - (\overline{u}, \overline{v})$ is a sub-supersolution pair for the initial value problem

$$\begin{cases} u_t - d_1 \Delta u = f(t, x, u, v), & x \in \Omega, \ s < t < T_0, \\ v_t - d_2 \Delta v = g(t, x, u, v), & x \in \Omega, \ s < t < T_0, \\ \mathcal{B}_1(u) = \mathcal{B}_2(v) = 0, & x \in \partial\Omega, \ s < t < T_0, \\ u(s) = u_s, & x \in \Omega, \\ v(s) = v_s, & x \in \Omega. \end{cases} \tag{7}$$

in the sense of Definition 8.9.1 of [15]. Indeed, consider for example $\underline{u}$. By definition we have that $\mathcal{B}_1(\underline{u}) \le 0$ on $\partial\Omega$, and $\underline{u}(s) \le u_s \le \overline{u}(s)$. Moreover,

$$\underline{u}_t - d_1 \Delta \underline{u} \le f(t, x, \underline{u}, v) \quad x \in \Omega, \quad s < t < T_0, \quad \forall v \in [\underline{v}, \overline{v}].$$

Similar inequalities can be shown for $\overline{u}$, $\underline{v}$ and $\overline{v}$. Hence, we can apply Theorem 8.9.3 of [15] and conclude that the unique solution of (7) satisfies

$$\underline{u}(t) \le u(t, s; u_s, v_s) \le \overline{u}(t), \quad \underline{v}(t) \le v(t, s; u_s, v_s) \le \overline{v}(t) \quad \text{for } s < t < T_0. \tag{8}$$

On the other hand, since the sub-supertrajectories pair is bounded at $-\infty$ there exist $t_1 < T_0$ and $C > 0$ such that $|\underline{u}(t, x)|, |\underline{v}(t, x)|, |\overline{u}(t, x)|, |\overline{v}(t, x)| \le C$ for all $t \le t_1$ and $x \in \overline{\Omega}$. In particular

$$|u(t, s, x; u_s, v_s)|, |v(t, s, x; u_s, v_s)| \le C_1, \quad \text{for all } x \in \overline{\Omega} \text{ and } s < t \le t_1 \tag{9}$$

and for any choice of initial data satisfying (6).

Fix now $T_1 < t_1$ and $\delta > 0$. Then for $s \le T_1 - \delta$, consider, as in (9), $(u(\cdot, s; u_s, v_s), v(\cdot, s; u_s, v_s))$ restricted to $[T_1, t_1]$.

Then, by (9) and the regularity of f and g we have that for each $s \le T_1 - \delta$, both $u(\cdot, s; u_s, v_s)$ and $v(\cdot, s; u_s, v_s)$ satisfy an equation of the form

$$\begin{cases} z_t - d\Delta z + \lambda z = h^s(t, x) & \text{in } \Omega,\ t \in [T_1, t_1] \\ \mathcal{B} z = 0 & \text{on } \partial\Omega, \end{cases}$$

with intial data $z(s)$ uniformly bounded in $\overline{\Omega}$ and and h^s is uniformly bounded in $[T_1, t_1] \times \overline{\Omega}$, both independent of s. Also, $\lambda > 0$ can be chosen large enough such that the linear semigroup $S_0(t)$, generated by $d\Delta - \lambda I$ and boundary conditions $\mathcal{B}$ decays exponentially. Hence for $t \in [T_1, t_1]$,

$$z(t) = S_0(t - T_1 + \delta) z(T_1 - \delta) + \int_{T_1 - \delta}^{t} S_0(t - r) h^s(r)\, dr,$$

and, from (9), $|z(T_1 - \delta, x)| \le C_1$ for all $x \in \overline{\Omega}$.

From parabolic smoothing estimates we get that for some $0 < \theta < 1$ we have

$$\|z(t)\|_{C^\theta(\overline{\Omega})} \le K \quad \text{for all } t \in [T_1, t_1]$$

and also

$$\|z\|_{C^\theta([T_1, t_1], C(\overline{\Omega}))} \le K$$

and the constant K does not depend on $s \le T_1 - \delta$.

Therefore, from Ascoli-Arzelá's theorem

$$\{(u(\cdot, s; u_s, v_s), v(\cdot, s; u_s, v_s)) = S(\cdot, s)(u_s, v_s), s \le T_1 - \delta\}$$

is relatively compact in $C([T_1, t_1], C(\overline{\Omega})^2)$, for any family of initial data satisfying (6).

In what follows we denote, for short,

$$(u(\cdot, s), v(\cdot, s)) = (u(\cdot, s; u_s, v_s), v(\cdot, s; u_s, v_s)).$$

Now take any sequence $s_n \to -\infty$ and any sequence $T_k \to -\infty$.

First, there exists a subsequence $s_{n_1} \to -\infty$ with $s_{n_1} \le T_1 - \delta$ such that

$$\lim_{n_1 \to \infty} (u(\cdot, s_{n_1}), v(\cdot, s_{n_1})) \to (u^1_\infty(\cdot), v^1_\infty(\cdot)) \quad \text{in} \quad C([T_1, t_1] \times \overline{\Omega})^2$$

and

$$\underline{u}(t) \le u^1_\infty(t) \le \overline{u}(t), \quad \underline{v}(t) \le v^1_\infty(t) \le \overline{v}(t) \quad \text{for } t \in [T_1, t_1].$$

Using the variations of constants formula, it is not difficult to obtain that (u^1_∞, v^1_∞) is solution of (7) in the interval $[T_1, t_1]$ with initial condition

$$(u^1_\infty(T_1), v^1_\infty(T_1)).$$

Now, we repeat the argument in the interval $[T_2, t_1]$, and so there exists a subsequence of s_{n_1} such that $s_{n_2} \to -\infty$ such that $s_{n_2} \le T_2 - \delta$ and

$$\lim_{n_2\to\infty} (u(\cdot, s_{n_2}), v(\cdot, s_{n_2})) \to (u^2_\infty(\cdot), v^2_\infty(\cdot)) \quad \text{in} \quad C([T_2, t_1] \times \overline{\Omega})^2$$

$$\underline{u}(t) \le u^2_\infty(t) \le \overline{u}(t), \quad \underline{v}(t) \le v^2_\infty(t) \le \overline{v}(t) \quad \text{for } t \in [T_2, t_1],$$

and

$$(u^2_\infty, v^2_\infty) = (u^1_\infty, v^1_\infty) \quad \text{in } [T_1, t_1].$$

After some induction, using the intervals $[T_k, t_1]$, $k \ge 3$, we get a function $(u^*(t), v^*(t))$, defined for all $t \le t_1$, which is limit, uniformly on compact sets of $(-\infty, t_1]$, of a subsequence of $(u(\cdot, s_n), v(\cdot, s_n))$ and satisfying

$$\underline{u}(t) \le u^*(t) \le \overline{u}(t), \quad \underline{v}(t) \le v^*(t) \le \overline{v}(t) \quad \text{for } t \le t_1.$$

Moreover, we can prolong this function for all $t_1 < t$, as the unique solution of (7) with initial data $(u(t_1), u(t_1)) = (u^*(t_1), u^*(t_1))$. Therefore $(u^*(t), v^*(t))$ is defined for all $t \in \mathbb{R}$ and satisfies

$$\underline{u}(t) \le u^*(t) \le \overline{u}(t), \quad \underline{v}(t) \le v^*(t) \le \overline{v}(t) \quad \text{for } t < T_0.$$

It remains to prove then that (u^*, v^*) is a complete trajectory. Take $t > s$ and the initial data $(u^*(s), v^*(s))$. We distinguish several cases:
1. If $s \ge t_1$ it is clear that $(u^*(t), v^*(t)) = S(t, s)(u^*(s), v^*(s))$ by construction.
2. Assume that $s < t \le t_1$. Consider $k \in \mathbb{N}$ such that $s, t \in [T_k, t_1]$, and hence $(u^*(\cdot), v^*(\cdot)) = (u^k_\infty(\cdot), v^k_\infty(\cdot))$ on $[T_k, t_1]$. Therefore,

$$\begin{aligned} S(t,s)(u^*(s), v^*(s)) = {} & S(t,s)(u^k_\infty(s), v^k_\infty(s)) = \\ = {} & S(t,s) \lim_{n_k\to\infty} (u(s, s_{n_k}), v(s, s_{n_k})) = \\ = {} & \lim_{n_k\to\infty} (u(t, s_{n_k}), v(t, s_{n_k})) = (u^k_\infty(t), v^k_\infty(t)) \\ = {} & (u^*(t), v^*(t)). \end{aligned}$$

3. Assume $s \leq t_1 < t$. Then, by the second case above we have that $S(t_1,s)(u^*(s),v^*(s)) = (u^*(t_1),v^*(t_1))$. Hence

$$\begin{aligned} S(t,s)(u^*(s),v^*(s)) = & \; S(t,t_1)S(t_1,s)(u^*(s),v^*(s)) = S(t,t_1)(u^*(t_1),v^*(t_1)) = \\ = & \; (u^*(t),v^*(t)). \end{aligned}$$

■

Remark 2.6. *i) The proof above shows that for families of initial data satisfying (6), the evolution process is pullback asymptotically compact (cf. Caraballo et al. [5]).*
ii) In particular we have that for any fixed $t \leq t_1$

$$\{S(t,s)(\underline{u}(s),\underline{v}(s)), \quad s \leq t-\delta\}$$

and

$$\{S(t,s)(\overline{u}(s),\overline{v}(s)), \quad s \leq t-\delta\}$$

are relatively compact in $C(\overline{\Omega})$. In particular, for any sequence $s_n \to -\infty$, there is a subsequence, that we denote the same, such that

$$S(t,s_n)(\underline{u}(s_n),\underline{v}(s_n)) \quad \textit{and} \quad S(t,s_n)(\overline{u}(s_n),\overline{v}(s_n))$$

converge in $C(\overline{\Omega})$.

Compare with (14), in the case of monotonicity in the system.

Note that if f is increasing in v and g in u, part 3 in Definition 2.4 for complete sub-supertrajectory pair reads

$$\begin{aligned} \underline{u}_t - d_1\Delta\underline{u} - f(t,x,\underline{u},\underline{v}) \leq 0 \leq \overline{u}_t - d_1\Delta\overline{u} - f(t,x,\overline{u},\overline{v}), \\ \underline{v}_t - d_2\Delta\underline{v} - g(t,x,\underline{u},\underline{v}) \leq 0 \leq \overline{v}_t - d_2\Delta\overline{v} - g(t,x,\overline{u},\overline{v}). \end{aligned}$$

Also, thanks to the monotonicity properties of f and g, it is easy to show that for two ordered initial data in (1), we have

$$\text{if} \quad \left. \begin{array}{l} u_s^1 \leq u_s^2 \\ v_s^1 \leq v_s^2 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} u(t,s;u_s^1,v_s^1) \leq u(t,s;u_s^2,v_s^2) \\ v(t,s;u_s^1,v_s^1) \leq v(t,s;u_s^2,v_s^2) \end{array} \right. . \tag{10}$$

Hence, we define the natural order

$$(u_1,v_1) \leq (u_2,v_2) \Longleftrightarrow u_1 \leq u_2 \quad \text{and} \quad v_1 \leq v_2 \tag{11}$$

and then (10) reads

$$(u_s^1, v_s^1) \leq (u_s^2, v_s^2) \Longrightarrow S(t,s)(u_s^1, v_s^1) \leq S(t,s)(u_s^2, v_s^2),$$

i.e. the evolution process associated to (1) is order preserving for the order (11).

Finally, observe that given ordered functions in Ω, $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ the set of pairs of functions

$$\mathcal{I} := [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}] = \{(u,v),\ \underline{u} \leq u \leq \overline{u}, \quad \underline{v} \leq v \leq \overline{v}\}$$

is described in terms of the order (11) as

$$\mathcal{I} := [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}] = \{(u,v),\ (\underline{u}, \underline{v}) \leq (u,v) \leq (\overline{u}, \overline{v})\},$$

which is the order interval between $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ for the order (11).

Using these monotonicity properties, in this case of being f and g monotonic we get (cf. Arnold and Chueshov [1]).

Corollary 2.7. *Under the assumptions of Theorem 2.5, assume moreover that f is increasing in v and g in u. Then, there exist two complete trajectories (u_*, v_*) and (u^*, v^*) of (1) with $(u_*, v_*), (u^*, v^*) \in \mathcal{I} := [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ such that they are minimal and maximal in $\mathcal{I}$ in the following sense: for any other complete trajectory $(u,v) \in \mathcal{I}$ we have:*

$$\underline{u}(t) \leq u_*(t) \leq u(t) \leq u^*(t) \leq \overline{u}(t),\ \underline{v}(t) \leq v_*(t) \leq v(t) \leq v^*(t) \leq \overline{v}(t),\ t < T_0. \tag{12}$$

If moreover f, g, $\underline{u}, \underline{v}, \overline{u}$ and $\overline{v}$ are T-periodic, then, the complete trajectories (u_*, v_*) and (u^*, v^*) above are also T-periodic.

In particular, if f and g and $\underline{u}, \underline{v}, \overline{u}$ and $\overline{v}$ are time independent, then (u_*, v_*) and (u^*, v^*) are equilibria of (1).

Proof. By the monotonicity properties of f and g, it is not hard to prove that if $(\underline{u}, \underline{v}) - (\overline{u}, \overline{v})$ is a pair of complete sub-supertrajectory, then

$$(\underline{u}(t), \underline{v}(t)) \leq S(t,s)(\underline{u}(s), \underline{v}(s)) \text{ and } S(t,s)(\overline{u}(s), \overline{v}(s)) \leq (\overline{u}(t), \overline{v}(t))\ \forall t \geq s. \tag{13}$$

In what follows denote $\underline{\phi}(t) := (\underline{u}(t), \underline{v}(t))$ and $\overline{\phi}(t) := (\overline{u}(t), \overline{v}(t))$ for $t < T_0$. Hence, from (10) and (13) we have

$$\underline{\phi}(t) \leq S(t,s)\underline{\phi}(s) \leq S(t,s)\overline{\phi}(s) \leq \overline{\phi}(t), \quad \text{for all } s < t < T_0.$$

In particular, for all $\varepsilon > 0$ we have

$$S(s+\varepsilon, s)\overline{\phi}(s) \leq \overline{\phi}(s+\varepsilon)$$

which implies

$$S(t,s)\overline{\phi}(s) = S(t,s+\varepsilon)S(s+\varepsilon,s)\overline{\phi}(s) \leq S(t,s+\varepsilon)\overline{\phi}(s+\varepsilon).$$

Therefore for any fixed $t < T_0$

$$\{S(t,s)\overline{\phi}(s)\}_{s\leq t} \quad \text{is monotonically increasing in } s.$$

Analogously, for any fixed $t < T_0$

$$\{S(t,s)\underline{\phi}(s)\}_{s\leq t} \quad \text{is monotonically decreasing in } s.$$

The monotonicty above, combined with Theorem 2.5, gives the existence of the following limits

$$\begin{aligned} \varphi_*(t) &:= \lim_{s\to-\infty} S(t,s)\underline{\phi}(s) = (u_*, v_*)(t) \\ \varphi^*(t) &:= \lim_{s\to-\infty} S(t,s)\overline{\phi}(s) = (u^*, v^*)(t) \end{aligned} \tag{14}$$

uniformly in $\overline{\Omega}$, and φ^*, φ_* are complete trajectories of (1).

Finally, if $\varphi = (u,v) \in \mathcal{I}$ is another complete trajectory of (1), we have for any $s < t < T_0$

$$\underline{\phi}(s) \leq \varphi(s) \leq \overline{\phi}(s),$$

and using the monotonicity property (10) we get

$$\underline{\phi}(t) \leq S(t,s)\underline{\phi}(s) \leq \varphi(t) = S(t,s)\varphi(s) \leq S(t,s)\overline{\phi}(s) \leq \overline{\phi}(t)$$

and taking the limit as $s \to -\infty$ and using (14), we conclude (12).

Finally, in case f, g and $\underline{\phi}$, $\overline{\phi}$ are T-periodic, observe that by periodicity we have $S(t+T, s+T) = S(t,s)$ for all $t \geq s$ and then

$$\varphi^*(t+T) = \lim_{s\to-\infty} S(t+T,s)\overline{\phi}(s) = \lim_{s\to-\infty} S(t,s-T)\overline{\phi}(s-T) = \varphi^*(t).$$

The periodicity of φ_* is obtained analogously.

The time independent case is obtained from the T-periodic case or any $T > 0$. ■

Note that the arguments above are quite general since they depend only on the monotonicity of the evolution process and the existence of the complete sub-supertrajecroy pairs $\underline{\phi}$ and $\overline{\phi}$. In fact Theorem 2.5 is only used to obtain the sufficient compactness to take the limits as $s \to -\infty$.

This implies that a completely analogous result to Corollary 2.7 can be obtained for (1) when f is decreasing in v and g in u, since in this case the evolution process is monotonic for a suitable order defined below. In this case part 3 in Definition 2.4 for complete sub-supertrajectory pair reads

$$\begin{array}{l} \underline{u}_t - d_1\Delta\underline{u} - f(t,x,\underline{u},\overline{v}) \le 0 \le \overline{u}_t - d_1\Delta\overline{u} - f(t,x,\overline{u},\underline{v}), \\ \underline{v}_t - d_2\Delta\underline{v} - g(t,x,\overline{u},\underline{v}) \le 0 \le \overline{v}_t - d_2\Delta\overline{v} - g(t,x,\underline{u},\overline{v}). \end{array}$$

Also, in this case, thanks to the monotonicity properties of f and g, it holds that

$$\left.\begin{array}{l} u_s^1 \le u_s^2 \\ v_s^1 \ge v_s^2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} u(t,s;u_s^1,v_s^1) \le u(t,s;u_s^2,v_s^2), \\ v(t,s;u_s^1,v_s^1) \ge v(t,s;u_s^2,v_s^2). \end{array}\right. \tag{15}$$

Then we define the following reverse order

$$(u_1,v_1) \preceq (u_2,v_2) \Longleftrightarrow u_1 \le u_2 \quad \text{and} \quad v_2 \le v_1 \tag{16}$$

and then (15) reads

$$(u_s^1,v_s^1) \preceq (u_s^2,v_s^2) \Longrightarrow S(t,s)(u_s^1,v_s^1) \preceq S(t,s)(u_s^2,v_s^2),$$

i.e. the evolution process associated to (1) is order preserving for the order (16).

Finally, observe that given ordered functions in Ω, $\underline{u} \le \overline{u}$ and $\underline{v} \le \overline{v}$ the set of pairs of functions

$$\mathcal{I} := [\underline{u},\overline{u}] \times [\underline{v},\overline{v}] = \{(u,v),\ \underline{u} \le u \le \overline{u}, \quad \underline{v} \le v \le \overline{v}\}$$

is now described in terms of the order (16) as

$$\mathcal{I} := [\underline{u},\overline{u}] \times [\underline{v},\overline{v}] = \{(u,v),\ (\underline{u},\overline{v}) \preceq (u,v) \preceq (\overline{u},\underline{v})\},$$

which is the order interval between $(\underline{u},\overline{v})$ and $(\overline{u},\underline{v})$ for the order (16).

Thus we get

Corollary 2.8. *Under the assumptions of Theorem 2.5, assume moreover that f is decreasing in v and g in u. Then, there exist two complete trajectories (u_*, v^*) and (u^*, v_*) of (1) with $(u_*, v^*), (u^*, v_*) \in \mathcal{I} := [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ and such that they are minimal-maximal and maximal-minimal in the following sense: for any other complete trajectory $(u, v) \in \mathcal{I}$ we have:*

$$\begin{aligned} &\underline{u}(t) \le u_*(t) \le u(t) \le u^*(t) \le \overline{u}(t), \\ &\underline{v}(t) \le v_*(t) \le v(t) \le v^*(t) \le \overline{v}(t), \text{ for all } t < T_0. \end{aligned} \tag{17}$$

If in addition f, g, $\underline{u}, \underline{v}, \overline{u}$ and $\overline{v}$ are T-periodic, then, the complete trajectories (u_*, v^*) and (u^*, v_*) above are also T-periodic.

In particular, if f and g and $\underline{u}, \underline{v}, \overline{u}$ and $\overline{v}$ are time independent, then (u_*, v^*) and (u^*, v_*) are equilibria of (1).

Proof. With the order (16), it is not hard to show that the definition of complete sub-supertrajectory implies that

$$(\underline{u}(t), \overline{v}(t)) \preceq S(t,s)(\underline{u}(s), \overline{v}(s)) \text{ and } S(t,s)(\overline{u}(s), \underline{v}(s)) \preceq (\overline{u}(t), \underline{v}(t)). \tag{18}$$

The proof runs then as in Corollary 2.7 using monotonicity with respect to the order (16). The compactness is obtained from Theorem 2.5. ■

2.3. The scalar case

In fact since the compactness argument in Theorem 2.5 is based on scalar equation, the arguments above give the following, cf. [12]. Consider the scalar problem

$$\begin{cases} u_t - d\Delta u = f(t,x,u) & x \in \Omega,\ t > s \\ \mathcal{B}u = 0, & x \in \partial\Omega,\ t > s \\ u(s) = u_s, & \end{cases} \tag{19}$$

with $d > 0$, $\mathcal{B}$ as in (2) and a smooth f. Hence the solution $u(t, s; u_s) = S(t, s)u_s$ is well defined.

Definition 2.9. *Let $T_0 \le \infty$ and $\underline{u}, \overline{u} \in \mathcal{X} = C^{1,2}_{t,x}((-\infty, T_0) \times \overline{\Omega})$. We say that $\underline{u}, \overline{u}$ is a complete sub-supertrajectory pair of (1) if*

1. *$\underline{u}(t) \le \overline{u}(t)$ in Ω, for all $t < T_0$.*
2. *$\mathcal{B}(\underline{u}) \le 0 \le \mathcal{B}(\overline{u})$ on $\partial\Omega$, for all $t < T_0$.*

3. *For all* $x \in \Omega$*,* $t < T_0$

$$\underline{u}_t - d\Delta \underline{u} - f(t,x,\underline{u}) \le 0 \le \overline{u}_t - d\Delta \overline{u} - f(t,x,\overline{u}).$$

Corollary 2.10. *Assume that there exists a complete sub-supertrajectory pair of (19),* $\underline{u}, \overline{u}$*, in the sense of Definition 2.9. In addition, assume* $\underline{u}$ *and* $\overline{u}$ *are bounded at* $-\infty$*.*

For any $s < t < T_0$*, consider initial data* u_s *in (1) such that*

$$\underline{u}(s) \le u_s \le \overline{u}(s). \tag{20}$$

Then there exists some $t_1 < T_0$ *such that for any sequence* $s_n \to -\infty$ *there is a subsequence of*

$$u(\cdot, s_n; u_{s_n}) = S(\cdot, s_n) u_{s_n}$$

that we denote the same, converging uniformly in compact sets of $(-\infty, t_1]$ *to a complete solution of (19).*

In particular, there exist two complete trajectories u_* *and* u^* *of (19) such that for any other complete trajectory such that* $\underline{u}(t) \le u(t) \le \overline{u}(t)$ *for* $t < T_0$*, we have:*

$$\underline{u}(t) \le u_*(t) \le u(t) \le u^*(t) \le \overline{u}(t), \quad \textit{for all } t < T_0.$$

If moreover f*,* $\underline{u}$ *and* $\overline{u}$ *are* T*-periodic, then, the complete trajectories* u_* *and* u^* *above are also* T*-periodic.*

In particular, if f*,* $\underline{u}$ *and* $\overline{u}$ *are time independent, then* u_* *and* u^* *are equilibria of (19).*

3. The non-autonomous logistic equation

Note that (3) always admits semi-trivial trajectories of the form $(u, 0)$ or $(0, v)$. In this case, when one species is not present, the other one satisfies the logistic equation

$$\begin{cases} u_t - d\Delta u = h(t,x)u - g(t,x)u^2 & \text{in } \Omega,\ t > s \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \\ u(s) = u_s \ge 0 & \text{in } \Omega, \end{cases} \tag{21}$$

where $d > 0$ and $\mathcal{B}$ as in (2), that is,

$$\mathcal{B}u = \begin{cases} u, & \text{Dirichlet case, or} \\ d\dfrac{\partial u}{\partial \vec{n}} + \sigma(x)u, & \text{Robin case,} \end{cases}$$

$0 \leq u_s \in C(\overline{\Omega})$, $h, g \in C^{\theta}(\overline{Q})$, with $Q = \mathbb{R} \times \Omega$, $\sigma \in C^1(\partial\Omega)$ and $g \geq 0$. Formally, we will consider Dirichlet boundary conditions as corresponding to the limit case $\sigma(x) = \infty$ on $\partial\Omega$. Also, note that we will always restrict ourselves here to nonnegative solutions of (21).

Now we review some results on the scalar logistic equation (21) that will be used for the study of the Lotka–Volterra system (3).

For $m \in L^{\infty}(\Omega)$ we denote by $\Lambda_{\mathcal{B}}(d, m)$, the first eigenvalue of

$$\begin{cases} -d\Delta u = \lambda u + m(x)u & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{22}$$

In particular, we denote by $\Lambda_{0,\mathcal{B}}(d) = \Lambda_{\mathcal{B}}(d, 0) = d\Lambda_{\mathcal{B}}(1, 0)$ the first eigenvalue of the operator $-d\Delta$ with boundary conditions $\mathcal{B}$. It is well known that $\Lambda_{\mathcal{B}}(d, m)$ is a simple eigenvalue with a positive eigenfunction, and a continuous and decreasing function of m. Also note that if m_1 is constant then

$$\Lambda_{\mathcal{B}}(d, m_1 + m_2) = \Lambda_{\mathcal{B}}(d, m_2) - m_1. \tag{23}$$

We write $\varphi_{1,\mathcal{B}}(d, m)$ for the positive eigenfunction associated to $\Lambda_{\mathcal{B}}(d, m)$, normalized such that $\|\varphi_{1,\mathcal{B}}(d, m)\|_{L^{\infty}(\Omega)} = 1$.

If there is no possible confusion we will suppress the dependence on d and $\mathcal{B}$ in the notations above. When we need to distinguish these quantities with respect to $\mathcal{B}_i$, or d_i, $i = 1, 2$, we will employ superscripts as $\Lambda^i(m)$ or Λ^i_0.

Finally, for $h, g \in L^{\infty}(\Omega)$ with $g_L := \inf\{g(x),\ x \in \overline{\Omega}\} > 0$ consider the elliptic equation

$$\begin{cases} -d\Delta u = h(x)u - g(x)u^2 & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{24}$$

In the following result we show the existence of solutions for (21) and (24), see [11], [18] and [2]:

Proposition 3.1. 1. *Assume that in (24) we have* $g_L > 0$. *If* $\Lambda_{\mathcal{B}}(h) < 0$ *there exits a unique positive solution of (24), which we denote by* $\omega_{[h,g]}(x)$. *Moreover,* $0 < \omega_{[h,g]}(x) \le \Psi(x)$ *in* Ω, *where*

$$\Psi(x) = \begin{cases} \frac{h_M}{g_L} & \text{for Dirichlet BCs,} \\ -\frac{\Lambda_{\mathcal{B}}(h)}{\varphi_L g_L}\varphi(x) & \text{for Robin BCs,} \end{cases}$$

with $\varphi = \varphi_{1,\mathcal{B}}(d,h)$ *and where* $h_M := \sup\{h(x),\ x \in \overline{\Omega}\}$.
On the other hand, if $\Lambda_{\mathcal{B}}(h) \ge 0$, *the unique non-negative solution of (24) is the trivial one, i.e.* $\omega_{[h,g]}(x) = 0$.

2. *Assume that in (21)*

$$h_M := \sup_{\overline{Q}} h(t,x) < \infty \quad \text{and} \quad g_L := \inf_{\overline{Q}} g(t,x) > 0. \tag{25}$$

Then, for every non-trivial $u_s \in C(\overline{\Omega})$, $u_s \ge 0$, *there exists a unique positive solution of* (21) *denoted by* $\Theta_{[h,g]}(t,s;u_s)$. *Moreover,*

$$0 \le \Theta_{[h,g]}(t,s;u_s) \le K, \quad t > s, \tag{26}$$

where

$$K := \begin{cases} \max\left\{(u_s)_M, \frac{h_M}{g_L}\right\} & \text{for Dirichlet BCs,} \\ \max\left\{(\frac{u_s}{\varphi})_M, \frac{-\Lambda_{\mathcal{B}}(h_M)}{\varphi_L g_L}\right\} & \text{for Robin BCs,} \end{cases}$$

and φ *is the positive eigenfunction associated to* $\Lambda(h_M)$ *with* $\|\varphi\|_{L^\infty(\Omega)} = 1$.

In the following result we show the existence and properties of a complete nonnegative trajectory for (21). For this we will assume henceforth that $h(t,x)$ and $g(t,x)$ satisfy (25) and there exist bounded functions $h_0^\pm(x)$, $H_0^\pm(x)$, $g_0^\pm(x)$ and $G_0^\pm(x)$ defined in Ω such that

$$\limsup_{t\to\pm\infty} \sup_{x\in\Omega} \Big(h(t,x) - H_0^\pm(x)\Big) \le 0, \quad 0 \le \liminf_{t\to\pm\infty} \inf_{x\in\Omega} \Big(h(t,x) - h_0^\pm(x)\Big) \tag{27}$$

and

$$\limsup_{t\to\pm\infty} \sup_{x\in\Omega} \Big(g(t,x) - G_0^\pm(x)\Big) \le 0, \quad 0 \le \liminf_{t\to\pm\infty} \inf_{x\in\Omega} \Big(g(t,x) - g_0^\pm(x)\Big). \tag{28}$$

Note that these conditions imply that, for every $\varepsilon > 0$, as $t \to \pm\infty$ we have,

$$h_0^{\pm}(x) - \varepsilon \leq h(t,x) \leq H_0^{\pm}(x) + \varepsilon, \quad \text{for all } x \in \Omega, \text{ and}$$

$$g_0^{\pm}(x) - \varepsilon \leq g(t,x) \leq G_0^{\pm}(x) + \varepsilon, \quad \text{for all } x \in \Omega\ .$$

Note also that from (25) we can assume

$$(g_0^{\pm}(x))_L := \inf\{g_0^{\pm}(x),\ x \in \overline{\Omega}\} > 0.$$

Then, we have

Proposition 3.2. *Assume (25), (27) and (28). Then:*
i) There exists a maximal bounded complete trajectory, denoted by $\varphi_{[h,g]}(t)$, of (21), in the sense that, for any other non-negative complete bounded trajectory $\xi(t)$ of (21) we have

$$0 \leq \xi(t) \leq \varphi_{[h,g]}(t), \qquad t \in \mathbb{R}.$$

Moreover, for any bounded set of nonnegative and non-trivial initial data $B \subset C(\overline{\Omega})$ we have

$$0 \leq \limsup_{s \to -\infty} \Theta_{[h,g]}(t,s,x;u_0) \leq \varphi_{[h,g]}(t,x) \tag{29}$$

uniformly in $x \in \Omega$ and for $u_0 \in B$. That is, $\varphi_{[h,g]}(t,x)$ gives a pullback asymptotic bound for all solutions of (21).

Finally, if $\varphi_{[h,g]}(t,x)$ is nondegenerate at $-\infty$ then it is the only one of such solutions.
ii) If $\Lambda(H_0^-) > 0$, then $\varphi_{[h,g]}(t) = 0$ for all $t \in \mathbb{R}$. Therefore all non-negative solutions of (21) converge to 0, uniformly in Ω, in the pullback sense.
iii) If $\Lambda(h_0^-) < 0$ then $\varphi_{[h,g]}$ is the unique complete bounded and non-degenerate trajectory at $-\infty$ of (21), and for t in compact sets of $\mathbb{R}$, if $s \mapsto u_s \geq 0$ is bounded and non-degenerate, then

$$\Theta_{[h,g]}(t,s;u_s) - \varphi_{[h,g]}(t) \to 0 \quad \text{as } s \to -\infty$$

uniformly in Ω. That is, $\varphi_{[h,g]}$ describes the pullback behaviour of all non-trivial non-negative solutions of (21).

Moreover for sufficiently negative t and all $x \in \Omega$ we have

$$\omega_{[h_0^-, G_0^-]}(x) \leq \varphi_{[h,g]}(t,x). \tag{30}$$

iv) If, as $t \to \infty$, a positive solution of (21) goes to zero in Ω then all positive solutions behave the same. In particular, if $\Lambda(H_0^+) > 0$, then for all $u_s \in C(\overline{\Omega})$, $u_s \geq 0$, the positive solution of (21) satisfies $\Theta_{[h,g]}(t,s;u_s) \to 0$ uniformly in Ω as $t \to \infty$. In particular, $\varphi_{[h,g]}(t) \to 0$ uniformly in Ω as $t \to \infty$.
v) If a positive solution of (21) is nondegenerate at ∞, then all positive solutions are nondegenerate at ∞. Moreover, in such a case, any two nontrivial solutions of (21) satisfy

$$\lim_{t\to\infty} (u_1(t,x) - u_2(t,x)) = 0, \quad \textit{uniformly in } \overline{\Omega}.$$

In particular, if $\Lambda(h_0^+) < 0$ and $\varphi_{[h,g]} \neq 0$, then $\varphi_{[h,g]}$ is non degenerate at ∞ and for any s and any non-trivial initial data $u_s \geq 0$,

$$\Theta_{[h,g]}(t,s;u_s) - \varphi_{[h,g]}(t) \to 0 \quad \textit{in } C^1(\overline{\Omega}) \textit{ as } t \to \infty.$$

That is, $\varphi_{[h,g]}$ describes the forwards behaviour of all solutions.
vi) If h, g are independent of t and $\Lambda(h) < 0$, then $\varphi_{[h,g]}(t,x) = \omega_{[h,g]}(x)$ is the unique positive solution of (24) and for all $t > s$ and nontrivial $u_0 \geq 0$

$$\Theta_{[h,g]}(t,s;u_0) = \Theta_{[h,g]}(t-s,u_0) \to \omega_{[h,g]} \quad \textit{in } C^1(\overline{\Omega}) \textit{ as } t-s \to \infty$$

uniformly for bounded sets of initial data $u_s \geq 0$ bounded away from zero. In particular, there exist $m \leq 1 \leq M$ such that

$$m\omega_{[h,g]} \leq \Theta_{[h,g]}(t,s;u_s) \leq M\omega_{[h,g]},$$

for $t-s$ large.
Moreover, the convergence in iii), v) and vi) is exponentially fast.

Proof.
i) This part follows from [17] and [18], see in particular Proposition 8 in the last reference. The uniqueness follows from Theorem 2 in [18].
ii) If $\Lambda(H_0^-) > 0$, then for sufficiently small $\varepsilon > 0$ we also have $\Lambda(H_0^- + \varepsilon) > 0$ and from (27) we have, for $u \geq 0$, $x \in \Omega$ and sufficiently negative t,

$$h(t,x)u - g(t,x)u^2 \leq (H_0^-(x) + \varepsilon)u.$$

Now for all $u_s \in C(\overline{\Omega})$, $u_s \geq 0$, the nonnegative solution of (21) satisfies

$$0 \leq \Theta_{[h,g]}(t,s;u_s) \leq \Theta_{[H_0^- + \varepsilon, 0]}(t,s;u_s) = w(t-s,u_s)$$

where the latter function is the solution of a linear parabolic equation with positive first eigenvalue. In particular, we take $u_s = \varphi_{[h,g]}(s)$ to obtain

$$0 \leq \varphi_{[h,g]}(t) \leq w(t-s, \varphi_{[h,g]}(s))$$

and the right hand side above converges to 0 as $s \to -\infty$. Therefore $\varphi_{[h,g]}(t) = 0$ for sufficiently negative t and then for all $t \in \mathbb{R}$. The rest follows from (29).
iii) Now if $\Lambda(h_0^-) < 0$, then for sufficiently small $\varepsilon > 0$ we also have $\Lambda(h_0^- - \varepsilon) < 0$ and from (27) and (28) we have, for $u \geq 0$, $x \in \Omega$ and sufficiently negative t,

$$(h_0^-(x) - \varepsilon)u - (G_0^-(x) + \varepsilon)u^2 \leq h(t,x)u - g(t,x)u^2.$$

Now the rest follows from Theorem 7 and Theorem 2 in [18]. Note that the former in particular implies that for sufficiently negative t and all $x \in \Omega$ we have

$$\omega_{[h_0^- - \varepsilon, G_0^- + \varepsilon]}(x) \leq \varphi_{[h,g]}(t,x).$$

Now letting $\varepsilon \to 0$ gives the result.
iv) The first part follows from Corollary 2 in [18]. Now if $\Lambda(H_0^+) > 0$, then for sufficiently small $\varepsilon > 0$ we also have $\Lambda(H_0^+ + \varepsilon) > 0$ and from (27) we have, for $u \geq 0$, $x \in \Omega$ and sufficiently large t,

$$h(t,x)u - g(t,x)u^2 \leq (H_0^+(x) + \varepsilon)u.$$

Now for all $u_s \in C(\overline{\Omega})$, $u_s \geq 0$, the positive solution of (21) satisfies

$$\Theta_{[h,g]}(t,s;u_s) \leq \Theta_{[H_0^+ + \varepsilon, 0]}(t,s;u_s) = w(t-s, u_s)$$

where the latter function is the solution of a linear parabolic equation with positive first eigenvalue. Hence the result follows.
v) The first part follows from Corollary 2 and Theorem 3 in [18].

Now if $\Lambda(h_0^+) < 0$, then for sufficiently small $\varepsilon > 0$ we also have $\Lambda(h_0^+ - \varepsilon) < 0$ and from (27) and (28) we have, for $u \geq 0$, $x \in \Omega$ and sufficiently large t,

$$(h_0^+(x) - \varepsilon)u - (G_0^+(x) + \varepsilon)u^2 \leq h(t,x)u - g(t,x)u^2.$$

Now for all $u_s \in C(\overline{\Omega})$, $u_s \geq 0$, the positive solution of (21) satisfies

$$\Theta_{[h_0^+ - \varepsilon, G_0^+ + \varepsilon]}(t,s;u_s) = w(t-s, u_s) \leq \Theta_{[h,g]}(t,s;u_s).$$

Now, part vi) below implies $\Theta_{[h_0^+-\varepsilon, G_0^++\varepsilon]}(t,s;u_s) \to \omega_{[h_0^+-\varepsilon, G_0^++\varepsilon]}$ as $t \to \infty$. In particular, any nontrivial solution is nondegenerate at ∞. Hence, if $\varphi_{[h,g]} \neq 0$, then it is nondegenerate at ∞ and the rest follows.
vi) This follows from the uniqueness of both $\omega_{[h,g]}$ and $\varphi_{[h,g]}$, the previous results and the C^1 regularity. The asymptotic behavior of the solutions follows from [2], see also [19].

Finally the fact that the convergence in iii), v) and vi) is exponentially fast, follows from Theorems 5.3 and 5.4 in [16]. ■

In particular we have the following

Corollary 3.3. *Assume (25), (27) and (28).*
i) If

$$\Lambda(H_0^-) > 0, \qquad \Lambda(H_0^+) > 0$$

Then 0 *is the only global bounded solution of (21) and all solutions converge to* 0 *in the pullback sense and forwards senses, that is, for any* $u_0 \neq 0$ *we have*

$$\lim_{s\to-\infty} \Theta_{[h,g]}(t,s;u_0) = 0 \quad \textit{uniformly in } \overline{\Omega}$$

and

$$\lim_{t\to\infty} \Theta_{[h,g]}(t,s;u_0) = 0 \quad \textit{uniformly in } \overline{\Omega}.$$

ii) If

$$\Lambda(H_0^-) > 0, \qquad \Lambda(h_0^+) < 0.$$

Then 0 *is the only global bounded solution of (21) and all solutions converge to* 0 *in the pullback sense, that is, for any* $u_0 \neq 0$ *we have*

$$\lim_{s\to-\infty} \Theta_{[h,g]}(t,s;u_0) = 0 \quad \textit{uniformly in } \overline{\Omega}.$$

At the same time all nontivial solutions are nondegenerate and bounded at ∞ *and have the same asymptotic behavior as* $t \to \infty$*. In particular, assume* $h(t,x) = h_0^+(x)$ *and* $g(t,x) = g_0^+(x)$ *for all* $x \in \Omega$ *for* $t \geq t_0$*, then for any* $u_0 \neq 0$ *we have*

$$\lim_{t\to\infty} \Theta_{[h,g]}(t,s;u_0) = \omega_{[h_0^+, g_0^+]} \quad \textit{uniformly in } \overline{\Omega}.$$

iii) If

$$\Lambda(h_0^-) < 0, \qquad \Lambda(h_0^+) < 0.$$

then for any $u_0 \neq 0$ *we have*

$$\Theta_{[h,g]}(t,s;u_0) - \varphi_{[h,g]}(t) \to 0 \quad \text{as } s \to -\infty \text{ or } t \to \infty.$$

Moreover for sufficiently negative t *and all* $x \in \Omega$ *we have*

$$\omega_{[h_0^-,G_0^-]}(x) \le \varphi_{[h,g]}(t,x) \le \omega_{[H_0^-,g_0^-]}(x). \tag{31}$$

while

$$\omega_{[h_0^+,G_0^+]}(x) \le \liminf_{t\to\infty} \varphi_{[h,g]}(t,x) \le \limsup_{t\to\infty} \varphi_{[h,g]}(t,x) \le \omega_{[H_0^+,g_0^+]}(x) \tag{32}$$

uniformly in $\overline{\Omega}$.
iv) If

$$\Lambda(h_0^-) < 0, \qquad \Lambda(H_0^+) > 0$$

then for any $u_0 \neq 0$ *we have*

$$\Theta_{[h,g]}(t,s;u_0) - \varphi_{[h,g]}(t) \to 0 \quad \text{as } s \to -\infty$$

and (31) holds.

Also, for any $s \in \mathbb{R}$ *we have*

$$\lim_{t\to\infty} \Theta_{[h,g]}(t,s;u_0) = 0 \quad \text{uniformly in } \overline{\Omega}.$$

Proof. From Proposition 3.2 it only remains to prove (31) and (32). For large $|t|$, we have

$$(h_0^\pm(x)-\varepsilon)u-(G_0^\pm(x)+\varepsilon)u^2 \le h(t,x)u-g(t,x)u^2 \le (H_0^\pm(x)+\varepsilon)u-(g_0^\pm(x)-\varepsilon)u^2$$

and then for any $u_0 \neq 0$, we have

$$\Theta_{[h_0^\pm-\varepsilon,G_0^\pm+\varepsilon]}(t-s;u_0) \le \Theta_{[h,g]}(t,s;u_0) \le \Theta_{[H_0^\pm+\varepsilon,g_0^\pm-\varepsilon]}(t-s;u_0).$$

Now for t negative and the $-$ sign, we take $u_0 = \varphi_{[h,g]}(s)$ to get

$$\Theta_{[h_0^--\varepsilon,G_0^-+\varepsilon]}(t-s;\varphi_{[h,g]}(s)) \le \varphi_{[h,g]}(t) \le \Theta_{[H_0^-+\varepsilon,g_0^--\varepsilon]}(t-s;\varphi_{[h,g]}(s)).$$

Since $\Lambda(h_0^-) < 0$, using iii) and vi) of Proposition 3.2 and letting $s \to -\infty$ we get

$$\omega_{[h_0^--\varepsilon,G_0^-+\varepsilon]} \le \varphi_{[h,g]}(t) \le \omega_{[H_0^-+\varepsilon,g_0^--\varepsilon]}$$

and with $\varepsilon \to 0$ we conclude.

Now for t positive and the $+$ sign, we take $u_0 = \varphi_{[h,g]}(s)$ to get

$$\Theta_{[h_0^+-\varepsilon, G_0^++\varepsilon]}(t-s;\varphi_{[h,g]}(s)) \leq \varphi_{[h,g]}(t) \leq \Theta_{[H_0^++\varepsilon, g_0^+-\varepsilon]}(t-s;\varphi_{[h,g]}(s)).$$

Since $\Lambda(h_0^+) < 0$, using vi) of Proposition 3.2 and letting $t \to \infty$ we get

$$\omega_{[h_0^+-\varepsilon, G_0^++\varepsilon]}(x) \leq \liminf_{t\to\infty} \varphi_{[h,g]}(t,x) \leq \limsup_{t\to\infty} \varphi_{[h,g]}(t,x) \leq \omega_{[H_0^++\varepsilon, g_0^+-\varepsilon]}(x)$$

uniformly in $\overline{\Omega}$, and with $\varepsilon \to 0$ we conclude. ■

Note that the first part of the Corollary gives examples such that the pullback behavior of solutions is completely unrelated with the forwards behavior. On the other hand, the second part gives examples which can be phrased as saying that the pullback attractor is also the forwards one.

The next results state some monotonicity properties of the complete solution, $\varphi^{\sigma}_{[h,g]}(t)$, of (21) with respect of the coefficients h, g, σ of the problem in the line of (31).

Proposition 3.4. *Let $T_0 < \infty$ and assume $h_1(t,x), h_2(t,x), g_1(t,x)$ and $g_2(t,x)$ satisfy (25).*

Assume that for $t \leq T_0$ we have $h_1(t,x) \leq h_2(t,x)$, $g_1(t,x) \geq g_2(t,x)$ in Ω and $\sigma_2(x) \leq \sigma_1(x)$ on $\partial\Omega$. Then,

$$\varphi^{\sigma_1}_{[h_1,g_1]}(t) \leq \varphi^{\sigma_2}_{[h_2,g_2]}(t) \quad \textit{for } t \leq T_0.$$

Proof. Observe that $\varphi^{\sigma_1}_{[h_1,g_1]}(t)$ is a subsolution of (21) with $h = h_2$ and $g = g_2$, that is,

$$(\varphi^{\sigma_1}_{[h_1,g_1]})_t - d\Delta\varphi^{\sigma_1}_{[h_1,g_1]} \leq h_2(t,x)\varphi^{\sigma_1}_{[h_1,g_1]} - g_2(t,x)(\varphi^{\sigma_1}_{[h_1,g_1]})^2$$

and

$$\mathcal{B}_2\varphi^{\sigma_1}_{[h_1,g_1]} = d\frac{\partial}{\partial\vec{n}}\varphi^{\sigma_1}_{[h_1,g_1]} + \sigma_2(x)\varphi^{\sigma_1}_{[h_1,g_1]} = (\sigma_2(x) - \sigma_1(x))\varphi^{\sigma_1}_{[h_1,g_1]} \leq 0.$$

Then, for $s < t \leq T_0$

$$\varphi^{\sigma_1}_{[h_1,g_1]}(t) \leq \Theta_{[h_2,g_2]}(t,s;\varphi^{\sigma_1}_{[h_1,g_1]}(s)) \tag{33}$$

and letting $s \to -\infty$ and using (29), we get the result. ■

Now for large times we have, in a similar way as in (32)

Proposition 3.5. *Let $T_0 > -\infty$ and assume $h_1(t,x), h_2(t,x), g_1(t,x)$ and $g_2(t,x)$ satisfy (25).*

Assume that for $t \geq T_0$ we have $h_1(t,x) \leq h_2(t,x)$, $g_1(t,x) \geq g_2(t,x)$ in Ω and $\sigma_2(x) \leq \sigma_1(x)$ on $\partial\Omega$. Also, consider two nonnegative, nontrivial initial data u_0^1, u_0^2 and denote

$$u_i(t,s) = \Theta^{\sigma_i}_{[h_i,g_i]}(t,s;u_0^i), \quad i = 1,2,$$

for $t \geq s \geq T_0$.

Then, if either u_1 or u_2 are nondegenerate at ∞

$$\liminf_{t\to\infty} \Big(u_2(t,s) - u_1(t,s) \Big) \geq 0 \quad \textit{uniformly in } \overline{\Omega}. \tag{34}$$

Also, if $u_2(t,s) \to 0$ as $t \to \infty$ then $u_1(t,s) \to 0$ as $t \to \infty$, while if $u_1(t,s)$ is nondegenerate at ∞ then $u_2(t,s)$ is nondegenerate at ∞.

In particular, the above applies to $u_1(t) = \varphi^{\sigma_1}_{[h_1,g_1]}(t)$ and $u_2(t) = \varphi^{\sigma_2}_{[h_2,g_2]}(t)$ if they are nonzero.

Proof. Using (33), we have for $t > s > T_0$

$$u_1(t,s) \leq \Theta_{[h_2,g_2]}(t,s;u_0^1).$$

Now, from part iv) in Proposition 3.2, if $u_2(t,s) \to 0$ as $t \to \infty$ then the right hand side above tends to 0 as $t \to \infty$ and (34) is satisfied.

On the other hand, if $u_2(t,s)$ is nondegenerate at ∞ then, from part v) in Proposition 3.2, the right hand side above has the same asymptotic behavior as $t \to \infty$ than $u_2(t,s)$ and (34) follows.

Finally, if $u_1(t,s)$ is nondegenerate at ∞, we use that, analogously as above, for $t > s > T_0$

$$\Theta_{[h_1,g_1]}(t,s;u_0^2) \leq u_2(t,s)$$

and, since $u_0^2 \neq 0$, the left hand side above has the same asymptotic behavior as $t \to \infty$ than $u_1(t,s)$ and (34) follows. ■

Observe that if $u_1(t,s) \to 0$ as $t \to \infty$ then (34) is trivially satisfied.

The following result gives sufficient conditions for the robustness of the asymptotic behavior as $t \to \infty$ of the solutions of (21), when the coefficient $h(t,x)$ is perturbed slightly at ∞. Apart from being interesting by itself, this result will be very helpful in the next section.

For a linear operator $T(t,s) : X \to X$, we call its associated *exponential type* (see [16]) to the number

$$\beta_0(T) = \inf\{\beta \in \mathbb{R}, \text{ such that } \|T(t,s)\| \le Me^{\beta(t-s)} \text{ holds for some } M > 0\}.$$

Lemma 3.6. *Assume that $q(x,t) \to 0$ uniformly in Ω as $t \to \infty$, consider two nonnegative, nontrivial initial data u_0^1, u_0^2 and denote*

$$u_1(t,s) = \Theta^{\sigma}_{[h,g]}(t,s;u_0^1), \quad u_2(t,s) = \Theta^{\sigma}_{[h+q,g]}(t,s;u_0^1),$$

for $t \ge s \ge T_0$.

Assume either
i) u_1 and u_2 are nondegenerate at ∞, or
ii) u_1 is nondegenerate at ∞, that is $u_1(x,t) \ge \varphi_0(x)$ for $x \in \Omega$ and for sufficiently large t, and for some constant $k < 1$

$$q(t,x) \ge -kg_L\varphi_0(x), \quad \text{for } x \in \Omega \text{ and sufficiently large } t.$$

Then, uniformly in $\overline{\Omega}$

$$u_2(t) - u_1(t) \to 0 \quad \text{as } t \to \infty.$$

In particular, the above applies to $u_1(t) = \varphi^{\sigma_1}_{[h_1,g_1]}(t)$ and $u_2(t) = \varphi^{\sigma_2}_{[h_2,g_2]}(t)$ if they are nonzero.

Proof.
i) Define $w(t) = u_2(t) - u_1(t)$. Then

$$w_t - d\Delta w + (gu_2 - h - q + gu_1)w = qu_1. \tag{35}$$

Now, observe that u_2 is a bounded and nondegenerate at ∞ solution of

$$w_t - d\Delta w + (gu_2 - h - q)w = 0, \tag{36}$$

and so, the associated exponential type at ∞ for the potential $gu_2 - h - q$ is equal to zero (see Lemma 3.5 in [16] and Proposition 3 in [18]).

Also the perturbation gu_1 decreases the exponential type at ∞ since $g_L > 0$ and u_1 is non degenerate, see Proposition 4.7 in [16]. Then the exponential type at ∞ for the linear equation

$$w_t - d\Delta w + (gu_2 - h - q + gu_1)w = 0$$

is negative.

With this, going back to (35) and using that $\|qu_1\|_\infty \to 0$ as $t \to \infty$, we can apply Corollary 4.6 of [17] and conclude the result.

ii) In this case we show that actually u_2 is nondegenerate at ∞ and then i) applies. For this we show that $\varepsilon u_1 \leq u_2$ for some $\varepsilon > 0$ small enough and sufficiently large t. For this, in turn, we show that εu_1 is a subsolution of the equation for u_2. Indeed, we have

$$\varepsilon(u_1)_t - \varepsilon d\Delta u_1 = \varepsilon(hu_1 - gu_1^2)$$

and the right hand side is less than $\varepsilon(h+q)u_1 - \varepsilon^2 gu_1^2$ iff

$$-q(x,t) \leq (1-\varepsilon)gu_1.$$

But given $gu_1 \geq g_L\varphi_0$ and our assumptions, chose ε small such that the above condition is met for sufficiently large t. Hence, εu_1 is a subsolution of the equation for u_2 for sufficiently large t.

Now, using the smoothing of the differential equation, we can assume that s is large enough and $u_1(s), u_2(s) \in C^1(\overline{\Omega})$. Hence we can take ε such that $\varepsilon u_1(s) \leq u_2(s)$. Then, by comparison we get $\varepsilon u_1(t) \leq u_2(t)$ for all $t > s$. Thus u_2 is nondegenerate at ∞. ■

A similar result can be proved in $-\infty$ for complete solutions.

Lemma 3.7. *Assume that $q(x,t) \to 0$ uniformly in Ω as $t \to -\infty$. Assume either*

i) $\varphi_{[h,g]}$ and $\varphi_{[h+q,g]}$ are nondegenerate at $-\infty$, or

ii) $\varphi_{[h,g]}$ is nondegenerate at $-\infty$, that is $\varphi_{[h,g]}(x,t) \geq \varphi_0(x)$ for very negative large t, and for some constant $k < 1$

$$q(t,x) \geq -kg_L\varphi_0(x), \quad \textit{for } x \in \Omega \textit{ and very negative } t.$$

Then, uniformly in $\overline{\Omega}$

$$\varphi_{[h+q,g]}(t) - \varphi_{[h,g]}(t) \to 0 \quad \textit{as } t \to -\infty.$$

Proof. The proof of part i) follows as in Lemma 3.6 but using Lemma 3.7 in [16] instead of Lemma 3.5.

For the proof of ii) we show that $\varphi_{[h+q,q]}$ is nondegenerate at $-\infty$ and part i) applies. For this, note that arguing as in Lemma 3.6, we have that $\varepsilon\varphi_{[h,q]}$ is a complete subtrajectory for the problem with coefficient $h+q$. Now we take a large constant K as a complete supertrajectory of that problem and then Corollary 2.10 implies that there is a complete solution, u^*, for the problem with coefficient $h+q$ such that $\varepsilon\varphi_{[h,q]} \leq u^* \leq K$. Since $\varphi_{[h+q,q]}$ is the maximal complete trajectory, we get $\varepsilon\varphi_{[h,q]} \leq u^* \leq \varphi_{[h+q,q]}$ and the result follows. ■

Remark 3.8. *With the notations in Lemmas 3.6 and 3.7 we have*
i) If $\Lambda(h_0^+) < 0$ then both u_1 and u_2 are nondegenerate at ∞ and the conclusion of Lemma 3.6 is true.

Analogously, if $\Lambda(h_0^-) < 0$ then both $\varphi_{[h,g]}$ and $\varphi_{[h+q,g]}$ are nondegenerate at $-\infty$ and the conclusion of Lemma 3.7 is true.
ii) If $\Lambda(H_0^+) > 0$ then both u_1 and u_2 converge to zero as $t \to +\infty$ and the conclusion of Lemma 3.6 is true.

Analogously, if $\Lambda(H_0^-) > 0$ then both $\varphi_{[h,g]}$ and $\varphi_{[h+q,g]}$ converge to zero as $t \to -\infty$ and the conclusion of Lemma 3.7 is true.
iii) If $q(t,x) \geq 0$ the condition in case ii) of Lemma 3.6 or 3.7 is always satisfied.

Also, for Robin (or Neumann) boundary conditions φ_0 can always be taken as a positive constant. Therefore the condition in case ii) of Lemma 3.6 or 3.7 is always satisfied.

For Dirichlet boundary conditions the condition in case ii) of Lemma 3.6 or 3.7 restricts the way the negative part of $q(t,x)$ goes to zero near the boundary of Ω. In particular if $q(t,x)$ is nonnegative in a neighborhood of the boundary of Ω then the condition in case ii) is satisfied.
iv) Note that the condition in case ii) of Lemma 3.6 or 3.7 can be replaced by

$$q(t,x) \geq -kg_\infty\varphi_0(x), \quad \text{for } x \in \Omega \text{ and sufficiently large or negative } t$$

where g_∞ satisfies that $g(t,x) \geq g_\infty$ for $x \in \Omega$ and sufficiently large or negative t. Such constant can be much larger than g_L.

4. Applications to the Lotka-Volterra models

In this section we apply the above results to prove the existence of complete trajectories for the following Lotka-Volterra model:

$$\begin{cases} u_t - d_1\Delta u = u(\lambda(t,x) - a(t,x)u - b(t,x)v), & x \in \Omega,\ t > s \\ v_t - d_2\Delta v = v(\mu(t,x) - c(t,x)u - d(t,x)v), & x \in \Omega,\ t > s \\ \mathcal{B}_1 u = 0,\ \mathcal{B}_2 v = 0, & x \in \partial\Omega,\ t > s \\ u(s) = u_s \geq 0,\ v(s) = v_s \geq 0, & \end{cases} \tag{37}$$

with $d_1, d_2 > 0$; $\lambda, \mu, a, b, c, d \in C^\theta(\overline{Q})$, and $\overline{Q} = \mathbb{R} \times \overline{\Omega}$. Given a function $e \in C^\theta(\overline{Q})$, we define

$$e_L := \inf_{\overline{Q}} e(t,x) \qquad e_M := \sup_{\overline{Q}} e(t,x).$$

We assume from now on that

$$a_L, d_L > 0 \tag{38}$$

and consider the three classical cases depending on the signs of b and c:

1. *Competition*: $b_L, c_L > 0$ in $\overline{Q}$.
2. *Symbiosis*: $b_M, c_M < 0$ in $\overline{Q}$.
3. *Prey-predator*: $b_L > 0$, $c_M < 0$ in $\overline{Q}$.

Also note that in the results of this section we will use the quantities $\lambda_I^\pm \leq \lambda_S^\pm$, $\mu_I^\pm \leq \mu_S^\pm$, $a_I^\pm \leq a_S^\pm$, $b_I^\pm \leq b_S^\pm$, $c_I^\pm \leq c_S^\pm$ and $d_I^\pm \leq d_S^\pm$, to control the asymptotic sizes of the coefficients λ, μ, a, b, c, d as $t \to \infty$ or $t \to -\infty$, respectively. More precisely we will assume

$$\begin{aligned} &\lambda_I^\pm \leq \lambda(t,x) \leq \lambda_S^\pm, \quad \mu_I^\pm \leq \mu(t,x) \leq \mu_S^\pm, \quad a_I^\pm \leq a(t,x) \leq a_S^\pm, \\ &b_I^\pm \leq b(t,x) \leq b_S^\pm, \quad\ c_I^\pm \leq c(t,x) \leq c_S^\pm, \quad\ d_I^\pm \leq d(t,x) \leq d_S^\pm. \end{aligned} \tag{39}$$

for all $x \in \Omega$ and for all $t \geq t_0$ or $t \leq t_0$ with the convention that a, b, c and d have the same sign as their upper and lower bounds in (39).

Also, we will keep the notation $\varphi_{[h,g]}$ to denote the complete solution of the logistic equation (21), as in Proposition 3.2. Superscripts will be used to indicate the boundary conditions $\mathcal{B}_i$, $i = 1, 2$ in (37) used for (21).

The next results give the existence of complete solutions for (37) and also give sufficient conditions for such complete solutions to be nondegenerate at $\pm\infty$.

Starting with the case of competition, we have

Proposition 4.1. (Competitive case) *Assume (38) and* $b_L, c_L > 0$. *Then, there exists a complete trajectory* (u^*, v^*) *of (37) with*

$$\varphi^1_{[\lambda - b\varphi^2_{[\mu,d]}, a]}(t) \leq u^*(t) \leq \varphi^1_{[\lambda,a]}(t), \quad \varphi^2_{[\mu - c\varphi^1_{[\lambda,a]}, d]}(t) \leq v^*(t) \leq \varphi^2_{[\mu,d]}(t), \qquad t \in \mathbb{R}. \tag{40}$$

Moreover, if (39) is satisfied for very negative t and

$$\lambda_I^- > \Lambda^1(-b_S^- \omega^2_{[\mu_S^-, d_I^-]}) \qquad \text{and} \qquad \mu_I^- > \Lambda^2(-c_S^- \omega^1_{[\lambda_S^-, a_I^-]}), \tag{41}$$

then (u^*, v^*) is non-degenerate at $-\infty$.

If moreover (39) is satisfied for large and very negative t, (41) and

$$\lambda_I^+ > \Lambda^1(-b_S^+ \omega^2_{[\mu_S^+, d_I^+]}) \qquad \text{and} \qquad \mu_I^+ > \Lambda^2(-c_S^+ \omega^1_{[\lambda_S^+, a_I^+]}) \tag{42}$$

holds, then (u^*, v^*) is non-degenerate at ∞.

Proof. Note that in this case f is decreasing in v and g in u. Hence, we show that in this case we can apply Corollary 2.8 with

$$(\underline{u}, \overline{u}) = (\varphi^1_{[\lambda - b\varphi^2_{[\mu,d]}, a]}, \varphi^1_{[\lambda,a]}) \quad \text{and} \quad (\underline{v}, \overline{v}) = (\varphi^2_{[\mu - c\varphi^1_{[\lambda,a]}, d]}, \varphi^2_{[\mu,d]}).$$

First, observe that by Proposition 3.4, with T_0 arbitrary, we have that $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ for $t \in \mathbb{R}$ since $b, c \geq 0$.

In this case, by the monotonicity of f and g, the definition of complete sub-supertrajectory pair of Definition 2.4 is equivalent to

$$\begin{aligned} &\underline{u}_t - d_1 \Delta \underline{u} - f(t, x, \underline{u}, \overline{v}) \leq 0 \leq \overline{u}_t - d_1 \Delta \overline{u} - f(t, x, \overline{u}, \underline{v}), \quad x \in \Omega,\ t \in \mathbb{R}, \\ &\underline{v}_t - d_2 \Delta \underline{v} - g(t, x, \overline{u}, \underline{v}) \leq 0 \leq \overline{v}_t - d_2 \Delta \overline{v} - g(t, x, \underline{u}, \overline{v}), \quad x \in \Omega,\ t \in \mathbb{R}. \end{aligned}$$

We check now these inequalities. We only prove the second inequality. Observe that

$$\begin{gathered} 0 \leq \overline{u}_t - d_1 \Delta \overline{u} - f(t, x, \overline{u}, \underline{v}) \Longleftrightarrow \\ 0 \leq \varphi^1_{[\lambda,a]}(\lambda - a\varphi^1_{[\lambda,a]}) - \varphi^1_{[\lambda,a]}(\lambda - a\varphi^1_{[\lambda,a]} - b\varphi^2_{[\mu - c\varphi^1_{[\lambda,a]}, d]}) \end{gathered}$$

which is obviously satisfied.

Now, assume (39) for $t \leq t_0$. Then, using Proposition 3.4 we get $\varphi^2_{[\mu,d]} \leq \varphi^2_{[\mu_S^-, d_I^-]} = \omega^2_{[\mu_S^-, d_I^-]}$ for $t \leq t_0$. Then, again Proposition 3.4 gives

$$\varphi^1_{[\lambda - b\varphi^2_{[\mu,d]}, a]} \geq \varphi^1_{[\lambda_I^- - b_S^- \varphi^2_{[\mu_S^-, d_I^-]}, a]} = \varphi^1_{[\lambda_I^- - b_S^- \omega^2_{[\mu_S^-, d_I^-]}, a]},$$

which is non-degenerate, by case iii) in Proposition 3.2, if $\lambda_I^- > \Lambda^1(-b_S^- \omega^2_{[\mu_S^-, d_I^-]})$. An analogous reasoning can be made for $\underline{v}$.

Finally, assume (39) and (42) are satisfied for very large t. Then Proposition 3.7 in [10] gives the result. ■

Observe that condition (42) is the same as the one in Proposition 3.7 in [10], while condition (41) is the one in Proposition 3.8 in [10] which is here shown to guarantee that nondegenerate complete trajectories actually exist.

Now for the case of symbiosis, we have

Proposition 4.2. (Symbiotic case) *Assume (38), $b_M, c_M < 0$ and*

$$b_L c_L < a_L d_L.$$

Then there exists a complete trajectory (u^*, v^*) of (37) with

$$\varphi^1_{[\lambda - b\varphi^2_{[\mu,d]}, a]}(t) \leq u^*(t), \quad \varphi^2_{[\mu - c\varphi^1_{[\lambda,a]}, d]}(t) \leq v^*(t), \qquad t \in \mathbb{R}. \tag{43}$$

Moreover, if (39) is satisfied for very negative t and

$$\lambda_I^- > \Lambda^1(-b_S^- \omega^2_{[\mu_I^-, d_S^-]}) \qquad and \qquad \mu_I^- > \Lambda^2(-c_S^- \omega^1_{[\lambda_I^-, a_S^-]}) \tag{44}$$

holds, then (u^*, v^*) is non-degenerate at $-\infty$.

If moreover (39) is satisfied for large and very negative t, (44) and

$$\lambda_I^+ > \Lambda^1(-b_S^+ \omega^2_{[\mu_I^+, d_S^+]}) \qquad and \qquad \mu_I^+ > \Lambda^2(-c_S^+ \omega^1_{[\lambda_I^+, a_S^+]}), \tag{45}$$

then (u^*, v^*) is non-degenerate at ∞.

Proof. Note that in this case f is increasing in v and g in u. Consider

$$(\underline{u}, \underline{v}) = (\varphi^1_{[\lambda - b\varphi^2_{[\mu,d]}, a]}, \varphi^2_{[\mu - c\varphi^1_{[\lambda,a]}, d]}),$$

and

$$(\overline{u}, \overline{v}) = (M_1\xi, M_2\xi)$$

where M_1, M_2 are positive constants to be chosen, and ξ is a positive eigenfunction associated to the problem

$$-\Delta\xi = \lambda\xi \quad \text{in } \Omega, \quad \mathcal{B}\xi = 0 \quad \text{on } \partial\Omega,$$

where

$$\mathcal{B}\xi := \frac{\partial\xi}{\partial\vec{n}} + \sigma(x)\xi$$

and $\sigma(x) := \min\{\sigma_1(x)/d_1, \sigma_2(x)/d_2\}$ considering $\sigma_i(x) = +\infty$ if $\mathcal{B}_i$ is a Dirichlet operator. Denote by $\Sigma = \Lambda_{\mathcal{B}}(1,0)$ the principal eigenvalue associated to this problem.

If both boundary conditions are Dirichlet, take $\xi = 1$.

Now, take M_1 and M_2 large enough such that $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$. Note that this is always possible, even for Dirichlet boundary conditions. Moreover, it is clear that $\mathcal{B}_1(\overline{u}) \geq 0$ and $\mathcal{B}_2(\overline{v}) \geq 0$. On the other hand, by the monotonicity of f and g, the Definition 2.4 is equivalent to

$$\begin{array}{ll} \underline{u}_t - d_1\Delta\underline{u} - f(t,x,\underline{u},\underline{v}) \leq 0 \leq \overline{u}_t - d_1\Delta\overline{u} - f(t,x,\overline{u},\overline{v}), & x \in \Omega,\ t \in \mathbb{R}, \\ \underline{v}_t - d_2\Delta\underline{v} - g(t,x,\underline{u},\underline{v}) \leq 0 \leq \overline{v}_t - d_2\Delta\overline{v} - g(t,x,\overline{u},\overline{v}), & x \in \Omega,\ t \in \mathbb{R}. \end{array}$$

The inequalities refereed to $\underline{u}$ and $\underline{v}$ are easy to check. For example, for $\underline{u}$ we need to show that

$$\varphi^2_{[\mu - c\varphi^1_{[\lambda,a]},d]} \geq \varphi^2_{[\mu,d]}, \quad t \in \mathbb{R},$$

which is true by Proposition 3.4, with arbitrary T_0 since $c \leq 0$.

On the other hand, $(\overline{u},\overline{v})$ is a super-trajectory of (37) if

$$d_1\Sigma \geq \lambda - aM_1\xi - bM_2\xi, \quad \text{and} \quad d_2\Sigma \geq \mu - dM_2\xi - cM_1\xi.$$

For that, it suffices that

$$\frac{1}{-b_L}\left[\left(\frac{-\lambda + d_1\Sigma}{\xi}\right)_L + a_L M_1\right] \geq M_2 \geq \frac{1}{d_L}\left[\left(\frac{\mu - d_2\Sigma}{\xi}\right)_M - c_L M_1\right]$$

Thanks to $b_L c_L < a_L d_L$ it suffices to take M_1 and M_2 large enough.

Now, assuming (39) is satisfied for $t \leq t_0$, (44) and using Proposition 3.4 we get $\varphi^2_{[\mu,d]} \geq \varphi^2_{[\mu_I^-,d_S^-]} = \omega^2_{[\mu_I^-,d_S^-]}$ for $t \leq t_0$. Then again Proposition 3.4 gives

$$\varphi^1_{[\lambda - b\varphi^2_{[\mu,d]},a]} \geq \varphi^1_{[\lambda_I^- - b_S^-\varphi^2_{[\mu_I^-,d_S^-]},a]} = \varphi^1_{[\lambda_I^- - b_S^-\omega^2_{[\mu_I^-,d_S^-]},a]}$$

which is non-degenerate, by case iii) in Proposition 3.2, if $\lambda_I^- > \Lambda^1(-b_S^-\omega^2_{[\mu_I^-,d_S^-]})$. Analogously for $\underline{v}$,

$$\varphi^2_{[\mu - c\varphi^1_{[\lambda,a]},d]} \geq \varphi^2_{[\mu_I^- - c_S^-\varphi^1_{[\lambda_I^-,a_S^-]},d]} = \varphi^2_{[\mu_I^- - c_S^-\omega^1_{[\lambda_I^-,a_S^-]},d]}$$

which is non-degenerate, by case iii) in Proposition 3.2, if $\mu_I^- > \Lambda^2(-c_S^-\omega^1_{[\lambda_I^-,a_S^-]})$.

Finally, assume (39) and (45) are satisfied for very large t. Then Proposition 3.9 in [10] gives the result. ■

Observe that condition (45) is the same as the one in Proposition 3.9 in [10], while condition (44) is the one in Proposition 3.10 in [10] which is here shown to guarantee that nondegenerate complete trajectories actually exist.

Then, we conclude with the prey–predator case.

Proposition 4.3. (Prey-predator case) *Assume (38), $b_L > 0$ and $c_M < 0$. Then there exists a complete trajectory of (37), with*

$$\varphi^1_{[\lambda - b\varphi^2_{[\mu - c\varphi^1_{[\lambda,a]},d]},a]}(t) \le u^*(t) \le \varphi^1_{[\lambda,a]}(t), \quad \varphi^2_{[\mu,d]}(t) \le v^*(t) \le \varphi^2_{[\mu - c\varphi^1_{[\lambda,a]},d]}(t), \tag{46}$$

with $t \in \mathbb{R}$. If moreover (39) is satisfied for very negative t and

$$\lambda^-_I > \Lambda^1(-b^-_S \omega^2_{[\mu^-_S - c^-_I \omega^1_{[\lambda^-_S, a^-_I]}, d^-_I]}) \qquad \text{and} \qquad \mu^-_I > \Lambda^2_0 \tag{47}$$

then (u^*, v^*) is non-degenerate at $-\infty$.

If moreover (39) is satisfied for large t and very negative t, (47) and

$$\lambda^+_I > \Lambda^1(-b^+_S \omega^2_{[\mu^+_S - c^+_I \omega^1_{[\lambda^+_S, a^+_I]}, d^+_I]}) \qquad \text{and} \qquad \mu^+_I > \Lambda^2_0 \tag{48}$$

then (u^*, v^*) is non-degenerate at ∞.

Proof. Note that in this case f is decreasing in v and g increasing u. It suffices to take

$$(\underline{u}, \underline{v}) = (\varphi^1_{[\lambda - b\varphi^2_{[\mu - c\varphi^1_{[\lambda,a]},d]},a]}, \varphi^2_{[\mu,d]}), \qquad (\overline{u}, \overline{v}) = (\varphi^1_{[\lambda,a]}, \varphi^2_{[\mu - c\varphi^1_{[\lambda,a]},d]}).$$

Observe that now $\underline{u} \le \overline{u}$ by Proposition 3.4 since $b \ge 0$, while $\underline{v} \le \overline{v}$ by Proposition 3.4 since $c \le 0$. Also, in this case, by the monotonicity of f and g, the Definition 2.4 is equivalent to

$$\begin{aligned}
&\underline{u}_t - d_1\Delta\underline{u} - f(t,x,\underline{u},\overline{v}) \le 0 \le \overline{u}_t - d_1\Delta\overline{u} - f(t,x,\overline{u},\underline{v}), \quad x \in \Omega, \ t \in \mathbb{R}, \\
&\underline{v}_t - d_2\Delta\underline{v} - g(t,x,\underline{u},\underline{v}) \le 0 \le \overline{v}_t - d_2\Delta\overline{v} - g(t,x,\overline{u},\overline{v}), \quad x \in \Omega, \ t \in \mathbb{R}.
\end{aligned}$$

The inequalities for $\underline{v}$ and $\overline{u}$ are clear. Let us check the other ones. First the inequality $\underline{u}_t - d_1\Delta\underline{u} - f(x,t,\underline{u},\overline{v}) \le 0$ is equivalent to

$$\begin{aligned}
&\varphi^1_{[\lambda - b\varphi^2_{[\mu - c\varphi^1_{[\lambda,a]},d]},a]}(\lambda - b\varphi^2_{[\mu - c\varphi^1_{[\lambda,a]},d]} - a\varphi^1_{[\lambda - b\varphi^2_{[\mu - c\varphi^1_{[\lambda,a]},d]},a]}) - \\
&\varphi^1_{[\lambda - b\varphi^2_{[\mu - c\varphi^1_{[\lambda,a]},d]},a]}(\lambda - a\varphi^1_{[\lambda - b\varphi^2_{[\mu - c\varphi^1_{[\lambda,a]},d]},a]} - b\varphi^2_{[\mu - c\varphi^1_{[\lambda,a]},d]}) \le 0,
\end{aligned}$$

which is obviously satisfied. On the other hand, $0 \le \overline{v}_t - d_2\Delta\overline{v} - g(x,t,\overline{u},\overline{v})$ is equivalent to

$$0 \le \varphi^2_{[\mu-c\varphi^1_{[\lambda,a]},d]}(\mu - c\varphi^1_{[\lambda,a]} - d\varphi^2_{[\mu-c\varphi^1_{[\lambda,a]},d]}) - \varphi^2_{[\mu-c\varphi^1_{[\lambda,a]},d]}(\mu - d\varphi^2_{[\mu-c\varphi^1_{[\lambda,a]},d]} - c\varphi^1_{[\lambda,a]}),$$

which again is clear.

The nondegeneracy in $-\infty$ is obtained as in the previous cases, using (47) and Proposition 3.4 several times.

Finally, assume (39) and (48) are satisfied for very large t. Then Proposition 3.11 in [10] gives the result.

■

Observe that condition (48) is the same as the one in Proposition 3.11 in [10], while condition (47) is the one in Proposition 3.12 in [10] which is here shown to guarantee that nondegenerate complete trajectories actually exist.

Now, we can summarize our main results for the solutions of the Lotka–Volterra system (37) (see Figure 1). For this we will assume (39) and we consider nonnegative nontrivial initial data u_s, v_s, both nonzero. Also, as s varies we assume u_s, v_s is bounded and nondegenerate.

For the competitive case we have then

Theorem 4.4. (Competitive case) *Assume (38) and $b_L, c_L > 0$.*

1. *If $\lambda^-_S < \Lambda^1_0$ and $\mu^-_S < \Lambda^2_0$*

$$\lim_{s\to-\infty}(u(t,s;u_s,v_s), v(t,s;u_s,v_s)) = (0,0).$$

On the other hand, if $\lambda^+_S \le \Lambda^1_0$ and $\mu^+_S \le \Lambda^2_0$, then

$$\lim_{t\to\infty}(u(t,s;u_s,v_s), v(t,s;u_s,v_s)) = (0,0).$$

2. *If $\lambda^+_S < \Lambda^1_0$ and $\mu^+_I > \Lambda^2_0$, then*

$$\lim_{t\to\infty} u(t,s;u_s,v_s) = 0,$$

and for every nonnegative nontrivial $\tilde{v}_s$ we have

$$\lim_{t\to\infty}(v(t,s;u_s,v_s) - \Theta^2_{[\mu,d]}(t,s;\tilde{v}_s)) = 0.$$

If additionally $\mu^-_I > \Lambda^2_0$, then

$$\lim_{t\to\infty}(v(t,s;u_s,v_s) - \varphi^2_{[\mu,d]}(t)) = 0.$$

3. *If* $\lambda_I^+ > \Lambda_0^1$ *and* $\mu_S^+ < \Lambda_0^2$ *, then*

$$\lim_{t\to\infty} v(t,s;u_s,v_s) = 0,$$

and for every nonnegative nontrivial $\tilde{v}_s$ *we have*

$$\lim_{t\to\infty} (u(t,s;u_s,v_s) - \Theta^1_{[\lambda,a]}(t,s;\tilde{v}_s)) = 0.$$

If additionally, $\lambda_I^- > \Lambda_0^1$*, then*

$$\lim_{t\to\infty} (u(t,s;u_s,v_s) - \varphi^1_{[\lambda,a]}(t)) = 0.$$

4. *If*

$$\lambda_I^- > \Lambda^1(-b_S^-\omega^2_{[\mu_S^-,d_I^-]}) \qquad \text{and} \qquad \mu_I^- > \Lambda^2(-c_S^-\omega^1_{[\lambda_S^-,a_I^-]}), \tag{49}$$

there exists a complete bounded non-degenerate at $-\infty$ *trajectory of (37)* $(u^*(t), v^*(t))$*. Moreover, if* b *or* c *are small at* $-\infty$*, that is,*

$$\limsup_{t\to-\infty} \|b\|_{L^\infty(\Omega)} \limsup_{t\to-\infty} \|c\|_{L^\infty(\Omega)} < \rho_0$$

for some suitable constant $\rho_0 > 0$*, then this is the unique bounded non-degenerate at* $-\infty$ *trajectory of (37) and it is pullback attracting, that is*

$$\lim_{s\to-\infty}(u(t,s;u_s,v_s) - u^*(s), v(t,s;u_s,v_s) - v^*(s)) = (0,0).$$

If moreover

$$\lambda_I^+ > \Lambda^1(-b_S^+\omega^2_{[\mu_S^+,d_I^+]}) \qquad \text{and} \qquad \mu_I^+ > \Lambda^2(-c_S^+\omega^1_{[\lambda_S^+,a_I^+]}), \tag{50}$$

then $(u(t,s;u_s,v_s), v(t,s;u_s,v_s))$ *is nondegenerate at* ∞*. If additionally* b *or* c *are small at* ∞*, that is,*

$$\limsup_{t\to\infty} \|b\|_{L^\infty(\Omega)} \limsup_{t\to\infty} \|c\|_{L^\infty(\Omega)} < \rho_0$$

for some suitable constant $\rho_0 > 0$*, then all solutions of (37) have the same asymptotic behavior as* $t \to \infty$*. If (49) is also satisfied, then* $(u^*(t), v^*(t))$ *is non-degenerate at* ∞ *and it is also forwards attracting, that is,*

$$\lim_{t\to\infty}(u(t,s;u_s,v_s) - u^*(t), v(t,s;u_s,v_s) - v^*(t)) = (0,0).$$

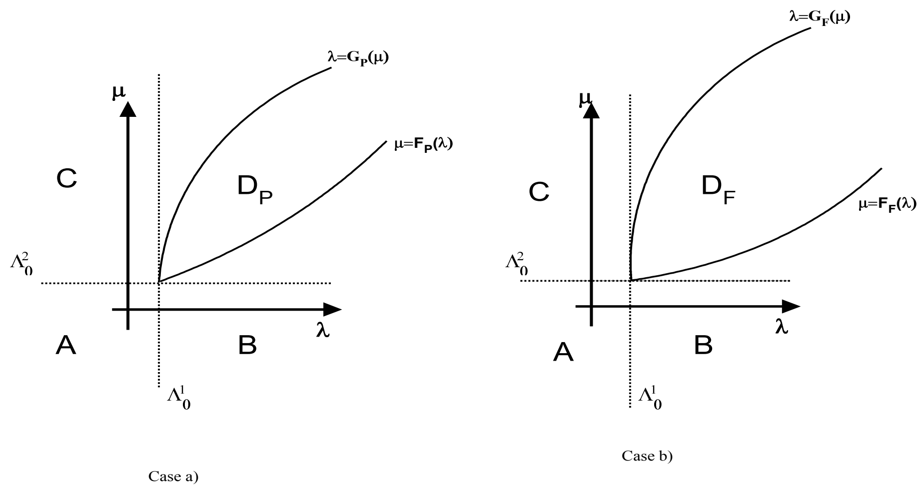

Figure 1: Description of the asymptotic dynamical regimes (pullback -Case a)- and forwards -Case b)) when λ and μ are constant functions: Region A: extinction of both species; Regions B and C: stability of semitrivial complete trajectories; Regions D_P and D_F: permanence regions (existence of global non-degenerate global solutions). The limiting curves are given in (49) and (50). Similar figures can be drawn for the prey-predator and symbiosis cases.

Proof.
1. Observe that using Proposition 3.2 i) we get

$$\limsup_{s\to-\infty} u(t,s;u_s,v_s) \le \varphi^1_{[\lambda,a]}(t).$$

Since $\lambda_S^- < \Lambda_0^1$ we have that $\Lambda^1(\lambda_S^-) > 0$ and then applying again Proposition 3.2 ii) we conclude that $\varphi^1_{[\lambda,a]}(t) = 0$ for all $t \in \mathbb{R}$. Analogously for $v(t,s;u_s,v_s)$ if $\mu_S^- < \Lambda_0^2$.

Now for large t,

$$u(t,s;u_s,v_s) \le \Theta^1_{[\lambda,a]}(t,s;u_s) \le \Theta^1_{[\lambda_S^+,a_I^+]}(t,s;u_s)$$

whence it follows that $u(t,s;u_s,v_s) \to 0$ as $t \to \infty$ when $\lambda_S^+ \le \Lambda_0^1$. Analogously, $v(t,s;u_s,v_s) \to 0$ as $t \to \infty$ when $\mu_S^+ \le \Lambda_0^2$.
2. Assume that $\lambda_S^+ < \Lambda_0^1$ and $\mu_I^+ > \Lambda_0^2$. Then $u \to 0$, for $t \to \infty$, as in case 1) and $v(t,s;u_s,v_s) = \Theta^2_{[\mu-cu,d]}(t,s;v_s)$ with $q = -cu \to 0$ uniformly in Ω,

as $t \to \infty$. Also, by the assumption and case v) in Proposition 3.2, for any nonegative nontrivial $\tilde{v}_s$ we have that $\Theta^2_{[\mu,d]}(t, s; \tilde{v}_s)$ is nondegenerate at ∞.

Also, $\mu_I^+ > \Lambda_0^2$ implies that for some $\varepsilon > 0$ we have for all $x \in \Omega$ and sufficiently large t

$$\mu(x,t) - c(x,t)u(x,t) \geq \mu(x,t) - \varepsilon \geq \mu_I^+ - \varepsilon > \Lambda_0^2$$

and $\Theta^2_{[\mu-cu,d]}(t, s; v_s)$ is nondegenerate at ∞. Then by Lemma 3.6 i), we get

$$\Theta^2_{[\mu,d]}(t, s; \tilde{v}_s) - v(t, s; u_s, v_s) \to 0, \quad \text{as } t \to \infty.$$

If additionally $\mu_I^- > \Lambda_0^2$, then by case v) in Proposition 3.2 implies that $\varphi^2_{[\mu,d]}(t)$ is also nonzero and nondegenerate at $\pm\infty$ and then

$$\Theta^2_{[\mu,d]}(t, s; \tilde{v}_s) - \varphi^2_{[\mu,d]}(t) \to 0, \quad \text{as } t \to \infty.$$

3. This case is symmetrical to case 2).
4. The existence of a complete bounded non-degenerate at $-\infty$ trajectory $(u^*(t), v^*(t))$ of (37) follows from by Proposition 4.1.

The results for $s \to -\infty$ follow from Theorem 6.2 in [10], while the results for $t \to \infty$ follow from Proposition 3.7 and Theorem 6.1 in [10]. ■

Now for the case of symbiosis we have the following theorem. Note that in part i) below we have not included the convergence to zero as $s \to -\infty$. This was already obtained in [10] under the additional assumption that $d_1 = d_2$; see Proposition 3.6 in [10].

Theorem 4.5. (Symbiotic case) *Assume (38), $b_M, c_M < 0$ and*

$$b_L c_L < a_L d_L.$$

1. *Denote by Σ the principal eigenvalue of $-\Delta$ under the boundary conditions $\mathcal{B}u := \partial u/\partial n + \sigma u$ where $\sigma(x) := \min\{\sigma_1(x)/d_1, \sigma_2(x)/d_2\}$ taking $\sigma_i = \infty$ if $\mathcal{B}_i$ is the Dirichlet BC.*
   When $\lambda_S^+ < d_1\Sigma$ and $\mu_S^+ < d_2\Sigma$, then

$$\lim_{t\to\infty} (u(t, s; u_s, v_s), v(t, s; u_s, v_s)) = (0, 0).$$

2. *If*

$$\lambda_I^- > \Lambda^1(-b_S^-\omega^2_{[\mu_I^-,d_S^-]}) \qquad \text{and} \qquad \mu_I^- > \Lambda^2(-c_S^-\omega^1_{[\lambda_I^-,a_S^-]}), \tag{51}$$

there exists a complete bounded non-degenerate at $-\infty$ *trajectory of (37)* $(u^*(t), v^*(t))$. *Moreover, if* b *or* c *are small at* $-\infty$, *that is,*

$$\limsup_{t\to-\infty} \|b\|_{L^\infty(\Omega)} \limsup_{t\to-\infty} \|c\|_{L^\infty(\Omega)} < \rho_0$$

for some suitable constant $\rho_0 > 0$, *then this is the unique bounded non-degenerate at* $-\infty$ *trajectory of (37) and it is pullback attracting, that is*

$$\lim_{s\to-\infty} (u(t,s;u_s,v_s) - u^*(s), v(t,s;u_s,v_s) - v^*(s)) = (0,0).$$

If moreover

$$\lambda_I^+ > \Lambda^1(-b_S^+\omega^2_{[\mu_I^+,d_S^+]}) \qquad \text{and} \qquad \mu_I^+ > \Lambda^2(-c_S^+\omega^1_{[\lambda_I^+,a_S^+]}), \tag{52}$$

then $(u(t,s;u_s,v_s), v(t,s;u_s,v_s))$ *is nondegenerate at* ∞. *If additionally* b *or* c *are small at* ∞, *that is,*

$$\limsup_{t\to\infty} \|b\|_{L^\infty(\Omega)} \limsup_{t\to\infty} \|c\|_{L^\infty(\Omega)} < \rho_0$$

for some suitable constant $\rho_0 > 0$, *then all solutions of (37) have the same asymptotic behavior as* $t \to \infty$. *If (51) is also satisfied, then* $(u^*(t), v^*(t))$ *is non-degenerate at* ∞ *and it is also forwards attracting, that is,*

$$\lim_{t\to\infty} (u(t,s;u_s,v_s) - u^*(t), v(t,s;u_s,v_s) - v^*(t)) = (0,0).$$

Proof. 1. Assume that $\lambda_S^+ < \Sigma$ and $\mu_S^+ < \Sigma$. Then, we can take

$$(\overline{u}, \overline{v}) = (M_1 e^{\gamma(t-s)}\xi, M_2 e^{\gamma(t-s)}\xi)$$

where $M_1, M_2 > 0$ are positive constant to be chosen, ξ is a positive eigenfunction associated to Σ and

$$\gamma := \max\{\lambda_S^+ - d_1\Sigma, \mu_S^+ - d_2\Sigma\} < 0.$$

It is not hard to show that $(\overline{u}, \overline{v})$ is a supersolution of (37) and so the first paragraph follows.
2. We can apply Proposition 4.2 and Theorems 6.1 and 6.2 in [10]. ■

Note that we could not obtain the semitrivial–case in the results above.

Theorem 4.6. (Prey-predator case) *Assume (38), $b_L > 0$ and $c_M < 0$.*

1. *If* $\lambda_S^- < \Lambda_0^1$ *and* $\mu_S^- < \Lambda_0^2$

$$\lim_{s\to-\infty}(u(t,s;u_s,v_s),v(t,s;u_s,v_s)) = (0,0).$$

On the other hand, if $\lambda_S^+ \leq \Lambda_0^1$ *and* $\mu_S^+ \leq \Lambda_0^2$*, then*

$$\lim_{t\to\infty}(u(t,s;u_s,v_s),v(t,s;u_s,v_s)) = (0,0).$$

2. *If* $\lambda_S^+ < \Lambda_0^1$ *and* $\mu_I^+ > \Lambda_0^2$*, then*

$$\lim_{t\to\infty} u(t,s;u_s,v_s) = 0,$$

and for every nonnegative nontrivial $\tilde{v}_s$ *we have*

$$\lim_{t\to\infty}(v(t,s;u_s,v_s) - \Theta^2_{[\mu,d]}(t,s;\tilde{v}_s)) = 0.$$

If additionally $\mu_I^- > \Lambda_0^2$*, then*

$$\lim_{t\to\infty}(v(t,s;u_s,v_s) - \varphi^2_{[\mu,d]}(t)) = 0.$$

3. *If* $\lambda_S^+ > \Lambda_0^1$ *and* $\mu_S^+ < \Lambda^2(-c_S^+\omega^1_{[\lambda_S^+,a_I^+]})$*, then then*

$$\lim_{t\to\infty} v(t,s;u_s,v_s) = 0,$$

and for every nonnegative nontrivial $\tilde{u}_s$ *we have*

$$\lim_{t\to\infty}(u(t,s;u_s,v_s) - \Theta^1_{[\lambda,a]}(t,s;\tilde{u}_s)) = 0.$$

If additionally $\lambda_I^- > \Lambda_0^1$*, then*

$$\lim_{t\to\infty}(u(t,s;u_s,v_s) - \varphi^1_{[\lambda,a]}(t)) = 0.$$

4. *If*

$$\lambda_I^- > \Lambda^1(-b_S^-\omega^2_{[\mu_S^- - c_I^-\omega^1_{[\lambda_S^-,a_I^-]},d_I^-]}) \qquad \text{and} \qquad \mu_I^- > \Lambda_0^2, \tag{53}$$

there exists a complete bounded non-degenerate at $-\infty$ trajectory of (37) $(u^*(t),v^*(t))$. Moreover, if b or c are small at $-\infty$, that is,

$$\limsup_{t\to-\infty}\|b\|_{L^\infty(\Omega)}\limsup_{t\to-\infty}\|c\|_{L^\infty(\Omega)} < \rho_0$$

for some suitable constant $\rho_0 > 0$, then this is the unique bounded non-degenerate at $-\infty$ trajectory of (37) and it is pullback attracting, that is

$$\lim_{s\to-\infty}(u(t,s;u_s,v_s)-u^*(s),v(t,s;u_s,v_s)-v^*(s))=(0,0).$$

If moreover

$$\lambda_I^+ > \Lambda^1(-b_S^+\omega^2_{[\mu_S^+-c_I^+\omega^1_{[\lambda_S^+,a_I^+]},d_I^+]}) \qquad \textit{and} \qquad \mu_I^+ > \Lambda_0^2, \tag{54}$$

then $(u(t,s;u_s,v_s),v(t,s;u_s,v_s))$ is nondegenerate at ∞. If additionally b or c are small at ∞, that is,

$$\limsup_{t\to\infty}\|b\|_{L^\infty(\Omega)}\limsup_{t\to\infty}\|c\|_{L^\infty(\Omega)}<\rho_0$$

for some suitable constant $\rho_0 > 0$, then all solutions of (37) have the same asymptotic behavior as $t \to \infty$. If (53) is also satisfied, then $(u^*(t),v^*(t))$ is non-degenerate at ∞ and it is also forwards attracting, that is,

$$\lim_{t\to\infty}(u(t,s;u_s,v_s)-u^*(t),v(t,s;u_s,v_s)-v^*(t))=(0,0).$$

Proof. The first and second paragraphs follow analogously to Theorem 4.4.

Assume $\lambda_S^+ > \Lambda_0^1$ and $\mu_S^+ < \Lambda^2(-c_S^+\omega^1_{[\lambda_S^+,a_I^s]})$. Then, since $u \leq \Theta^1_{[\lambda,a]}$ we get

$$v \leq \Theta^2_{[\mu-c\Theta^1_{[\lambda,a]},d]} \leq \Theta^2_{[\mu_S^+-c_I^+\Theta^1_{[\lambda_S^+,a_I^+]},d_I^+]}$$

whence the result follows.

Again, the last paragraph follows by Proposition 4.3 and Theorems 5.1 and 5.2 in [10].

■

5. Conclusions and open problems

We have proved, under some conditions for the parameters, the existence of bounded complete non-degenerate trajectories for Lotka-Volterra models. Note that the study of existence of complete bounded trajectories related to a system is always a difficult and interesting problem. A common tool to get this kind of results is by means of the existence of global attractors. However, we have adopted a different strategy, so that we generalize the classical sub-supersolution method for initial value problems to get bounded complete trajectories associated to non-autonomous dynamical systems. When we apply our abstract result to Lotka-Volterra symbiosis, competition or predator-prey models, we are able to give a complete description of the forwards and pullback dynamics inside the corresponding non-autonomous attractors. Indeed, we describe the geometrical structure of these attracting sets, generalizing in particular the existing results in the autonomous and periodic cases. The robustness of this structure under perturbations, which naturally leads to bifurcation phenomena in non-autonomous models, becomes as one of the natural important further steps from our results. We will pursue this direction in the near future. On the other hand, generalizing the autonomous case (see, for instance, [13, 14]) to obtain more accurate range for the parameter regions for the stability or instability for semitrivial and non-degenerate trajectories becomes a worthwhile open question to be analyzed.

References

[1] L. Arnold and I. Chueshov, Order-preserving random dynamical systems: equilibria, attractors, applications, Dyn. Stability of Systems, 13 (1998), 265-280.

[2] R. S. Cantrell and C. Cosner, Spatial Ecology via Reaction-Diffusion Equations, John Wiley & Sons. Ltd. 2003.

[3] R. S. Cantrell, C. Cosner and V. Hutson, Permanence in ecological systems with spatial heterogeneity, Proc. Royal Soc. Edin., 123A (1993) 533-559.

[4] R. S. Cantrell, C. Cosner and V. Hutson, Ecological models, permanence and spatial heterogeneity, Rocky Mountian J. of Math., 26 (1996) 1-35.

[5] T. Caraballo, G. Lukaszewicz and J. Real, Pullback attractors for asymptotically compact non-autonomous dynamical systems, Nonlinear Anal. 64 (2006) 484–498.

[6] A. N. Carvalho, J. A. Langa and J.C. Robinson, and A. Suárez, Characterization of non-autonomous attractors of a perturbed gradient system, J. Differential Equations 236 (2007) 570-603.

[7] I. Chueshov, Monotone random systems theory and applications. Lecture Notes in Mathematics, 1779. Springer-Verlag, Berlin, 2002.

[8] V. Hutson and K. Schmitt, Permanence in dynamical systems, Math. Biosci., 111 (1992) 1-71.

[9] J. A. Langa, J.C. Robinson, A. Rodríguez-Bernal, A. Suárez, and A.Vidal-López, Existence and nonexistence of unbounded forward attractor for a class of non-autonomous reaction diffusion equations, Discrete Contin. Dyn. Syst. 18 (2007) 483–497.

[10] J. A. Langa, J.C. Robinson, A. Rodríguez-Bernal and A. Suárez, Permanence and asymptotically stable complete trajectories for non-autonomous Lotka-Volterra models with diffusion, SIAM J. Math. Anal. 40 (2009) 2179–2216.

[11] J. A. Langa, J. C. Robinson, and A. Suárez, Permanence in the non-autonomous Lotka-Volterra competition model, J. Differential Equations 190 (2003) 214-238.

[12] J. A. Langa and A. Suárez, Pullback permanence for non-autonomous partial differential equations, Electron. J. Differential Equations 2002, 72, 20 pp.

[13] J. López-Gómez, On the structure of the permanence region for competing species models with general diffusivities and transport effects, Discrete and Continuous Dynamical Systems, 2 (1996) 525-542.

[14] J. López-Gómez and J. C. Sabina de Lis, Coexistence states and global attractivity for some convective diffusive competing species models, Trans. A.M.S., 347 (1995) 3797-3833.

[15] C. V. Pao, Nonlinear parabolic and elliptic equations, Plenum, New York, 1992.

[16] A. Rodríguez-Bernal, Perturbation of the exponential type of linear nonautonomous parabolic equations and applications to nonlinear equations, Serie de Prepublicaciones del Dept. de Matemática Aplicada U. Complutense, MA-UCM 2008–08. To appear in Discrete and Continuous Dynamical Systems A.

[17] J.C. Robinson, A. Rodríguez-Bernal, and A. Vidal-López, Pullback attractors and extremal complete trajectories for non-autonomous reaction-diffusion problems, J. Differential Equations 238 (2007) 289–337.

[18] A. Rodríguez-Bernal and A. Vidal-López, Existence, uniqueness and attractivity properties of positive complete trajectories for non-autonomous reaction-diffusion problems, Discrete Contin. Dyn. Syst. 18 (2007) 537–567.

[19] A. Rodríguez-Bernal and A. Vidal-López, Extremal equilibria for nonlinear parabolic equations in bounded domains and applications, J. Differential Equations 244 (2008) 2983-3030.