CHARACTERIZATION OF NON-AUTONOMOUS ATTRACTORS IN PERTURBED INFINITE-DIMENSIONAL GRADIENT SYSTEMS

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Abstract. In this paper we determine the exact structure of the pullback attractors in non-autonomous problems that are perturbations of autonomous gradient systems with attractors that are the union of the unstable manifolds of a finite set of hyperbolic equilibria. We show that the pullback attractors of the perturbed systems inherit this structure, and are given as the union of the unstable manifolds of a set of hyperbolic global solutions which are the non-autonomous analogues of the hyperbolic equilibria. We also prove, again parallel to the autonomous case, that all solutions converge as $t \to +\infty$ to one of these hyperbolic global solutions. We then show how to apply these results to systems that are asymptotically autonomous as $t \to -\infty$ and as $t \to +\infty$, and use these relatively simple test cases to illustrate a discussion of possible definitions of a forwards attractor in the non-autonomous case.

1. Introduction

1.1. **Overview.** The study of the global attractors that arise in many infinite-dimensional dynamical systems has been developed extensively over the past thirty years, and for autonomous systems much of the theory is now classical (see, for example, the books by Hale [9], Ladyzhenskaya [13], or Temam [24]). However, given the underlying models that arise in various branches of the sciences it is very natural to try to extend the theory to treat non-autonomous equations.

In the autonomous case the concept of a global attractor is settled and for gradient systems (those that possess a Liapunov function) the structure of the attractor is well understood: it is given as the union of the unstable manifolds of the equilibria. However, for non-autonomous dynamical systems the appropriate definition of 'a global attractor' is still not entirely settled, and there are few examples with attractors whose structure is known.

In this paper we identify a class of non-autonomous systems in which the structure of the pullback attractor can be determined exactly: these are uniformly small non-autonomous perturbations of gradient systems with a finite number of hyperbolic equilibria. Loosely speaking, the main result proved in this paper is that the structure of the attractor is unchanged by such non-autonomous perturbations. More precisely, we show that the pullback attractor is the union of the unstable manifolds of hyperbolic global solutions. These 'global hyperbolic solutions' are the non-autonomous analogue of hyperbolic equilibria, being solutions defined for all $t \in \mathbb{R}$, the linearizations around which enjoy exponential dichotomies.

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Even in the case of autonomous systems this provides new examples in which the structure of the attractor is known explicitly. (A proof of this result for a restricted class of finite-dimensional systems is given by Langa et al. [16], but the argument there uses time reversal and so is not applicable in the infinite-dimensional setting.)

We then show how to adapt our results to the case of asymptotically autonomous systems. As well as obtaining results that are interesting in their own right, we use these simple models as a basis for a discussion of the possible definitions of an attractor in non-autonomous systems.

1.2. **Semigroups and processes.** In order to describe the results of this paper in more detail we need to introduce some terminology. Although in the main body of the paper we choose to work with a particular model for which we are able to prove that some key properties hold (see Section 1.4, below) more generally we are interested in non-autonomous perturbations of an underlying autonomous process.

Taking a Banach space $\mathcal{Z}$ as our phase space, the underlying autonomous system is naturally described by a semigroup of nonlinear operators (or 'nonlinear semigroup'), i.e. a family $\{S(t):\ t \geq 0\}$ (or $S(\cdot)$ for short) consisting of continuous operators from $\mathcal{Z}$ into itself such that

$$1)\ S(t) = I,$$
$$2)\ S(t)S(s) = S(t+s),\ \text{for each } t, s \geq 0,\ \text{and}$$
$$3)\ t \mapsto S(t)z_0 \text{ is continuous for } t \geq 0,\ z_0 \in \mathcal{Z}.$$

If each $S(t)$ is linear then we call $\{S(t):\ t \geq 0\}$ a linear semigroup.

Upon addition of a non-autonomous perturbation the initial time becomes as important as the final time, and the dynamics is then described by a nonlinear process, i.e. a two parameter family $\{S(t,\tau):\ t \geq \tau \in \mathbb{R}\}$ (or $S(\cdot,\cdot)$ for short) of continuous operators from $\mathcal{Z}$ into itself such that

$$1)\ S(\tau,\tau) = I,$$
$$2)\ S(t,\sigma)S(\sigma,\tau) = S(t,\tau),\ \text{for each } t \geq \sigma \geq \tau,\ \text{and}$$
$$3)\ (t,\tau) \mapsto S(t,\tau)z_0 \text{ is continuous for } t \geq \tau,\ z_0 \in \mathcal{Z}.$$

Again, if each $S(t,\tau)$ is linear then we refer to $S(\cdot,\cdot)$ as a linear process.

For a nonlinear process $S(\cdot,\cdot)$ with the property that $S(t,\tau) = S_0(t-\tau)$ for all $t \geq \tau \in \mathbb{R}$, i.e. for a process that is really a nonlinear semigroup in disguise, the behaviour of solutions as $t \to \infty$, which we refer to as 'the forwards dynamics', is the same as the behaviour of solutions as $\tau \to -\infty$, 'the pullback dynamics'. However, for general processes these 'dynamical limits' are totally unrelated and can produce entirely different qualitative properties (see [6, 17]). We believe that this point is made forcibly in Section 4, where we consider asymptotically autonomous gradient systems and are able to describe both dynamical limits completely.

1.3. **Attractors.** Our main tool for describing the long-term dynamics of both the autonomous and non-autonomous systems we consider is the theory of attractors. Here we first recall the definition of a global attractor for a nonlinear semigroup $S(\cdot)$ (see [9] or [24]), and then discuss how this concept can be generalised to the attractor of a non-autonomous process $S(\cdot,\cdot)$.

If B and C are subsets of $\mathcal{Z}$, we say that the set B *attracts the set* C *under* $S(t)$ if $\mathrm{dist}(S(t)C, B) \to 0$ as $t \to \infty$, where $\mathrm{dist}(A,B) = \sup_{a\in A} \inf_{b\in B} |a-b|$.

A set $A \subset \mathcal{Z}$ is said to be *invariant* under $\{S(t) : t \geq 0\}$ if, for any $z \in A$, there is a complete orbit $\gamma(z)$ through z such that $\gamma(z) \subset A$ or equivalently if $S(t)A = A$ for any $t \geqslant 0$.

Definition 1.1. *A set $\mathcal{A} \subseteq \mathcal{Z}$ is said to be the* global attractor *for $S(\cdot)$ if it is*

(i) *compact,*
(ii) *invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and*
(iii) *it attracts bounded subsets B of $\mathcal{Z}$,*

$$\mathrm{dist}(S(t)B, \mathcal{A}) \to 0 \text{ as } t \to \infty.$$

The notion of a global attractor for a nonlinear process $S(\cdot, \cdot)$ requires much more care. Since any fixed set A will not, in general, be invariant in the above sense for a non-autonomous process, it is natural to define *invariance* in this context as follows:

- A family $\{A(t) \subset \mathcal{Z} : t \in [\sigma, \infty)\}$ is invariant under $S(\cdot, \cdot)$ if $S(t, \tau)A(\tau) = A(t)$ for all $t \geq \tau \geq \sigma$.

With this in mind one might think that a non-autonomous attractor should be defined as follows:

- A family $\{A(t) \subset \mathcal{Z} : t \in \mathbb{R}\}$ with $A(t)$ compact for all $t \in \mathbb{R}$ is a non-autonomous attractor if it is invariant under $S(\cdot, \cdot)$ and attracts bounded sets; that is, for each bounded set $B \subset \mathcal{Z}$ and $\tau \in \mathbb{R}$

$$\lim_{t \to \infty} \mathrm{dist}(S(t, \tau)B, A(t)) = 0.$$

Unfortunately such a definition is likely to be satisfied only in some very specific and restrictive situations (e.g. if $\{T(t, \tau) : t \geq \tau \in \mathbb{R}\}$ is uniformly asymptotically compact in the sense of [7]). Some very simple examples of systems that we would expect to possess a 'global non-autonomous attractor' will not have an attractor in the sense of this definition; this is essentially due to the fact that some of the forwards dynamics may be associated with solutions that blow up in finite backwards time (see Section 4.2).

Central to much of what follows is the concept of a globally-defined solution. In the autonomous case, a globally-defined solution (or simply a global solution) through z is a function $\xi : \mathbb{R} \to \mathcal{Z}$ such that $\xi(0) = z$ and for all $s \in \mathbb{R}$ and $t \geq 0$ we have $S(t)\xi(s) = \xi(t+s)$. In the autonomous case the attractor is exactly the union of all such orbits [24],

$$\mathcal{A} = \{z : \text{ there is a bounded global solution through } z\}. \tag{1.1}$$

In the non-autonomous case, the definition of an 'attractor' that has the same characterization as the union of all globally-defined bounded orbits,

$$\{A(t) : t \in \mathbb{R}\} = \{\xi(t) : \xi(\cdot) : \mathbb{R} \to \mathcal{Z} \text{ is bounded and } S(t, \tau)\xi(\tau) = \xi(t)\}, \tag{1.2}$$

is the pullback attractor:

Definition 1.2. *A family of compact sets $\{A(t) \subset \mathcal{Z} : t \in \mathbb{R}\}$ with $\overline{\cup_{t\in\mathbb{R}} A(t)}$ compact is a* pullback attractor *for $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ if it is invariant and attracts all bounded subsets of $\mathcal{Z}$ 'in the pullback sense', i.e.*

$$\lim_{\tau \to -\infty} \mathrm{dist}(S(t, \tau)B, A(t)) = 0, \ \forall t \in \mathbb{R}.$$

(See [7], where the sets $A(t)$ are referred to as *kernel sections*, and also [12, 22] who use the terminology 'pullback attractor'). It is shown in [7] that when a pullback attractor exists it is given by (1.2). Note that the requirement that $\overline{\cup_{t\in\mathbb{R}}A(t)}$ be compact is not a standard one in the literature on pullback attractors, and without it the definition still reduces to the familiar one in the autonomous case. Indeed, there are examples in which allowing $A(t)$ to be unbounded, particular as $t \to +\infty$, is a useful weakening of the definition. Nevertheless, the uniformity imposed here occurs in most interesting applications, and allows for stronger results, while ruling out some potentially pathological behaviour, e.g. unstable sets that do not belong to the attractor, see Theorem 5.2 in [14].)

For autonomous problems, it is clear that the concept of a pullback attractor coincides with the standard definition of the attractor, while the characterization in (1.2) shows that this notion is in some sense a 'natural' generalization. However, as is well-known and demonstrated here by the example presented in Section 4.2, the pullback attractor will not necessarily enjoy any kind of forward attraction. Except in very specific situations, for example when the non-autonomous nonlinear process is asymptotically autonomous backwards and forwards to the same nonlinear semigroup, the pullback behaviour and the forwards behaviour will not be related (see Theorems 4.2 and 4.5, and [6, 21, 15] for other specific cases).

Ideally, therefore, one would describe the pullback attractor of a non-autonomous system, and give some information on the limits of solutions as $t \to \infty$. We accomplish both aims in the particular class of systems that we consider here.

1.4. **Gradient systems and 'gradient-like' attractors.** Our result considers small non-autonomous perturbations of autonomous gradient systems. In order to make it clear where our work differs from previous results, we need to draw a distinction between gradient systems and systems with 'gradient-like' attractors.

If $T_0(\cdot)$ is a gradient nonlinear semigroup (i.e. $T_0(\cdot)$ has a Liapunov function, see Definition 2.4) that has a global attractor A_0 and a finite number of stationary solutions y_i^*, $1 \le i \le n$, then every solution converges to one of the equilibria as $t \to +\infty$, and every solution defined for all $t \le 0$ is also backwards asymptotic to one of the equilibria. This implies, in particular, that the attractor A_0 is the union of the unstable manifolds $W_0^{\rm u}(y_i^*)$ of the equilibria, i.e.

$$A_0 = \bigcup_{j=1}^{n} W^{\rm u}(y_j^*), \tag{1.3}$$

and so the structure of A_0 is completely understood. This is essentially the class of nonlinear semigroups in Banach spaces for which a detailed knowledge of the structure of the attractor is available. An attractor of the form (1.3) we term 'gradient-like'. Clearly the class of systems with gradient-like attractors is larger than those that are strictly gradient.

The argument that leads to our main result has two ingredients. We consider an underlying semigroup $T_0(\cdot)$, and a parametrized family $T_\eta(\cdot,\cdot)$ of non-autonomous processes that converge to $T_0(\cdot)$ (in a sense which will of course be made precise) as $\eta \to 0$. First, we assume that the equilibria of $T_0(\cdot)$ become hyperbolic global solutions for $T_\eta(\cdot,\cdot)$ for η sufficiently small, and that the corresponding stable and unstable manifolds change continuously (this is made precise in Section 2).

In this case, if one only assumes that the attractor of $T_0(\cdot)$ is 'gradient-like' and all the y_j^* are hyperbolic, then it is possible to show that the pullback attractors A_η of $T_\eta(\cdot,\cdot)$ behave continuously as $\eta \to 0$, i.e.

$$\sup_{t\in\mathbb{R}} \operatorname{dist}(A_\eta(t), A_0) \to 0 \qquad \text{as} \qquad \eta \to 0,$$

where $\operatorname{dist}(X, Y) = \max[\operatorname{dist}(X, Y), \operatorname{dist}(Y, X)]$, see [16, 5]. In fact this continuity result is proved by showing that the pullback attractors for the perturbed problem contain (possibly strictly) the union of the unstable manifolds of the global hyperbolic solutions, while the remaining part of the pullback attractor for the perturbed problem (if it exists) is small.

Our main result here is that under the additional assumption that the unperturbed problem is truly gradient, i.e. has a Liapunov function, then there is no 'remainder', and the pullback attractor has the same structure as the autonomous attractor, i.e.

$$A_\eta(t) = \bigcup_{j=1}^{n} W^{\mathrm{u}}(\xi_{j,\eta}(\cdot))(t),$$

where the $\xi_{j,\eta}(\cdot)$ are the hyperbolic global solutions corresponding to the hyperbolic equilibria y_j^* in the original problem. We also show that every solution converges to one of the $\xi_{j,\eta}(\cdot)$ as $t \to +\infty$. To obtain these results we make continual use of the Liapunov function for the unperturbed problem: the structure of the attractor for $T_\eta(\cdot,\cdot)$ cannot be deduced from the continuity of the attractor under perturbation.

Our results provide new classes of systems in which the exact structure of the attractor is known, even in the autonomous case. For example, if we consider an autonomous dynamical system that is gradient and perturb it in such a way that the perturbed dynamical system is still autonomous but no longer has a Liapunov function, the results in [5, 9] prove that the attractors behave continuously but do not ensure that the perturbed attractor is exactly the union of unstable manifolds of hyperbolic equilibria, which is what we are able to prove here. (Section 5 gives the striking example of a damped hyperbolic equation which is not gradient but whose attractor is nevertheless gradient-like.)

It is a natural question whether our results can be obtained for small perturbations of a larger class than autonomous gradient systems. One might hope to prove a similar result starting from a *"generalized gradient dynamical system"*, a reasonable definition of which is a dynamical system (autonomous or non-autonomous) that has a pullback attractor given as the union of the unstable manifolds of finitely many global hyperbolic solutions, and for which every solution is forwards asymptotic to one of the (finite) set of global hyperbolic solutions. However, our arguments are completely unable to treat this case, since we use the Liapunov function of the limiting system throughout our proof.

We like to think of the characterization result of this paper as midway between full structural stability (which one would expect to involve assumptions such as the transversality of stable and unstable manifolds) and the weaker property of continuity of attractors (as in [5, 16]) valid under less stringent conditions. We suspect that extending our results to treat generalized gradient systems will require techniques more akin to those involved in considerations of structural stability. For example, in a system whose vector field is periodic in time, a hypothesis such as the transversality of stable and unstable manifolds should lead to a similar characterization of the attractor and also guarantee the preservation of the connections between hyperbolic orbits, since if the associated Poincaré map is Morse-Smale one can apply the results due to Oliva (see Oliva [19] or Hale et al. [10]) to show that the system is topologically stable.

1.5. **Detailed summary of results.** We now specify the particular model that we will consider in detail, and give a formal summary of our main results. Our choice of model is motivated by the need to guarantee that the stable and unstable manifold structure near a hyperbolic equilibrium perturbs continuously. Such results were shown in [5] for a class of semilinear problems on a Banach space $\mathcal{Z}$, and it is these models that we consider in what follows. At the risk of labouring the point, our results could be stated and proved within a more abstract setting (an abstract process $T_\eta(\cdot,\cdot)$ that is a perturbation of a semigroup $T_0(\cdot)$ with the relevant additional properties), but here we choose to concentrate on this particular example for the sake of concreteness.

We will consider the semilinear autonomous problem

$$\dot{y} = \mathfrak{B}y + f_0(y) \quad \text{with} \quad y(\tau) = y_0 \in \mathcal{Z}, \tag{1.4}$$

and the non-autonomous family for $\eta \in (0,1]$

$$\dot{y} = \mathfrak{B}y + f_\eta(t,y) \quad \text{with} \quad y(\tau) = y_0 \in \mathcal{Z}, \tag{1.5}$$

where $\mathfrak{B} : D(\mathfrak{B}) \subset \mathcal{Z} \to \mathcal{Z}$ is the generator of a C^0-semigroup of bounded linear operators and, for $\eta \in [0,1]$, $f_\eta(t,\cdot)$ is a differentiable function which is Lipschitz continuous in bounded subsets of $\mathcal{Z}$ with Lipschitz constant independent of η and t. Assume that, for each $\tau \in \mathbb{R}$ and $y_0 \in \mathcal{Z}$, unique solutions of (1.4) and (1.5) exist for all $t \geq \tau$. Then the solution $t \to T_0(t-\tau)y_0$ of (1.4) defines a nonlinear semigroup on $\mathcal{Z}$, and the solution $t \mapsto T_\eta(t,\tau)y_0$ of (1.5) gives rise to a family of nonlinear processes on $\mathcal{Z}$.

Some authors have considered models with additional properties, e.g. Shen & Yi [23] have considered coefficients that are almost periodic, but we prefer to consider more general non-autonomous terms and, indeed, it seems that the extra properties of almost periodic equations would not help us in the direction we are pursuing here.

As remarked above, we choose this particular model because it is shown in [5] that if f_η is a C^1 perturbation of f_0 then to each hyperbolic equilibrium point of $T_0(\cdot)$ there corresponds a hyperbolic global solution $\xi^*_{i,\eta}(\cdot)$ of $T_\eta(\cdot,\cdot)$ and the corresponding stable and unstable manifolds behave continuously as $\eta \to 0$; these results are recalled in Section 2. Using these results and the assumption that $T_0(\cdot)$ is gradient with a finite number of equilibria y_i^*, all of which are hyperbolic, we show the following in our main theorem, Theorem 2.11.

- *Structure of the pullback attractors for the perturbed systems*

$$A_\eta(t) = \bigcup_{i=1}^{n} W^{\mathrm{u}}_\eta(\xi^*_{i,\eta})(t),$$

  for all $t \in \mathbb{R}$, where $W^{\mathrm{u}}_\eta(\xi^*_{i,\eta})(t)$ denotes the unstable manifold associated to the global hyperbolic solutions $\xi^*_{i,\eta}$ (these are shown in [5] to be given as a graph near each of the hyperbolic equilibrium points y_i^*).
- *Dimension of the pullback attractors for the perturbed systems*
  For each $t \in \mathbb{R}$,

$$\dim_H(A_\eta(t)) = \dim_H(A_0)$$

  and give an explicit expression for this dimension in (2.21).
- *Backwards and forwards limits of global solutions*
  For every bounded global solution of (1.5), $\xi_\eta(t)$, there are j,k with $1 \leq j \leq n$ and

$1 \le k \le n$ such that

$$\lim_{t\to\infty} \|\xi_\eta(t) - \xi^*_{j,\eta}(t)\|_{\mathcal{Z}} = 0 \text{ and } \lim_{t\to-\infty} \|\xi_\eta(t) - \xi^*_{k,\eta}(t)\|_{\mathcal{Z}} = 0.$$

- *Forwards limits of all solutions*
  For each $(\tau, y_0) \in \mathbb{R} \times \mathcal{Z}$ there is a $1 \le j \le n$ such that

$$\lim_{t\to\infty} \|T_\eta(t,\tau)y_0 - \xi^*_{j,\eta}(t)\|_{\mathcal{Z}} = 0.$$

In the second part (Section 4) of this paper we consider asymptotically autonomous dynamical systems in the case that the limiting system is gradient. Because asymptotically autonomous systems can be analysed by considering non-autonomous perturbations of an autonomous equation, we are able to take advantage of the above results to describe the structure of the attractors in this case. Moreover, we show that every solution converges to one of the hyperbolic global solutions of the non-autonomous problem; these are the true time-dependent (and invariant) attracting structures, rather than their asymptotic limits (which are invariant only for the limit system), cf. [3, 4]. We highlight the fact that if the backwards and forwards limit systems are different then, although both the forwards and pullback dynamics can be described in detail, they can be entirely unrelated.

Ideally, we would also like to characterize a forwards attractor for bounded sets (when possible), insisting on the requirement that this be invariant. However, there are non-trivial problems with defining such a forwards attractor, and these are also discussed in Section 4.

In Section 5 we present a number of examples that illustrate the broad applicability of our results, and finally we make some general comments and conjectures in Section 6.

2. Background Results and Statement of the Main Theorem

We start by describing some previous results that are central in the proof of our main theorem, namely results on the continuity of stable and unstable manifolds proved in [5], and classical results on the structure of attractors in gradient systems.

If $t \mapsto T_0(t-\tau)y_0$ denotes the solution of

$$\dot{y} = \mathfrak{B}y + f_0(y) \quad \text{with} \quad y(\tau) = y_0, \tag{2.1}$$

then

$$T_0(t-\tau)y_0 = e^{\mathfrak{B}(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} f_0(T_0(s-\tau)y_0)\, ds, \tag{2.2}$$

while if we denote by $t \mapsto T_\eta(t,\tau)y_0$ the solution of

$$\dot{y} = \mathfrak{B}y + f_\eta(t,y) \quad \text{with} \quad y(\tau) = y_0, \tag{2.3}$$

we have

$$T_\eta(t,\tau)y_0 = e^{\mathfrak{B}(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} f_\eta(s, T_\eta(s,\tau)y_0)\, ds. \tag{2.4}$$

The following result on the continuity of these solution operators as $\eta \to 0$ is easy to prove.

Theorem 2.1. *Assume that*

$$\lim_{\eta\to 0} \sup_{t\in\mathbb{R}} \sup_{z\in B(0,r)} \|f_\eta(t,z) - f_0(z)\|_{\mathcal{Z}} = 0, \quad \textit{for each } r > 0. \tag{2.5}$$

Then, for each $r > 0$ and $T > 0$,

$$\lim_{\eta\to 0} \sup\{\|T_\eta(t+\tau,\tau)z - T_0(t)z\|_{\mathcal{Z}}, \tau \in \mathbb{R},\ t \in [0,T] \ \textit{and} \ \|z\|_{\mathcal{Z}} \le r\} \to 0. \tag{2.6}$$

A solution of (2.1) is an equilibrium solution if it satisfies

$$\mathfrak{B}y + f_0(y) = 0. \tag{2.7}$$

Suppose that y_0^* is solution of (2.7). It follows that, if $\mathcal{A} = \mathfrak{B} + f_0'(y_0^*)$, then $\mathcal{A}$ generates a strongly continuous semigroup $\{e^{\mathcal{A}t} : t \geqslant 0\}$ of bounded linear operators.

Definition 2.2. *An equilibrium solution y_0^* to (2.1) is said to be hyperbolic if the following are satisfied:*

(1) *The spectrum of $\mathcal{A}$ does not intersect the imaginary axis and the set $\sigma^+ = \{\lambda \in \sigma(\mathcal{A}) : \mathrm{Re}\lambda > 0\}$ is compact.*
This allows us to choose a smooth closed simple curve γ in $\rho(\mathcal{A}) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ that is positively orientated and encloses σ^+; we can then define the projection

$$\mathcal{Q} = \mathcal{Q}(\sigma^+) = \frac{1}{2\pi \mathrm{i}} \int_\gamma (\lambda I - \mathcal{A})^{-1} \, d\lambda. \tag{2.8}$$

If $\mathcal{Z}^+ = \mathcal{Q}(\mathcal{Z})$, $\mathcal{Z}^- = (I - \mathcal{Q})(\mathcal{Z})$, and $\mathcal{A}^\pm = \mathcal{A}_{|_{\mathcal{Z}^\pm}}$, then $\mathcal{Z} = \mathcal{Z}^+ \oplus \mathcal{Z}^-$, $\mathcal{A}^-$ generates a strongly continuous semigroup on $\mathcal{Z}^-$ and $\mathcal{A}^+ \in L(\mathcal{Z}^+)$.

(2) *There are constants $\bar{M} \geq 1$ and $\beta > 0$ such that*

$$\begin{aligned} \|e^{\mathcal{A}^+ t}\|_{L(\mathcal{Z}^+)} &\leqslant \bar{M} e^{\beta t}, \quad t \leqslant 0, \\ \|e^{\mathcal{A}^- t}\|_{L(\mathcal{Z}^-)} &\leqslant \bar{M} e^{-\beta t}, \quad t \geqslant 0. \end{aligned} \tag{2.9}$$

The stable and unstable manifolds of an equilibrium y_0^*, $W^{\mathrm{s}}(y_0^*)$ and $W^{\mathrm{u}}(y_0^*)$ respectively, are defined as follows:

$$W^{\mathrm{s}}(y_0^*) = \{\, z \in \mathcal{Z} : \lim_{t \to +\infty} \|T_0(t)z - y_0^*\|_{\mathcal{Z}} = 0\}.$$

$$\begin{aligned} W^{\mathrm{u}}(y_0^*) = \{\, z \in \mathcal{Z} : &\text{ there is a backwards solution } y(t) \text{ of (2.1)} \\ &\text{satisfying } y(\tau) = z \text{ and such that } \lim_{t \to -\infty} \|y(t) - y_0^*\|_{\mathcal{Z}} = 0\}. \end{aligned}$$

The intersection of the unstable manifold with a neighbourhood of y_0^* is termed the local unstable manifold, which we write $W^{\mathrm{u}}_{\mathrm{loc}}(y_0^*)$. The existence of local stable and unstable manifolds as graphs near a hyperbolic equilibrium is well-known:

Theorem 2.3. *If y_0^* is a hyperbolic equilibrium then for suitably small $\epsilon > 0$ there are Lipschitz functions*

$$\begin{aligned} B(0, \epsilon) \ni w &\mapsto \Sigma^{*,u}(\mathcal{Q}w) \in (I - \mathcal{Q})\mathcal{Z} \\ B(0, \epsilon) \ni w &\mapsto \Sigma^{*,s}((I - \mathcal{Q})w) \in \mathcal{Q}\mathcal{Z} \end{aligned}$$

such that

$$\begin{aligned} W^{\mathrm{u}}_{\mathrm{loc}}(y_0^*) &= \{y_0^* + (\mathcal{Q}w, \Sigma^{*,u}(\mathcal{Q}w)), \|w\|_{\mathcal{Z}} \leq \epsilon\} \\ W^{\mathrm{s}}_{\mathrm{loc}}(y_0^*) &= \{y_0^* + (\Sigma^{*,s}((I - \mathcal{Q})w), (I - \mathcal{Q})w), \|w\|_{\mathcal{Z}} \leq \epsilon\}, \end{aligned}$$

where $\mathcal{Q}$ is the projection from (2.8).

Next we recall the definition of a gradient nonlinear semigroup. (Note that we have slightly strengthened the definition from that in Hale [9], since to ensure that $\xi(\cdot)$ is an equilibrium we only require $V(\xi(t))$ to be constant on a semi-infinite interval.)

Definition 2.4. *We say that a nonlinear semigroup $\{T_0(t) : t \geq 0\}$ is* gradient *if $\{T_0(t)z : t \geq 0\}$ is relatively compact for each $z \in \mathcal{Z}$ and there exists a continuous function $V : \mathcal{Z} \to \mathbb{R}$ such that*

- *$t \mapsto V(T_0(t)z) : [0,\infty) \to \mathcal{Z}$ is non-increasing for each $z \in \mathcal{Z}$.*
- *If $z \in \mathcal{Z}$ is such that there is a global solution $\xi(\cdot) : \mathbb{R} \to \mathcal{Z}$ through $\xi(0) = z$ and there exists a $t^* \in \mathbb{R}$ such that $V(\xi(t)) = V(z)$ for all $t \geq t^*$ or for all $t \leq t^*$, then z is a solution of* (2.7) *(and so in fact $V(\xi(t)) = V(z)$ for all $t \in \mathbb{R}$).*

The function $V : \mathcal{Z} \to \mathbb{R}$ is called a Liapunov function for $\{S(t) : t \geq 0\}$.

In a gradient system with a finite number of hyperbolic equilibria the attractor has a particularly simple structure, and all global solutions in it are both forwards and backwards asymptotic to an equilibrium, which we state formally in the following theorem (see [9]). Note that the assumptions in the theorem are satisfied (at least generically) for many examples that have a gradient structure.

Theorem 2.5. *If $T_0(\cdot)$ is a gradient system that has a global attractor A_0, $V : \mathcal{Z} \to \mathbb{R}$ is its Liapunov function and* (2.7) *has a finite number of solutions y_i^*, $1 \leq i \leq n$, then A_0 is given by*

$$A_0 = \bigcup_{i=1}^{n} W_0^{\mathrm{u}}(y_i^*), \tag{2.10}$$

i.e. the attractor is 'gradient-like'. Furthermore if we denote by $\{\mathfrak{n}_1, \cdots, \mathfrak{n}_p\}$ the set of all distinct values of $V(y_j^*)$, ordered so that $\mathfrak{n}_i < \mathfrak{n}_j$, $1 \leq i < j \leq p \leq n$, and define $\mathcal{E}_k = \{y_i^* \in \mathcal{E} : V(y_i^*) = \mathfrak{n}_k\}$, then if $y(\cdot) : \mathbb{R} \to \mathcal{Z}$ is a global solution for (2.1)*, there are k_1, k_2 with $1 \leq k_1 < k_2 \leq p$, $y_i^* \in \mathcal{E}_{k_1}$ and $y_j^* \in \mathcal{E}_{k_2}$, such that*

$$\lim_{t\to-\infty} y(t) = y_j^* \text{ and } \lim_{t\to+\infty} y(t) = y_i^*.$$

While a characterization of the pullback attractor for small non-autonomous perturbations of *finite-dimensional* gradient systems is given by Langa et al. [16], such a characterization is not currently available in the literature for any class of infinite-dimensional problems.

Our first task is to find a non-autonomous analogue of a hyperbolic equilibrium points. In [5] it is shown that near each of the hyperbolic equilibrium y_i^* there is a unique global solution $\xi_{i,\eta}^*$ which enjoys a hyperbolic structure. In order to be more precise we need the notion of an exponential dichotomy, which we now introduce.

Definition 2.6. *We say that a linear process $\{U(t,\tau) : t \geq \tau \in \mathbb{R}\}$ has an exponential dichotomy with exponent ω and constant M if there exists a family of projections $\{Q(t) : t \in \mathbb{R}\} \subset L(\mathcal{Z})$ such that*

(1) *$Q(t)U(t,s) = U(t,s)Q(s)$, for all $t \geq s$.*
(2) *The restriction $U(t,s)_{|R(Q(s))}$, $t \geq s$ is an isomorphism from $R(Q(s))$ into $R(Q(t))$; we denote its inverse by $U(s,t) : R(Q(t)) \to R(Q(s))$.*
(3) *There are constants $\omega > 0$ and $M \geq 1$ such that*

$$\begin{aligned} &\|U(t,s)(I - Q(s))\|_{L(\mathcal{Z})} \leq Me^{-\omega(t-s)} \quad t \geq s \\ &\|U(t,s)Q(s)\|_{L(\mathcal{Z})} \leq Me^{\omega(t-s)}, \quad t \leq s. \end{aligned} \tag{2.11}$$

Now we will define the analog of a hyperbolic equilibrium for non-autonomous problems (2.3). But first we need to introduce some more terminology. Consider the problem

$$\begin{aligned} &\dot{z} = \mathcal{A}z + B_\eta(t)z \\ &z(\tau) = z_0 \in \mathcal{Z}, \end{aligned} \tag{2.12}$$

where $\mathbb{R} \ni t \mapsto B_\eta(t) \in L(\mathcal{Z})$ is strongly continuous. It is well known that the problem (2.12) has a unique mild solution $U_\eta(t,\tau)z_0$ for each $z_0 \in \mathcal{Z}$ which satisfies

$$U_\eta(t,\tau)z_0 = e^{\mathcal{A}(t-\tau)}z_0 + \int_\tau^t e^{\mathcal{A}(t-s)}B_\eta(s)U_\eta(s,\tau)z_0\,ds. \tag{2.13}$$

The family $\{U_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ is a linear process. We say that (2.12) has an exponential dichotomy if $\{U_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ has an exponential dichotomy.

Now we are ready to define the analogue of a hyperbolic equilibrium for non-autonomous systems.

Definition 2.7. *Let $\xi_\eta^* : \mathbb{R} \to \mathcal{Z}$ be a global solution of* (2.3)*. We say that ξ_η^* is hyperbolic if*

$$\begin{aligned} \dot z &= \mathfrak{B}z + (f_\eta)_y(t,\xi_\eta^*(t))z \\ z(\tau) &= z_0 \in \mathcal{Z} \end{aligned}$$

has an exponential dichotomy. A global solution that has an exponential dichotomy will be called a global hyperbolic solution*.*

The following result can be adapted from Theorem 7.6.11 in [11].

Theorem 2.8. *Suppose that $\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\|B_\eta(t)\|_{L(\mathcal{Z})} = 0$ and that $\mathcal{A}$ is the generator of a C^0-semigroup such that* (2.9) *is satisfied for some $\beta > 0$ and $\bar M \geq 1$. Then, for each $M > \bar M$ and $\omega < \beta$, there is a $\eta_0 > 0$ such that, for all $\eta \leq \eta_0$,* (2.12) *has an exponential dichotomy with exponent ω and constant M.*

Definition 2.9. *The unstable manifold of a global hyperbolic solution ξ_η^* to* (2.3) *is the set*

$$\begin{aligned} W_\eta^{\mathrm{u}}(\xi_\eta^*) = \{\,(\tau,\zeta) \in \mathbb{R}\times\mathcal{Z} :\ &\text{there is a backwards solution } z(t,\tau,\zeta) \text{ of } (2.3) \\ &\text{satisfying } z(\tau,\tau,\zeta)=\zeta \text{ and such that } \lim_{t\to-\infty}\|z(t,\tau,\zeta)-\xi_\eta^*(t)\|_{\mathcal{Z}} = 0\}. \end{aligned}$$

The stable manifold of a hyperbolic solution ξ_η^* to (2.3) *is the set*

$$\begin{aligned} W_\eta^{\mathrm{s}}(\xi_\eta^*) = \{\,(\tau,\zeta) \in \mathbb{R}\times\mathcal{Z} :\ &\text{there is a forwards solution } z(t,\tau,\zeta) \text{ of } (2.3) \\ &\text{satisfying } z(\tau,\tau,\zeta)=\zeta \text{ and such that } \lim_{t\to+\infty}\|z(t,\tau,\zeta)-\xi_\eta^*(t)\|_{\mathcal{Z}} = 0\}. \end{aligned}$$

The intersection of the unstable (*stable*) *manifold with a neighbourhood of the curve $(\cdot,\xi(\cdot))$ in $\mathbb{R}\times\mathcal{Z}$ is called a local unstable* (*stable*) *manifold and is denoted by $W_{\eta,\mathrm{loc}}^{\mathrm{u}}$ ($W_{\eta,\mathrm{loc}}^{\mathrm{s}}$).*

Setting $z = y - y_i^*$, we rewrite equation (2.1) as

$$\begin{aligned} \dot z &= \mathcal{A}^i z + h_i(z) \\ z(\tau) &= z_0 = y_0 - y_i^*, \end{aligned} \tag{2.14}$$

where $\mathcal{A}^i = \mathfrak{B} + f_0'(y_i^*)$, $h_i(z) = f_0(y_i^* + z) - f_0(y_i^*) - f_0'(y_i^*)z$. Hence, 0 is an equilibrium solution for (2.14) and $h_i(0) = 0$, $h_i'(0) = 0 \in L(\mathcal{Z})$.

If $\xi_{i,\eta}^*$ is a a global hyperbolic solution of (2.3), proceeding as in the autonomous case we change variables $z(t) = y - \xi_{i,\eta}^*(t)$ in (2.3) and rewrite (2.3) as

$$\begin{aligned} \dot z &= (\mathcal{A}^i + B_\eta^i(t))z + h_\eta^i(t,z) \\ z(\tau) &= z_0 \end{aligned} \tag{2.15}$$

where $h^i_\eta(t,z) = f_\eta(t,\xi^*_{i,\eta}(t)+z) - f_\eta(t,\xi^*_{i,\eta}(t)) - (f_\eta)_y(t,\xi^*_{i,\eta}(t))z$ and $B^i_\eta(t) = (f_\eta)_y(t,\xi^*_{i,\eta}(t)) - f_0'(y^*_i)$. Hence, 0 is a globally defined bounded solution for (2.15) and $h^i_\eta(t,0)=0$, $(h^i_\eta)_z(t,0) = 0 \in L(\mathcal{Z})$.

The following proposition summarizes the main results proved in [5] which will be used in the proof of our main theorem.

Proposition 2.10. *Let $\eta \in (0,1]$, $f_\eta : \mathbb{R}\times\mathcal{Z} \to \mathcal{Z}$ differentiable. Consider the initial value problem* (2.3)*. Assume that all solutions of* (2.7) *are hyperbolic equilibrium solutions for* (2.1) *and that* (2.1) *has a global attractor A_0. Assume that, for any $r>0$,*

$$\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\sup_{y\in B(0,r)}\left\{\|f_\eta(t,y)-f_0(y)\|_{\mathcal{Z}} + \|(f_\eta)_y(t,y)-f_0'(y)\|_{L(\mathcal{Z})}\right\} = 0. \tag{2.16}$$

Under these assumptions the following hold:

(1) *If $\mathcal{A}_i = \mathfrak{B} + f_0'(y^*_i)$, the spectrum of $\mathcal{A}_i$ does not intersect the imaginary axis and the set $\sigma^+_i = \{\lambda\in\sigma(\mathcal{A}_i) : \mathrm{Re}\lambda > 0\}$ is compact. If γ_i is a smooth curve in $\rho(\mathcal{A}_i)\cap\{\lambda\in\mathbb{C} : \mathrm{Re}\lambda>0\}$ orientated anti-clockwise and enclosing σ^+_i, define*

$$\mathcal{Q}_i = \mathcal{Q}_i(\sigma^+_i) = \frac{1}{2\pi \mathrm{i}}\int_{\gamma_i}(\lambda I - \mathcal{A}_i)^{-1}\,d\lambda; \tag{2.17}$$

then there are constants $\bar{M}_i \geq 1$ and $\beta_i > 0$ such that

$$\|e^{\mathcal{A}^+_i t}\|_{L(\mathcal{Z}^+_i)} \leqslant \bar{M}_i e^{\beta_i t}, \quad t\leqslant 0,$$
$$\|e^{\mathcal{A}^-_i t}\|_{L(\mathcal{Z}^-_i)} \leqslant \bar{M}_i e^{-\beta_i t}, \quad t\geqslant 0,$$

where $\mathcal{Z}^+_i = R(\mathcal{Q}_i)$, $\mathcal{Z}^-_i = R(I-\mathcal{Q}_i)$, $\mathcal{A}^\pm_i = \mathcal{A}_{|_{\mathcal{Z}^\pm_i}}$, $\mathcal{A}^+_i \in L(Z^+_i)$.

(2) *For each η sufficiently small, there are globally defined solutions of* (2.15) *$\xi^*_{i,\eta} : \mathbb{R}\to\mathcal{Z}$ with $\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\|\xi^*_{i,\eta}(t) - y^*_i\|_{L(\mathcal{Z})} = 0$ and such that*

$$\dot{z} = \mathfrak{B}z + (f_\eta)_y(t,\xi^*_{i,\eta}(t))z \tag{2.18}$$

has an exponential dichotomy; that is, there is a family of projections $\{Q^i_\eta(t) :\ t\in\mathbb{R}\}$ such that the conditions in Definition 2.7 are satisfied, where $U^i_\eta(t,\tau)$ is the solution operator associated to (2.18)*.*

(3) *For any $\tau\in\mathbb{R}$*

$$\lim_{\eta\to 0}\sup_{t\geq\tau}\|U^i_\eta(t,\tau)(I-Q^i_\eta(\tau)) - e^{\mathcal{A}_i(t-\tau)}(I-\mathcal{Q}_i)\|_{L(\mathcal{Z})} \to 0,$$

$$\lim_{\eta\to 0}\sup_{t\leq\tau}\|U^i_\eta(t,\tau)Q^i_\eta(\tau) - e^{\mathcal{A}_i(t-\tau)}\mathcal{Q}_i\|_{L(\mathcal{Z})} \to 0.$$

Furthermore, for any $T>0$,

$$\lim_{\eta\to 0}\sup_{|t-\tau|\leq T}\|U^i_\eta(t,\tau) - e^{\mathcal{A}_i(t-\tau)}\|_{L(\mathcal{Z})} \to 0.$$

(4) *The projections $Q^i_\eta(t)$ and $\mathcal{Q}_i$ satisfy*

$$\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\|Q^i_\eta(t) - \mathcal{Q}_i\|_{L(\mathcal{Z})} = 0.$$

(5) *For $\epsilon > 0$ sufficiently small there are functions*

$$\mathbb{R} \times B(0,\epsilon) \ni (\tau, w) \mapsto \Sigma^{*,u}_{i,\eta}(\tau, Q^i_\eta(\tau)w) \in (I - Q^i_\eta(\tau))\mathcal{Z}$$

$$\mathbb{R} \times B(0,\epsilon) \ni (\tau, w) \mapsto \Sigma^{*,s}_{i,\eta}(\tau, (I - Q^i_\eta(\tau))w) \in Q^i_\eta(\tau)\mathcal{Z}$$

such that

$$W^{\rm u}_{\eta,{\rm loc}}(\xi^*_{i,\eta}) = \{(\tau, \xi^*_{i,\eta} + w) : w = (Q^i_\eta(\tau)w, \Sigma^{*,u}_{i,\eta}(\tau, Q^i_\eta(\tau)w)), \tau \in \mathbb{R}, \|w\|_{\mathcal{Z}} \le \epsilon\}$$

$$W^{\rm s}_{\eta,{\rm loc}}((\xi^*_{i,\eta})) = \{(\tau, \xi^*_{i,\eta} + w) : w = (\Sigma^{*,s}_{i,\eta}(\tau, (I - Q^i_\eta(\tau))w), (I - Q^i_\eta(\tau))w), \tau \in \mathbb{R}, \|w\|_{\mathcal{Z}} \le \epsilon\}.$$

(6) *Finally, the unstable and stable manifolds behave continuously at $\eta = 0$ in the sense that*

$$\sup_{t\le\tau} \sup_{\|w\|_{\mathcal{Z}}\le\epsilon} \left\{\|Q^i_\eta(t)w - \mathcal{Q}_i w\|_{\mathcal{Z}} + \|\Sigma^{*,u}_{i,\eta}(t, Q^i_\eta(t)w) - \Sigma^{*,u}_0(\mathcal{Q}_i w)\|_{\mathcal{Z}}\right\} \xrightarrow{\eta\to 0} 0.$$

$$\sup_{t\ge\tau} \sup_{\|w\|_{\mathcal{Z}}\le\epsilon} \left\{\|Q^i_\eta(t)w - \mathcal{Q}_i w\|_{\mathcal{Z}} + \|\Sigma^{*,s}_{i,\eta}(t, (I - Q^i_\eta(t))w) - \Sigma^{*,s}_0((I - \mathcal{Q}_i)w)\|_{\mathcal{Z}}\right\} \xrightarrow{\eta\to 0} 0.$$

In what follows we suppose the following:

i) For each $\eta \in (0, 1]$ there exists a pullback attractor $\{A_\eta(t)\}_{t\in\mathbb{R}}$ associated to (2.3).
ii) There exist η_0 and a compact attracting set $K \subset \mathcal{Z}$ such that, for all $B \subset \mathcal{Z}$ bounded

$$\lim_{t\to\infty} \sup_{\tau\in\mathbb{R}} \sup_{\eta\le\eta_0} {\rm dist}(T_\eta(t,\tau)B, K) = 0. \tag{2.19}$$

In particular, this implies (see [7]) that

$$\bigcup_{\eta\le\eta_0} \bigcup_{t\in\mathbb{R}} A_\eta(t) \subset K. \tag{2.20}$$

We are now ready to state our main theorem.

Theorem 2.11. *Let $\eta \in [0, 1]$, $f_\eta : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ be differentiable. Consider the initial value problem (2.3). Assume that (2.1) is a gradient system, that all solutions of (2.7) are hyperbolic equilibrium solutions for (2.1), and that (2.1) has a global attractor A_0 (which is consequently given by (2.10)). Assume in addition that (2.16) is satisfied for any $r > 0$.*

(1) *If we write*

$$W^{\rm u}_\eta(\xi^*_{i,\eta})(\tau) = \{\,\zeta \in \mathcal{Z} : \text{ there is a backwards solution } z(t,\tau,\zeta) \text{ of } (2.3)$$
$$\text{satisfying } z(\tau,\tau,\zeta) = \zeta \text{ and such that } \lim_{t\to-\infty} \|z(t,\tau,\zeta) - \xi^*_{i,\eta}(t)\|_{\mathcal{Z}} = 0\},$$

then the attractor $\{A_\eta(\tau) : \tau \in \mathbb{R}\}$ of (2.3) is given by

$$A_\eta(\tau) = \bigcup_{i=1}^n W^{\rm u}_\eta(\xi^*_{i,\eta})(\tau).$$

As a consequence for each $\tau \in \mathbb{R}$ we have

$$\dim_H(A_\eta(\tau)) = \dim_H(A_0) = \max_{j=1,\dots,n} {\rm rank}(\mathcal{Q}_j), \tag{2.21}$$

where $\mathcal{Q}_j$ is the projection defined in (2.17).

(2) *For each globally defined bounded solution* $\xi_\eta(\cdot)$ *of* (2.3) *and* $\eta \leq \eta_0$, *there are* j, k *with* $1 \leq j \leq k \leq n$ *such that*

$$\lim_{t\to\infty} \|\xi_\eta(t) - \xi^*_{j,\eta}(t)\|_{\mathcal{Z}} = 0 \text{ and } \lim_{t\to-\infty} \|\xi_\eta(t) - \xi^*_{k,\eta}(t)\|_{\mathcal{Z}} = 0. \tag{2.22}$$

(3) *For each* $(\tau, y_0) \in \mathbb{R} \times \mathcal{Z}$ *there is a* j *with* $1 \leq j \leq n$ *such that*

$$\lim_{t\to\infty} \|T_\eta(t,\tau)y_0 - \xi^*_{j,\eta}(t)\|_{\mathcal{Z}} = 0.$$

In Section 4.1 we use this theorem to give a characterization of the pullback attractors in the case of asymptotically autonomous problems at $-\infty$ and in Section 4.2 we define forwards invariant attractors and give a characterization of them for the case of asymptotically autonomous problems at $+\infty$ (Section 4.3).

Finally, we note that the characterization result (the structure of the attractor in part (1) of the above theorem) does not rely on the dynamics of the equation on any semi-infinite interval $[\tau, \infty)$. A similar result therefore holds for 'pullback attractors' for which we only assume that

$$\bigcup_{-\infty}^{\tau} A_0(t)$$

is compact (cf. remarks after Definition 1.2).

3. Proof of Theorem 2.11

Before we can start the proof of Theorem 2.11 we need the following very important lemma.

Lemma 3.1. *Let* η_k *be a sequence of positive numbers such that* $\eta_k \to 0$ *as* $k \to \infty$. *Assume that there is a sequence* $\xi_{\eta_k}(\cdot) : \mathbb{R} \to \mathcal{Z}$ *of solutions* (2.3) *such that* $\overline{\cup_{k\in\mathbb{N}} \cup_{t\in\mathbb{R}} \xi_{\eta_k}(t)}$ *is compact. Then, for any sequence* $\{t_k\}$ *in* $\mathbb{R}$, *there is a subsequence which we again denote by* ξ_{η_k} *and a globally defined bounded solution* $y(\cdot)$ *of* (2.1) *such that*

$$\lim_{k\to\infty} \xi_{\eta_k}(t + t_k) \to y(t) \tag{3.1}$$

uniformly for t *in compact subsets of* $\mathbb{R}$.

Proof: Since $\overline{\cup_{k\in\mathbb{N}} \cup_{t\in\mathbb{R}} \xi_{\eta_k}(t)}$ is compact, there is a subsequence which we again denote by η_k and $y_0 \in \mathcal{Z}$ such that $\xi_{\eta_k}(t_k) \to y_0$. Let $y(\cdot) : [0, \infty) \to \mathcal{Z}$ be the solution of (2.1) such that $y(0) = y_0$. Of course, this solution is bounded. If $t > t_k$ it follows from the continuity of the nonlinear process $T_{\eta_k}(t + t_k, t_k)$, uniformly for $t_k \in \mathbb{R}$ and for t in compact subsets of $[0, \infty)$ (equation (2.6)), that

$$\xi_{\eta_k}(t + t_k) = T_{\eta_k}(t + t_k, t_k)\xi_{\eta_k}(t_k) \to T_0(t)y_0 = y(t)$$

uniformly for t in compact subsets of $[0, \infty)$. Proceeding similarly, $\xi_{\eta_k}(t_k - 1)$ has a convergent subsequence $\xi_{\eta_k^1}(t_k - 1)$ with limit y_{-1}. Defining $y(t) := T_0(t + 1)y_{-1}$, $t \in [-1, \infty)$, we have that $y(0) = T_0(t_k - (t_k - 1))y_{-1} = \lim_{k\to\infty} T_{\eta_k^1}(t_k, t_k - 1)\xi_{\eta_k^1}(t_k - 1) = \lim_{k\to\infty} \xi_{\eta_k^1}(t_k) = y_0$ and $y(t) = T_0(t + 1)y_{-1}$ for all $t \geq -1$. From this we have that $y(\cdot) : [-1, \infty) \to \mathcal{Z}$ is a solution of (2.1) with $y(-1) = y_{-1}$, $y(0) = y_0$ and

$$\xi_{\eta_k^1}(t + t_k) = T_{\eta_k}(t + t_k, t_k - 1)\xi_{\eta_k^1}(t_k - 1) \to T_0(t + 1)y_{-1} = y(t)$$

uniformly for t in compact subsets of $[-1,\infty)$. Suppose that:

- We have obtained subsequences $\{\xi_{\eta_k^i}\}_{n=1}^\infty$ for $1\le i\le m-1$ with the property that $\{\eta_k^i\}_{n=1}^\infty$ is a subsequence of $\{\eta_k^{i-1}\}_{n=1}^\infty$ and such that $\xi_{\eta_k^i}(t_k-i)\to y_{-i}$, $1\le i\le m-1$.
- We have defined $y(t)$ by $\lim_{n\to\infty}\xi_{\eta_k^i}(t+t_k)=y(t)$, in $[-i,-i+1]$ and, consequently, $y(t):[-i,\infty)\to\mathcal{Z}$ is a solution of (2.1) with $y(-j)=y_{-j}$, $0\le j\le i$ and $\xi_{\eta_k^i}(t+t_k)$ converges to $y(t)$ uniformly for t in compact subsets of $[-i,\infty)$, $1\le i\le m-1$.

Now construct $\{\eta_k^m\}_{n=1}^\infty$ a subsequence of $\{\eta_k^{m-1}\}_{n=1}^\infty$ such that $\xi_{\eta_k^m}(t_k-m)$ is convergent. Let y_{-m} be its limit and define $y(t)=T_0(t+m)y_{-m}$ for $t\in[-m,-m+1]$. Then $y(t)$ is a solution of (2.1) with $y(-i)=y_{-i}$, $0\le i\le m$ and $\xi_{\eta_k^m}(t+t_k)$ converges to $y(t)$ uniformly for t in compact subsets of $[-m,\infty)$.

With this we have constructed a sequence $\{\xi_{\eta_k^k}\}_{k=1}^\infty$ and a solution $y(\cdot):\mathbb{R}\to\mathcal{Z}$ of (2.1) with $y(-i)=y_{-i}$ for all $i\in\mathbb{N}$ and such that $\xi_{\eta_k^k}(t+t_k)\to y(t)$ uniformly for t in compact subsets of $\mathbb{R}$. This concludes the proof.□

The following lemma is taken from [9] and its proof is added here for completeness only.

Lemma 3.2. *Assume that $\{T_0(t):t\ge 0\}$ is a gradient nonlinear semigroup which has a global attractor A_0 and such that (2.7) has a finite number of solutions $\mathcal{E}=\{y_i^*:1\le i\le n\}$, all of them hyperbolic. Let $V:\mathcal{Z}\to\mathbb{R}$ be the Liapunov function associated with $\{T_0(t):t\ge 0\}$ and $V(\mathcal{E})=\{\mathfrak{n}_1,\cdots,\mathfrak{n}_p\}$ with $\mathfrak{n}_i<\mathfrak{n}_{i+1}$, $1\le i\le p-1$.*

If $1\le j\le p$ and $\mathfrak{n}_{j-1}<r<\mathfrak{n}_j$, then $\mathcal{Z}_r^j=\{z\in\mathcal{Z}:V(z)\le r\}$ is positively invariant under $\{T_0(t):t\ge 0\}$ and $\{T_{0,r}^j(t):t\ge 0\}$, the restriction of $\{T_0(t):t\ge 0\}$ to $\mathcal{Z}_r^j$, has a global attractor A_0^j given by

$$A_0^j=\cup\{W^{\mathrm{u}}(y_\ell^*):V(y_\ell^*)\le\mathfrak{n}_{j-1}\}.$$

In particular, $V(z)\le\mathfrak{n}_{j-1}$ for all $z\in A_0^j$.

Proof: It is clear from the definition of the Liapunov function that $\mathcal{Z}_r^j$ is invariant under $\{T_0(t):t\ge 0\}$. To prove the existence of an attractor for $\{T_{0,r}^j(t):t\ge 0\}$ we note that it inherits from $\{T_0(t):t\ge 0\}$ the properties required to obtain the existence of a global attractor; namely, orbits of bounded subsets of $\mathcal{Z}_r$ are bounded, $\{T_{0,r}^j(t):t\ge 0\}$ is bounded dissipative and $\{T_{0,r}^j(t):t\ge 0\}$ is asymptotically compact. Hence, $\{T_{0,r}^j(t):t\ge 0\}$ has a global attractor A_0^j. The restriction V_r^j of V to $\mathcal{Z}_r^j$ is a Liapunov function for $\{T_{0,r}^j(t):t\ge 0\}$ and the characterization of A_0^j follows. The last statement is an immediate consequence of the characterization of A_0^j.□

Proposition 3.3. *Assume that the hypotheses of Theorem 2.11 are satisfied. Let $V:\mathcal{Z}\to\mathbb{R}$ be the Liapunov function associated with the nonlinear semigroup $\{T_0(t):t\ge 0\}$, let $\mathcal{E}=\{y_1^*,\cdots,y_n^*\}$ and $V(\mathcal{E})=\{\mathfrak{n}_1,\cdots,\mathfrak{n}_p\}$ with $\mathfrak{n}_i<\mathfrak{n}_j$, $i<j$. Set $\nu_0=\frac{1}{2}\min\{\mathfrak{n}_k-\mathfrak{n}_{k-1}:2\le k\le p\}$ and let*

$$\mathcal{O}_\nu^j:=V^{-1}(\mathfrak{n}_j-\nu,\mathfrak{n}_j+\nu),\ 1\le j\le p,\ \nu\le\nu_0.$$

Then, for each $\nu\le\nu_0$, there exists an $\eta_\nu>0$ such that, for each globally defined bounded solution $\xi_\eta(\cdot)$ of (2.3) with $\eta\le\eta_\nu$, there are j,k with $1\le j<k\le n$ and $t^*=t^*(\xi_\eta(\cdot))>0$ such that

$$\xi_\eta(t)\in\mathcal{O}_\nu^j,\quad \textit{for all } t\ge t^* \tag{3.2}$$

and

$$\xi_\eta(t) \in \mathcal{O}_\nu^k, \quad \textit{for all } t \le -t^*. \tag{3.3}$$

(Note that while each $\mathcal{O}_\nu^j$ is a neighbourhood of the equilibrium y_j^*, it is not necessarily the case that $\mathcal{O}_\nu^j \to y_j^*$ as $\nu \to 0$.)

Proof: Both (3.2) and (3.3) will be proved by contradiction. We present a complete proof of (3.2) and since the proof of (3.3) is similar we present an abridged version of the proof highlighting the main differences.

Proof of (3.2)*:* If (3.2) does not hold then there exists a sequence $\eta_k \to 0$ and corresponding bounded solutions $\xi_{\eta_k}(\cdot)$ of (2.3) (with $\eta = \eta_k$) such that

$$\text{for any } t \in \mathbb{R}, \text{ there is a } \tau > t \text{ such that } \xi_{\eta_k}(\tau) \notin \mathcal{O}_\nu(\mathcal{E}) := \cup_{j=1}^p \mathcal{O}_\nu^j. \tag{3.4}$$

We deduce a contradiction using Lemma 3.1 and the fact that (2.1) is gradient. Before giving all the details, we briefly summarize the argument.

The uniform convergence of $T_\eta(t,s)$ to $T_0(t-s)$ (via Lemma 3.1) together with the fact that every solution in A_0 must tend to a single point of $\mathcal{E}$ guarantees that for some t_k we must have $\xi_{\eta_k}(t_k) \in \mathcal{O}_\nu^j$. While our hypothesis means that $\xi_{\eta_k}(\cdot)$ must leave $\mathcal{O}_\nu^j$, we can repeat the above argument to deduce that $\xi_{\eta_k}(\cdot)$ must then enter $\mathcal{O}_\nu^l$ for some l. By showing that $V(y_l^*) < V(y_j^*)$ we deduce a contradiction, since this process must terminate.

Now we formalize this procedure. Choose ν and ν' with $\nu_0 > \nu' > \nu > 0$. The first stage of the argument is to move every solution into $\mathcal{O}_\nu^j$ for some $1 \le j \le n$. Using our hypothesis, there exists a sequence t_k^1 such that $\xi_{\eta_k}(t_k^1) \notin \mathcal{O}_\nu(\mathcal{E})$. It follows from Lemma 3.1 that there is a subsequence, which we again denote by ξ_{η_k}, such that

$$\lim_{k\to\infty} \xi_{\eta_k}(t + t_k^1) \to y^1(t)$$

uniformly for t in compact subsets of $\mathbb{R}$, where $y^1(\cdot)$ is a solution of (2.1). Since $y^1(\cdot)$ must enter $\mathcal{O}_{\frac{\nu}{2}}(\mathcal{E})$ there is a $T > 0$ such that $\xi_{\eta_k}(t_k^1 + T) \in \mathcal{O}_\nu^j$ for some $1 \le j \le n$. We therefore set $t_k = t_k^1 + T$ and begin our argument with a sequence of solutions $\xi_{\eta_k}(t)$ of (2.3) (with $\eta = \eta_k$) and times t_k such that

$$\xi_{\eta_k}(t_k) \in \mathcal{O}_\nu^j. \tag{3.5}$$

However, since by assumption $\xi_{\eta_k}(\cdot)$ does not stay in $\mathcal{O}_\nu^j$, it must leave $\mathcal{O}_\nu^j$, and so there is a sequence $t_k^2 > t_k$ such that

$$\xi_{\eta_k}(t_k^2) \in \mathcal{O}_{\nu'}^j \backslash \mathcal{O}_\nu^j; \tag{3.6}$$

that is,

$$V(\xi_{\eta_k}(t_k^2)) \in (\mathfrak{n}_j - \nu', \mathfrak{n}_j - \nu] \cup [\mathfrak{n}_j + \nu, \mathfrak{n}_j + \nu').$$

We now embark on a case-by-case analysis. In each case we show that we can restart from (3.5) with j replaced by l for some $l < j$. Given that there are only a finite number of distinct values of the Liapunov function at the equilibria, this produces a contradiction.

Case (a). For infinitely many k we have $V(\xi_{\eta_k}(t_k^2)) \le \mathfrak{n}_j - \nu$.

We can use Lemma 3.1 to obtain a subsequence, which we again denote by $\xi_{\eta_k}(\cdot)$, such that

$$\lim_{k\to\infty} \xi_{\eta_k}(t + t_k^2) \to y^2(t)$$

uniformly for t in compact subsets of $\mathbb{R}$. It follows that $V(y^2(0)) \le \mathfrak{n}_j - \nu$, and since $V(y(\cdot))$ is non-increasing and $y^2(\cdot)$ must enter $\mathcal{O}_\nu(\mathcal{E})$ as t increases, it follows using the uniform convergence of T_η to T_0 that there is a fixed $T_2 > 0$ such that $\xi_{\eta_k}(t_k^2 + T_2) \in \mathcal{O}_\nu^\ell$ for some $\ell < j$.

So we can restart from (3.5) but with $\mathcal{O}_\nu^j$ replaced by $\mathcal{O}_\nu^\ell$ for some $\ell < j$.

Case (*b*). *There exists a* $k_\nu \in \mathbb{N}$ *such that for all* $k \geq k_\nu$,

$$\mathfrak{n}_j + \nu \leq V(\xi_{\eta_k}(t_k^2)) < \mathfrak{n}_j + \nu'.$$

Note first that we may assume that

$$V(\xi_{\eta_k}(t)) > \mathfrak{n}_j - \nu \qquad \forall\, t \geq t_k^2,\ k \in \mathbb{N} \tag{3.7}$$

for otherwise, we must have

$$V(\xi_{\eta_k}(t_k^3)) \leq \mathfrak{n}_j - \nu$$

for some $t_k^3 > t_k^2$ and for infinitely many values k. In this case would can return to case (a), but with t_k^2 replaced by t_k^3.

Now set

$$B = \{\xi_{\eta_k}(\theta) : V(\xi_{\eta_k}(\theta)) \leq \mathfrak{n}_j + \nu',\ \text{for some } \theta \in \mathbb{R} \text{ and some } k \in \mathbb{N}\}.$$

Using Lemma 3.2, we can find $t_\nu^* > 0$ such that

$$\sup\{V(T_0(t)B)\} < \mathfrak{n}_j + \frac{\nu}{2} \qquad \forall\, t \geq t_\nu^*. \tag{3.8}$$

Now use the continuity of V and the uniform convergence of T_η to T_0 to choose k_ν sufficiently large that

$$\sup\{V(T_{\eta_k}(t+s,t)B)\} < \mathfrak{n}_j + \nu \qquad \forall\, s \in [t_\nu^*, 2t_\nu^*] \qquad \forall\, k \geq k_\nu. \tag{3.9}$$

It follows by induction that if $\xi_{\eta_k}(t_k^2 + t) \in B$ then $\xi_{\eta_k}(t_k^2 + t + \tau) \in B$ for all $\tau \geq t_\nu^*$, and hence that for all $k \geq k_\nu$ we must have

$$V(\xi_{\eta_k}(t_k^2 + t)) < \mathfrak{n}_j + \nu \qquad \forall\, t \geq t_\nu^*. \tag{3.10}$$

Now, (3.10) together with (3.7) contradicts our assumption in (3.4).

It follows, therefore, that case (b) is impossible and that case (a) must always occur. But case (a) can only occur a finite number of times, and so we obtain a contradiction and (3.2) must hold.

Proof of (3.3)*:* The argument to show that every bounded global solution must end backwards in $\mathcal{O}_\nu(\mathcal{E})$ is similar to the forwards case. We start from the assumption that there exists a sequence $\eta_k \to 0$ and corresponding bounded solutions $\xi_{\eta_k}(\cdot)$ of (2.3) (with $\eta = \eta_k$) such that

$$\text{for any } t \in \mathbb{R},\ \text{there is a } \tau < t \text{ such that } \xi_{\eta_k}(\tau) \notin \mathcal{O}_\nu(\mathcal{E}). \tag{3.11}$$

Case (*a*). *For infinitely many* k *we have* $V(\xi_{\eta_k}(t_k^2)) \geq \mathfrak{n}_j + \nu$ is almost identical, except for the obvious changes and its proof will be omitted.

Case (*b*). *There exists a* k_ν *such that for all* $k \geq k_\nu$,

$$\mathfrak{n}_j - \nu' < V(\xi_{\eta_k}(t_k^2)) < \mathfrak{n}_j - \nu.$$

We may assume that

$$V(\xi_{\eta_k}(t)) \leq \mathfrak{n}_j + \nu \qquad \text{for all} \qquad t \leq t_k^2,\ k \in \mathbb{N}, \tag{3.12}$$

for otherwise, using the continuity of V, we must have

$$V(\xi_{\eta_k}(t_k^3)) \geq \mathfrak{n}_j + \nu$$

for some $t_k^3 < t_k^2$ and for infinitely many values of k. Hence, we can return to case (a) with t_k^2 replaced by t_k^3. Since case (a) can only be repeated a finite number of times, we eventually find ourselves in case (b) with (3.12) valid.

In this situation we choose ν'' with $\nu_0 > \nu'' > \nu' > \nu$ and set

$$B = \{\xi_{\eta_k}(\theta) : V(\xi_{\eta_k}(\theta)) \leq \mathfrak{n}_j - \nu, \text{ for some } \theta \in \mathbb{R} \text{ and some } k \in \mathbb{N}\}.$$

Using Lemma 3.2, we can find $t_\nu^* > 0$ such that

$$\sup\{V(T_0(t)B)\} < \mathfrak{n}_j - \nu'' \qquad \forall\, t \geq t_\nu^*.$$

Now use the continuity of V and the uniform convergence of T_η to T_0 to choose k_ν sufficiently large that

$$\sup\{V(T_{\eta_k}(t, t-s)B)\} < \mathfrak{n}_j - \nu' \qquad \forall\, s \in [t_\nu^*, 2t_\nu^*] \qquad \forall\, k \geq k_\nu. \tag{3.13}$$

It follows by induction that if $\xi_{\eta_k}(t_k^2 - t) \in B$ then $\xi_{\eta_k}(t_k^2 - t + \tau) \in B$ for all $\tau \geq t_\nu^*$.

We now claim that for all $k \geq k_\nu$ we must have

$$V(\xi_{\eta_k}(t_k^2 - t)) > \mathfrak{n}_j - \nu \qquad \forall\, t \geq t_\nu^*.$$

Indeed, suppose not. Then it follows that for some $t \geq t_\nu^*$ that

$$V(\xi_{\eta_k}(t_k^2 - t)) \leq \mathfrak{n}_j - \nu,$$

i.e. $\xi_{\eta_k}(t_k^2 - t) \in B$. But then (3.13) shows that $V(\xi_{\eta_k}(t_k^2)) < \mathfrak{n}_j - \nu'$, a contradiction.

It follows that, there exist k_ν and t_ν^* such that

$$\mathfrak{n}_j - \nu < V(\xi_{\eta_k}(t_k^2 - t)) < \mathfrak{n}_j + \nu \qquad \forall\, k \geq k_\nu,\ t \geq t_\nu^*.$$

As before, this contradicts our initial assumption (3.11), and the argument is concluded as before.□

We are now ready to prove the main result of this paper.

Proof of Theorem 2.11: The proof of (1) follows from (2) and the proof of (3) follows in the same way as the forwards argument in (2).

To prove (2), let $\epsilon > 0$ and $\eta_0 > 0$ be such that in $\mathcal{B}_\epsilon^j = B(y_j^*, \epsilon)$ there is a unique global hyperbolic solution $\xi_{j,\eta}^*$ with the stable and unstable manifolds given as graphs for all $0 \leq \eta \leq \eta_0$. Hence, if a global solution $\xi_\eta(\cdot)$ is such that $\xi_\eta(t) \in \mathcal{B}_\epsilon^j$ for all $t \geq t_\epsilon$ (or for all $t \leq -t_\epsilon$) for some $t_\epsilon > 0$, then $\xi_\eta(t) \in W_j^{\rm s}(\xi_\eta(\cdot))$ (or $\xi_\eta(t) \in W_j^{\rm u}(\xi_\eta(\cdot))$). Hence, to prove (2.22), it is enough to show that there is a $\eta_0 > 0$ such that every globally defined bounded solution of (2.3) ends forwards and backwards in $\mathcal{B}_\epsilon$.

Suppose that $\eta_k \overset{k\to\infty}{\longrightarrow} 0$ and that $\xi_{\eta_k}(\cdot)$ does not end forwards or backwards in $\mathcal{B}_\epsilon = \cup_{j=1}^n \mathcal{B}_\epsilon^j$. Taking a subsequence if necessary, we may assume that given $\nu \leq \nu_0$ there exists k_ν such that ξ_{η_k} ends forward (respectively backward) in $\mathcal{O}_\nu^j$ for some fixed j with $1 \leq j \leq p$ whenever $k \geq k_\nu$. Hence we have a sequence t_k such that $\|\xi_{\eta_k}(t_k) - y_i^*\|_{\mathcal{Z}} \geq \epsilon$ for all y_i^* with $V(y_i^*) = \mathfrak{n}_j$. Consequently, there is a subsequence (which we again denote by ξ_{η_k}) such that

$$\lim_{k\to\infty} \xi_{\eta_k}(t + t_k) \to y(t)$$

uniformly for t in compact subsets of $\mathbb{R}$, where $y(\cdot)$ is a solution of (2.1). It is clear that $V(y(0)) = \mathfrak{n}_j$ and since $y(0)$ is not an equilibrium of (2.1) and V is non-increasing it follows (using the convergence of T_η to T_0) that for a suitable choice of $T > 0$ (respectively $T < 0$) we must have $V(\xi_{\eta_k}(t_k + T)) \notin (\mathfrak{n}_j - \nu_0, \mathfrak{n}_j + \nu_0)$ which leads to a contradiction.

Finally we prove (2.21). We will use the following two properties of the Hausdorff dimension (see Falconer [8], for example): it is non-increasing under Lipschitz mappings,

$$\Sigma : \mathcal{Z} \to \mathcal{Z} \text{ with } \|\Sigma(z_1) - \Sigma(z_2)\|_{\mathcal{Z}} \le L\|z_1 - z_2\|_{\mathcal{Z}} \qquad \Rightarrow \qquad \dim_H(\Sigma(X)) \le \dim_H(X),$$

and it is stable under countably infinite unions,

$$\dim_H\left(\bigcup_{j=1}^{\infty} X_j\right) = \sup_{1\le j\le\infty} \dim_H(X_j). \tag{3.14}$$

First, observe that $d_H(W^{\mathrm{u}}_{\mathrm{loc}}(\xi^*_{i,\eta})(\tau)) = \mathrm{rank}(Q^i_\eta(\tau)) = \mathrm{rank}(\mathcal{Q}_i)$, since sufficiently close to $\xi^*_{i,\eta}$ the unstable manifold is given as a Lipschitz graph over $Q^i_\eta(\tau)\mathcal{Z}$ and from the continuity of the projections we have that $\mathrm{rank}(Q^i_\eta(\tau)) = \mathrm{rank}(\mathcal{Q}_i)$. Note that

$$W^{\mathrm{u}}_\eta(\xi^*_{i,\eta})(t) = \bigcup_{n=0}^{\infty} T_\eta(t, t-n) W^{\mathrm{u}}_{\mathrm{loc},\eta}(t-n).$$

Since each $T_\eta(t,\tau) : \mathcal{Z} \to \mathcal{Z}$ is Lipschitz it follows that

$$\dim_H(T_\eta(t, t-n) W^{\mathrm{u}}_{\mathrm{loc}}(\xi^*_{i,\eta})(t-n)) \le \dim_H(W^{\mathrm{u}}_{\mathrm{loc}}(\xi^*_{i,\eta})(t-n)),$$

and hence, using (3.14), that $\dim_H(W^{\mathrm{u}}(\xi^*_{i,\eta})) = \mathrm{rank}(\mathcal{Q}_i)$. The equality in (2.21) then follows using (3.14) once again. □

4. Asymptotically autonomous problems

As in Section 1 we consider a Banach space $\mathcal{Z}$ and the semilinear problem

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f(t,y) \\ y(\tau) &= y_0, \end{aligned} \tag{4.1}$$

where $\mathfrak{B} : D(\mathfrak{B}) \subset \mathcal{Z} \to \mathcal{Z}$ is the generator of a C^0-semigroup of bounded linear operators and $f(t,\cdot)$ is a differentiable function that is Lipschitz continuous in bounded subsets of $\mathcal{Z}$ with Lipschitz constant independent of t. If we denote by $t \mapsto T(t,\tau)y_0$ the solution for (2.3), then $\{T(t,\tau) : t \ge \tau \in \mathbb{R}\}$ defines a nonlinear process. We will assume that the problem (4.1) has a pullback attractor $\{A(t) : t \in \mathbb{R}\}$.

4.1. Asymptotically Autonomous Problems at $-\infty$.

Assume that

$$\lim_{t\to-\infty} \sup_{z\in B(0,r)} \{\|f(t,z) - f_0(z)\|_{\mathcal{Z}} + \|f_y(t,z) - f_0'(z)\|_{L(\mathcal{Z})}\} = 0, \qquad \text{for each } r > 0, \tag{4.2}$$

and that (2.1) has an autonomous attractor A_0.

Suppose that (2.7) is gradient and has a finite number of solutions, all of them hyperbolic: then A_0 is given by (2.10) (Theorem 2.5). We now prove that the non-autonomous system possesses global solutions that are backwards asymptotic to the equilibria of the limiting autonomous problem.

Proposition 4.1. *Assume that* (4.2) *holds and that all solutions of* (2.7) *are hyperbolic. Then, there are solutions* $\xi^*_{i_-} : \mathbb{R} \to \mathcal{Z}$, $1 \le i_- \le n^-$, *such that*

$$\lim_{t\to-\infty} \max_{1\le i_-\le n^-} \|\xi^*_{i_-}(t) - y^*_{i_-}\|_{\mathcal{Z}} = 0. \tag{4.3}$$

Furthermore, there is a $\tau \in \mathbb{R}$ *such that*

$$\dot{y} = \mathcal{A}_{i_-} y + B_{i_-}(t) y \tag{4.4}$$

has an exponential dichotomy in $(-\infty, \tau]$, *where* $\mathcal{A}_{i_-} = \mathfrak{B} + f_0'(y^*_{i_-})$ *and* $B_{i_-}(t) = f_y(t, \xi^*_{i_-}(t)) - f_0'(y^*_{i_-})$.

Proof: The proof of this result reduces to the proof of (2) in Proposition 2.10, cutting the nonlinearities f and f_0 around $y^*_{i_-}$ in such a way that the fixed point argument works. To be more specific, we fix $1 \le i_- \le n^-$ and consider the change of variables $z = y - y^*_{i_-}$ in (4.1). In this new variable (4.1) becomes

$$\dot{z} = \mathcal{A}_{i_-} z + \tilde{g}_{i_-}(t, z) \tag{4.5}$$

where $\tilde{g}_{i_-}(t,z) = f(t, z + y^*_{i_-}) - f_0(y^*_{i_-}) - f_0'(y^*_{i_-})z$. Cut $\tilde{g}_{i_-}$ outside a small neighbourhood of $z = 0$ and suitably large negative times $t \le \tau$ in such a way that it becomes globally Lipschitz and bounded with very small Lipschitz constant and bound. Denote by g_{i_-} the new nonlinearity and consider, for $t \le \tau$,

$$z(t) = e^{\mathcal{A}_{i_-}(t-\tau)} z(\tau) + \int_\tau^t e^{\mathcal{A}_{i_-}(t-s)} g_{i_-}(s, z(s))\, ds.$$

Hence

$$\mathcal{Q}_{i_-} z(t) = \int_\infty^t e^{\mathcal{A}_{i_-}(t-s)} \mathcal{Q} g_{i_-}(s, z(s))\, ds$$

and

$$(I - \mathcal{Q}_{i_-}) z(t) = \int_{-\infty}^t e^{\mathcal{A}_{i_-}(t-s)} (I - \mathcal{Q}) g_{i_-}(s, z(s))\, ds.$$

Consequently, there exists in a small neighbourhood of $z = 0$ a globally defined solution of (4.1) if and only if

$$T_{i_-}(z)(t) = \int_\infty^t e^{\mathcal{A}_{i_-}(t-s)} \mathcal{Q}_{i_-} g_{i_-}(s, z(s))\, ds + \int_{-\infty}^t e^{\mathcal{A}_{i_-}(t-s)} (I - \mathcal{Q}_{i_-}) g_{i_-}(s, z(s))\, ds$$

has a unique fixed point in the set

$$\{z : \mathbb{R} \to \mathcal{Z} : \sup_{t\in\mathbb{R}} \|z(t)\|_{\mathcal{Z}} \le \epsilon\}$$

for ϵ sufficiently small. This follows assuming that, for $z, z_1, z_2 \in B(0, \epsilon)$, $\|g_{i_-}(t,z)\|_{\mathcal{Z}} \le \delta$ and that $\|g_{i_-}(t, z_1) - g_{i_-}(t, z_2)\|_{\mathcal{Z}} \le \delta \|z_1 - z_2\|_{\mathcal{Z}}$, with $\delta > 0$ sufficiently small. As a consequence of this it follows that $\xi^*_{i_-}(\cdot)$ is uniformly close to $y^*_{i_-}$. This solution is hyperbolic on $\mathbb{R}$. Hence, $\xi^*_{i_-}$ is a hyperbolic solution of (4.5) for all t large and negative. Hence, $y^*_{i_-} + \xi^*_{i_-}$ is a hyperbolic solution of (4.1) in $(-\infty, \tau]$ with $-\tau > 0$ suitably large.□

This also ensures that (1) below holds, and we can show that all globally defined bounded solutions are backwards asymptotic to one of the solutions from Proposition 4.1 (themselves asymptotic to the equilibria of the limiting autonomous system).

Theorem 4.2. *Let $f : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ be a differentiable function that satisfies* (4.2)*. Consider the initial value problem* (4.1)*. Assume that* (2.1) *is gradient and that all solutions of* (2.7) *are hyperbolic equilibrium solutions for* (2.1)*.*

(1) *If we write*

$$W^{\mathrm{u}}(\xi_i^*)(\tau) = \{\, (\tau,\zeta) \in \mathbb{R} \times \mathcal{Z} : \text{ there is a backwards solution } z(t,\tau,\zeta)$$
$$\text{of (4.1) satisfying } z(\tau,\tau,\zeta) = \zeta \text{ and such that } \lim_{t\to-\infty} \|z(t,\tau,\zeta) - \xi_i^*(t)\|_{\mathcal{Z}} = 0\},$$

then the attractor $\{A(\tau) : \tau \in \mathbb{R}\}$ of (4.1) *is given by*

$$A(\tau) = \cup_{i=1}^n W^{\mathrm{u}}(\xi_i^*)(\tau).$$

(2) *For each globally defined bounded solution $\xi(\cdot)$ of* (4.1) *there is an i_- with $1 \le i_- \le n^-$ such that*

$$\lim_{t\to-\infty} \|\xi(t) - \xi_{i_-}^*(t)\|_{\mathcal{Z}} = 0. \tag{4.6}$$

Proof: The proof of (1) is a consequence of Proposition 4.1 and Theorem 2.11 if we analyse (4.1) by considering the small non-autonomous perturbations of (2.1) obtained by replacing $f(t,y)$ by

$$f_\nu(t,y) = \begin{cases} f(t,y), & \text{if } t \le -\nu \\ f(\nu,y), & \text{if } t > -\nu. \end{cases}$$

¿From Theorem 2.11, for suitably large ν, there exists a pullback attractor $\{A_\nu(s) : s \in \mathbb{R}\}$ for

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f_\nu(t,y) \\ y(\tau) &= y_0 \end{aligned} \tag{4.7}$$

given by $A_\nu(s) = \cup_{i=1}^n W_\nu^{\mathrm{u}}(\xi_{i,\nu}^*)(s)$. To obtain the pullback attractor for (4.1) we first note that (4.7) and (4.1) coincide for $t \le \tau \le -\nu$. Hence $A(t) = A_\nu(t)$ for $t \le -\nu$. To recover $A(t)$ for $t \ge -\nu$ we only have take advantage of the invariance to see that $A(t) = T(t,\tau)A(\nu)$, for all $\tau \le -\nu \le t$.

Now, (2) is also essentially proved since, by (4.3), every global solution approaches one of the equilibria $y_{i_-}^*$ as $t \to -\infty$, so that, in particular, (2) holds.□

It is clear from the above proof that in order to characterize the pullback attractor $\{A(t) : t \ge 0\}$ it is not necessary that $A(t)$ remains bounded as $t \to \infty$. This accounts for many cases in the existing literature where the pullback attractors do not remain bounded as $t \to \infty$ (see [12, 22]) (cf. comments after Definition 1.2).

4.2. **Time-dependent forwards attractors.** Before considering problems that are asymptotically autonomous as $t \to +\infty$ we consider in general the problem of defining forwards attractors in non-autonomous problems.

It is relatively easy to give a definition of an attractor for individual solutions ('the point attractor') in the non-autonomous setting:

Definition 4.3. *For any fixed $t_0 \in \mathbb{R}$, a family $\{A(t) : t \ge t_0\}$ is the forwards point attractor of a process $S(\cdot,\cdot)$ for $t \ge t_0$ if*

- *$A(t)$ is non-empty and compact for each $t \ge t_0$;*
- *$A(t)$ is invariant, in the sense that*

$$S(t,s)A(s) = A(t) \qquad \text{for all} \quad t \ge s \ge t_0;$$

- *$A(t)$ attracts each individual solution,*
$$\mathrm{dist}(S(t,s)z_0, A(t)) \to 0 \qquad \textit{as} \qquad t \to \infty$$
for all $s \in \mathbb{R}$, $z_0 \in \mathcal{Z}$; and
- *$A(t)$ is the minimal set with this property, in that if $C(t)$ is another such family, we have $A(t) \subseteq C(t)$ for all $t \geq t_0$.*

A somewhat simpler definition would have $A(t)$ defined for all $t \in \mathbb{R}$, but this does not seem appropriate for the asymptotically autonomous problems that we are considering here. Allowing for attractors that are only defined on semi-infinite intervals makes them more widely applicable, and we can use the minimality to show that even given this freedom the forwards attractor is essentially unique.

Indeed, suppose that $t_1 > t_0$, and $\{A_0(t): \ t \geq t_0\}$ is a point attractor for $t \geq t_0$ and $\{A_1(t): \ t \geq t_1\}$ is a point attractor for $t \geq t_1$, then due to the minimality property it is immediate that we have
$$A_1(t) \subseteq A_0(t) \qquad \text{for all } t \geq t_1,$$
while the reverse inclusion follows if we define $\{\tilde{A}_1(t): \ t \geq t_0\}$ by setting $\tilde{A}_1(t) = A_1(t)$ for all $t \geq t_1$ and
$$\tilde{A}_1(t) = \{z \in \mathcal{Z}: \ S(t_1,t)z \in A_1(t_1)\}, \text{ for } t \in [t_0, t_1).$$
That $\tilde{A}_1(\cdot)$ so defined is compact and invariant follows since $A_1(t) \subseteq A_0(t)$, and so solutions starting in $A_1(t_1)$ can be extended back to $t = t_0$. It follows that
$$A_0(t) = A_1(t) \qquad \text{for all} \quad t \geq t_1,$$
showing that 'asymptotic behaviour' of the point attractor is uniquely specified by this definition.

Identifying the correct concept of a forwards global attractor (i.e. a forwards attractor of bounded sets) for non-autonomous problems is still something that requires further reflection. One would certainly desire that any definition of such a global attractor would include all globally defined bounded solutions, and as discussed in the introduction this is sufficient to define the global attractor in autonomous problems, and gives rise to the pullback attractor in non-autonomous problems. So the pullback attractor should certainly be a subset of the 'global attractor' in a non-autonomous problem.

However, to see that the pullback attractor will not in general describe all the interesting asymptotic behaviour of a truly non-autonomous problem, consider the equation
$$\dot{x} = \lambda(t)x - x^3 \tag{4.8}$$
with $\lambda: \mathbb{R} \to \mathbb{R}$ being a smooth function with the property that $0 \leq \lambda(t) \leq 1$, $\lambda(t) = 0$ for all $t \leq 0$ and $\lambda(t) = 1$ for all $t \geq 1$.

While the pullback attractor for (4.8) is $A(t) = \{0\}$ for all $t \in \mathbb{R}$, for $t \geq 1$ the equation is
$$\dot{x} = x - x^3,$$
which has three stationary solutions $x_0 = 0$, $x_- = -1$ and $x_+ = 1$ with the equilibrium $x_0 = 0$ being unstable. If we look at the solution with $x(1) = 1$ and solve the equation for $t \leq 1$ we see that $x(0,1,1) = x_1 > 0$ and therefore
$$x(t,1,1) = \frac{1}{\sqrt{2t + x_1^2}} \quad \text{for} \quad -\frac{x_1^2}{2} < t \leq 0,$$

which blows up as $t \to -x_1^2/2$. We also see that the set $\{-1, 0, 1\}$ attracts points of $\mathbb{R}$ and that $[-1, 1]$ attracts bounded subsets of $\mathbb{R}$ forwards in time.

In this case it is natural to define $A^+(t) = [-1, 1]$ for each $t > 1$. This set has the property that $T(t, \tau)A^+(\tau) = A^+(t)$ for all $t \geq \tau \geq 1$ and one can easily see that $\mathrm{dist}(T(t, \tau)B, A^+(t)) \to 0$ as $t \to \infty$. We now give a general definition along these lines:

Definition 4.4. *We say that a family* $\{A^+(t) \subset \mathcal{Z} : t \geq t_0\}$ *is a* time-dependent forwards attractor *for* (4.1) *if:*

- $A^+(t)$ *is compact for each* $t \geq t_0$*,*
- $\{A^+(t) : t \geq \tau\}$ *is invariant in the sense that* $T(t, \tau)A^+(\tau) = A^+(t)$ *for all* $t \geq \tau \geq t_0$*, and*
- $\mathrm{dist}(T(t, \tau)B, A^+(t)) \to 0$ *as* $t \to \infty$ *for each bounded set* $B \subset \mathcal{Z}$ *and for any* $\tau \geq t_0$.

However, it is important to note that even in our simple example, the 'natural' choice $A^+(t) = [-1, 1]$ for $t > 1$ is not the only possibility that satisfies our definition. Indeed, if K is any compact set whose interior contains $\{0\}$ and $t_0 \in \mathbb{R}$ is fixed then it is easy to see that

$$A^+(t) = T(t, t_0)K$$

has the properties required by our definition. This implies, in particular, that we cannot impose uniqueness by requiring either maximality or minimality of the forwards attractor.

Whether there can be a definitive notion of a forwards attractor for bounded sets is an outstanding open problem. Equally important would be to determine conditions under which the pullback attractor also attracts solutions forwards in time (for examples where this does occur, see [15] and [17]).

In the next section we discuss forwards attractors in the context of asymptotically autonomous problems. In this case we can identify the forwards point attractor, and also find a candidate set that satisfies our definition of a forwards global attractor.

4.3. **Asymptotically Autonomous Problems at** $+\infty$.

Assume that

$$\lim_{t \to +\infty} \sup_{z \in B(0,r)} \{\|f(t, z) - f_0(z)\|_{\mathcal{Z}} + \|f_y(t, z) - f_0'(z)\|_{L(\mathcal{Z})}\} = 0, \qquad \text{for each } r > 0, \tag{4.9}$$

and that (2.1) has an autonomous attractor A_0. We note that the nonlinearity f_0 in this subsection may be different from that in the previous subsection and consequently the attractor A_0 in this subsection may be different from that in the previous one. We assume in addition (and crucially) that is gradient, and that (2.7) has a finite number of solutions, all of them hyperbolic: it follows from Theorem 2.5 that A_0 is given by (2.10).

Assume that (4.1) gives rises to a nonlinear process $\{T(t, \tau) : t \geq \tau \in \mathbb{R}\}$ for which there is an absorbing ball $B(0, r_0)$. Consider $f_k(t, z)$ the function which coincides with f in $[k, \infty) \times \mathcal{Z}$ and which is equal to $f(k, z)$ for all $t < k$ and $z \in \mathcal{Z}$. Then

$$\lim_{k \to +\infty} \sup_{t \in \mathbb{R}} \sup_{z \in B(0,r_0)} \{\|f_k(t, z) - f_0(z)\|_{\mathcal{Z}} + \|(f_k)_y(t, z) - f_0'(z)\|_{L(\mathcal{Z})}\} = 0. \tag{4.10}$$

It has been proved in [5] that the family of attractors for

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f_k(t, y) \\ y(\tau) &= y_0 \end{aligned} \tag{4.11}$$

behaves upper and lower semicontinuously as $k \to \infty$ with the limit attractor being the attractor for (2.1), i.e.

$$\sup_{t\in\mathbb{R}} \mathrm{dist}(A_k(t), A_0) \to 0 \qquad \text{as} \ \ k \to \infty,$$

where $\mathrm{dist}(A, B)$ is the symmetric Hausdorff distance defined in Section 1.4.

If we denote by $\{A_k(t) : t \in \mathbb{R}\}$ the pullback attractor for (4.11) then

$$A_j(t) = A_k(t), \quad \text{for all } j > k \text{ and } t \le k.$$

Let k_0 be such that for $k \ge k_0$ the pullback attractor of (4.11) coincides with the union of the unstable manifolds of all those $\{\xi^*_{i,k}\}$ with $\sup_{t\in\mathbb{R}} \|\xi^*_{i,k}(t) - y^*_i\|_{\mathcal{Z}} \to 0$ as $k \to \infty$. Define $A^+(t) = A_{k_0}(t)$ for $t \ge k_0$. Note that $A^+(t)$ is in fact the forwards image of the global attractor of the autonomous system $\dot{y} = \mathfrak{B}y + f(k_0, y)$ under the non-autonomous process $T(\cdot, \cdot)$.

If we define $T_\infty(t, \tau) = T_0(t - \tau)$, it follows from the fact that

$$\sup_{t\ge\tau} \|T_k(t, \tau)B - T_\infty(t, \tau)B\|_{\mathcal{Z}} \to 0, \ \text{as } k \to \infty$$

and from the lower semicontinuity of attractors that, given $\epsilon > 0$ there is a $T_\epsilon > 0$ such that, for all $t \ge T_\epsilon$, $T_\infty(t, \tau)B \subset O_\epsilon(A_0)$ and an $N \in \mathbb{N}$ such that $T_k(t, \tau)B \subset O_\epsilon(T_\infty(t, \tau)B) \subset O_{2\epsilon}(A_0) \subset O_{3\epsilon}(A^+(t))$ for all $t \ge k \ge N$. This proves the following result:

Theorem 4.5. *There is a $t_0 \in \mathbb{R}$ and a time dependent forwards attractor $\{A^+(t) : t \ge t_0\}$ for* (4.1)*.*

We now show that there is a finite number of hyperbolic solutions that attract all other solutions as $t \to \infty$. First we show that there are hyperbolic solutions asymptotic (as $t \to \infty$) to each of the equilibria of (2.1)

Proposition 4.6. *Assume that* (4.9) *holds. Then, there are solutions $\xi^*_{j+} : \mathbb{R} \to \mathcal{Z}$, $1 \le j \le n^+$, such that*

$$\lim_{t\to+\infty} \max_{1\le j+ \le n^+} \|\xi^*_{j+}(t) - y^*_{j+}\|_{\mathcal{Z}} = 0. \tag{4.12}$$

Furthermore, there is a $t_0 \in \mathbb{R}$ such that

$$\dot{y} = \mathcal{A}^{j+}y + B^{j+}(t)y \tag{4.13}$$

has an exponential dichotomy in $[t_0, +\infty)$, where $\mathcal{A}^{j+} = \mathfrak{B} + f_0'(y^*_{j+})$ and $B^{j+}(t) = f_y(t, \xi^*_{j+}(t)) - f_0'(y^*_{j+})$.

Proof: Again, the proof of this result reduces to the proof of (2) in Proposition 2.10, cutting the nonlinearities f in the same way as before to make (4.10) hold. To be more specific, we fix $1 \le j+ \le n^+$ and consider the change variables $z = y - y^*_{j+}$ in (4.1). In this new variable (4.1) becomes

$$\dot{z} = \mathcal{A}^{j+}z + \tilde{g}_{j+}(t, z) \tag{4.14}$$

where $\tilde{g}_{j+}(t, z) = f(t, z + y^*_{j+}) - f_0(y^*_{j+}) - f_0'(y^*_{j+})z$. Cut $\tilde{g}_{j+}$ outside a small neighbourhood of $z = 0$ and sufficiently large times in such a way that it becomes globally Lipschitz and bounded with very small Lipschitz constant and bound. Let g_{j+} be the new nonlinearity and proceed exactly as in the previous section (asymptotically autonomous in $-\infty$) to obtain the existence of a global hyperbolic solution $\xi^*_{j+}(\cdot)$ for the modified equation which is uniformly close to y^*_{j+}. Now, ξ^*_{j+} is a solution of (4.14) for all t large enough. Hence, $y^*_{j+} + \xi^*_{j+}$ is a solution of (4.1) in $[\tau, \infty)$ with $\tau > 0$ suitably large. This solution is hyperbolic on $\mathbb{R}$. Hence, ξ^*_{j+} is a hyperbolic solution of (4.5) for all t large enough. Hence, $y^*_{j+} + \xi^*_{j+}$ is a hyperbolic solution of (4.1) in $[\tau, +\infty)$ with $\tau > 0$ suitably large. □

Ball and Peletier [4] (see also [3]) prove that, further to (4.12), given each $(\tau, y_0) \in \mathbb{R} \times \mathcal{Z}$, there exists a $j+$ with $1 \le j+ \le n$ such that

$$\lim_{t\to\infty} \|T(t,\tau)y_0 - y^*_{j+}\|_{\mathcal{Z}} = 0. \tag{4.15}$$

For us, this is a corollary of the following:

Corollary 4.7. *Let $f : \mathbb{R}\times\mathcal{Z} \to \mathcal{Z}$ be a differentiable function which satisfies* (4.9)*. Consider the initial value problem* (4.1)*. Assume that* (2.1) *is gradient and has a global attractor A_0, and that all solutions of* (2.7) *are hyperbolic equilibrium solutions for* (2.1)*.*

Then, for each $(\tau, y_0) \in \mathbb{R} \times \mathcal{Z}$, there exists a j_+ with $1 \le j_+ \le n$ such that

$$\lim_{t\to\infty} \|T(t,\tau)y_0 - \xi^*_{j_+}(t)\|_{\mathcal{Z}} = 0. \tag{4.16}$$

In particular, for each globally defined bounded solution $\xi(\cdot)$ of (4.1) *there is a j_+ with $1 \le j_+ \le n$ such that*

$$\lim_{t\to\infty} \|\xi(t) - \xi^*_{j_+}(t)\|_{\mathcal{Z}} = 0. \tag{4.17}$$

Note that results on asymptotically autonomous systems in the literature usually show that the forwards asymptotic behaviour of the equations tends to limiting structures within the limit attractor, for instance equilibria of the limit equations, which, in general, are not solutions of the non-autonomous system. (Although there are non-gradient examples showing that the limiting behaviour can differ from that of the limit system, e.g. [20, 25].) Corollary 4.7 goes a little further, since it describes the forwards long time dynamics by means of hyperbolic solutions of the non-autonomous equations. Observe that we also get (4.15) from (4.16) and (4.12).

5. Examples

In this section we give three examples to illustrate the wide applicability of our results: an autonomous damped wave equation (a striking example of an autonomous system in which the attractor is still gradient-like even though the underlying system is not gradient), an asymptotically autonomous parabolic equation that illustrates some of the peculiarities of non-autonomous systems, and a simple non-autonomous scalar ordinary differential equation whose pullback attractor we can describe very fully.

We hope that our results will further the understanding of non-autonomous attractors in an even wider array of examples.

5.1. A gradient-like attractor for a damped wave equation.

Let Ω be a bounded smooth domain in $\mathbb{R}^3$. For $\eta \in [0, 1]$, assume that $g : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function that is bounded with bounded derivatives up to second order.

For $a \in C(\bar{\Omega}, \mathbb{R}^3)$ and $\eta \ge 0$, consider the damped hyperbolic equation

$$u_{tt} + \beta u_t - \Delta u = \eta\, a(x) \cdot \nabla u + g(u) \qquad \text{in} \quad \Omega \tag{5.1}$$

with the boundary condition

$$u = 0 \quad \text{in} \quad \partial\Omega. \tag{5.2}$$

The initial data for (5.1), (5.2) will be taken in the space $\mathcal{Z} = H^1_0(\Omega) \times L^2(\Omega)$, where the norm in $H^1_0(\Omega)$ is defined by $\|\varphi\|_{H^1_0(\Omega)} = \|\nabla\varphi\|_{L^2(\Omega)}$, $\varphi \in H^1_0(\Omega)$.

It is easy to (see [1, 2]) that (5.1), (5.2) defines a nonlinear semigroup $\{T_\eta(t), t \geq 0\}$ on $\mathcal{Z}$ where $T_\eta(t)(\varphi, \psi) = (u(t), u_t(t))$ with $(u(t), u_t(t))$ being the solution of (5.1), (5.2) such that $u(0) = \varphi$ and $u_t(0) = \psi$.

If we let $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ be $-\Delta$ with homogeneous Dirichlet boundary conditions, then $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. We consider (5.1), (5.2) as an abstract evolutionary equation in $\mathcal{Z}$:

$$\begin{aligned} \dot{z} &= \mathcal{C}z + f_\eta(z), \\ z(0) &= z_0 \in \mathcal{Z} \end{aligned} \tag{5.3}$$

where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathcal{Z},$$

$$\mathcal{C} = \begin{pmatrix} 0 & I \\ -A & -\beta \end{pmatrix},$$

and

$$f_\eta(z)(x) = \begin{pmatrix} 0 \\ \eta\, a(x) \cdot \nabla z_1(x) + g(z_1(x)) \end{pmatrix}, \quad \text{for } x \in \Omega.$$

Under these assumptions f_η is continuously differentiable (see [1]), and it is not difficult to see that (2.16) is satisfied.

Using the energy $V : \mathcal{Z} \to \mathbb{R}$ defined by

$$V(z) = \frac{1}{2}\int_\Omega |\nabla z_1|^2 + \delta \int_\Omega z_1 z_2 + \frac{1}{2}\int_\Omega z_2^2 + \int_\Omega G(z_1),$$

where $\delta > 0$ is chosen suitably and

$$G(z_1) = \int_0^{z_1} g(s)\, ds,$$

it follows in a similar way as in [1] that (5.3) has a global attractor in $\mathcal{Z}$.

We note that the equilibrium points of (5.1) with $\eta = 0$ are of the form $z_0^* = (u_0^*, 0)$ where u_0^* is a solution of

$$Au = g(u). \tag{5.4}$$

Furthermore, if u_0^* is a solution of (5.4) such that $0 \notin \sigma(-\Delta - g'(u_0^*)I)$ (which is true generically) then $(u_0^*, 0)$ is a hyperbolic equilibrium point of (5.1) with $\eta = 0$.

As a consequence of Theorem 2.11 the following result holds:

Theorem 5.1. *Assume that g is twice continuously differentiable with bounded derivatives up to second order and that $0 \notin \sigma(A - g'(u_0^*)I)$ whenever u_0^* is a solution of* (5.4)*. Then, the nonlinear semigroup associated to* (5.3) *has a global attractor A_η, $\eta \in [0, 1]$, and from the results in* [5] *this family of attractors is upper and lower semicontinuous at $\eta = 0$. Additionally, as a consequence of Theorem 2.11, for suitably small $\eta > 0$, A_η has a gradient-like structure, i.e. it is exactly the union of the unstable manifolds of hyperbolic equilibria.*

Proof. The only thing we need to prove is that in each small neighbourhood of a hyperbolic equilibrium point $z^* = (u^*, 0)$ of (5.1) with $\eta = 0$ there is a unique equilibrium point of (5.1) with $\eta > 0$ suitably small. This will ensure that the global hyperbolic solutions for (5.1) are equilibrium solutions and the result then follows from Theorem 2.11.

We want to prove that in a neighbourhood of the solution u^* of (5.4) there is a unique solution u^*_η of

$$Au = \eta\, a(x) \cdot \nabla u + g(u). \tag{5.5}$$

This will follow from proving that

$$\Phi_\eta(v) = (A - g'(u^*))^{-1}[g(v + u^*) - g(u^*) - g'(u^*)v + \eta\, a(x) \cdot \nabla(v + u^*)]$$

has a unique fixed point in a small $H^1_0(\Omega)$ neighbourhood of u^*_0 and noting that if v^*_η is a fixed point of Φ_η then $u^*_\eta = v^*_\eta + u^*$ is a solution of (5.5).

To prove that Φ_η has a unique fixed point in a closed ball $\bar{B}_\epsilon(0)$ of $H^1_0(\Omega)$ with ϵ sufficiently small, note that

- $(A - g'(u^*))^{-1} \in L(L^2(\Omega), H^1_0(\Omega))$,
- $\|g(v + u^*) - g(u^*) - g'(u^*)v\|_{L^2(\Omega)} \leq C\|v\|^2_{L^4(\Omega)}$,
- $\eta\|a(x) \cdot \nabla(v + u^*)\|_{L^2(\Omega)} \leq \eta C(\|u^*\|_{H^1_0(\Omega)} + \epsilon))$, $\forall\, v \in \bar{B}_\epsilon(0)$

and from the fact that $L^4(\Omega) \subset H^1_0(\Omega)$ $(\Omega \subset \mathbb{R}^3)$ we can choose $\epsilon > 0$ and $\eta > 0$ sufficiently small that

$$\|(A - g'(u^*))^{-1}[g(v + u^*) - g(u^*) - g'(u^*)v + \eta\, a(x) \cdot \nabla(v + u^*)]\|_{H^1_0(\Omega)} \leq \epsilon,\ \forall v \in \bar{B}_\epsilon(0).$$

A similar reasoning proves that Φ_η is a contraction in $\bar{B}_\epsilon$ and the result follows. □

5.2. **An asymptotically autonomous parabolic problem.** Let $\lambda \in [0, \infty)$ be a parameter and consider the problem

$$\begin{aligned} &u''(x) + \lambda(u - u^3) = 0, \quad x \in (0, \pi), \\ &u(0) = u(\pi) = 0. \end{aligned} \tag{5.6}$$

It is well known that if $\lambda \in (n^2, (n+1)^2)$, $n \geq 0$, then (5.6) has exactly $2n+1$ equilibrium solutions, namely $\mathcal{E} = \{u_0, u^\pm_1, \cdots u^\pm_n\}$, all of them hyperbolic. If u is a continuous function, denote by $\ell(u)$ the number of sign changes of u. For each $1 \leq i \leq n$, $u^\pm_i$ changes signs $i - 1$ times. The corresponding parabolic initial boundary value problem

$$\begin{aligned} &u_t(t, x) = u_{xx}(t, x) + \lambda(u(t, x) - u(t, x)^3), \quad x \in (0, \pi), \\ &u(0, \cdot\,) = u_0(\cdot\,) \in H^1_0(\Omega), \\ &u(t, 0) = u(t, \pi) = 0, \end{aligned} \tag{5.7}$$

has a global attractor $\mathcal{A}_\lambda$ which is given by

$$\mathcal{A}_\lambda = W^{\mathrm{u}}(u_0) \cup \left(\cup^n_{i=1} W^u(u^+_i)\right) \cup \left(\cup^n_{i=1} W^u(u^-_i)\right).$$

We also know that any solution $\xi(\cdot) : \mathbb{R} \to H^1_0(0, \pi)$ must satisfy $\lim_{t\to-\infty} \xi(t) = u^*$ and $\lim_{t\to+\infty} \xi(t) = v^*$ with $\ell(u^*) > \ell(v^*)$.

Let $\lambda_k \in (n^2_k, (n_k + 1)^2)$, $k = 1, 2$, with $n_1 < n_2$ and let $h : \mathbb{R} \to \mathbb{R}$ be a smooth function with the property that $0 \leq h(t) \leq 1$, $h(t) = 0$ for all $t \leq 0$ and $h(t) = 1$ for all $t \geq 1$.

Consider the problem

$$\begin{aligned} &u_t(t,x) = u_{xx}(t,x) + (h(t)\lambda_2 + (1-h(t))\lambda_1)(u(t,x) - u(t,x)^3), \quad x \in (0,\pi),\\ &u(0,\cdot\,) = u_0(\cdot\,) \in H_0^1(0,\pi),\\ &u(0) = u(\pi) = 0. \end{aligned} \tag{5.8}$$

Our results in Section 4.1 (Section 4.3) ensure that any globally defined solution must converge as $t \to -\infty$ to an equilibrium point of (5.7) with $\lambda = \lambda_1$, and as $t \to +\infty$ to an equilibrium point of (5.7) with $\lambda = \lambda_2$.

However, as a consequence of the results in [18], if $\xi : \mathbb{R} \to H_0^1(0,\pi)$ is a global bounded solution of (5.8) then $t \mapsto \ell(\xi(t))$ is decreasing. Combining these results it follows that any solution that converges to an equilibrium point u^* of (5.7) with $\lambda = \lambda_2$ and $l(u^*) > n_1 - 1$ cannot be a solution in the pullback attractor associated to (5.8). It follows that such solutions either do not exist globally or must blow up backwards in a finite time.

5.3. **An ordinary differential equation.** Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a smooth function with the property that $0 \le \lambda(t) \le 1$, $\lambda(t) = 0$ for all $t \le 0$ and $\lambda(t) = 1$ for all $t \ge 1$. Consider the non-autonomous scalar equation

$$\dot{u} = \lambda(t)(u - u^3) + (1-\lambda(t))(u(u^2-1)(4-u^2)). \tag{5.9}$$

If we set $u_j^* = j - 2$ of (5.9), $j = 0,1,2,3,4$, then the pullback attractor $\{A(t) : t \in \mathbb{R}\}$ for (5.9) is given by

$$A(t) = \cup_{j=0}^{4} W^{\mathrm{u}}(u_j^*)(t),\ t \in \mathbb{R}.$$

The interval $[-1,1]$ is a forwards attractor for (5.9), and the set $\{-1,0,1\}$ is a point attractor for (5.9).

Note that the solutions u_j^* are constant for $j = 1,2,3$ and $A(t) = [u_0(t), u_4(t)]$ is the pullback attractor and a forwards attractor for (5.9). Unfortunately, such simple geometry is not present in higher dimensions. Many examples of this kind can be produced with different asymptotically autonomous problems at $-\infty$ and $+\infty$, each having its own particular structure.

6. Conclusion

Given an infinite-dimensional autonomous gradient system, we have shown here that results on the continuity of local stable and unstable manifolds of hyperbolic equilibria under non-autonomous perturbations have significant consequences for the structure of the attractors in these systems.

By showing that all bounded global solutions are backwards asymptotic to a hyperbolic global solution, we have extended the characterization of attractors given in Langa et al. [16] to the more significant infinite-dimensional case. In addition we have shown that all solutions are also forwards asymptotic to a hyperbolic global solution.

Not only is the structure of the attractor preserved, but its Hausdorff dimension is unchanged, and equal to the maximum dimension of the local unstable manifolds near the equilibria of the autonomous equation. It is not clear – even in the autonomous case – whether a similar result holds for the box-counting (‘fractal’) dimension.

We note that if the results on perturbations of stable and unstable manifolds in [5] can be generalized to treat singular perturbations, the results in the present paper will hold with the same proofs (maintaining, of course, the assumption that the limiting system is gradient).

Such a generalization is not straightforward, however, since this requires generalization of existing results on the roughness of exponential dichotomies.

It is natural to conjecture that the connections between equilibria are also preserved under perturbation: i.e. if there exists a solution $y(t)$ of (2.1) such that

$$\lim_{t\to-\infty} y(t) = y_j^* \qquad \text{and} \qquad \lim_{t\to+\infty} y(t) = y_k^*$$

then for every η sufficiently small there exists a global solution of (2.3) such that

$$\lim_{t\to-\infty} \|y(t) - \xi_{j,\eta}^*(t)\| = 0 \qquad \text{and} \qquad \lim_{t\to+\infty} \|y(t) - \xi_{k,\eta}^*(t)\| = 0.$$

Such a result will probably require the assumption that the stable and unstable manifolds in the limiting system intersect transversally.

Finally, we wish to highlight once more the problems associated with the definition of a forwards attractor for bounded sets, and the interesting open problem of finding conditions under which the pullback attractor is also a forwards attractor.

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