Positive solutions for some indefinite nonlinear eigenvalue elliptic problems with Robin boundary conditions

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Abstract

We consider a nonlinear eigenvalue problem with indefinite weight under Robin boundary condition. We prove the existence and multiplicity of positive solutions. To this end, we carry out a detailed study of some linear eigenvalues problems and we use mainly bifurcation and sub-supersolution methods.

Key Words. Elliptic equations, Indefinite weight, Robin boundary conditions.

1 Introduction and main results

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with a $C^{2,\gamma}$ boundary, $0 < \gamma < 1$. We are interested in the study of positive solutions for the problem

$$(P) \qquad \begin{cases} -\Delta u = \lambda m(x)(u - u^2) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \alpha u & \text{on } \partial\Omega, \end{cases}$$

where $\lambda, \alpha \in \mathbb{R}$, $m \in C^1(\overline{\Omega})$ changes sign and ν is the outward unit normal to $\partial\Omega$.

Throughout this article we assume that

$$\int_\Omega m < 0, \tag{1.1}$$

since the case $\int_\Omega m > 0$ reduces to (1.1) changing λ by $-\lambda$. The case $\int_\Omega m = 0$ is singular and will be treated elsewhere.

We shall treat (P) by a bifurcation approach, so we shall consider the linear eigenvalue problem

$$(E) \qquad \begin{cases} -\Delta u = \lambda m(x) u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \alpha u & \text{on } \partial\Omega. \end{cases}$$

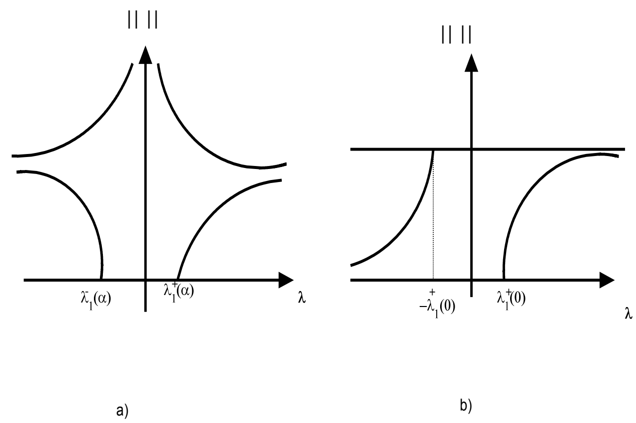

Figure 1: Bifurcation diagrams of (P): Case a) $\alpha < 0$ and Dirichlet boundary conditions. Case b) $\alpha = 0$.

It is shown in [1] that there exists $\alpha_0^* > 0$ such that, for $\alpha < \alpha_0^*$, (E) possesses two principal eigenvalues, denoted by $\lambda_1^-(\alpha)$ and $\lambda_1^+(\alpha)$. In the homogeneous Dirichlet boundary conditions case, we denote them by $\lambda_1^\pm(D)$. In Section 2 we recall the results from [1] and complement them providing an expression for α_0^*.

(P) has already been studied in different cases. For the cases $\alpha < 0$ [7] and Dirichlet boundary conditions [2, 11], it has been proved that (P) has a positive solution for all $\lambda \neq 0$ and, under further conditions for a priori bounds, at least two positive solutions for $\lambda \in (-\infty, \lambda_1^-(\alpha)) \cup (\lambda_1^+(\alpha), +\infty)$ and $\lambda \in (-\infty, \lambda_1^-(D)) \cup (\lambda_1^+(D), +\infty)$, respectively. See Figure 1 (a) for the bifurcation diagram in these cases.

The case $\alpha = 0$, which has been analyzed in [6] (see also [14, 17]), is singular in the following sense: the trivial solutions $u \equiv 0$ and $u \equiv 1$ exist for all $\lambda \in \mathbb{R}$, and for $\lambda = 0$ the positive constants are solutions. Moreover, for $\lambda \in (-\infty, -\lambda_1^+(0)) \cup (\lambda_1^+(0), +\infty)$ there exists a stable solution $u < 1$, which is the only positive solution of (P) less than one, see Figure 1 (b). Recall that in this case $\lambda_1^-(0) = 0$.

Finally, the case $\alpha > 0$ and small was studied in [7]. Assuming $2 < (N+2)/(N-2)$ and using variational methods, the authors proved that if $0 < \alpha < \alpha_0^*$ and $\lambda \in (\lambda_1^-(\alpha), \lambda_1^+(\alpha))$ then (P) possesses at least a positive solution.

In this article, we adopt a different viewpoint, namely, we consider λ fixed and look at α as a bifurcation parameter. Consequently, we improve some results of [6] for $\alpha = 0$, and complement the study of (P) when $\alpha > 0$.

We shall assume that

$$M_\pm := \{x \in \Omega : m^\pm > 0\}$$

are open and regular sets; here $m^\pm$ denote the positive and negative part of m respectively. We shall also assume that $m^\pm(x) \approx [\mathrm{dist}(x, \partial M_\pm)]^{\gamma^\pm}$ for x close to $\partial M_\pm$ and some $\gamma_\pm \geq 0$. Let

$$M_0 := \Omega \setminus (\overline{M_+} \cup \overline{M_-}). \tag{1.2}$$

We assume the following conditions on $M_\pm$ and M_0:

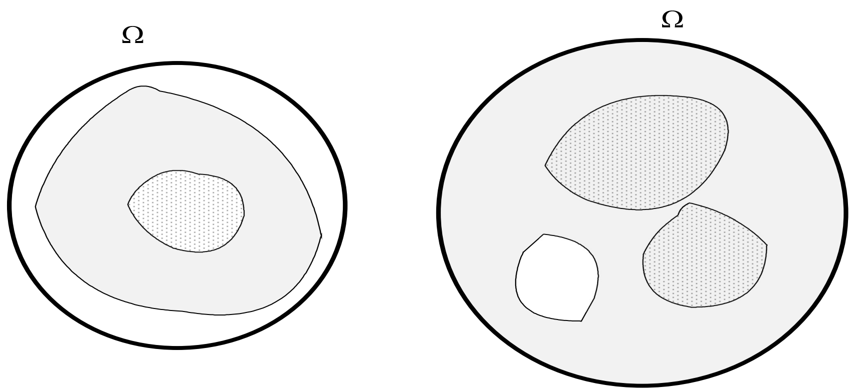

Figure 2: Two examples of admissible domains. The white, shady and lined sets represent M_0, M_+ and M_-, respectively.

(H_{M_0}) M_0 is a proper subdomain of Ω, i.e. $dist(\partial\Omega, \partial M_0 \cap \Omega) > 0$.

$(H_{M_\pm})$ $\partial M_\pm = \Gamma_1^\pm \cup \Gamma_2^\pm$, with $\Gamma_1^\pm = \partial\Omega \cap \partial M_\pm$ and $\Gamma_2^\pm \subset \Omega$.

In fact, $(H_{M_\pm})$ is assumed to avoid regularity issues, see [12]. In Figure 2 we have represented two different admissible domains.

Our first result is related to *a priori* bounds for positive solutions of (P). We show that if

$$2 < \min\left\{\frac{N+2}{N-2}, \frac{N+1+\gamma^\pm}{N-1}\right\}, \tag{1.3}$$

then, there exist *a priori* bounds for positive solutions of (P) whenever α varies in compact sets of $\mathbb{R}$.

In order to show our main results, we need to introduce some further notation. We denote by $\lambda_1(-\Delta - \lambda m, N)$ and $\lambda_1(-\Delta - \lambda m, D)$ the principal eigenvalues of the problem

$$-\Delta\varphi - \lambda m(x)\varphi = \sigma\varphi \quad \text{in } \Omega,$$

under homogenous Neumann and Dirichlet boundary conditions, respectively. In Section 2, we show that given $\lambda \in \mathbb{R}$, there exists a principal eigenvalue of (E) with respect to α, denoted by $\alpha_1(\lambda)$, if and only if $\lambda_1(-\Delta - \lambda m, D) > 0$. Furthermore, $sign(\alpha_1(\lambda)) = sign(\lambda_1(-\Delta - \lambda m, N))$.

Note that if $\lambda = 0$ then (P) has no positive solutions unless if $\alpha = 0$, in which case, all the positive constants are solutions. So we assume that $\lambda \neq 0$ along this article.

We state now our main result (see Figure 3):

Theorem 1.1. *Assume (1.1) and (1.3).*

1. *Assume $\lambda_1(-\Delta - \lambda m, D) > 0$. Then there exists $\alpha_* \geq \alpha_1(\lambda)$ such that (P) has a positive solution if $\alpha < \alpha_*$ and no positive solution for $\alpha > \alpha_*$. Moreover, there exists $\alpha_{**} \in (\alpha_1(\lambda), \alpha_*]$ such that (P) has at least two positive solutions for $\alpha \in (\alpha_1(\lambda), \alpha_{**})$. In addition:*

(a) *If* $\lambda_1(-\Delta - \lambda m, N) > 0$ *then* $0 < \alpha_1(\lambda) \leq \alpha_{**}$.
(b) *If* $\lambda_1(-\Delta - \lambda m, N) = 0$ *then* $0 = \alpha_1(\lambda) < \alpha_{**}$.
(c) *If* $\lambda_1(-\Delta - \lambda m, N) < 0$ *and* $\lambda \neq -\lambda_1^+(0)$ *then* $\alpha_1(\lambda) < 0 < \alpha_{**}$.
(d) *If* $\lambda = -\lambda_1^+(0)$ *then* $\alpha_1(\lambda) < 0 \leq \alpha_{**}$.

2. *Assume* $\lambda_1(-\Delta - \lambda m, D) \leq 0$. *Then there exist* $\alpha_* > 0$ *such that* (P) *has a positive solution if and only if* $\alpha \leq \alpha_*$. *Moreover, there exists* $\alpha_{**} \in (0, \alpha_*]$ *such that* (P) *has at least two positive solutions for* $\alpha < \alpha_{**}$.

As a consequence, we obtain (see Figure 4 (a)):

Theorem 1.2. *Assume (1.1) and* $\alpha = 0$.

1. *For all* $\lambda \in \mathbb{R}$, $u \equiv 1$ *is a positive solution of* (P), *which is stable for* $\lambda \in (-\lambda_1^+(0), 0)$.
2. (P) *has a second (and stable) positive solution for* $\lambda > \lambda_1^+(0)$ *and* $\lambda < -\lambda_1^+(0)$.
3. *Assume (1.3).* (P) *has a second positive solution for* $-\lambda_1^+(0) < \lambda < 0$.

In the case $\alpha > 0$, we get:

Theorem 1.3. *Assume (1.1), (1.3) and* $\alpha > 0$.

1. *There exists* $\alpha_0 > 0$, *such that* (P) *has no positive solution for* $\alpha \geq \alpha_0$.
2. *Let* $\lambda < 0$ *and* $\lambda \neq -\lambda_1^+(0)$. *Then there exists* $\alpha^*(\lambda)$ *such that* (P) *has at least two positive solutions for* $\alpha < \alpha^*(\lambda)$.
3. *Let* $\lambda \in (\lambda_1^-(\alpha), \lambda_1^+(\alpha))$ *and* $0 < \alpha < \alpha_0^*$. *Then* (P) *has at least a positive solution.*
4. *Let* $\lambda \geq \lambda_1^+(0)$. *Then* (P) *has at least two positive solutions for* α *sufficiently small.*

We stress that we do not know what the bifurcation diagram looks like in the case $\lambda \in [\lambda_1^+(\alpha), \lambda_1^+(0))$. However, since $\lambda_1^+(\alpha) \to \lambda_1^+(0)$ as $\alpha \to 0$, we have represented in Figure 4 (b) the suggested bifurcation diagram in the case $\alpha > 0$ and small.

The outline of this article is as follows: in Section 2 we study in detail the eigenvalue problems related to (P). In Section 3 we consider (P) with α as the bifurcation parameter. Finally, Section 4 is devoted to prove our main results.

2 Eigenvalue problems

Given $m \in L^\infty(\Omega)$ and $h \in C^1(\partial\Omega)$, we denote by $\lambda_1(-\Delta + m, N + h)$ the principal eigenvalue (the notation N refers to the *Neumann* boundary condition) of the problem

$$\begin{cases} -\Delta u + m(x)u = \lambda u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + h(x)u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us summarize the main properties of $\lambda_1(-\Delta + m, N + h)$. For a proof, we refer to [8]:

Lemma 2.1. $\lambda_1(-\Delta+m, N+h)$ *is a simple eigenvalue, and any eigenfunction* φ *associated to* $\lambda_1(-\Delta + m, N + h)$ *satisfies* $\varphi \in C^{1,\gamma}(\overline{\Omega}) \cap H^2(\Omega)$, $\gamma \in (0, 1)$. *In addition:*

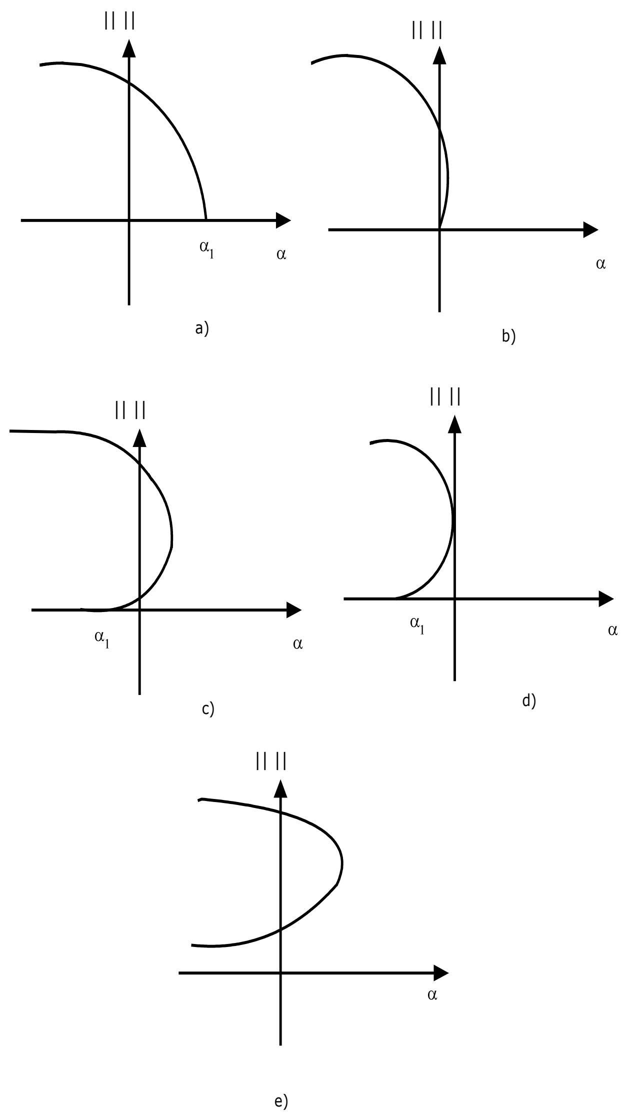

Figure 3: Bifurcation diagrams of (P): Case a) $\lambda_1(-\Delta - \lambda m, N) > 0$. Case b) $\lambda_1(-\Delta - \lambda m, N) = 0$. Case c) $\lambda_1(-\Delta - \lambda m, N) < 0 < \lambda_1(-\Delta - \lambda m, D)$ and $\lambda \neq -\lambda_1^+(0)$. Case d) $\lambda = -\lambda_1^+(0)$. Case e) $\lambda_1(-\Delta - \lambda m, D) \leq 0$.

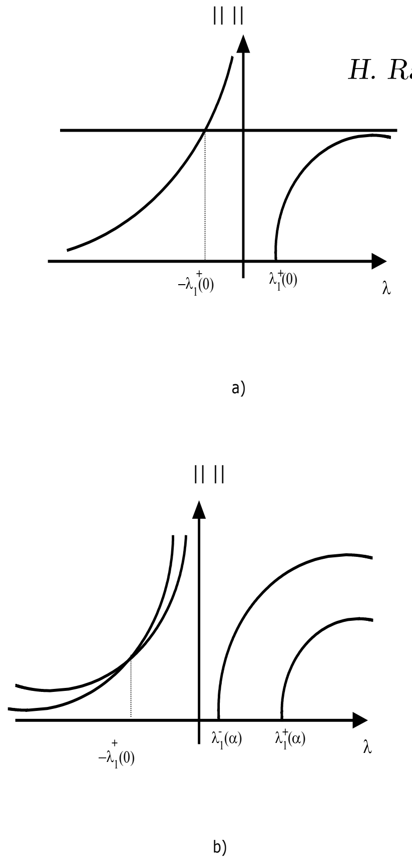

Figure 4: Bifurcation diagrams of (P): Case a) $\alpha = 0$. Case b) $\alpha > 0$ and small.

1. *$\lambda_1(-\Delta + m, N + h)$ is separately increasing with respect to m and h.*

2. *$\lambda_1(-\Delta+m, N+h) < \lambda_1(-\Delta+m, D)$ where $\lambda_1(-\Delta+m, D)$ stands for the principal eigenvalue of $-\Delta + m$ with homogeneous Dirichlet boundary conditions.*

3. *Assume that $G \subset \Omega$ is a proper regular subdomain of Ω, that is,*

$$dist(\partial\Omega, \partial G \cap \Omega) > 0,$$

and denote by $\lambda_1^G(-\Delta + m, N + h, D)$ the principal eigenvalue of

$$\begin{cases} -\Delta u + m(x)u = \lambda u & in\ G, \\ \dfrac{\partial u}{\partial \nu} + h(x)u = 0 & on\ \partial G \cap \partial\Omega, \\ u = 0 & on\ \partial G \cap \Omega. \end{cases}$$

Then

$$\lambda_1(-\Delta + m, N + h) < \lambda_1^G(-\Delta + m, N + h, D).$$

4. *There holds*

$$\begin{aligned} &\lim_{K\to-\infty} \lambda_1(-\Delta + m, N + K) = -\infty, \\ &\lim_{K\to+\infty} \lambda_1(-\Delta + m, N + K) = \lambda_1(-\Delta + m, D). \end{aligned} \tag{2.4}$$

Given $\lambda, \alpha \in \mathbb{R}$, we set

$$\mu(\lambda, \alpha) := \lambda_1(-\Delta - \lambda m, N - \alpha) \tag{2.5}$$

and

$$I_{\lambda,\alpha}(u) = \int_\Omega \left(|\nabla u|^2 - \lambda m(x)u^2\right) - \alpha \int_{\partial\Omega} u^2 \quad \text{for} \quad u \in H^1(\Omega).$$

Recall that

$$\mu(\lambda, \alpha) = \min\left\{ I_{\lambda,\alpha}(u);\ u \in H^1(\Omega), \int_\Omega u^2 = 1 \right\}.$$

This map has the following properties, which follow from Lemma 2.1 and [1, Lemma 2]:

Lemma 2.2.

1. *The map $\alpha \mapsto \mu(\lambda, \alpha)$ is decreasing on $\mathbb{R}$ and*

$$\begin{aligned} \lim_{\alpha \to +\infty} \mu(\lambda, \alpha) &= -\infty, \\ \lim_{\alpha \to -\infty} \mu(\lambda, \alpha) &= \lambda_1(-\Delta - \lambda m, D). \end{aligned} \tag{2.6}$$

2. *Assume that m changes sign. Then the map $\lambda \mapsto \mu(\lambda, \alpha)$ is concave on $\mathbb{R}$ and $\lim_{|\lambda| \to \infty} \mu(\lambda, \alpha) = -\infty$. Moreover, it is differentiable and*

$$\frac{d\mu}{d\lambda}(\lambda, \alpha) = -\int_\Omega m(x)\phi_{\lambda,\alpha}^2,$$

where $\phi_{\lambda,\alpha}$ is the eigenfunction achieving $\mu(\lambda, \alpha)$.

3. *For every $\alpha \in \mathbb{R}$ the map $\lambda \mapsto \mu(\lambda, \alpha)$ has an unique maximum point.*

We shall first consider (E) as an eigenvalue problem with respect to α. It is clear that, given $\lambda \in \mathbb{R}$, $\alpha_1(\lambda)$ is a principal eigenvalue of (E) if and only if $\mu(\lambda, \alpha_1(\lambda)) = 0$. From Lemma 2.2 we deduce:

Lemma 2.3. *Given $\lambda \in \mathbb{R}$, (E) has a principal eigenvalue $\alpha_1(\lambda)$ if and only if $\lambda_1(-\Delta - \lambda m, D) > 0$. In this case we have*

$$\alpha_1(\lambda) = \min\left\{ I_{\lambda,0}(u);\ u \in H^1(\Omega),\ \int_{\partial\Omega} u^2 = 1 \right\} \tag{2.7}$$

and

$$\alpha_1(\lambda) > 0 \quad (\textit{respect.} = 0, < 0) \quad \Longleftrightarrow \quad \lambda_1(-\Delta - \lambda m, N) > 0 \quad (\textit{respect.} = 0, < 0).$$

In Figure 5 we have depicted the map $\alpha \mapsto \mu(\lambda, \alpha)$ depending on the values of λ.

On the other hand, when dealing with (E) as an eigenvalue problem with respect to λ, we shall consider the maximum of the map $\lambda \mapsto \mu(\lambda, \alpha)$. We complement Lemma 2.2 providing an expression for this maximum, namely:

$$\mu_0(\alpha) := \inf\left\{ I_{0,\alpha}(u);\ u \in H^1(\Omega), \int_\Omega u^2 = 1, \int_\Omega m(x)u^2 = 0 \right\}. \tag{2.8}$$

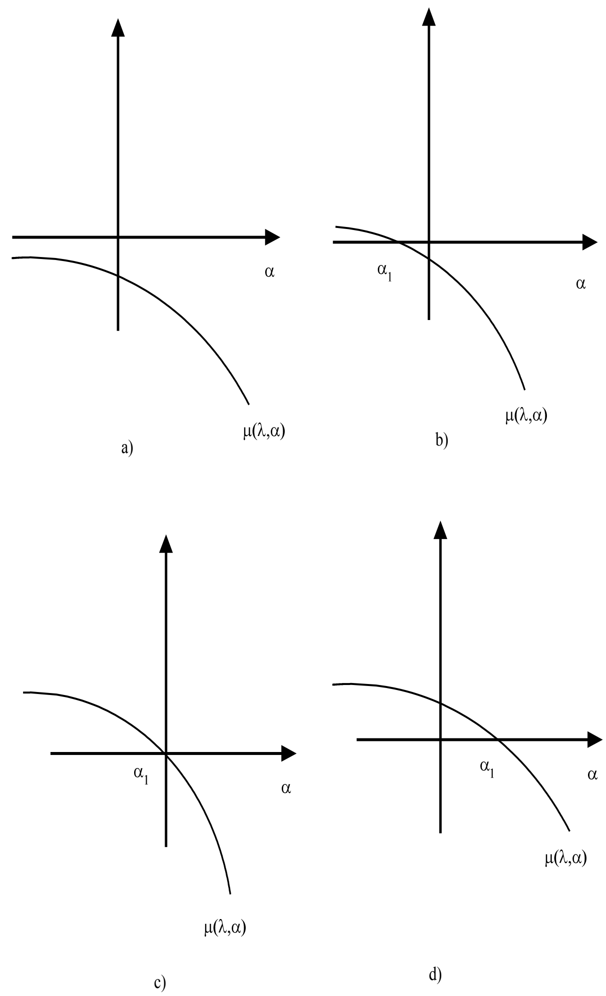

Figure 5: The map $\alpha \mapsto \mu(\lambda, \alpha)$: Case a) $\lambda_1(-\Delta - \lambda m, D) \leq 0$. Case b) $\lambda_1(-\Delta - \lambda m, N) < 0 < \lambda_1(-\Delta - \lambda m, D)$. Case c) $\lambda_1(-\Delta - \lambda m, N) = 0$. Case d) $\lambda_1(-\Delta - \lambda m, N) > 0$.

Lemma 2.4. *For every $\alpha \in \mathbb{R}$ there holds*

$$\max_{\lambda \in \mathbf{R}} \mu(\lambda, \alpha) = \mu_0(\alpha).$$

Proof. We know that $\lim_{|\lambda| \to \infty} \mu(\lambda, \alpha) = -\infty$ and $\lambda \mapsto \mu(\lambda, \alpha)$ is continuous, so that it has a global maximum achieved by some λ_0, i.e.

$$\max_{\lambda \in \mathbf{R}} \mu(\lambda, \alpha) = \mu(\lambda_0, \alpha).$$

We shall prove that $\mu(\lambda_0, \alpha) = \mu_0(\alpha)$. Since $\lambda \mapsto \mu(\lambda, \alpha)$ is differentiable and

$$\frac{d\mu}{d\lambda}(\lambda, \alpha) = -\int_\Omega m(x)\phi_{\lambda,\alpha}^2,$$

where $\phi_{\lambda,\alpha}$ is the eigenfunction achieving $\mu(\lambda, \alpha)$, we must have

$$\int_\Omega m(x)\phi_0^2 = 0,$$

where $\phi_0 = \phi_{\lambda_0,\alpha}$. Consequently

$$\mu(\lambda_0, \alpha) = I_{\lambda_0,\alpha}(\phi_0) = I_{0,\alpha}(\phi_0) \geq \mu_0(\alpha).$$

On the other hand, it is easily seen that $\mu_0(\alpha)$ is achieved by some u_0. Hence

$$\mu(\lambda_0, \alpha) \leq I_{\lambda_0,\alpha}(u_0) = I_{0,\alpha}(u_0) = \mu_0(\alpha),$$

and we get the conclusion. □

We are now in position to analyse the existence of zeros for the map $\lambda \mapsto \mu(\lambda, \alpha)$. The case $\alpha = 0$ (Neumann) is well-known, whereas the other cases were considered in [1], but we shall provide them a complete and unified description. We set

$$\alpha_0^* := \inf \left\{ \int_\Omega |\nabla u|^2;\ u \in H^1(\Omega), \int_\Omega m(x)u^2 = 0, \int_{\partial\Omega} u^2 = 1 \right\}. \tag{2.9}$$

Lemma 2.5. *Assume (1.1).*

1. *If $\alpha > \alpha_0^*$ then (E) has no principal eigenvalues.*
2. *If $\alpha = \alpha_0^*$ then (E) has a unique principal eigenvalue $\lambda_1(\alpha)$.*
3. *If $\alpha < \alpha_0^*$ then (E) has two principal eigenvalues $\lambda_1^-(\alpha) < \lambda_1^+(\alpha)$, given by*

$$\lambda_1^\pm(\alpha) = \pm \min \left\{ I_{0,\alpha}(u);\ u \in H^1(\Omega), \int_\Omega m(x)u^2 = \pm 1 \right\}. \tag{2.10}$$

Moreover:

(a) If $\alpha < 0$ then $\lambda_1^-(\alpha) < 0 < \lambda_1^+(\alpha)$.
(b) If $\alpha = 0$ then $\lambda_1^-(\alpha) = 0 < \lambda_1^+(\alpha)$.
(c) If $0 < \alpha < \alpha_0^*$ then $0 < \lambda_1^-(\alpha) < \lambda_1^+(\alpha)$.

Proof. Since λ is a principal eigenvalue of (E) if and only if $\mu(\lambda,\alpha)=0$, we shall look for the zeros of the map $\lambda\mapsto\mu(\lambda,\alpha)$. From Lemma 2.4, we know that $\max_{\lambda\in\mathbf{R}}\mu(\lambda,\alpha)=\mu_0(\alpha)$. Thus the condition $\mu_0(\alpha)\geq 0$ is necessary for the existence of principal eigenvalues of (E).

Note also from (2.8) that $\mu_0(\alpha)>0$ if $\alpha\leq 0$. Now, if $\alpha>0$ then $\mu_0(\alpha)\geq 0$ if and only if and only if $I_{0,\alpha}(u)>0$ for every $u\neq 0$ such that $\int_\Omega m(x)u^2=0$, i.e. if and only if $\alpha\leq\alpha_0^*$. Moreover, if $\alpha=\alpha_0^*$ then $\mu_0(\alpha)=0$ and, by Lemma 2.4, there is an unique λ_0 such that $\mu(\lambda_0,\alpha)=\mu_0(\alpha)$. We set $\lambda_1(\alpha)=\lambda_0$. On the other hand, if $\alpha<\alpha_0^*$ then $\lambda\mapsto\mu(\lambda,\alpha)$ vanishes at some $\lambda_1^-(\alpha)<\lambda_1^+(\alpha)$. Since

$$\mu(\lambda_1^-(\alpha),\alpha)=0 \quad\text{and}\quad \frac{d\mu}{d\lambda}(\lambda_1^-(\alpha),\alpha)>0,$$

we have, denoting $\phi=\phi_{\lambda_1^-(\alpha),\alpha}$,

$$I_{\lambda_1^-(\alpha),\alpha}(u)\geq 0 \text{ for every } u\in H^1(\Omega),\quad I_{\lambda_1^-(\alpha),\alpha}(\phi)=0 \quad\text{and}\quad \int_\Omega m(x)\phi^2<0.$$

Let $\psi=\left(-\int_\Omega m(x)\phi^2\right)^{-\frac{1}{2}}\phi$. Then $\int_\Omega m(x)\psi^2=-1$ and, from $I_{\lambda_1^-(\alpha),\alpha}(\phi)=0$, we get

$$-\lambda_1^-(\alpha)=I_{0,\alpha}(\psi).$$

Moreover, since $I_{\lambda_1^-(\alpha),\alpha}(u)\geq 0$ for every $u\in H^1(\Omega)$, we have in particular

$$-\lambda_1^-(\alpha)\leq I_{0,\alpha}(u) \quad\text{for every } u \text{ such that } \int_\Omega m(x)u^2=-1.$$

Thus

$$-\lambda_1^-(\alpha)=\min\left\{I_{0,\alpha}(u);\ u\in H^1(\Omega),\int_\Omega m(x)u^2=-1\right\}. \tag{2.11}$$

In a similar way, we can prove that

$$\lambda_1^+(\alpha)=\min\left\{I_{0,\alpha}(u);\ u\in H^1(\Omega),\int_\Omega m(x)u^2=1\right\}.$$

Finally, note from (2.11) that the map $\alpha\mapsto-\lambda_1^-(\alpha)$ is decreasing on $(-\infty,\alpha_0^*)$ and $\lambda_1^-(0)=0$, in view of (1.1). Therefore $\lambda_1^-(\alpha)>0$ if and only if $0<\alpha<\alpha_0^*$. In a similar way, $\alpha\mapsto\lambda_1^+(\alpha)$ is decreasing on $(-\infty,\alpha_0^*)$ and $\lambda_1^+(\alpha)>\lambda_1^-(\alpha)>0$ if $0<\alpha<\alpha_0^*$, so that $\lambda_1^+(\alpha)>0$ for every $\alpha<\alpha_0^*$. □

In the following result, we compare the maps $\alpha_1(\lambda)$ and $\lambda_1^\pm(\alpha)$. It can be easily proved using Lemmas 2.2 and 2.3, and the fact that, whenever $\alpha_1(\lambda)$ and $\lambda_1^\pm(\alpha)$ exist, we have

$$\alpha<\alpha_1(\lambda)\Leftrightarrow\mu(\lambda,\alpha)>0\Leftrightarrow\lambda_1^-(\alpha)<\lambda<\lambda_1^+(\alpha).$$

Lemma 2.6. *Assume (1.1) and $\alpha<\alpha_0^*$. Then*

1. $\alpha<\alpha_1(\lambda)\Leftrightarrow\lambda\in(\lambda_1^-(\alpha),\lambda_1^+(\alpha))$.
2. *If $\alpha>\alpha_1(\lambda)$ and $\lambda<0$, then $\lambda<\lambda_1^-(\alpha)$.*
3. *If $\alpha>\alpha_1(\lambda)$, $\lambda>0$ and:*
   (a) *$\alpha\leq 0$, then $\lambda>\lambda_1^+(\alpha)$.*
   (b) *$\alpha>0$, then either $0<\lambda<\lambda_1^-(\alpha)$ or $\lambda>\lambda_1^+(\alpha)$.*

3 Bifurcation with respect to α

Let us recall that a positive solution u_0 of (P) is stable if the principal eigenvalue of the linearisation of (P) at u_0 is positive, i.e.

$$\lambda_1(-\Delta - \lambda m + 2u_0\lambda m, N - \alpha) > 0.$$

Since $\alpha_1(\lambda)$ is a simple eigenvalue whenever it exists, i.e., if $\lambda \in (\lambda_1^-(D), \lambda_1^+(D))$, we can apply the classical Crandall-Rabinowitz Theorem [9] to deduce the following result:

Lemma 3.1. *Assume that there exists $\alpha_1(\lambda)$. Then:*

1. *The trivial solution $u \equiv 0$ is stable for $\alpha < \alpha_1(\lambda)$ and unstable for $\alpha > \alpha_1(\lambda)$.*
2. *The point $(\alpha, u) = (\alpha_1(\lambda), 0)$ is a bifurcation point from the trivial solution of (P). Moreover, there exist $\varepsilon > 0$ and two C^1 maps*

$$\alpha : (-\varepsilon, \varepsilon) \mapsto \mathbb{R} \quad \text{and} \quad v : (-\varepsilon, \varepsilon) \mapsto \langle \varphi_1 \rangle^\perp,$$

where φ_1 is a positive eigenfunction associated to $\alpha_1(\lambda)$, satisfying $\alpha(0) = \alpha_1(\lambda)$, $v(0) = 0$ and

$$\alpha(s) = \alpha_1(\lambda) + s\alpha_2 + o(s), \quad u(s) = s(\varphi_1 + v(s))$$

are such that $(\alpha(s), u(s))$ is the only solution of (P) in a neighborhood of $(\alpha_1(\lambda), 0)$. Moreover,

$$\alpha_2 = \frac{\lambda \displaystyle\int_\Omega m\varphi_1^3}{\displaystyle\int_\Omega \varphi_1^2}.$$

Consequently, for $\lambda \neq 0$, the bifurcation direction is supercritical (resp. subcritical) if $\alpha_2 > 0$ (resp. $\alpha_2 < 0$).

3. *If the bifurcation direction is supercritical (respect. subcritical) the new solution $u(s)$ is stable (respect. unstable).*
4. *There exists $\delta > 0$ such that $\alpha_2 > 0$ if $\lambda \in (\lambda_1^-(D) - \delta, 0) \cup (\lambda_1^+(0) - \delta, \lambda_1^+(D) + \delta)$. In particular $\alpha_2 > 0$ if $\alpha_1(\lambda) = 0$.*

Proof. Observe that $u \equiv 0$ is stable if $\lambda_1(-\Delta - \lambda m, N - \alpha) > 0$, that is $\alpha < \alpha_1(\lambda)$. The existence and properties of the maps $\alpha(s)$ and $u(s)$ follow by the Crandall-Rabinowitz Theorem. In addition, since $(\alpha(s), u(s))$ solve (P), we have

$$\begin{cases} -\Delta((\varphi_1 + v(s))) = \lambda m(x)(\varphi_1 + v(s))(1 - s(\varphi_1 + v(s))) & \text{in } \Omega, \\ \dfrac{\partial(\varphi_1 + v(s))}{\partial\nu} = (\alpha_1(\lambda) + s\alpha_2 + o(s))(\varphi_1 + v(s)) & \text{on } \partial\Omega. \end{cases}$$

We can write $v(s) = sv_1 + s^2 v_2 + o(s^2)$ for $s \simeq 0$. Plugging this expression in the above equation and rearranging the terms in s, we get

$$-\Delta v_1 - \lambda m(x)v_1 = -\lambda m(x)\varphi_1^2 \quad \text{in } \Omega, \quad \frac{\partial v_1}{\partial\nu} - \alpha_1(\lambda)v_1 = \alpha_2\varphi_1 \quad \text{on } \partial\Omega,$$

and multiplying by φ_1, we get

$$\alpha_2 = \frac{\lambda \displaystyle\int_\Omega m\varphi_1^3}{\displaystyle\int_\Omega \varphi_1^2}.$$

Now, since

$$-\Delta\varphi_1 - \lambda m(x)\varphi_1 = 0 \quad \text{in } \Omega, \quad \frac{\partial \varphi_1}{\partial \nu} = \alpha_1(\lambda)\varphi_1 \quad \text{on } \partial\Omega,$$

multiplying by φ_1^2, we get

$$2\int_\Omega \varphi_1|\nabla\varphi_1|^2 - \alpha_1(\lambda)\int_{\partial\Omega}\varphi_1^3 = \lambda\int_\Omega m\varphi_1^3.$$

Recall that $\alpha_1(\lambda) < 0$ for $\lambda \in (\lambda_1^-(D), 0 = \lambda_1^-(0)) \cup (\lambda_1^+(0), \lambda_1^+(D))$. Thus, by continuity of $\alpha_1(\lambda)$ and φ_1 with respect to λ (see for instance [8]), there exists $\delta > 0$ such that $\alpha_2 > 0$ for $\lambda \in (\lambda_1^-(D) - \delta, 0) \cup (\lambda_1^+(0) - \delta, \lambda_1^+(D) + \delta)$. In particular, $\alpha_2 > 0$ when $\alpha_1(\lambda) = 0$, that is, when $\lambda = \lambda_1^+(0)$.

Finally, the stability of the new solution $u(s)$ follows by [10].

□

The following result has a global character (see [15]):

Lemma 3.2. *Whenever $\alpha_1(\lambda)$ exists, there is an unbounded continuum $\mathcal{C}$ of positive solutions of (P) emanating from $(\alpha, u) = (\alpha_1(\lambda), 0)$.*

In the following result, we prove that (P) has no positive solutions for α large and independent of λ. Let α_0 be the principal eigenvalue of

$$\begin{cases} -\Delta u = 0 & \text{in } M_0, \\ \dfrac{\partial u}{\partial \nu} = \alpha u & \text{on } \partial M_0 \cap \partial\Omega, \\ u = 0 & \text{on } \partial M_0 \setminus \partial\Omega, \end{cases} \tag{3.12}$$

where M_0 is given in (1.2).

Remark 3.3. *Note that (3.12) has indeed a principal eigenvalue α_0. This can be proved in the same way as the existence of $\alpha_1(\lambda)$ in Lemma 2.3. As a matter of fact, if we denote by $\mu_1(\alpha)$ the principal eigenvalue of the problem*

$$\begin{cases} -\Delta u = \mu_1(\alpha)u & \textit{in } M_0, \\ \dfrac{\partial u}{\partial \nu} - \alpha u = 0 & \textit{on } \partial M_0 \cap \partial\Omega, \\ u = 0 & \textit{on } \partial M_0 \setminus \partial\Omega, \end{cases} \tag{3.13}$$

then α_0 is a principal eigenvalue of (3.12) if and only if $\mu_1(\alpha_0) = 0$. Now, since $\alpha \mapsto \mu_1(\alpha)$ is decreasing, $\mu_1(0) > 0$ and $\lim_{\alpha\to\infty}\mu_1(\alpha) = -\infty$, we deduce the existence and uniqueness of α_0.

Lemma 3.4. *If $\alpha \geq \alpha_0$ then (P) has no positive solution.*

Proof. Let u be a positive solution of (P). Since M_0 is a proper subdomain of Ω, by Lemma 2.1 we have

$$0 = \lambda_1(-\Delta - \lambda m + \lambda m u, N - \alpha) \leq \lambda_1^{M_0}(-\Delta, N - \alpha, D),$$

which implies $\alpha < \alpha_0$. □

We shall take advantage of the results known for (P) when $\alpha = 0$, as shown in [6]:

Lemma 3.5. *Assume (1.1) and $\alpha = 0$. Then:*

1. *(P) has two trivial solutions, $u \equiv 0$ and $u \equiv 1$, for all $\lambda \in \mathbb{R}$. Moreover, $u \equiv 0$ is stable for $\lambda \in (0, \lambda_1^+(0))$ and $u \equiv 1$ is stable for $\lambda \in (-\lambda_1^+(0), 0)$.*
2. *(P) has a stable positive solution u_λ for $\lambda \in (-\infty, -\lambda_1^+(0)) \cup (\lambda_1^+(0), +\infty)$. Moreover, $u_\lambda < 1$ and this is the only positive solution of (P) satisfying $u < 1$.*

Proof.

1. It is clear that $u = 0$ and $u = 1$ solve (P). The stability of $u = 0$ follows by Theorem 3 in [6], whereas the linearized problem around $u = 1$ is

$$-\Delta w = (-\lambda) m(x) w \quad \text{in } \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

   for which the first eigenvalue is positive if and only $-\lambda \in (0, \lambda_1^+(0))$.
2. The case $\lambda \in (\lambda_1^+(0), +\infty)$ follows by [6]. On the other hand, observe that $w = 1 - u$ verifies

$$-\Delta w = (-\lambda) m(x) w(1 - w) \quad \text{in } \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

   so that, by [6], there exists a unique stable solution $0 < w_\lambda < 1$ for $-\lambda \in (\lambda_1^+(0), +\infty)$, that is, for $\lambda \in (-\infty, -\lambda_1^+(0))$.

□

Let us set $\mathcal{E} := \mathcal{C}(\overline{\Omega})$, $\mathcal{P} := \{u \in \mathcal{E}; u \geq 0\}$ and

$$\Sigma =: \{(\alpha, u) \in \mathbb{R} \times \mathcal{P}; u \text{ is a positive solution of } (P)\}.$$

Lemma 3.6. *Assume that u_0 is a positive stable solution of (P) for $\alpha = 0$. Then:*

1. *There exist $\varepsilon > 0$ and a neighborhood $\mathcal{U} \subset \mathbb{R} \times \mathcal{P}$ such that $\mathcal{U} \cap \Sigma = \{(\alpha, u_\alpha); \alpha \in (-\varepsilon, \varepsilon)\}$. Morever, u_α is stable for $\alpha \in (-\varepsilon, \varepsilon)$.*
2. *There exists an unbounded continuum $\mathcal{C}_0$ of positive solutions of (P) containing $(0, u_0)$. Moreover, if we assume that there exist a priori bounds for positive solutions of (P) whenever α varies in a compact set, then there exists $\mathcal{F} \subset \mathcal{P}$ such that*

$$\mathcal{C}_0 \cap (\{0\} \times \mathcal{F}) \neq \emptyset, \quad \mathcal{C}_0 \cap (\{0\} \times (\mathcal{P} \setminus \mathcal{F})) \neq \emptyset.$$

Proof. Since u_0 is stable, the first result follows by Proposition 20.6 in [3] and the existence of $\mathcal{C}_0$ containing $(0, u_0)$ by Theorem 17.1 in [3]. The second paragraph is a consequence of the first one and Lemma 3.4. See also [5], Theorems 4.4.1 and 4.4.2 for the first and second paragraphs, respectively. □

3.1 A priori bounds

In this section we get *a priori* bounds for positive solutions of (P) when α belongs to a compact set of $\mathbb{R}$.

Proposition 3.7. *Let $\lambda > 0$. Assume that there exist a function $h^- : \overline{M_-} \mapsto \mathbb{R}^+$, continuous and bounded away from zero in a neighborhood of ∂M_-, and a constant $\gamma_- \geq 0$ such that*

$$m^-(x) = h^-(x)(dist(x, \partial M_-))^{\gamma_-} \qquad \text{in } M_-.$$

Assume in addition

$$2 < \min\left\{\frac{N+1+\gamma_-}{N-1}, \frac{N+2}{N-2}\right\} \qquad \text{if } N \geq 3. \tag{3.14}$$

Then, for every compact interval $\Lambda \subset \mathbb{R}$ there exists a positive constant C such that

$$\|u\|_\infty \leq C,$$

for any positive solution u of (P) with $\alpha \in \Lambda$.

Proof. First note that if (P) has positive solution u then, by Lemma 3.4, we must have $\alpha < \alpha_0$.

We split the proof in two steps.

Step 1: A priori bounds on $\overline{M_-}$. For this step, we use (3.14), an adequate rescaling Gidas-Spruck argument and a Liouville type theorem, see exactly Lemma 4.2 and Theorem 4.3 of [4].

Step 2: A priori bounds on Ω. Define

$$R := \sup_{\alpha \in \Lambda} \sup_{x \in \overline{M}_-} u(x) < \infty.$$

We consider the problem

$$\begin{cases} -\Delta u = \lambda m(x) u(1-u) & \text{in } \Omega \backslash \overline{M_-}, \\ u = R & \text{on } \partial(M_-)\backslash \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = \alpha u & \text{on } \partial(\Omega \setminus \overline{M}_-) \cap \partial\Omega. \end{cases} \tag{3.15}$$

We claim that there exists a unique positive solution U of (3.15) for all $\alpha < \alpha_0$. In this case it is clear that a solution u of (P) is a subsolution of (3.15) in $\Omega\backslash\overline{M_-}$. By the uniqueness of the positive solution of (3.15) we get

$$\|u\|_{L^\infty(\Omega\backslash\overline{M}_-)} \leq \|U\|_{L^\infty(\Omega\backslash\overline{M}_-)},$$

whence the result follows.

It remains to prove the claim. We use the sub-supersolution method to obtain U. Indeed, $\underline{u} := 0$ is a subsolution of (3.15). Now, set

$$M_\delta := \{x \in \Omega : dist(x, M_0) < \delta\},$$

for $\delta > 0$, and consider the eigenvalue problem

$$\begin{cases} -\Delta u = 0 & \text{in } M_\delta, \\ u = 0 & \text{on } \partial M_\delta \cap \Omega, \\ \dfrac{\partial u}{\partial \nu} = \alpha u & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

Thanks to Remark 3.3 there exists a principal eigenvalue $\alpha_1(\delta)$ of (3.16), and φ_δ a positive eigenfunction associated to $\alpha_1(\delta)$. Now, we can show that M_δ is a sequence of bounded and regular domains converging to M_0 from the exterior in the sense of [8]. So, by Theorem 7.1 in [8], we conclude that

$$\alpha_1(\delta) \uparrow \alpha_0 \quad \text{as } \delta \uparrow 0.$$

Take $\alpha < \alpha_0$ and consider δ such that $\alpha < \alpha_1(\delta) < \alpha_0$. Now, define

$$\Psi := \begin{cases} \varphi_\delta & \text{in } M_{\delta/2} \cap (\Omega \setminus \overline{M}_-), \\ \psi & \text{in } M_+ \setminus M_{\delta/2}, \end{cases}$$

where ψ is a smooth and positive function such that Ψ is smooth. Then, $\overline{u} := K\Psi$ is a supersolution of (3.15) for K sufficiently large. Indeed, it is clear that $K\Psi$ is supersolution in M_0 because $-\Delta(K\Psi) = 0$ in M_0. In $M_+ \cap M_{\delta/2}$ we have

$$-\Delta(K\Psi) = 0 \geq \lambda m(x) K\varphi_\delta(1 - K\varphi_\delta) \quad \text{for } K \text{ large.}$$

Moreover, in $M_+ \setminus M_{\delta/2}$ we get

$$-\Delta(K\Psi) = K(-\Delta(\psi)) \geq \lambda m(x) K\psi(1 - K\psi) \quad \text{for } K \text{ large.}$$

On ∂M_-, we take K such that $K\Psi \geq R$. Thus, it is clear that

$$\frac{\partial \overline{u}}{\partial \nu} = \frac{\partial(K\Psi)}{\partial \nu} = K\frac{\partial \varphi_d}{\partial \nu} = K\alpha_1\varphi_\delta > \alpha\overline{u} \quad \text{on } \partial\Omega.$$

Finally, the uniqueness follows by Theorem 1.2 in [16]. □

By symmetry on λ, we deduce the following *a priori bounds* result for positive solutions of (P).

Theorem 3.8. *Let $\lambda \neq 0$. Assume that there exist two functions $h^\pm : \overline{M_\pm} \mapsto \mathbb{R}^+$, continuous and bounded away from zero in a neighborhood of $\partial M_\pm$, and constants $\gamma_\pm \geq 0$ such that*

$$m^\pm(x) = h^\pm(x)(dist(x, \partial M_\pm))^{\gamma_\pm} \qquad \text{in } M_\pm.$$

Assume in addition

$$2 < \min\left\{\frac{N+1+\gamma_\pm}{N-1}, \frac{N+2}{N-2}\right\} \qquad \text{if } N \geq 3.$$

Then, for every compact interval $\Lambda \subset \mathbb{R}$ there exists a positive constant C such that

$$\|u\|_\infty \leq C,$$

for any positive solution u of (P) with $\alpha \in \Lambda$.

4 Proof of the main results

Before proving our main results, we need the following result (recall the definition of $\mu(\lambda, \alpha)$ in (2.5)).

Lemma 4.1.

1. *Assume that there exists a positive solution u_* of (P) for $\alpha = \alpha_*$. Then, there exists a positive solution for every $\alpha < \alpha_*$ such that $\mu(\lambda, \alpha) < 0$.*

2. *Assume that there exists a positive solution u_0 of (P) for $\alpha = 0$. Then, there exists a positive solution u_α for all $\alpha < 0$ such that $\mu(\lambda, \alpha) < 0$. Moreover, if $u_0 \le 1$ then u_α is stable.*

Proof.

1. We use the sub-supersolution method. Consider φ a positive eigenfunction associated to $\mu(\lambda, \alpha)$. Take as pair of sub-supersolution $(\underline{u}, \overline{u}) = (\varepsilon\varphi, u_*)$, with $\varepsilon > 0$. It is easily seen that $\underline{u}$ is a sub-solution of (P) if
$$\mu(\lambda, \alpha) + \lambda m(x)\varepsilon\varphi \le 0,$$
which holds for ε small enough. Then, there exists a positive solution $u_\alpha \in (\underline{u}, u_*)$.

2. The existence of u_α follows by the previous item. Assume that $u_0 \le 1$. Since $u \equiv 1$ is not solution of (P) for $\alpha < 0$, we have $u_\alpha < 1$. Now, we show that u_α is stable, i.e.
$$\lambda_1(-\Delta - \lambda m(x)(1 - 2u_\alpha), N - \alpha) > 0. \tag{4.17}$$
To this end, we prove the existence of a positive supersolution for the operator $(-\Delta - \lambda m(x)(1 - 2u_\alpha), N - \alpha)$. Take $\overline{u} := f(u_\alpha)$ where $f(u_\alpha) = u_\alpha(1 - u_\alpha) > 0$, see [6]. Then, it is clear that
$$-\Delta\overline{u} - \lambda m(x)(1 - 2u_\alpha)\overline{u} = -f''(u_\alpha)|\nabla u_\alpha|^2 > 0 \quad \text{in } \Omega,$$
and
$$\frac{\partial \overline{u}}{\partial \nu} - \alpha\overline{u} = -2\alpha u_\alpha^2 > 0 \quad \text{on } \partial\Omega.$$
This proves that u_α is stable.

□

We are now ready to prove Theorem 1.1:

Proof of Theorem 1.1.

1. Assume that $\lambda_1(-\Delta - \lambda m, D) > 0$. By Lemma 2.3 we know that (E) has a principal eigenvalue $\alpha_1(\lambda)$ and from Lemma 3.2 there exists an unbounded continuum $\mathcal{C}$ of positive solutions of (P) emanating from $(\alpha, u) = (\alpha_1(\lambda), 0)$. On the other hand, by Lemma 3.4 there is no positive solution of (P) for $\alpha \ge \alpha_0$. Moreover, Theorem 3.8 provides us with *a priori bounds* for positive solutions of (P), so we conclude the existence of positive solutions of (P) for all $\alpha < \alpha_1$.

Now, we set

$$\alpha_* := \sup\{\alpha \in \mathbb{R} : \ (P) \text{ has a positive solution}\}$$

It is clear that $\alpha_* < \infty$. Thanks to the *a priori bounds*, we infer the existence of a non-negative solution of (P) for $\alpha = \alpha_*$, which we denote by u_*.

If $\alpha_* = \alpha_1(\lambda)$ we conclude the existence of positive solution for $\alpha < \alpha_*$ and no positive solution for $\alpha > \alpha_*$.

Assume that $\alpha_* > \alpha_1(\lambda)$. In this case, since $\alpha_1(\lambda)$ is the unique bifurcation point from the trivial solution, we can show that $u_* > 0$. Now, by Lemma 4.1, we know that (P) has a positive solution u_α for every $\alpha < \alpha_*$ such that $\mu(\lambda, \alpha) < 0$, that is, for $\alpha \in (\alpha_1, \alpha_*)$.

On the other hand, by Lemma 3.1 the solution u_α is stable for $\alpha \in (\alpha_1(\lambda), \alpha_1(\lambda)+\delta)$ for some $\delta > 0$. This implies the existence of two positive solutions of (P) for $\alpha \in (\alpha_1(\lambda), \alpha_1(\lambda) + \delta)$ and the existence of positive solution for all $\alpha \leq \alpha_*$.

(a) Assume that $\lambda_1(-\Delta - \lambda m, N) > 0$. Then, in this case $\alpha_1(\lambda) > 0$ and so $\alpha_* > 0$.

(b) If $\lambda_1(-\Delta - \lambda m, N) = 0$ then $\alpha_1(\lambda) = 0$. In this case, by Lemma 3.1, the direction of bifurcation is supercritical, and so again $\alpha_* > 0 = \alpha_1(\lambda)$.

(c) Assume now that $\lambda_1(-\Delta - \lambda m, N) < 0$, that is, $\lambda > \lambda_1^+(0)$ or $\lambda < \lambda_1^-(0) = 0$ and $\lambda \neq -\lambda_1^+(0)$. Recall that in this case $\alpha_1 < 0$. Hence, by Lemma 3.5, for $\alpha = 0$ there exists a stable solution $u_0 \leq 1$ of (P). By Lemma 4.1, we have a stable positive solution u_α for all $\alpha \in (\alpha_1, 0]$. By continuity, we have a stable solution, still denoted u_α, for $\alpha \in (\alpha_1, \alpha_{**})$. Now, in view of non-existence of solutions for large α, the continuum $\mathcal{C}_0$ has to turn backwards, and so we conclude the existence of a second solution, w_α, for $\alpha \in (\alpha_1, \alpha_{**})$.

(d) Finally, assume that $\lambda = -\lambda_1^+(0)$. Again we have $\alpha_1(\lambda) < 0$ and by Lemma 3.5 for $\alpha = 0$ there exists the trivial solution $u_0 \equiv 1$ of (P). By Lemma 4.1, we have a stable positive solution u_α for all $\alpha \in (\alpha_1(\lambda), 0]$. Hence, in this case, $\alpha_{**} \geq 0$.

2. Assume now that $\lambda_1(-\Delta - \lambda m, D) \leq 0$, which implies that $\lambda_1(-\Delta - \lambda m, N) < 0$. In this case, $\alpha_1(\lambda)$ does not exist. However, by Lemma 3.5, for $\alpha = 0$ there exists a stable solution $u_0 \leq 1$ of (P), and consequently, by Lemma 3.6, there exists an unbounded continuum $\mathcal{C}_0$ containing $(0, u_0)$ and at least a positive solution for $\alpha \in (-\varepsilon, \varepsilon)$. We set α_*as in the previous case. Now, by Lemma 4.1, there exists a positive solution for every $\alpha < \alpha_*$ such that $\mu(\lambda, \alpha) < 0$, that is, for all $\alpha < \alpha_*$. Indeed, since $\lambda_1(-\Delta - \lambda m, D) \leq 0$, we have $\mu(\lambda, \alpha) < 0$ for all α. Moreover, there exists a stable solution for all $\alpha \in (-\infty, \alpha_{**})$. This implies the existence of a second solution in this interval.

□

Proof of Theorem 1.2.

Items 1 and 2 follow by Lemma 3.5. Moreover, for $\lambda \in (-\lambda_1^+(0), 0)$ there holds

$$\lambda_1(-\Delta - \lambda m, N) < 0 < \lambda_1(-\Delta - \lambda m, D)$$

so that, by Theorem 1.1, there exist two positive solutions for $\alpha = 0$. □

Proof of Theorem 1.3.

1. This item follows directly from Lemma 3.4.

2. Assume that $\lambda < 0$, $\lambda \neq -\lambda_1^+(0)$. In this case, $\lambda_1(-\Delta - \lambda m, N) < 0$ and applying Theorem 1.1 (in both cases $\lambda_1(-\Delta - \lambda m, D) \leq 0$ and $\lambda_1(-\Delta - \lambda m, D) > 0$) we conclude that there exist two positive solutions of (P) for α small enough.

3. Assume now $\lambda \in (\lambda_1^-(\alpha), \lambda_1^+(\alpha))$. Then $\lambda_1(-\Delta - \lambda m, N) > 0$, so $\alpha_1(\lambda) > 0$ and there exists at least a positive solution for $\alpha < \alpha_1(\lambda)$, that is, for $\lambda \in (\lambda_1^-(\alpha), \lambda_1^+(\alpha))$, by Lemma 2.6.

4. Assume that $\lambda > \lambda_1^+(0)$. In this case $\lambda_1(-\Delta - \lambda m, N) < 0$ and again by Theorem 1.1 we conclude that (P) has at least two positive solutions for α small enough. Finally, for $\lambda = \lambda_1^+(0)$ we have $\alpha_1(\lambda) = 0$, so that for α sufficiently small, (P) has at least two positive solutions.

□

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