The Oseen and Navier-Stokes equations in a non-solenoidal framework.

Chérif Amrouche & María Ángeles Rodríguez-Bellido
Laboratoire de Mathématiques et de Leurs Applications, CNRS UMR 5142,
Université de Pau et des Pays de l'Adour, IPRA, Pau (France)
and
Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Aptdo. 1160, 41080 Sevilla (Spain).
cherif.amrouche@univ-pau.fr, angeles@us.es

Abstract

The very weak solution for the Stokes, Oseen and Navier-Stokes equations has been studied by several authors in the last decades in domains of $\mathbb{R}^n$, $n \geq 2$. The authors studied the Oseen and Navier-Stokes problems assuming a solenoidal convective velocity in a bounded domain $\Omega \subset \mathbb{R}^3$ of class $\mathcal{C}^{1,1}$ for $\boldsymbol{v} \in \mathbf{L}^s(\Omega)$ for $s \geq 3$ in some previous papers. The results for the Navier-Stokes equations were obtained by using a fixed-point argument over the Oseen problem. These results improve those of Galdi et al. , Farwig et al. and Kim for the Navier-Stokes equations, because a less regular domain $\Omega \subset \mathbb{R}^3$ and more general hypothesis on the data are considered. In particular, the external forces must not be small.

In this work, existence of weak, strong, regularised and very weak solution for the Oseen problem are proved, mainly assuming that $\boldsymbol{v} \in \mathbf{L}^3(\Omega)$ and its divergence $\nabla \cdot \boldsymbol{v}$ is sufficiently small in the $W^{-1,3}(\Omega)$-norm. In this sense, one extends the analysis made by the authors for a given solenoidal $\boldsymbol{v}$ in some previous papers. As a consequence, the existence of very weak solution for the Navier-Stokes problem $(\mathbf{u}, \pi) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)/\mathbb{R}$ for a non-zero divergence condition is obtained in the 3D case.

Keywords: Oseen equations, Navier-Stokes equations; Very weak solutions; Stationary Solutions.

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1 Introduction

Let Ω be a bounded domain (an open connected set) of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$, with boundary Γ. We want to study the regularity for the solution $(\boldsymbol{u}, \pi)$ for the Oseen (O) and Navier-Stokes (NS)

equations:

$$(O) - \Delta \boldsymbol{u} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u} = h \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma,$$

$$(NS) - \Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u} = h \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma,$$

where $\boldsymbol{u}$ denotes the velocity and π the pressure and both are unknown, $\boldsymbol{f}$ the external forces, h the compressibility condition and $\boldsymbol{g}$ the boundary condition for the velocity, the three functions being known. In the case of the Oseen equation, the given velocity $\boldsymbol{v}$ belongs to $\mathbf{L}^s(\Omega)$ ($s \geq 3$).

The vector fields and matrix fields (and the corresponding spaces) defined over Ω or over $\mathbb{R}^3$ are respectively denoted by boldface Roman and special Roman.

In the case of incompressible fluids, $h = 0$, it has been well-known since Leray [19] (see also [20]) that if $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ with $p \geq 2$ and for any $i = 0, \ldots, I$,

$$\int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{n} \, d\sigma = 0, \tag{1.1}$$

where Γ_i denote the connected components of the boundary Γ of the open set Ω, then there exists a solution $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfying (NS). In [26], Serre proved the existence of weak solution $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ for any $\frac{3}{2} < p < 2$ when $h = 0$ and $\boldsymbol{g}$ satisfy the above conditions. More recently, Kim [18] improves Serre's existence and regularity results on weak solutions of (NS) for any $\frac{3}{2} \leq p < 2$ (including the case $p = \frac{3}{2}$), when the boundary of Ω is connected ($I = 0$) provided h is small in an appropriate norm (due to the compatibility condition between h and $\boldsymbol{g}$, then $\boldsymbol{g}$ is also small in the corresponding appropriate norm).

As to our knowledge, the notion of very weak solutions $(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ for Stokes or Navier-Stokes equations, corresponding to very irregular data, has been developed in the last years by Giga [17] (in a domain Ω of class $\mathcal{C}^\infty$), Amrouche & Girault [4] (in a domain Ω of class $\mathcal{C}^{1,1}$) and by Galdi et al. [16], Farwig et al. [13] (in a domain Ω of class $\mathcal{C}^{2,1}$, see also Schumacher [25]) and Kim [18] (in a domain of class $\mathcal{C}^2$). The choice of the space for the boundary condition $\boldsymbol{g}$ is made differently: $\boldsymbol{g} \in \mathbf{L}^p(\Gamma)$ (see Brown & Shen [10], Conca [11], Fabes et al. [12], Moussaoui [22], Shen [27], Savaré [24], Marusic-Paloka [21]) or more generally $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$. For the non-stationary case, the existence, uniqueness and regularity of very weak solutions for the Navier-Stokes equations have been investigated (among other authors) by Amann [1, 2].

In the Navier-Stokes case, the existence of very weak solution $\boldsymbol{u} \in \mathbf{L}^{2n/(n-1)}(\Omega)$, $n = 2, 3$, for arbitrary large external forces $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$, $h = 0$, arbitrary large boundary condition $\boldsymbol{g} \in \mathbf{L}^2(\Gamma)$ and without assuming condition (1.1), was proved first by Marusic-Paloka in Theorem 5 of [21] with Ω a bounded simply-connected open set of class $\mathcal{C}^{1,1}$. But the proof of such theorem becomes correct only if either condition (1.1) or smallness condition similar to (3.95) hold. The result of existence of very weak solution $(\boldsymbol{u}, \pi) \in \mathbf{L}^q(\Omega) \times W^{-1,q}(\Omega)$ was proved by Kim [18] in $\mathcal{C}^2$-domains of $\mathbb{R}^n$, $n = 2, 3, 4$, for arbitrary large external forces $\boldsymbol{f} \in [\mathbf{W}_0^{1,q'}(\Omega) \cap W^{2,q'}(\Omega)]'$, for small $h \in [W^{1,q'}(\Omega)]'$ and $\boldsymbol{g} \in \mathbf{W}^{-1/q,q}(\Gamma)$ for

$$q_0 \leq q < \infty, \quad q_0 = n \quad \text{if } n \geq 3, \quad 2 < q_0 < 3 \quad \text{if } n = 2$$

and where the boundary of Ω is supposed connected ($I = 0$). Our results improve those of Kim considering best spaces for the data $\boldsymbol{f}$ and $\boldsymbol{g}$ (see Remark 2 in [7]). Moreover, the very weak solution $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$ for the 2-dimensional case is obtained in [9] for the solenoidal case.

Similar results on the existence of very weak solution for Stokes and Navier-Stokes equations were obtained by Galdi et al. in [16], Farwig et al. in [14] for the n-dimensional case ($n \geq 3$) and in [13] for the 2-dimensional case. They consider a more regular domain $\mathcal{C}^{2,1}$ and the hypothesis on the data $\boldsymbol{f} = \nabla \cdot \mathbf{F}$ with $\mathbf{F} \in \mathbb{L}^r(\Omega)$, $h \in L^r(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/q,q}(\partial\Omega)$ for

$$\left.\begin{array}{ll} n \leq q < \infty & \text{if } n \geq 3, \\ & \\ 2 < q < \infty & \text{if } n = 2, \end{array}\right\} \quad q' < r \leq q, \quad \frac{1}{r} \leq \frac{1}{q} + \frac{1}{n}$$

and smallness assumptions for all the data $\boldsymbol{f}$, h and $\boldsymbol{g}$. In our case, the data are more general, the smallness assumptions are only demanded for h and $\boldsymbol{g}$.

In some previous papers ([7, 8, 9]), the authors studied the regularity for the Stokes, Oseen and Navier-Stokes equations for regular and singular data in the 2-dimensional and 3-dimensional cases (the case of the Stokes problem in a bounded domain of $\mathbb{R}^n$, $n \geq 2$, was treated in [8]). In all these works, the convective velocity $\boldsymbol{v}$ and the Navier-Stokes velocity field $\boldsymbol{u}$ were considered solenoidal.

In this work, we want to analyse the Oseen equation in the 3-dimensional case for a non-solenoidal convective velocity $\boldsymbol{v}$ whose divergence $\nabla \cdot \boldsymbol{v}$ is sufficiently small in an adequate norm (smallness condition will be necessary in order to obtain the existence of solution). The existence of solution in this framework is not known for the authors, and generalizes the results existing for the solenoidal case ($\nabla \cdot \boldsymbol{v} = 0$). As a consequence, using a fixed point argument, the existence of very weak solution for the Navier-Stokes equations is proved. This result was also treated in [7], but here the estimates are improved due to the best knowledge about the Oseen equation for the solenoidal and non-solenoidal cases. This new knowledge of the non-solenoidal case is very interesting when studying compressible Navier-Stokes equations. In the proofs of such results, we will use the ideas developed in [4] (for bounded domains) and in [6, 5] (for the half-space and whole space $\mathbb{R}^3$) about the existence of very weak solutions for the stationary Stokes equations and linearized Navier-Stokes equations. For questions related to the rigorous definition for the traces of the vector functions living in subspaces of $\mathbf{L}^p(\Omega)$ and the density lemmas, the reader can consult the results appearing in [7].

The work is organised as follows: In the rest of this section, we will set the space framework, including space definitions and trace spaces, together with compatibility conditions for the Oseen and Navier-Stokes problems. Section 2 is devoted to the existence of solution $(\boldsymbol{u}, \pi)$ for the Oseen problem in $\mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$, strong solution in $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ for $p > 1$, generalised solution in $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ and very weak solution in $\mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$, assuming that the velocity $\boldsymbol{v}$ (in almost all the cases) belongs to $\mathbf{L}^3(\Omega)$ and $\nabla \cdot \boldsymbol{v}$ belongs to $W^{-1,3}(\Omega)$ whose norm in its respective spaces is sufficiently small. The Navier-Stokes case will be studied in Section 3 using a fixed point argument on the Oseen problem, correcting the proof made in [7] which was only valid for solenoidal velocities $\boldsymbol{v}$. As in [7], the case of small data will be considered first, and the smallness hypothesis on $\boldsymbol{f}$ will be removed later.

1.1 Space framework

The space related to the existence of very weak solution is:

$$\mathbf{X}_{r,p}(\Omega) = \{\boldsymbol{\varphi} \in \mathbf{W}_0^{1,r}(\Omega);\ \nabla \cdot \boldsymbol{\varphi} \in W_0^{1,p}(\Omega)\}, \quad 1 < r,\, p < \infty,$$

and we set $\mathbf{X}_{p,p}(\Omega) = \mathbf{X}_p(\Omega)$,being $\mathbf{X}_p(\Omega)$ the space appearing in [3, 4]. Their dual space $[\mathbf{X}_{r,p}(\Omega)]'$ is characterised by the following result:

Lemma 1.1 *(See [7]) Let* $\boldsymbol{f} \in [\mathbf{X}_{r,p}(\Omega)]'$. *Then, there exist* $\mathbb{F}_0 = (f_{ij})_{1\leq i,j\leq 3}$ *such that* $\mathbb{F}_0 \in \mathbb{L}^{r'}(\Omega)$, $f_1 \in W^{-1,p'}(\Omega)$ *and satisfying:*

$$\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1. \tag{1.2}$$

Moreover,

$$\|\boldsymbol{f}\|_{[\mathbf{X}_{r,p}(\Omega)]'} = \max\{\|f_{ij}\|_{L^{r'}(\Omega)}, 1 \leq i,j \leq 3,\ \|f_1\|_{W^{-1,p'}(\Omega)}\}.$$

Conversely, if $\boldsymbol{f}$ *satisfies (1.2), then* $\boldsymbol{f} \in [\mathbf{X}_{r,p}(\Omega)]'$.

In particular, we have the following embeddings:

$$\mathbf{W}^{-1,r}(\Omega) \hookrightarrow (\mathbf{X}_{r',p'}(\Omega))' \hookrightarrow \mathbf{W}^{-2,p}(\Omega), \tag{1.3}$$

where the second embedding holds if $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{n}$, for n the space dimension of $\mathbb{R}^n$ ($n = 2$ or 3).

In the search of a very weak solution (primal problem), we will study the dual problem which will need strong regularity. Concretely, we will need to handle with the space $\mathbf{Y}_{p'}(\Omega)$ which can be defined in two different ways (see [4]):

$$\mathbf{Y}_{p'}(\Omega) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ (\nabla \cdot \boldsymbol{\psi})|_\Gamma = 0\}$$

$$\text{and} \tag{1.4}$$

$$\mathbf{Y}_{p'}(\Omega) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}\Big|_\Gamma = 0\}.$$

Observe that the range space of its normal derivative $\gamma_1 : \mathbf{Y}_{p'}(\Omega) \to \mathbf{W}^{1/p,p'}(\Gamma)$ is defined by:

$$\mathbf{Z}_{p'}(\Gamma) = \{\boldsymbol{z} \in \mathbf{W}^{1/p,p'}(\Gamma);\ \boldsymbol{z} \cdot \boldsymbol{n} = 0\}.$$

And finally, the spaces where the traces for the very weak solution belong will be defined as:

$$\mathbf{T}_{p,r}(\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \Delta \boldsymbol{v} \in [\mathbf{X}_{r',p'}(\Omega)]'\},$$

endowed with the topology given by the norm $\|\boldsymbol{v}\|_{\mathbf{T}_{p,r}(\Omega)} = \|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)} + \|\Delta \boldsymbol{v}\|_{[\mathbf{X}_{r',p'}(\Omega)]'}$. Observe that when $p = r$, these spaces are denoted as $\mathbf{T}_p(\Omega)$ and $\mathbf{T}_{p,\sigma}(\Omega)$, respectively, the σ-subscript denotes the subspace of solenoidal fields. The tangential trace of functions $\boldsymbol{v}$ of $\mathbf{T}_{p,r,\sigma}(\Omega)$ belongs to the dual space of $\mathbf{Z}_{p'}(\Gamma)$, which is:

$$(\mathbf{Z}_{p'}(\Gamma))' = \{\boldsymbol{\mu} \in \mathbf{W}^{-1/p,p}(\Gamma);\ \boldsymbol{\mu} \cdot \boldsymbol{n} = 0\}.$$

The proof can be seen in [7].

We treat the Stokes, Oseen and Navier-Stokes equations under the compatibility condition:

$$\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = \langle \boldsymbol{g} \cdot \boldsymbol{n}, 1 \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)}. \tag{1.5}$$

The results for the Stokes problem, defined in a domain $\Omega \subset \mathbb{R}^n$ for $n \geq 2$, were studied in [8]. We recall that in [7, 8] we studied the case of singular data satisfying precisely the following assumptions:

$$\boldsymbol{f} \in [\mathbf{X}_{r',p'}(\Omega)]',\ h \in L^r(\Omega),\ \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma),\ \text{with } \frac{1}{r} \leq \frac{1}{p} + \frac{1}{n} \text{ and } r \leq p.$$

2 The Oseen problem for a non-solenoidal given $\boldsymbol{v}$

The Oseen problem can be described as:

$$(O) \quad -\Delta \boldsymbol{u} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = h \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma$$

for some given $\boldsymbol{v}$, $\boldsymbol{f}$, h and $\boldsymbol{g}$ functions or distributions. In [7, 9], the study was made for a given $\boldsymbol{v}$ belonging to the space $\mathbf{L}^s_\sigma(\Omega)$, for $s \geq 3$. Concretely (see [7]), the following result was proved:

Theorem 2.1 (Existence of solution for (O)) *Let*

$$\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega), \quad \boldsymbol{v} \in \mathbf{L}^3_\sigma(\Omega), \quad h \in L^2(\Omega) \quad \text{and } \boldsymbol{g} \in \mathbf{H}^{1/2}(\Gamma)$$

verify the compatibility condition (1.5). Then, the problem (O) has a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C = C(\Omega) > 0$ such that:

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\Big(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\Big)\Big),$$

$$\|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C\ \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\ \Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\Big(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\Big)\Big).$$

In the case of $\nabla \cdot \boldsymbol{v} = 0$, the problem (O) can also be described by:

$$(O') \qquad -\Delta \boldsymbol{u} + \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = h \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma.$$

However, when $\nabla \cdot \boldsymbol{v} \neq 0$, both terms $\nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{u})$ and $\boldsymbol{v} \cdot \nabla \boldsymbol{u}$ do not coincide.

The problem (O') appears in the study of the very weak solution of problem (O), because is the system appearing for the dual problem associated to (O). This is the reason for which the study of problem (O') is being done here.

Theorem 2.2 (Weak regularity for (O)) *Let*

$$\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega), \quad \boldsymbol{v} \in \mathbf{L}^3(\Omega), \quad h \in L^2(\Omega) \quad \textit{and} \quad \boldsymbol{g} \in \mathbf{H}^{1/2}(\Gamma)$$

verify the compatibility condition (1.5). Then, there exists a constant $\delta_0 > 0$ (defined in (2.9)) such that if $\|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)} \leq \delta_0$, the problem (O) has a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending on Ω and δ_0 such that:

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \leq C \left(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + (1 + \|\boldsymbol{v}\|_{L^3(\Omega)})(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}) \right) \tag{2.6}$$

$$\|\pi\|_{L^2(\Omega)} \leq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \left(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + (1 + \|\boldsymbol{v}\|_{L^3(\Omega)})(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}) \right) \tag{2.7}$$

Proof. Following the proof of Theorem 13 of [7], we lift the data by $\boldsymbol{u}_0 \in \mathbf{H}^1(\Omega)$ such that $\nabla \cdot \boldsymbol{u}_0 = h$ and $\boldsymbol{u}_0|_\Gamma = \boldsymbol{g}$, satisfying:

$$\|\boldsymbol{u}_0\|_{\mathbf{H}^1(\Omega)} \leq C \left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)} \right). \tag{2.8}$$

The initial problem is equivalent to finding $\boldsymbol{z} = \boldsymbol{u} - \boldsymbol{u}_0 \in \mathbf{H}^1_0(\Omega)$ with $\nabla \cdot \boldsymbol{z} = 0$ such that:

$$\forall \boldsymbol{\varphi} \in \mathbf{H}^1_0(\Omega) \text{ such that } \nabla \cdot \boldsymbol{\varphi} = 0, \qquad a(\boldsymbol{z}, \boldsymbol{\varphi}) + b(\boldsymbol{v}, \boldsymbol{z}, \boldsymbol{\varphi}) = \langle \widetilde{\boldsymbol{f}}, \boldsymbol{\varphi} \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}^1_0(\Omega)},$$

with $\widetilde{\boldsymbol{f}} = \boldsymbol{f} + \Delta \boldsymbol{u}_0 - (\boldsymbol{v} \cdot \nabla) \boldsymbol{u}_0$. The bilinear form is given by:

$$a(\boldsymbol{z}, \boldsymbol{\varphi}) = \int_\Omega \nabla \boldsymbol{z} : \nabla \boldsymbol{\varphi} \, d\boldsymbol{x} - \frac{1}{2} \langle \nabla \cdot \boldsymbol{v}, \boldsymbol{z} \cdot \boldsymbol{\varphi} \rangle_{W^{-1,3}(\Omega) \times W^{1,3/2}_0(\Omega)}.$$

Taking into account that $\boldsymbol{z} \in \mathbf{H}^1_0(\Omega)$, as $\nabla \cdot \boldsymbol{v} \in W^{-1,3}(\Omega)$, then:

$$\begin{aligned} \langle \nabla \cdot \boldsymbol{v}, |\boldsymbol{z}|^2 \rangle_{W^{-1,3}(\Omega) \times W^{1,3/2}_0(\Omega)} &\leq \|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)} \| |\boldsymbol{z}|^2 \|_{W^{1,3/2}(\Omega)} \leq C_0 \|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)} \|\nabla |\boldsymbol{z}|^2\|_{L^{3/2}(\Omega)} \\ &\leq 2 C_0 C_1 \|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)} \|\nabla \boldsymbol{z}\|^2_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

where C_0 is the Poincaré constant associated to the Sobolev space $\mathbf{W}^{1,3/2}_0(\Omega)$ and C_1 is the product of the constant of the Sobolev embedding $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$ and the Poincaré constant associated to $\mathbf{H}^1_0(\Omega)$. Therefore

$$a(\boldsymbol{z}, \boldsymbol{z}) = \|\nabla \boldsymbol{z}\|^2_{\mathbf{L}^2(\Omega)} - \frac{1}{2} \langle \nabla \cdot \boldsymbol{v}, |\boldsymbol{z}|^2 \rangle_{W^{-1,3}(\Omega) \times W^{1,3/2}_0(\Omega)} \geq \left(1 - C_0 C_1 \|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)}\right) \|\nabla \boldsymbol{z}\|^2_{\mathbf{L}^2(\Omega)}.$$

If we choose $\boldsymbol{v}$ such that:

$$\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,3}(\Omega)} \le \frac{1}{2\,C_0\,C_1} = \delta_0, \tag{2.9}$$

(being δ_0 a constant only depending on Ω) the bilinear form $a(\cdot,\cdot)$ is then coercive. Moreover, the trilinear form:

$$b(\boldsymbol{v},\boldsymbol{z},\boldsymbol{\varphi}) = \int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{z}\cdot\boldsymbol{\varphi}\,d\boldsymbol{x} + \frac{1}{2}\,\langle\nabla\cdot\boldsymbol{v},\boldsymbol{z}\cdot\boldsymbol{\varphi}\rangle_{W^{-1,3}(\Omega)\times W_0^{1,3/2}(\Omega)},$$

well-defined for $\boldsymbol{v}\in\mathbf{L}^3(\Omega)$ and $\boldsymbol{z}$, $\boldsymbol{\varphi}\in\mathbf{H}_0^1(\Omega)$, is an antisymmetric form with respect to the last two variables.

By Lax-Milgram's Theorem, we can deduce the existence of a unique $\boldsymbol{z}\in\mathbf{H}_0^1(\Omega)$ with $\nabla\cdot\boldsymbol{z}=0$ in Ω verifying the estimate:

$$\|\boldsymbol{z}\|_{\mathbf{H}^1(\Omega)} \le C\,\|\widetilde{\boldsymbol{f}}\|_{\mathbf{H}^{-1}(\Omega)} \le C\left(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + (1+\|\boldsymbol{v}\|_{L^3(\Omega)})\left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\right)\right)$$

because of:

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{u}_0\|_{\mathbf{H}^{-1}(\Omega)} \le \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\,\|\boldsymbol{u}_0\|_{\mathbf{H}^1(\Omega)}.$$

This estimate together with (2.8) implies (2.6). By De Rham'Lemma (see [23] or Lemma 6 in [7]) there exists a pressure $\pi\in L^2(\Omega)$ and using that $\nabla\pi = \boldsymbol{f}+\Delta\boldsymbol{u}-\boldsymbol{v}\cdot\nabla\boldsymbol{u}$ we obtain (2.7). □ ▮

Theorem 2.3 (Weak regularity for (O')) *Under the assumptions of Theorem 2.2, the problem (O') has a unique solution $(\boldsymbol{u},\pi)\in\mathbf{H}^1(\Omega)\times L^2(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C>0$ depending on Ω and δ_0 such that estimates (2.6) and (2.7) are satisfied.*

Proof. The proof follows the same scheme but this time the bilinear form is defined by:

$$a'(\boldsymbol{z},\boldsymbol{\varphi}) = \int_\Omega \nabla\boldsymbol{z} : \nabla\boldsymbol{\varphi}\,d\boldsymbol{x} + \frac{1}{2}\,\langle\nabla\cdot\boldsymbol{v},\boldsymbol{z}\cdot\boldsymbol{\varphi}\rangle_{W^{-1,3}(\Omega)\times W_0^{1,3/2}(\Omega)},$$

for $\boldsymbol{v}\in\mathbf{L}^3(\Omega)$, $\boldsymbol{z}$, $\varphi\in\mathbf{H}_0^1(\Omega)$. Taking into account that $\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u}) = \boldsymbol{v}\cdot\nabla\boldsymbol{u} + (\nabla\cdot\boldsymbol{v})\,\boldsymbol{u}$, there is no need to redefine the trilinear form $b(\cdot,\cdot,\cdot)$ in Theorem 2.2. □ ▮

Regularity results given for the solenoidal case (see [9]) can be generalised to the case of $\boldsymbol{v}\in\mathbf{L}^3(\Omega)$ with $\nabla\cdot\boldsymbol{v}\neq 0$ and $\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ sufficiently small. All those can be summarised, making separately both problems (O) and (O'), as follows:

Theorem 2.4 (Strong solution for (O') when $p\ge 6/5$) *Let $p\ge 6/5$ and*

$$\boldsymbol{f}\in\mathbf{L}^p(\Omega),\quad \boldsymbol{v}\in\mathbf{L}^3(\Omega),\quad h\in W^{1,p}(\Omega)\quad \textit{and}\quad \boldsymbol{g}\in\mathbf{W}^{2-1/p,p}(\Gamma)$$

verify the compatibility condition (1.5) and $\|\nabla\cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ *being sufficiently small (in the sense of (2.9)). If moreover* $\nabla\cdot\boldsymbol{v}\in L^{3/2}(\Omega)$*, then there exists a unique solution* $(\boldsymbol{u},\pi)$ *of* (O') *belonging to* $\mathbf{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$*. Moreover, there exists a constant* $C>0$ *satisfying the following estimate:*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)}+\|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \quad\leq\quad C\left(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)}+\big(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\left(\|h\|_{W^{1,p}(\Omega)}+\|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\right)\right), \tag{2.10}$$

where $C=C(\Omega,p,\delta_0)\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\boldsymbol{v}\|_{L^{3/2}(\Omega)}\right)$*. Moreover,* $\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u})\in\mathbf{L}^p(\Omega)$ *is true when* $p\geq 3$*.*

Proof. The proof is based on Theorem 14 in [7] and Theorem 2.2 in [9]. We have the following embeddings:

$$\mathbf{L}^p(\Omega)\hookrightarrow\mathbf{H}^{-1}(\Omega)\quad W^{1,p}(\Omega)\hookrightarrow L^2(\Omega)\quad\text{and}\quad\mathbf{W}^{2-1/p,p}(\Gamma)\hookrightarrow\mathbf{H}^{1/2}(\Gamma)$$

which, thanks to Theorem 2.3, guarantee the existence of a unique solution $(\boldsymbol{u},\pi)$ of (O') belonging to $\mathbf{H}^1(\Omega)\times L^2(\Omega)/\mathbb{R}$. But, this regularity is not sufficient to deduce regularity in $\mathbf{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ using the regularity for the Stokes problem. As in Theorem 2.2 in [9], the proof assume first that $\boldsymbol{v}$ and $\nabla\cdot\boldsymbol{v}$ are more regular. This regularity will be removed in a second step.

(a) The case of $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\overline{\Omega})$. Let $(\boldsymbol{u},\pi)\in\mathbf{H}^1(\Omega)\times L^2(\Omega)/\mathbb{R}$ be the solution of (O'). Using the Stokes regularity, we prove that $(\boldsymbol{u},\pi)\in\mathbf{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)/\mathbb{R}$ and we have the following estimate:

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)}+\|\pi\|_{W^{1,p}(\Omega)}\leq C\,\left(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)}+\|h\|_{W^{1,p}(\Omega)}+\|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}+\|\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u})\|_{\mathbf{L}^p(\Omega)},\right) \tag{2.11}$$

The bound for the term $\|\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u})\|_{\mathbf{L}^p(\Omega)}$ does not coincide with $\|\boldsymbol{v}\cdot\nabla\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}$ because:

$$\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u})=\boldsymbol{v}\cdot\nabla\boldsymbol{u}+(\nabla\cdot\boldsymbol{v})\,\boldsymbol{u}.$$

Let $\varepsilon>0$ and set:

$$\boldsymbol{v}=\boldsymbol{v}_1^\varepsilon+\boldsymbol{v}_2^\varepsilon\quad\textit{where}\quad\boldsymbol{v}_1^\varepsilon=\widetilde{\boldsymbol{v}}\star\rho_{\varepsilon/2}\quad\text{and}\quad\boldsymbol{v}_2^\varepsilon=\boldsymbol{v}-\widetilde{\boldsymbol{v}}\star\rho_{\varepsilon/2}, \tag{2.12}$$

being $\widetilde{\boldsymbol{v}}$ the extension by zero of $\boldsymbol{v}$ to $\mathbb{R}^3$ and $\rho_{\varepsilon/2}$ the classical mollifier.

i) Estimate of the term $\|\boldsymbol{v}\cdot\nabla\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}$. Using the Hölder inequality and the Sobolev embedding, we have:

$$\|\boldsymbol{v}_2^\varepsilon\cdot\nabla\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}\leq\|\boldsymbol{v}_2^\varepsilon\|_{L^m(\Omega)}\|\nabla\boldsymbol{u}\|_{\mathbf{L}^q(\Omega)}\leq C\,\varepsilon\,\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} \tag{2.13}$$

where $\mathbf{W}^{2,p}(\Omega)\hookrightarrow\mathbf{W}^{1,q}(\Omega)$ for $q=p^*$ with $\frac{1}{p^*}=\frac{1}{p}-\frac{1}{3}$ if $p<3$, for any $q\in(1,+\infty)$ if $p=3$ and $q=\infty$ if $p>3$, and m is defined in (2.17).

The estimate depending on $\boldsymbol{v}_1^\varepsilon$, is divided into 2 steps (similar to Theorem 14 in [7]):

- Case $p \le 2$: Assuming $\frac{1}{p} = \frac{1}{r} + \frac{1}{2}$ and $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$ (which implies that $r \ge 3$ and $t \in [1, 3/2]$), we bound:

$$\|\boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} \le \|\boldsymbol{v}_1^\varepsilon\|_{\mathbf{L}^r(\Omega)} \|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \le \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)} \|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \le C_\varepsilon \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \tag{2.14}$$

for C_ε the constant absorbing the norm of the mollifier.

- Case $p > 2$: First we choose an exponent $2 < q < p^*$ such that $\mathbf{W}^{2,p}(\Omega) \hookrightarrow \mathbf{W}^{1,q}(\Omega)$. Therefore, for any $\varepsilon' > 0$, we know the existence of a constant $C_{\varepsilon'} > 0$ in such a way that the following interpolation inequality holds:

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,q}(\Omega)} \le \varepsilon' \|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)}. \tag{2.15}$$

In the case of $p < 3$, using $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ and $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$ (which implies $r \in]3, \infty]$ and $t \ge 1$), we obtain:

$$\begin{aligned} \|\boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} &\le \|\boldsymbol{v}_1^\varepsilon\|_{\mathbf{L}^r(\Omega)} \|\nabla \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)} \le \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)} \|\nabla \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)} \\ &\le C_\varepsilon \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \left(\varepsilon' \|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)}\right). \end{aligned} \tag{2.16}$$

In the case of $p \ge 3$, the previous estimate is also verified using $\frac{1}{p} = \frac{1}{m} + \frac{1}{p^*}$ for m given by:

$$m = \max\{3, p\} \quad \text{if } p \ne 3 \quad \text{and} \quad m > 3 \quad \text{if } p = 3, \tag{2.17}$$

and $1 + \frac{1}{m} = \frac{1}{3} + \frac{1}{t}$ (which implies that $t > 1$), we obtain:

$$\|\boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} \le \|\boldsymbol{v}_1^\varepsilon\|_{\mathbf{L}^m(\Omega)} \|\nabla \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)} \le C_\varepsilon \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \left(\varepsilon' \|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)}\right). \tag{2.18}$$

Note that (2.15) is satisfied for $q = p^*$.

Thus, choosing $\varepsilon' > 0$ small enough, we can deduce from (2.14), (2.16) or (2.18) that:

$$\|\boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} \le C_\varepsilon \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \left(\varepsilon' \|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)}\right). \tag{2.19}$$

ii) Estimate of the term $\|(\nabla \cdot \boldsymbol{v})\, \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}$. We consider:

$$\nabla \cdot \boldsymbol{v} = w_1^\varepsilon + w_2^\varepsilon \quad \textit{where} \quad w_1^\varepsilon = (\widetilde{\nabla \cdot \boldsymbol{v}}) \star \rho_{\varepsilon/2} \quad \text{and} \quad w_2^\varepsilon = \nabla \cdot \boldsymbol{v} - (\widetilde{\nabla \cdot \boldsymbol{v}}) \star \rho_{\varepsilon/2}.$$

being $\widetilde{\nabla \cdot \boldsymbol{v}}$ the extension by zero of $\nabla \cdot \boldsymbol{v}$ to $\mathbb{R}^3$. It is easy to see that:

$$\|w_2^\varepsilon\|_{L^s(\Omega)} = \|\nabla \cdot \boldsymbol{v} - (\widetilde{\nabla \cdot \boldsymbol{v}}) \star \rho_{\varepsilon/2}\|_{L^s(\Omega)} \le \varepsilon.$$

The influence of w_2^ε in the bound of $\|(\nabla \cdot \boldsymbol{v})\, \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}$ is given by:

$$\|w_2^\varepsilon\, \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} \le \|w_2^\varepsilon\|_{L^s(\Omega)} \|\boldsymbol{u}\|_{\mathbf{L}^k(\Omega)} \le C\, \varepsilon\, \|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} \tag{2.20}$$

where s is defined as:

$$s = \max\left\{\frac{3}{2}, p\right\} \quad \text{if } p \neq \tfrac{3}{2} \quad \text{and} \quad s > \frac{3}{2} \quad \text{if } p = \tfrac{3}{2}$$

because of $\mathbf{W}^{2,p}(\Omega) \hookrightarrow \mathbf{L}^k(\Omega)$ for $k = \frac{3p}{3-2p}$ if $p < 3/2$, for $k = \frac{3s}{2s-3}$ if $p = 3/2$ and $k = \infty$ if $p > 3/2$.

The analysis of the influence of w_1^ε in the bound of $\|(\nabla \cdot \boldsymbol{v})\, \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}$ is made considering several cases:

- If $\frac{6}{5} \leq p \leq 6$, let r be defined by $\frac{1}{p} = \frac{1}{r} + \frac{1}{6}$ and t defined by $1 + \frac{1}{r} = \frac{2}{3} + \frac{1}{t}$. Thus, $r \in [3/2, \infty]$ and $t \in [1, 3]$ and the following estimate holds:

$$\begin{aligned} \|w_1^\varepsilon\, \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} &\leq \|w_1^\varepsilon\|_{L^r(\Omega)} \|\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)} \leq C_2 \|\nabla \cdot \boldsymbol{v}\|_{L^{3/2}(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)} \|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \\ &\leq C_2\, C_\varepsilon \,\|\nabla \cdot \boldsymbol{v}\|_{L^{3/2}(\Omega)} \left(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + (1 + \|\boldsymbol{v}\|_{L^3(\Omega)})(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}) \right), \end{aligned} \tag{2.21}$$

  where C_2 is the constant of the Sobolev embedding ~~of~~ $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$.

- If $p > 6$, we choose an exponent $6 < q < +\infty$ such that for any $\varepsilon' > 0$, we known that there exists a constant $C_{\varepsilon'} > 0$ such that (2.15) is satisfied. Now, for $\frac{1}{p} + 1 = \frac{2}{3} + \frac{1}{t}$, $t > 2$, we have:

$$\|w_1^\varepsilon\, \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} \leq C\, \|w_1^\varepsilon\|_{L^p(\Omega)} \|\boldsymbol{u}\|_{\mathbf{W}^{1,q}(\Omega)} \leq C\, \|\nabla \cdot \boldsymbol{v}\|_{\mathbf{L}^{3/2}(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)} \|\boldsymbol{u}\|_{\mathbf{W}^{1,q}(\Omega)}. \tag{2.22}$$

  Replacing (2.15) in (2.22), we have that:

$$\begin{aligned} \|w_1^\varepsilon\, \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} &\leq C \|\nabla \cdot \boldsymbol{v}\|_{\mathbf{L}^{3/2}(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)} \|\boldsymbol{u}\|_{\mathbf{W}^{1,q}(\Omega)} \\ &\leq C\, C_\varepsilon \|\nabla \cdot \boldsymbol{v}\|_{\mathbf{L}^{3/2}(\Omega)} \left(\varepsilon' \|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \right). \end{aligned} \tag{2.23}$$

From (2.11), (2.13), (2.19), (2.20) and (2.21) or (2.23), an adequate choice of the smallness parameters ε and ε' and the weak estimate (2.6) lead to (2.10).

(b) The case of $\boldsymbol{v} \in \mathbf{L}^3(\Omega)$ and $\nabla \cdot \boldsymbol{v} \in L^{3/2}(\Omega)$. In order to apply step **(a)**, we approach $\boldsymbol{v}$ by $\boldsymbol{v}_\lambda \in \mathcal{D}(\overline{\Omega})$ such that $\boldsymbol{v}_\lambda \to \boldsymbol{v}$ in $\mathbf{L}^3(\Omega)$ and $\nabla \cdot \boldsymbol{v}_\lambda \to \nabla \cdot \boldsymbol{v}$ in $L^{3/2}(\Omega)$. Therefore, the solution $(\boldsymbol{u}_\lambda, \pi_\lambda)$ of problem:

$$(O'_\lambda) \quad -\Delta \boldsymbol{u}_\lambda + \nabla \cdot (\boldsymbol{v}_\lambda \otimes \boldsymbol{u}_\lambda) + \nabla \pi_\lambda = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u}_\lambda = h \quad \text{in } \Omega, \quad \boldsymbol{u}_\lambda = \boldsymbol{g} \quad \text{on } \Gamma$$

belongs to $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ and satisfies (2.10). Letting λ tend to 0, the limit $(\boldsymbol{u}, \pi)$ is the solution of (O'), belongs to $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ and satisfies the required estimate. □ ∎

Theorem 2.5 (Strong solution for (O) when $p \geq 6/5$) *Let $p \geq 6/5$ and assume that the hypotheses of Theorem 2.4 are verified. Then, there exists a unique solution $(\boldsymbol{u}, \pi)$ of (O) belonging to $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Moreover, there exists a constant $C > 0$ satisfying estimate (2.10) where $C = C(\Omega, p, \delta_0)\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$ and $p \geq 6/5$. Moreover, $\boldsymbol{v} \cdot \nabla \boldsymbol{u} \in \mathbf{L}^p(\Omega)$ is true when $p \geq 3$.*

Proof. Let $\boldsymbol{v}_\lambda \in \mathcal{D}(\Omega)$ be satisfying $\boldsymbol{v}_\lambda \to \boldsymbol{v}$ in $\mathbf{L}^3(\Omega)$ as $\lambda \to 0$. If $\lambda > 0$ is sufficiently small, then $\|\nabla \cdot \boldsymbol{v}_\lambda\|_{W^{-1,3}(\Omega)}$ is also sufficiently small. By Theorem 2.2, there exists a unique $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ satisfying the problem:

$$(O_\lambda) \quad -\Delta \boldsymbol{u}_\lambda + \boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda + \nabla \pi_\lambda = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u}_\lambda = h \quad \text{in } \Omega, \quad \boldsymbol{u}_\lambda = \boldsymbol{g} \quad \text{on } \Gamma.$$

Using the regularity estimates for the Stokes problem, we obtain that $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies:

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\left(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)} + \|\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}\right), \tag{2.24}$$

where the control over the last term is the main difficulty. Following the same proof of Theorem 14 and Corollary 7 of [7] (see also the improved result in [9]) the bounds of $\|\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}$ are given by :

$$\|\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leq C\,\varepsilon\,\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + C_\varepsilon \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \varepsilon' \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + C_\varepsilon \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\, C_{\varepsilon'} \|\boldsymbol{u}_\lambda\|_{\mathbf{H}^1(\Omega)}, \tag{2.25}$$

for any $\varepsilon > 0$ and $\varepsilon' > 0$ (a detailed explanation can also be seen in the proof Theorem 2.4, point i)). Combining (2.24) and (2.25) for an adequate choice of the ε and ε', and using the weak estimate (2.6) (in which the smallness hypothesis for the norm of $\nabla \cdot \boldsymbol{v}$ stated in (2.9) must be satisfied), we obtain:

$$\begin{aligned} \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} &\leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \\ &\times \left(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\right)\right). \end{aligned}$$

Taking $\lambda \to 0$, we can deduce the convergence of $(\boldsymbol{u}_\lambda, \pi_\lambda) \rightharpoonup (\boldsymbol{u}, \pi)$ in $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and $\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda \rightharpoonup \boldsymbol{v} \cdot \nabla \boldsymbol{u}$ at least in $\mathbf{L}^{6/5}(\Omega)$, the limit pair $(\boldsymbol{u}, \pi)$ being the solution of (O), which belongs to $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies the required estimate. □ ∎

Theorem 2.6 (Generalised solution for (O) and $p \geq 2$) *Let $2 \leq p < \infty$,*

$$\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega), \quad h \in L^p(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma) \qquad \textit{verifying the compatibility condition (1.5)}$$

together with $\boldsymbol{v} \in \mathbf{L}^3(\Omega)$. Then, there exists $\delta_1 = \delta_1(\Omega, p)$ (δ_1 defined in (2.36) if $p \in (2, 3)$ and $\delta_1 = \delta_0$ defined in (2.9) in the other case) such that if $\|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)} \leq \delta_1$, then the problem

(O) *has a unique solution* $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. *Moreover, there exists a constant* $C > 0$ *depending on* Ω, p *and* δ_1 *such that the following inequality holds:*

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \quad &\leq \quad C\, \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big) \Big(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big) \\ &\times \quad \Big(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\Big)\Big). \end{aligned} \tag{2.26}$$

Proof. Unlike the Navier-Stokes equations, the regularity for the Stokes problem cannot be used because of $\boldsymbol{v} \in \mathbf{L}^3(\Omega)$ only implies that $\boldsymbol{v} \cdot \nabla \boldsymbol{u} \in \mathbf{L}^{6/5}(\Omega) \hookrightarrow \mathbf{H}^{-1}(\Omega)$.

We separate the proof into existence and estimates.

A) Existence. In order to obtain the existence of a solution $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$, we use the proof of Theorem 15 in [7].

We lift $\boldsymbol{f}$, h and $\boldsymbol{g}$ by using some functions $(\boldsymbol{u}_0, \pi_0) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that:

$$-\Delta \boldsymbol{u}_0 + \nabla \pi_0 = \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u}_0 = h \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{u}_0|_\Gamma = \boldsymbol{g} \quad \text{on } \Gamma,$$

verifying the estimate:

$$\|\boldsymbol{u}_0\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi_0\|_{L^p(\Omega)/\mathbb{R}} \leq C\, \Big(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\Big). \tag{2.27}$$

Let $(\boldsymbol{z}, \theta) \in \mathbf{W}^{2,t}(\Omega) \times W^{1,t}(\Omega)$ be the solution of the problem:

$$-\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla \theta = -\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 \quad \text{and} \quad \nabla \cdot \boldsymbol{z} = 0 \quad \text{in } \Omega, \quad \boldsymbol{z} = \mathbf{0} \quad \text{on } \Gamma,$$

with $t \in [6/5, 3)$ defined by $\frac{1}{t} = \frac{1}{3} + \frac{1}{p}$, satisfying the strong estimate:

$$\|\boldsymbol{z}\|_{\mathbf{W}^{2,t}(\Omega)} + \|\theta\|_{W^{1,t}(\Omega)/\mathbb{R}} \leq C\, \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big) \|\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0\|_{\mathbf{L}^t(\Omega)} \leq C\, \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big) \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\boldsymbol{u}_0\|_{\mathbf{W}^{1,p}(\Omega)}. \tag{2.28}$$

Then, $(\boldsymbol{u}, \pi) = (\boldsymbol{z} + \boldsymbol{u}_0, \theta + \pi_0)$ is the solution of (O), which thanks to (2.27) and (2.28) satisfies the bound:

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq \widetilde{C}_4\, \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)^2\, \Big(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\Big).$$

This estimate will be improved below.

B) Estimates. In order to improve the estimates obtained in the existence part, we approximate function $\boldsymbol{v} \in \mathbf{L}^3(\Omega)$, which implies that $\nabla \cdot \boldsymbol{v} \in W^{-1,3}(\Omega)$, by $\boldsymbol{v}_\lambda \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ in such a way that:

$$\boldsymbol{v}_\lambda \to \boldsymbol{v} \quad \text{in } \mathbf{L}^3(\Omega) \quad \text{and} \quad \nabla \cdot \boldsymbol{v}_\lambda \to \nabla \cdot \boldsymbol{v} \quad \text{in } W^{-1,3}(\Omega)$$

(the proof can be made in a similar manner to Lemma 13, point i) in [7]) and we study the following problem:

$$(O_\lambda) \quad -\Delta \boldsymbol{u}_\lambda + \boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda + \nabla \pi_\lambda = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u}_\lambda = h \quad \text{in } \Omega, \quad \boldsymbol{u}_\lambda = \boldsymbol{g} \quad \text{on } \Gamma.$$

Let $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ with $\|\nabla \cdot \boldsymbol{v}_\lambda\|_{W^{-1,3}(\Omega)} < \delta_0$ be the solution of (O_λ). Using the Stokes regularity estimates in the space $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$, we have:

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)}+\|\pi_\lambda\|_{L^p(\Omega)} \leq C \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} + \|\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{W}^{-1,p}(\Omega)}\right). \tag{2.29}$$

In the solenoidal case, $\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda = \nabla \cdot (\boldsymbol{v}_\lambda \otimes \boldsymbol{u}_\lambda)$, but now:

$$\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda = \nabla \cdot (\boldsymbol{v}_\lambda \otimes \boldsymbol{u}_\lambda) - (\nabla \cdot \boldsymbol{v}_\lambda)\, \boldsymbol{u}_\lambda.$$

We separate the proof into several steps, which depend on the p-index for the Sobolev spaces $\mathbf{W}^{1,p}(\Omega)$:

i) $\mathbf{p} \in (\mathbf{2}, \mathbf{3})$: The Stokes regularity estimate in the space $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ is given by (2.29). In order to bound the last term of (2.29), we use the decomposition (2.12) in such a way that:

$$\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda = \nabla \cdot (\boldsymbol{v}_1^\varepsilon \otimes \boldsymbol{u}_\lambda) + \nabla \cdot (\boldsymbol{v}_{\lambda,2}^\varepsilon \otimes \boldsymbol{u}_\lambda) - (\nabla \cdot \boldsymbol{v}_\lambda)\, \boldsymbol{u}_\lambda.$$

For the term depending on $\boldsymbol{v}_1^\varepsilon$, using:

$$\|\boldsymbol{v}_1^\varepsilon\|_{\mathbf{L}^3(\Omega)} \leq \|\widetilde{\boldsymbol{v}} \star \rho_{\varepsilon/2}\|_{\mathbf{L}^3(\mathbb{R}^3)} \leq C_\varepsilon\, \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)},$$

for C_ε the constant absorbing the norm of the mollifier, we have:

$$\|\nabla\cdot(\boldsymbol{v}_1^\varepsilon\otimes\boldsymbol{u}_\lambda)\|_{\mathbf{W}^{-1,p}(\Omega)} \leq \|\boldsymbol{v}_1^\varepsilon\otimes\boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leq \|\boldsymbol{v}_1^\varepsilon\|_{\mathbf{L}^r(\Omega)}\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^b(\Omega)} \leq \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)}\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^b(\Omega)},$$

where $\frac{1}{p} = \frac{1}{r} + \frac{1}{b}$ and $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$. If we choose $6 < b < p*$, when $\mathbf{W}^{1,p}(\Omega) \hookrightarrow \mathbf{L}^{p^*}(\Omega)$ for $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$, then $t \in]1, \frac{2p}{p+2}[$ and $r \in]3, \frac{6p}{6-p}[$. Then, for any $\varepsilon' > 0$, we known that there exists $C_{\varepsilon'} > 0$ such that

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^b(\Omega)} \leq \varepsilon' \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)} \; + C_{\varepsilon'}\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^6(\Omega)}, \tag{2.30}$$

and therefore:

$$\|\nabla \cdot (\boldsymbol{v}_1^\varepsilon \otimes \boldsymbol{u}_\lambda)\|_{\mathbf{W}^{-1,p}(\Omega)} \leq \varepsilon'\, C_\varepsilon\, \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)} + C_2\, C_\varepsilon\, C_{\varepsilon'}\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\|\boldsymbol{u}_\lambda\|_{\mathbf{H}^1(\Omega)} \tag{2.31}$$

where C_2 is the constant of the Sobolev embedding $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$. For the term depending on $\boldsymbol{v}_{\lambda,2}^\varepsilon$, we use: for $\lambda < \varepsilon/2$

$$\|\boldsymbol{v}_{\lambda,2}^\varepsilon\|_{\mathbf{L}^3(\Omega)} \leq \|\boldsymbol{v}_\lambda - \boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} + \|\boldsymbol{v} - \boldsymbol{v}_1^\varepsilon\|_{\mathbf{L}^3(\Omega)} \leq \lambda + \varepsilon/2 < \varepsilon, \tag{2.32}$$

together with the Hölder inequality:

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^{p^*}(\Omega)} \leq C_3\, \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)}$$

where C_3 is the constant of the Sobolev embedding $\mathbf{W}^{1,p}(\Omega) \hookrightarrow \mathbf{L}^{p^*}(\Omega)$, obtaining:

$$\|\nabla \cdot (\boldsymbol{v}_{\lambda,2}^\varepsilon \otimes \boldsymbol{u}_\lambda)\|_{\mathbf{W}^{-1,p}(\Omega)} \leq \|\boldsymbol{v}_{\lambda,2}^\varepsilon \otimes \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leq \|\boldsymbol{v}_{\lambda,2}^\varepsilon\|_{\mathbf{L}^3(\Omega)}\, \|\boldsymbol{u}_\lambda\|_{\mathbf{L}^{p^*}(\Omega)} \leq C_3\, \varepsilon\, \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)}. \tag{2.33}$$

In order to bound the term $\|(\nabla\cdot\boldsymbol{v}_\lambda)\,\boldsymbol{u}_\lambda\|_{\mathbf{W}^{-1,p}(\Omega)}$, we must test by using functions $\boldsymbol{\varphi}\in\mathbf{W}_0^{1,p'}(\Omega)$, in such a way that:

$$\langle(\nabla\cdot\boldsymbol{v}_\lambda)\,\boldsymbol{u}_\lambda,\boldsymbol{\varphi}\rangle_{\mathbf{W}^{-1,p}(\Omega)\times\mathbf{W}_0^{1,p'}(\Omega)}=\langle\nabla\cdot\boldsymbol{v}_\lambda,\boldsymbol{u}_\lambda\cdot\boldsymbol{\varphi}\rangle_{W^{-1,3}(\Omega)\times W_0^{1,3/2}(\Omega)},$$

provided that $\boldsymbol{u}_\lambda\cdot\boldsymbol{\varphi}\in W_0^{1,3/2}(\Omega)$. Observe that $\boldsymbol{u}_\lambda\cdot\nabla\varphi\in\mathbf{L}^{3/2}(\Omega)$ and $\nabla\boldsymbol{u}_\lambda\cdot\boldsymbol{\varphi}\in\mathbf{L}^{3/2}(\Omega)$ because of $\frac{1}{p^*}+\frac{1}{p'}=\frac{1}{p}+\frac{1}{(p')^*}=\frac{2}{3}$, which implies that $\boldsymbol{u}_\lambda\cdot\boldsymbol{\varphi}\in W^{1,3/2}(\Omega)$ and:

$$\begin{aligned}\langle(\nabla\cdot\boldsymbol{v}_\lambda)\,\boldsymbol{u}_\lambda,\boldsymbol{\varphi}\rangle_{\mathbf{W}^{-1,p}(\Omega)\times\mathbf{W}_0^{1,p'}(\Omega)} &= \langle\nabla\cdot\boldsymbol{v}_\lambda,\boldsymbol{u}_\lambda\cdot\boldsymbol{\varphi}\rangle_{W^{-1,3}(\Omega)\times W_0^{1,3/2}(\Omega)}\\ &\le \|\nabla\cdot\boldsymbol{v}_\lambda\|_{W^{-1,3}(\Omega)}\|\boldsymbol{u}_\lambda\cdot\boldsymbol{\varphi}\|_{W^{1,3/2}(\Omega)}\\ &\le C_5\,\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,3}(\Omega)}\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,p'}(\Omega)}\end{aligned}\tag{2.34}$$

where C_5 is a constant, which depends on $\mathbf{W}^{1,p}(\Omega)\hookrightarrow\mathbf{L}^{p^*}(\Omega)$ and $\mathbf{W}^{1,p'}(\Omega)\hookrightarrow\mathbf{L}^{(p')^*}(\Omega)$. Consequently, we obtain:

$$\|(\nabla\cdot\boldsymbol{v}_\lambda)\,\boldsymbol{u}_\lambda\|_{\mathbf{W}^{-1,p}(\Omega)}\le C_5\,\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,3}(\Omega)}\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)}.\tag{2.35}$$

Looking at (2.29), (2.31), (2.33), (2.35), we choose ε, ε' and $\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ sufficiently small in such a way that

$$\varepsilon'\,C_\varepsilon\,\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}+C_3\,\varepsilon+C_5\,\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,3}(\Omega)}=\frac{1}{2}.$$

If we want to use (2.6) in order to bound $\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)}$, we also need that $\nabla\cdot\boldsymbol{v}$ satisfies (2.9). Thus, we chose (for instance):

$$\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,3}(\Omega)}\le\delta_1=\min\left\{\frac{1}{6}(C_5)^{-1},\delta_0\right\}.\tag{2.36}$$

Therefore, we deduce the existence of a constant $C_6>0$ such that:

$$\begin{aligned}\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)}+\|\pi_\lambda\|_{L^p(\Omega)/\mathbb{R}} &\le C_6\,\big(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\Big(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)}+\big(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\\ &\times\ \big(\|h\|_{L^p(\Omega)}+\|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\big)\Big)\end{aligned}\tag{2.37}$$

ii) $\mathbf{p}\in[\mathbf{3},\mathbf{6})$: For these values, we can reproduce the proof of Proposition 3 of [7]: By using (2.12) and (2.32), we have:

$$\|\boldsymbol{v}^\varepsilon_{\lambda,2}\cdot\nabla\boldsymbol{u}_\lambda\|_{\mathbf{W}^{-1,p}(\Omega)}\le C\|\boldsymbol{v}^\varepsilon_{\lambda,2}\cdot\nabla\boldsymbol{u}_\lambda\|_{\mathbf{L}^r(\Omega)}\le C\,\|\boldsymbol{v}^\varepsilon_{\lambda,2}\|_{\mathbf{L}^3(\Omega)}\|\nabla\boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}\le C\,\varepsilon\|\nabla\boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)},\tag{2.38}$$

where $\frac{1}{r}=\frac{1}{3}+\frac{1}{p}$, and:

$$\begin{aligned}\|\boldsymbol{v}^\varepsilon_1\cdot\nabla\boldsymbol{u}_\lambda\|_{\mathbf{W}^{-1,p}(\Omega)} &\le C\|\boldsymbol{v}^\varepsilon_1\cdot\nabla\boldsymbol{u}_\lambda\|_{\mathbf{L}^r(\Omega)}\le C\|\boldsymbol{v}^\varepsilon_1\|_{\mathbf{L}^k(\Omega)}\|\nabla\boldsymbol{u}_\lambda\|_{\mathbf{L}^2(\Omega)}\\ &\le C\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)}\|\nabla\boldsymbol{u}_\lambda\|_{\mathbf{L}^2(\Omega)},\end{aligned}\tag{2.39}$$

where $k = \frac{6p}{6-p}$ and $t = \frac{2p}{p+2} \in \, [\frac{6}{5}, \frac{3}{2}[$. Bounding the last term in (2.39) by using (2.6), and replacing the resulting estimate and (2.38) into (2.29), estimate (2.37) can be deduced.

iii) $\mathbf{p \geq 6}$: Estimate (2.38) is still true. Estimate (2.39) is bounded in a slightly different way:

$$\begin{aligned}\|\boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{W}^{-1,p}(\Omega)} &\leq C\|\boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^r(\Omega)} \leq C\|\boldsymbol{v}_1^\varepsilon\|_{\mathbf{L}^k(\Omega)}\|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)} \\ &\leq C\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)}\|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)},\end{aligned}$$

where $\frac{1}{r} = \frac{1}{k} + \frac{1}{q}$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ and $\frac{1}{t} = 1 + \frac{1}{p} - \frac{1}{q}$. Choosing $3 < q < p$, we have $k \in]3, p[$ and $t \in]1, \frac{3p}{2p+3}[$. The interpolation estimate of $\mathbf{W}^{1,q}(\Omega)$ between $\mathbf{H}^1(\Omega)$ and $\mathbf{W}^{1,p}(\Omega)$:

$$\|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)} \leq \|\boldsymbol{u}_\lambda\|_{\mathbf{H}^1(\Omega)}^{\frac{2(p-q)}{q(p-2)}}\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)}^{\frac{p(q-2)}{q(p-2)}} \leq \varepsilon' \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)} + C_{\varepsilon'}\|\boldsymbol{u}_\lambda\|_{\mathbf{H}^1(\Omega)}$$

with an adequate choice of the small parameters ε and $\varepsilon' > 0$, allow us to obtain (2.37).

Taking $\lambda \to 0$, we can deduce the convergence of $(\boldsymbol{u}_\lambda, \pi_\lambda) \rightharpoonup (\boldsymbol{u}, \pi)$ in $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ and $\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda \rightharpoonup \boldsymbol{v} \cdot \nabla \boldsymbol{u}$ at least in $\mathbf{L}^{6/5}(\Omega)$, the limit pair $(\boldsymbol{u}, \pi)$ being the solution of (O), which belongs to $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ and satisfies (2.26) for $\|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)} \leq \delta_1$ with $\delta_1 = \delta_0$ and δ_0 defined in (2.9) for $p \geq 3$, and with δ_1 defined in (2.36) for $p < 3$. □ ∎

Theorem 2.7 (Generalised solution for (O') and $p \geq 2$) *Let $2 \leq p < \infty$ and assume that the hypotheses of Theorem 2.6 are satisfied (the smallness assumption for $\|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ is only needed in the sense of (2.9)). Then the problem (O') has a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Moreover, there exist some constant $C > 0$ depending on Ω, p and δ_0 such that inequality (2.26) holds.*

Proof. In order to obtain the existence of a solution $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$, let $(\boldsymbol{v}_\lambda)_\lambda \subset \mathcal{D}(\Omega)$ be such that $\boldsymbol{v}_\lambda \to \boldsymbol{v}$ in $\mathbf{L}^3(\Omega)$ as $\lambda \to 0$. Then, we can suppose that $\lambda > 0$ is sufficiently small and therefore $\|\nabla \cdot \boldsymbol{v}_\lambda\|_{W^{-1,3}(\Omega)}$ is as small as $\|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)}$. Then, we can use Theorem 2.3 to deduce the existence of $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$. From the regularity estimates for the Stokes problem, we have that $(\boldsymbol{u}_\lambda.\pi_\lambda) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ and

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi_\lambda\|_{L^p(\Omega)/\mathbb{R}} \leq C \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} + \|\boldsymbol{v}_\lambda \otimes \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}\right). \tag{2.40}$$

Using decomposition (2.12) for $\boldsymbol{v}_\lambda$, for m defined in (2.17),

$$\|\boldsymbol{v}_{\lambda,2}^\varepsilon \otimes \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leq \|\boldsymbol{v}_{\lambda,2}^\varepsilon\|_{L^m(\Omega)}\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^{p^*}(\Omega)} \leq C\,\varepsilon\,\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)} \tag{2.41}$$

where $\mathbf{W}^{1,p}(\Omega) \hookrightarrow \mathbf{L}^{p^*}(\Omega)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$ if $p < 3$, for $p^* = \frac{3m}{m-3}$ if $p = 3$ and $p^* = \infty$ if $p > 3$. Using that $p \geq 2$, we can bound the term on $\boldsymbol{v}_1^\varepsilon$ as follows:

$$\|\boldsymbol{v}_1^\varepsilon \otimes \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leq \|\boldsymbol{v}_1^\varepsilon\|_{\mathbf{L}^r(\Omega)}\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^b(\Omega)} \leq \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)}\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^b(\Omega)}$$

for $\frac{1}{p} = \frac{1}{r} + \frac{1}{b}$, $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$ and $6 < b < p^*$. As for (2.31) , we have

$$\|\boldsymbol{v}_1^\varepsilon \otimes \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leq \varepsilon' C_\varepsilon \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)} + C_2 C_\varepsilon C_{\varepsilon'} \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\boldsymbol{u}_\lambda\|_{\mathbf{H}^1(\Omega)}, \tag{2.42}$$

for $C_2 > 0$ the constant of the embedding $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$. From (2.40), (2.41) and (2.42), choosing ε and ε' conveniently and using the weak estimate (2.6) furnished by the smallness of $\|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ in the sense of (2.9), we deduce the existence of a constant $C_4 > 0$ such that:

$$\begin{aligned} \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi_\lambda\|_{L^p(\Omega)} \quad &\leq \quad C_4 \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \Big(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \\ &\times \quad \Big(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} \Big) \Big). \end{aligned}$$

Taking $\lambda \to 0$, we can deduce the convergence of $(\boldsymbol{u}_\lambda, \pi_\lambda) \rightharpoonup (\boldsymbol{u}, \pi)$ in $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ and $\nabla \cdot (\boldsymbol{v}_\lambda \otimes \boldsymbol{u}_\lambda) \rightharpoonup \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{u})$ at least in $\mathbf{H}^{-1}(\Omega)$, the limit pair $(\boldsymbol{u}, \pi)$ being the solution of (O'), which belongs to $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ and satisfies (2.26). □ ∎

Theorem 2.8 (Generalised solution for (O) and $p < 2$) *Let $1 < p < 2$,*

$$\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega), \quad h \in L^p(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$$

satisfy (1.5), together with $\boldsymbol{v} \in \mathbf{L}^3(\Omega)$ and $\|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ being sufficiently small (in the sense of (2.9)). Then, the problem (O) has a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ and there exists some constant $C > 0$ such that the following inequality holds:

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C \Big(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \Big(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} \Big) \Big) \tag{2.43}$$

where $C = C(\Omega, p, \delta_0) \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2$. Moreover, there exists $\delta_2 > 0$ (defined in (2.55)) such that if $\|\nabla \cdot \boldsymbol{v}\|_{W^{-1,s}(\Omega)} \leq \delta_2$ for s defined by:

$$s = \max\{3, p'\} \quad \text{if } p \neq \tfrac{3}{2} \quad \text{and} \quad s > 3 \quad \text{if } p = \tfrac{3}{2}, \tag{2.44}$$

then estimate (2.43) is satisfied for $C = C(\Omega, p, \delta_2) \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$.

Proof. We separate the proof into existence and estimates.

A) Existence. In order to obtain the existence of a solution $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$, we use the proof of Theorem 15 in [7].

Using a duality method, first we suppose that $h = 0$ and $\boldsymbol{g} = \mathbf{0}$. The problem (O) is equivalent to find $(\boldsymbol{u}, \pi) \in \mathbf{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ such that: $\forall (\boldsymbol{w}, \chi) \in \mathbf{W}_0^{1,p'}(\Omega) \times L^{p'}(\Omega)$

$$\langle \boldsymbol{u}, -\Delta \boldsymbol{w} - \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{w}) + \nabla \chi \rangle_{\mathbf{W}_0^{1,p}(\Omega) \times \mathbf{W}^{-1,p'}(\Omega)} - \langle \pi, \nabla \cdot \boldsymbol{w} \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\mathbf{W}^{-1,p}(\Omega) \times \mathbf{W}_0^{1,p'}(\Omega)}.$$

Provided that $\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ is small enough in the sense of (2.9) and thanks to Theorem 2.7, for any pair $(\mathbf{F},\varphi)\in\mathbf{W}^{-1,p'}(\Omega)\times L^{p'}_0(\Omega)$, there exists a unique $(\boldsymbol{w},\chi)\in\mathbf{W}^{1,p'}_0(\Omega)\times L^{p'}(\Omega)/\mathbb{R}$ such that:

$$-\Delta\boldsymbol{w}-\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{w})+\nabla\chi=\mathbf{F}\quad\text{and}\quad\nabla\cdot\boldsymbol{w}=\varphi\quad\text{in }\Omega,\quad\boldsymbol{w}=\mathbf{0}\quad\text{on }\Gamma$$

and satisfying the estimate:

$$\|\boldsymbol{w}\|_{\mathbf{W}^{1,p'}(\Omega)}+\|\chi\|_{L^{p'}(\Omega)/\mathbb{R}}\le C\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|\mathbf{F}\|_{\mathbf{W}^{-1,p}(\Omega)}+\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\|\varphi\|_{L^{p'}(\Omega)}\right).$$

Therefore, we have:

$$\langle\boldsymbol{u},\mathbf{F}\rangle_{\mathbf{W}^{1,p}_0(\Omega)\times\mathbf{W}^{-1,p'}(\Omega)}-\langle\pi,\varphi\rangle_{L^p(\Omega)\times L^{p'}(\Omega)}=\langle\boldsymbol{f},\boldsymbol{w}\rangle_{\mathbf{W}^{-1,p}(\Omega)\times\mathbf{W}^{1,p'}_0(\Omega)}$$

with

$$\left|\langle\boldsymbol{f},\boldsymbol{w}\rangle_{\mathbf{W}^{-1,p}(\Omega)\times\mathbf{W}^{1,p'}_0(\Omega)}\right|\le C\,\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)}\,\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2\left(\|\mathbf{F}\|_{\mathbf{W}^{-1,p}(\Omega)}+\|\varphi\|_{L^{p'}(\Omega)}\right).$$

In other words, the mapping $(\mathbf{F},\varphi)\to\langle\boldsymbol{f},\boldsymbol{w}\rangle$ defines an element $(\boldsymbol{u},\pi)$ of the dual space of $\mathbf{W}^{-1,p'}(\Omega)\times L^{p'}_0(\Omega)$ solution of (O') and satisfying the estimate:

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)}+\|\pi\|_{L^p(\Omega)/\mathbb{R}}\le C\,\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)}.\tag{2.45}$$

The case of $h\neq0$ and $\boldsymbol{g}\neq\mathbf{0}$ can be treated lifting first these data by using:

$$\boldsymbol{z}\in\mathbf{W}^{1,p}(\Omega)\quad\text{such that }\nabla\cdot\boldsymbol{z}=h\text{ and }\boldsymbol{z}|_\Gamma=\boldsymbol{g}\text{ with}\quad\|\boldsymbol{z}\|_{\mathbf{W}^{1,p}(\Omega)}\le C\,\left(\|h\|_{L^p(\Omega)}+\|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\right).\tag{2.46}$$

Therefore, it remains to study the regularity for the solution $(\boldsymbol{u}_0,\pi_0)$ of the problem:

$$-\Delta\boldsymbol{u}_0+\boldsymbol{v}\cdot\nabla\boldsymbol{u}_0+\nabla\pi_0=\boldsymbol{f}+\Delta\boldsymbol{z}-\boldsymbol{v}\cdot\nabla\boldsymbol{z}\quad\text{and}\quad\nabla\cdot\boldsymbol{u}_0=0\quad\text{in }\Omega,\quad\boldsymbol{u}_0=\mathbf{0}\quad\text{on }\Gamma,$$

which using (2.45) satisfies:

$$\begin{aligned}\|\boldsymbol{u}_0\|_{\mathbf{W}^{1,p}(\Omega)}+\|\pi_0\|_{L^p(\Omega)/\mathbb{R}}&\le C\,\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2\left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)}+\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\right.\\&\times\quad\left.\left(\|h\|_{L^p(\Omega)}+\|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\right)\right).\end{aligned}\tag{2.47}$$

Finally, the solution of (O) is given by $(\boldsymbol{z}+\boldsymbol{u}_0,\pi_0)$ which satisfies estimate (2.43) for $C=C(\Omega,p,\delta_0)\,\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2$. This estimate will be improved below.

B) Estimates.

In order to improve the estimates obtained in the existence part, we consider first the case that $h=0$ and $\boldsymbol{g}=\mathbf{0}$.

(a) The case of $h=0$ and $\boldsymbol{g}=\mathbf{0}$: As the norm $\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ is sufficiently small, then the step **A)** guarantees the existence of $(\boldsymbol{u}_k,\pi_k)\in\mathbf{W}^{1,p}(\Omega)\times L^p(\Omega)$ solution for the problem:

$$(P)\quad-\Delta\boldsymbol{u}_k+\boldsymbol{v}\cdot\nabla\boldsymbol{u}_k+\nabla\pi_k=\boldsymbol{f}_k\quad\text{and}\quad\nabla\cdot\boldsymbol{u}_k=0\quad\text{in }\Omega,\quad\boldsymbol{u}_k=\mathbf{0}\quad\text{on }\Gamma,$$

for a given $\boldsymbol{f}_k$ defined as follows:

Construction of $\boldsymbol{f}_k$: Let $\rho \in \mathcal{D}(\mathbb{R}^3)$, be a smooth $\mathcal{C}^\infty$ function with compact support in $B(0,1)$, such that $\rho \geq 0$, $\int_{\mathbb{R}^3} \rho(x)\,dx = 1$. For $t \in (0,1)$, let ρ_t denote the function $x \longmapsto (\frac{1}{t^3})\rho(\frac{x}{t})$. Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$ be such that $0 \leqslant \varphi(x) \leqslant 1$ for any $x \in \mathbb{R}^3$, and

$$\varphi(x) = \begin{cases} 1 & \text{if} \quad 0 \leqslant \mid x \mid \leqslant 1, \\ 0 & \text{if} \quad \mid x \mid \geqslant 2. \end{cases}$$

We begin with applying the cut off functions φ_k defined on $\mathbb{R}^3$ for any $k \in \mathbb{N}^*$, as $\varphi_k(x) = \varphi(\frac{x}{k})$.

As $\boldsymbol{f} = \nabla \cdot \mathbb{F}$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$, we consider $\mathbb{F}_k = \varphi_k \widetilde{\mathbb{F}}$ (for $\widetilde{\mathbb{F}}_k$ the extension by zero of $\mathbb{F}$ to $\mathbb{R}^3$). Thus we obtain

$$\mathbb{G}_{t,k} = \rho_t * \mathbb{F}_k \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3), \quad \lim_{t\to 0}\lim_{k\to\infty} \mathbb{G}_{t,k} = \widetilde{\mathbb{F}} \quad \text{in} \quad \mathbb{L}^p(\mathbb{R}^3), \quad \mathbb{G}_{t,k}|_\Omega \to \mathbb{F} \quad \text{in } \mathbb{L}^p(\Omega) \text{ when } t \to 0,\ k \to +\infty.$$

Moreover, for $q = \frac{2p}{3p-2}$,

$$\|\rho_t \star \mathbb{F}_k\|_{L^2(\mathbb{R}^3)} \leq \|\rho_t\|_{L^q(\mathbb{R}^3)} \|\mathbb{F}_k\|_{L^p(\mathbb{R}^3)} \leq \frac{4}{3}\,\pi\, t^{-3/q'}\, \|\mathbb{F}_k\|_{L^p(\mathbb{R}^3)}. \tag{2.48}$$

We choose $t = k^{-\alpha}$ with $\alpha > 0$ which will be precised later. Then $\boldsymbol{f}_k = \nabla \cdot (\mathbb{G}_{t,k}|_\Omega) \in \mathbf{H}^{-1}(\Omega)$ and $\boldsymbol{f}_k \to \boldsymbol{f}$ in $\mathbf{W}^{-1,p}(\Omega)$. Observe that $(\boldsymbol{u}_k, \pi_k)$ also belongs to $\mathbf{H}^1_0(\Omega) \times L^2(\Omega)$. Using the Stokes regularity in $\mathbf{W}^{1,p}_0(\Omega) \times L^p(\Omega)$, we obtain:

$$\|\boldsymbol{u}_k\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi_k\|_{L^p(\Omega)/\mathbb{R}} \leq C\, \left(\|\boldsymbol{f}_k\|_{\mathbf{W}^{-1,p}(\Omega)} + \|\boldsymbol{v} \cdot \nabla \boldsymbol{u}_k\|_{\mathbf{W}^{-1,p}(\Omega)} \right). \tag{2.49}$$

The last term can be bounded using the following decomposition:

$$\boldsymbol{v} \cdot \nabla \boldsymbol{u}_k = \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{u}_k) - \boldsymbol{u}_k\, (\nabla \cdot \boldsymbol{v}).$$

Given $\varepsilon > 0$, let $\boldsymbol{v}_\varepsilon \in \mathcal{D}(\Omega)$ be a smooth function with $B_\varepsilon = \text{supp}(\boldsymbol{v}_\varepsilon)$, which approaches $\boldsymbol{v}$ in the sense that:

$$\|\boldsymbol{v} - \boldsymbol{v}_\varepsilon\|_{\mathbf{L}^3(\Omega)} < \varepsilon. \tag{2.50}$$

Then, we obtain:

$$\begin{aligned} \|\nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{u}_k)\|_{\mathbf{W}^{-1,p}(\Omega)} &\leq \|\boldsymbol{v} \otimes \boldsymbol{u}_k\|_{\mathbb{L}^p(\Omega)} \leq \|\boldsymbol{v} - \boldsymbol{v}_\varepsilon\|_{\mathbf{L}^3(\Omega)} \|\boldsymbol{u}_k\|_{\mathbf{L}^{p^*}(\Omega)} + \|\boldsymbol{v}_\varepsilon\|_{\mathbf{L}^3(B_\varepsilon)} \|\boldsymbol{u}_k\|_{\mathbf{L}^{p^*}(B_\varepsilon)} \\ &\leq \varepsilon\, C_3 \|\boldsymbol{u}_k\|_{\mathbf{W}^{1,p}(\Omega)} + \|\boldsymbol{v}_\varepsilon\|_{\mathbf{L}^3(B_\varepsilon)} \|\boldsymbol{u}_k\|_{\mathbf{L}^{p^*}(B_\varepsilon)} \end{aligned} \tag{2.51}$$

In order to treat the term $\boldsymbol{u}_k\,(\nabla \cdot \boldsymbol{v})$, we consider $\nabla \cdot \boldsymbol{v} \in W^{-1,s}(\Omega)$ for s defined by (2.44). We must test by using functions $\boldsymbol{\varphi} \in \mathbf{W}^{1,p'}_0(\Omega)$, in such a way that:

$$\langle (\nabla \cdot \boldsymbol{v})\, \boldsymbol{u}_k, \boldsymbol{\varphi} \rangle_{\mathbf{W}^{-1,p}(\Omega) \times \mathbf{W}^{1,p'}_0(\Omega)} = \langle \nabla \cdot \boldsymbol{v}, \boldsymbol{u}_k \cdot \boldsymbol{\varphi} \rangle_{W^{-1,s}(\Omega) \times W^{1,s'}_0(\Omega)},$$

holds for s defined in (2.44), providing that $\boldsymbol{u}_k \cdot \boldsymbol{\varphi} \in W_0^{1,s'}(\Omega)$. Observe that $\boldsymbol{u}_k \cdot \nabla\varphi \in \mathbf{L}^{3/2}(\Omega)$ because of $\frac{1}{p^*} + \frac{1}{p'} = \frac{2}{3}$. However, $\nabla \boldsymbol{u}_k \cdot \varphi \in \mathbf{L}^s(\Omega)$ for $\frac{1}{s} = \frac{1}{p} + \frac{1}{(p')^*}$. Depending on the value of $(p')^*$, we separate the proof into several steps, which depend on the p-index for the Sobolev spaces $\mathbf{W}^{1,p}(\Omega)$:

i) If $p > 3/2$, we obtain that $\nabla \boldsymbol{u}_k \cdot \boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\Omega)$, therefore $\boldsymbol{u}_k \cdot \boldsymbol{\varphi} \in W_0^{1,3/2}(\Omega)$ and (2.34) and (2.35) hold.

ii) If $p < 3/2$, we obtain that $\nabla \boldsymbol{u}_k \cdot \boldsymbol{\varphi} \in \mathbf{L}^p(\Omega)$, therefore $\boldsymbol{u}_k \cdot \boldsymbol{\varphi} \in W_0^{1,p}(\Omega)$ and:

$$\begin{aligned} |\langle(\nabla\cdot\boldsymbol{v})\,\boldsymbol{u}_k, \boldsymbol{\varphi}\rangle_{\mathbf{W}^{-1,p}(\Omega)\times\mathbf{W}_0^{1,p'}(\Omega)}| &= |\langle\nabla\cdot\boldsymbol{v}, \boldsymbol{u}_k\cdot\boldsymbol{\varphi}\rangle_{W^{-1,p'}(\Omega)\times W_0^{1,p}(\Omega)}| \\ &\leq \|\nabla\cdot\boldsymbol{v}\|_{W^{-1,p'}(\Omega)}\|\boldsymbol{u}_k\cdot\boldsymbol{\varphi}\|_{W^{1,p}(\Omega)} \\ &\leq C_5\,\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,p'}(\Omega)}\|\boldsymbol{u}_k\|_{\mathbf{W}^{1,p}(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,p'}(\Omega)}. \end{aligned}$$

Therefore,

$$\|(\nabla\cdot\boldsymbol{v})\,\boldsymbol{u}_k\|_{\mathbf{W}^{-1,p}(\Omega)} \leq C_5\,\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,p'}(\Omega)}\|\boldsymbol{u}_k\|_{\mathbf{W}^{1,p}(\Omega)}. \tag{2.52}$$

iii) If $p = 3/2$, we obtain that $\nabla \boldsymbol{u}_k \cdot \boldsymbol{\varphi} \in \mathbf{L}^r(\Omega)$ (for any $r < \frac{3}{2}$), therefore $\boldsymbol{u}_k \cdot \boldsymbol{\varphi} \in W_0^{1,r}(\Omega)$ and we can verify that:

$$\|(\nabla\cdot\boldsymbol{v})\,\boldsymbol{u}_k\|_{\mathbf{W}^{-1,p}(\Omega)} \leq C_5\,\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,a}(\Omega)}\|\boldsymbol{u}_k\|_{\mathbf{W}^{1,p}(\Omega)}, \tag{2.53}$$

for any $a > 3$.

In summary, from (2.35), (2.52) and (2.53), we can deduce:

$$\|(\nabla\cdot\boldsymbol{v})\,\boldsymbol{u}_k\|_{\mathbf{W}^{-1,p}(\Omega)} \leq C_5\,\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,s}(\Omega)}\|\boldsymbol{u}_k\|_{\mathbf{W}^{1,p}(\Omega)} \tag{2.54}$$

for $1 < p < 2$ with s defined by (2.44). Putting together (2.54) and (2.51) into (2.49) and choosing $\varepsilon\, C\, C_3 = \frac{1}{4}$ and:

$$\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,s}(\Omega)} \leq \delta_2 = \frac{1}{4\,C\,C_5}, \tag{2.55}$$

we obtain the existence of a constant $C_4 > 0$ such that:

$$\|\boldsymbol{u}_k\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi_k\|_{L^p(\Omega)/\mathbb{R}} \leq 2\,C_4\left(\|\boldsymbol{f}_k\|_{\mathbf{W}^{-1,p}(\Omega)} + \|\boldsymbol{v}_\varepsilon\|_{\mathbf{L}^3(B_\varepsilon)}\,\|\boldsymbol{u}_k\|_{\mathbf{L}^{p^*}(B_\varepsilon)}\right). \tag{2.56}$$

From (2.56), we prove that there exists $C > 0$ not depending on k and $\boldsymbol{v}$ such that for any $k \in \mathbb{N}^*$ we have

$$||\boldsymbol{u}_k||_{\boldsymbol{L}^{p^*}(B_\varepsilon)} \leqslant C||\boldsymbol{f}_k||_{\boldsymbol{W}^{-1,p}(\Omega)}. \tag{2.57}$$

Indeed, assuming, per absurdum, the invalidity of (2.57). Then for any $m \in \mathbb{N}^*$ there exists $\ell_m \in \mathbb{N}$, $\boldsymbol{f}_{\ell_m} \in \boldsymbol{H}^{-1}(\Omega)$ and $\boldsymbol{v}_m \in \mathbf{L}^3(\Omega)$ with $\|\nabla \cdot \boldsymbol{v}_m\|_{W^{-1,3}(\Omega)} \leq \delta_0$ (δ_0 defined in (2.9)) such that, if $(\boldsymbol{u}_{\ell_m}, \pi_{\ell_m}) \in \boldsymbol{H}_0^1(\Omega) \times L^2(\Omega)$ denotes the corresponding solution to the following problem:

$$-\Delta \boldsymbol{u}_{\ell_m} + \boldsymbol{v}_m \cdot \nabla \boldsymbol{u}_{\ell_m} + \nabla \pi_{\ell_m} = \boldsymbol{f}_{\ell_m}, \quad \nabla \cdot \boldsymbol{u}_{\ell_m} = 0 \quad \text{in} \quad \Omega, \quad \boldsymbol{u}_{\ell_m} = \mathbf{0} \quad \text{on} \quad \Gamma$$

the inequality

$$||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\epsilon)} > m||\boldsymbol{f}_{\ell_m}||_{\boldsymbol{W}^{-1,p}(\Omega)}, \tag{2.58}$$

would hold. Now, we set:

$$\boldsymbol{w}_m = \frac{\boldsymbol{u}_{\ell_m}}{||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\varepsilon)}}, \quad \theta_m = \frac{\pi_{\ell_m}}{||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\varepsilon)}} \quad \text{and} \quad \mathbf{R}_m = \frac{\boldsymbol{f}_{\ell_m}}{||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\varepsilon)}}.$$

Then for any $m \in \mathbb{N}^*$ we have

$$-\Delta \boldsymbol{w}_m + \boldsymbol{v}_m \cdot \nabla \boldsymbol{w}_m + \nabla \theta_m = \mathbf{R}_m \quad \text{and} \quad \nabla \cdot \boldsymbol{w}_m = 0 \quad \text{in} \quad \Omega, \quad \boldsymbol{w}_m = \mathbf{0} \quad \text{on} \quad \Gamma. \tag{2.59}$$

Now, using (2.59), the smallness assumption (2.9) for $\|\nabla \cdot \boldsymbol{v}_m\|_{W^{-1,3}(\Omega)}$ and Theorem 2.2, we can apply estimate (2.6) obtaining for any $m \in \mathbb{N}^*$ and $t > 0$

$$\|\boldsymbol{w}_m\|_{\mathbf{H}^1(\Omega)} \leq C\,\|\mathbf{R}_m\|_{\mathbf{H}^{-1}(\Omega)} = \frac{C}{||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\varepsilon)}}\,\|\boldsymbol{f}_{\ell_m}\|_{\mathbf{H}^{-1}(\Omega)} \leq \frac{C}{||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\varepsilon)}}\,\|\rho_t * \mathbb{F}_{\ell_m}\|_{\mathbb{L}^2(\Omega)}\,.$$

From (2.58), we have

$$||\boldsymbol{w}_m||_{\boldsymbol{H}^1(\Omega)} < \frac{C}{m||\boldsymbol{f}_{\ell_m}||_{\boldsymbol{W}^{-1,p}(\Omega)}}||\rho_t * \mathbb{F}_{\ell_m}||_{\boldsymbol{L}^2(\mathbb{R}^3)}.$$

Using (2.48) and choosing $t = \dfrac{1}{m^\alpha}$ with $0 < \alpha < \frac{q'}{3}$, we deduce that

$$||\boldsymbol{w}_m||_{\boldsymbol{H}^1(\Omega)} \leqslant \frac{4\,\pi\,C}{3m^{1-\frac{3\alpha}{q'}}||\boldsymbol{f}_{\ell_m}||_{\boldsymbol{W}^{-1,p}(\Omega)}}||\mathbb{F}_{\ell_m}||_{\boldsymbol{L}^p(\mathbb{R}^3)}.$$

Because the right hand side of the last inequality tends to zero when m goes to ∞, we deduce that

$$\boldsymbol{w}_m \to \mathbf{0} \quad \text{in} \quad \boldsymbol{H}^1(\Omega).$$

Then, $\boldsymbol{w}_m \to \mathbf{0}$ in $\boldsymbol{L}^6(\Omega)$ and in particular in $\boldsymbol{L}^{p*}(B_\varepsilon)$. On the other hand, we have:

$$||\boldsymbol{w}_m||_{\boldsymbol{L}^{p*}(B_\varepsilon)} = 1,$$

leading to a contradiction. Inequality (2.57) is therefore established. From (2.56), (2.57) and (2.50) we obtain for any $k \in \mathbb{N}^*$

$$||\boldsymbol{u}_k||_{\boldsymbol{W}^{1,p}(\Omega)} + ||\pi_k||_{L^p(\Omega)/\mathbb{R}} \leqslant 2\,C_4\left(1 + C||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}\right)||\boldsymbol{f}_k||_{\boldsymbol{W}^{-1,p}(\Omega)}.$$

Thus we can extract subsequences of $\boldsymbol{u}_k$ and π_k, still denoted by $\boldsymbol{u}_k$ and π_k, such that

$$\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{W}^{1,p}(\Omega) \quad \text{and} \quad \pi_k + C_k \rightharpoonup \pi \quad \text{in} \quad L^p(\Omega),$$

which implies $\boldsymbol{v} \otimes \boldsymbol{u}_k \rightharpoonup \boldsymbol{v} \otimes \boldsymbol{u}$ in $\mathbb{L}^p(\Omega)$. Assuming that $\nabla \cdot \boldsymbol{v} \in W^{-1,s}(\Omega)$ for s defined in (2.44) and following the same argument that proves that $(\nabla \cdot \boldsymbol{v})\, \boldsymbol{u}_k \in \mathbf{W}^{-1,p}(\Omega)$, we can deduce that $(\nabla \cdot \boldsymbol{v})\, \boldsymbol{u} \in \mathbf{W}^{-1,p}(\Omega)$. Thus, the convergence:

$$(\nabla \cdot \boldsymbol{v})\, \boldsymbol{u}_k \rightharpoonup (\nabla \cdot \boldsymbol{v})\, \boldsymbol{u} \quad \text{in} \quad \mathbf{W}^{-1,p}(\Omega)$$

can be deduced. Indeed, for any $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,p'}(\Omega)$

$$\langle (\nabla \cdot \boldsymbol{v})\, \boldsymbol{u}_k - (\nabla \cdot \boldsymbol{v})\, \boldsymbol{u}, \boldsymbol{\varphi} \rangle_{\mathbf{W}^{-1,p}(\Omega) \times \mathbf{W}_0^{1,p'}(\Omega)} = \langle \nabla \cdot \boldsymbol{v}, (\boldsymbol{u}_k - \boldsymbol{u}) \cdot \boldsymbol{\varphi} \rangle_{W^{-1,s}(\Omega) \times W_0^{1,s'}(\Omega)}.$$

Observe that $(\boldsymbol{u}_k - \boldsymbol{u}) \cdot \boldsymbol{\varphi} \rightharpoonup 0$ in $W_0^{1,s'}(\Omega)$, which implies that the term tends to zero. As a consequence, the limit $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ satisfies the problem:

$$-\Delta \boldsymbol{u} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \boldsymbol{u} = \mathbf{0} \quad \text{on} \;\; \Gamma,$$

and there exists a constant $C_4 > 0$ such that:

$$||\boldsymbol{u}||_{\boldsymbol{W}^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)/\mathbb{R}} \leqslant 2\, C_4 \left(1 + C||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}\right) ||\boldsymbol{f}||_{\boldsymbol{W}^{-1,p}(\Omega)}.$$

(b)The case of $h \neq 0$ and $\boldsymbol{g} \neq \mathbf{0}$: The data are lifted by using $(\boldsymbol{u}_0, \pi_0) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ the solution of the Stokes problem:

$$-\Delta \boldsymbol{u}_0 + \nabla \pi_0 = \mathbf{0}, \quad \nabla \cdot \boldsymbol{u}_0 = h, \quad \boldsymbol{u}_0|_\Gamma = \boldsymbol{g}$$

satisfying

$$\|\boldsymbol{u}_0\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi_0\|_{L^p(\Omega)/\mathbb{R}} \leq C \left(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p}(\Gamma)} \right).$$

Therefore, $(\breve{\boldsymbol{u}}, \breve{\pi}) = (\boldsymbol{u} - \boldsymbol{u}_0, \pi - \pi_0)$ is the solution of the Oseen problem:

$$-\Delta \breve{\boldsymbol{u}} + \boldsymbol{v} \cdot \nabla \breve{\boldsymbol{u}} + \nabla \pi = \breve{\boldsymbol{f}} \quad \text{and} \quad \nabla \cdot \breve{\boldsymbol{u}} = 0 \quad \text{in } \Omega, \quad \breve{\boldsymbol{u}} = \mathbf{0} \quad \text{on} \;\; \Gamma$$

with $\breve{\boldsymbol{f}} = \boldsymbol{f} - \boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 \in \mathbf{W}^{-1,p}(\Omega)$, which is a problem treated above. Therefore,

$$\|\breve{\boldsymbol{u}}\|_{\mathbf{W}^{1,p}(\Omega)} \quad + \quad \|\breve{\pi}\|_{\mathbf{L}^p(\Omega)/\mathbb{R}} \leq C \, \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \|\breve{\boldsymbol{f}}\|_{\mathbf{W}^{-1,p}(\Omega)}$$

and finally

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{\mathbf{L}^p(\Omega)/\mathbb{R}} \leq C \, \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \, \left(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p}(\Gamma)} \right) \right).$$

□ ▮

Theorem 2.9 (Generalised solution for (O') and $p < 2$) *Let $1 < p < 2$,*

$$\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega), \quad h \in L^p(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$$

together with $\boldsymbol{v} \in \mathbf{L}^3(\Omega)$ and $\|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ being sufficiently small (in the sense of (2.36) if $p \in (3/2, 2)$ and in the sense of (2.9) in the other case), verifying the compatibility condition (1.5). Then, the problem (O') has a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Moreover, there exists some constant $C > 0$ such that inequality (2.43) holds for $C = C(\Omega, p, \delta_1)\ \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)$ if $p \in (3/2, 3)$ and for $C = C(\Omega, p, \delta_0)\ \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)$ in the other case.

Proof. As in Theorem 2.6, we separate the proof into existence and estimates.
A) Existence. Using a duality method, first we suppose that $h = 0$ and $\boldsymbol{g} = \mathbf{0}$. The problem (O') is equivalent to find $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that: $\forall (\boldsymbol{w}, \chi) \in \mathbf{W}^{1,p'}(\Omega) \times L^{p'}(\Omega)$

$$\langle \boldsymbol{u}, -\Delta \boldsymbol{w} - \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \nabla \chi \rangle_{\mathbf{W}_0^{1,p}(\Omega) \times \mathbf{W}^{-1,p'}(\Omega)} - \langle \pi, \nabla \cdot \boldsymbol{w} \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\mathbf{W}^{-1,p}(\Omega) \times \mathbf{W}_0^{1,p'}(\Omega)}.$$

Therefore, for any pair $(\mathbf{F}, \varphi) \in \mathbf{W}^{-1,p'}(\Omega) \times L_0^{p'}(\Omega)$, let $(\boldsymbol{w}, \chi) \in \mathbf{W}_0^{1,p'}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ be the solution of the problem of type (O) described as:

$$-\Delta \boldsymbol{w} - \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \nabla \chi = \mathbf{F} \quad \text{and} \quad \nabla \cdot \boldsymbol{w} = \varphi \quad \text{in } \Omega, \quad \boldsymbol{w} = \mathbf{0} \quad \text{on } \Gamma,$$

which exists thanks to Theorem 2.6 (provided that $\|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ is sufficiently small in the sense of (2.36) if $p \in (3/2, 2)$ and in the sense of (2.9) in the other case). Moreover, we know that:

$$\|\boldsymbol{w}\|_{\mathbf{W}^{1,p'}(\Omega)} + \|\chi\|_{L^{p'}(\Omega)/\mathbb{R}} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \Big(\|\mathbf{F}\|_{\mathbf{W}^{-1,p}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \|\varphi\|_{L^{p'}(\Omega)}\Big).$$

Furthermore, we have:

$$\Big|\langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\mathbf{W}^{-1,p}(\Omega) \times \mathbf{W}_0^{1,p'}(\Omega)}\Big| \leq C \|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2 \Big(\|\mathbf{F}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|\varphi\|_{L^{p'}(\Omega)}\Big).$$

In other words, the mapping $(\mathbf{F}, \varphi) \to \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\mathbf{W}^{-1,p}(\Omega) \times \mathbf{W}_0^{1,p'}(\Omega)}$ defines an element of the dual space of $\mathbf{W}^{-1,p'}(\Omega) \times L_0^{p'}(\Omega)$. From Riesz's Representation Theorem, we deduce that there exists a unique $(\boldsymbol{u}, \pi) \in \mathbf{W}_0^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ solution of (O) (for $h = 0$ and $\boldsymbol{g} = \mathbf{0}$) that satisfies (2.45).

As in Theorem 2.8, the case of $h \neq 0$ or $\boldsymbol{g} \neq \mathbf{0}$ can be treated using the lifting (2.46). Therefore, it remains to study the regularity for the solution $(\boldsymbol{u}_0, \pi_0)$ of the problem:

$$-\Delta \boldsymbol{u}_0 + \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{u}_0) + \nabla \pi_0 = \boldsymbol{f} + \Delta \boldsymbol{z} - \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{z}) \quad \text{and} \quad \nabla \cdot \boldsymbol{u}_\lambda = 0 \quad \text{in } \Omega, \quad \boldsymbol{u}_\lambda = \mathbf{0} \quad \text{on } \Gamma,$$

which using (2.45) satisfies estimate (2.47). The bound (2.43) for $C = C(\Omega, p, \delta_0)\ \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2$ is satisfied by $(\boldsymbol{u}, \pi) = (\boldsymbol{u}_0 + \boldsymbol{z}, \pi_0)$. This estimate will be improved below.

B) Estimates. In order to improve the estimates, we can adapt the same argument used in the proof of Theorem 2.8 **B)**. Concretely, problem (P) is replaced by:

$$(P') \quad -\Delta \boldsymbol{u}_k + \nabla\cdot(\boldsymbol{v}\otimes \boldsymbol{u}_k) + \nabla \pi_k = \boldsymbol{f}_k \quad \text{and} \quad \nabla\cdot \boldsymbol{u}_k = 0 \quad \text{in } \Omega, \quad \boldsymbol{u}_k = \boldsymbol{0} \quad \text{on } \Gamma,$$

and therefore the study of the term $(\nabla\cdot \boldsymbol{v})\, \boldsymbol{u}_k$ and its correspondent estimate in (2.52) are not necessary. As a consequence, estimate (2.43) is obtained, for $C = C(\Omega, p, \delta_1)\, \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)$ if $p \in (3/2, 3)$ and for $C = C(\Omega, p, \delta_0)\, \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)$ in the other case, without assuming neither regularity for $\nabla\cdot \boldsymbol{v}$ in $W^{-1,s}(\Omega)$ for s defined in (2.44) nor smallness for this norm. □ ∎

Proceeding as in Corollary 2.4 in [9] (see also Corollary 7 in [7]), we prove that:

Theorem 2.10 (Strong solution for (O) **when** $1 < p < 6/5$**)** *Let* $1 < p < 6/5$ *and*

$$\boldsymbol{f} \in \mathbf{L}^p(\Omega), \quad \boldsymbol{v} \in \mathbf{L}^3(\Omega), \quad h \in W^{1,p}(\Omega) \quad \textit{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$$

verify the compatibility condition (1.5) and $\|\nabla\cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)} \le \delta_2$ *(in the sense of (2.55)). Then, there exists a unique solution* $(\boldsymbol{u}, \pi)$ *of* (O) *belonging to* $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$*. Moreover, there exists a constant* $C > 0$ *such that:*

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} &\le C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \\ &\times \Big(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\Big)\Big) \end{aligned} \tag{2.60}$$

is satisfied for $C = C(\Omega, p, \delta_2)\, \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)$.

Proof. First, taking into account that:

$$\mathbf{L}^p(\Omega) \hookrightarrow \mathbf{W}^{-1,p^*}(\Omega) \quad W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \quad \text{and} \quad \mathbf{W}^{2-1/p,p}(\Gamma) \hookrightarrow \mathbf{W}^{1-1/p^*,p^*}(\Gamma) \tag{2.61}$$

for $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$ and $3/2 < p^* < 2$, from Theorem 2.8 we can deduce the existence of a solution $(\boldsymbol{u}, \pi)$ in $\mathbf{W}^{1,p^*}(\Omega) \times L^{p^*}(\Omega)$ satisfying the estimate:

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{1,p^*}(\Omega)} + \|\pi\|_{L^{p^*}(\Omega)/\mathbb{R}} &\le C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\Big(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p^*}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \\ &\times \Big(\|h\|_{L^{p^*}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p^*,p^*}(\Gamma)}\Big)\Big). \end{aligned} \tag{2.62}$$

Second, the previous regularity provides that $\boldsymbol{v}\cdot\nabla \boldsymbol{u} \in \mathbf{L}^p(\Omega)$. As a consequence, the regularity for the Stokes problem allows to obtain the strong regularity for $(\boldsymbol{u}, \pi)$ in $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Moreover, the regularity estimate:

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \le C\left(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)} + \|\boldsymbol{v}\cdot\nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}\right)$$

together with the inequality:

$$\|\boldsymbol{v}\cdot\nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} \le \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\|\nabla \boldsymbol{u}\|_{\mathbf{L}^{p^*}(\Omega)} \le \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\|\boldsymbol{u}\|_{\mathbf{W}^{1,p^*}(\Omega)} \tag{2.63}$$

and (2.62) lead to the bound (2.60). □ ∎

Theorem 2.11 (Strong solution for (O') when $1 < p < 6/5$) *Under the hypotheses of Theorem 2.10, if $\nabla\cdot\boldsymbol{v} \in L^{3/2}(\Omega)$ and $\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ is sufficiently small in the sense of (2.36), then there exists a unique solution $(\boldsymbol{u}, \pi)$ of (O') belonging to $\mathbf{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$. Moreover, there exists a constant $C > 0$ such that estimate (2.60) is satisfied for $C = C(\Omega, p, \delta_1)\,\Big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} + \|\nabla\cdot\boldsymbol{v}\|_{L^{3/2}(\Omega)}\Big)$.*

Proof.
Observe first that (2.61) is satisfied for $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$ and $3/2 < p^* < 2$. Then, thanks to the smallness of $\|\nabla\cdot\boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ in the sense of (2.36), Theorem 2.9 guarantees the existence of a solution $(\boldsymbol{u}, \pi)$ of (O') belonging to $\mathbf{W}^{1,p^*}(\Omega)\times L^{p^*}(\Omega)/\mathbb{R}$ and satisfying (2.62):

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathbf{W}^{1,p^*}(\Omega)} + \|\pi\|_{L^{p^*}(\Omega)/\mathbb{R}} \;&\leq\; C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\\ &\times\;\Big(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\Big)\Big).\end{aligned} \tag{2.64}$$

Again based on the regularity over the Stokes problem, we easily verify that $(\boldsymbol{u}, \pi) \in \mathbf{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ and:

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \;&\leq\; C\,\Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\\ &+\; \|\boldsymbol{v}\cdot\nabla\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|(\nabla\cdot\boldsymbol{v})\,\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}\Big),\end{aligned} \tag{2.65}$$

where we have to bound the last two terms. Using that $\boldsymbol{u} \in \mathbf{W}^{1,p^*}(\Omega)$, we obtain (2.63) and

$$\|(\nabla\cdot\boldsymbol{v})\,\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} \leq \|\nabla\cdot\boldsymbol{v}\|_{\mathbf{L}^{3/2}(\Omega)}\|\boldsymbol{u}\|_{\mathbf{L}^q(\Omega)} \leq C_7\,\|\nabla\cdot\boldsymbol{v}\|_{\mathbf{L}^{3/2}(\Omega)}\|\boldsymbol{u}\|_{\mathbf{W}^{1,p^*}(\Omega)} \tag{2.66}$$

for C_7 the constant of the embedding $\mathbf{W}^{1,p^*}(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$ with $\frac{1}{q} = \frac{1}{p^*} - \frac{1}{3}$. Using (2.64) in the bounds of (2.63) and (2.66), and replacing the resulting estimate in (2.65), we found the required estimate. □ ∎

As a consequence of $\nabla\cdot\boldsymbol{v} \neq 0$, the definition of very weak solution for the Oseen problem given in [7] must be rewritten. In fact, this definition should be given for the problems (O) and (O'), which are not equivalent if $\nabla\cdot\boldsymbol{v} \neq 0$. We start the study for the weak solution of (O') because it is easier (in fact, it corresponds with the study about the very weak solution made in [7]).

Definition 2.12 (Very weak solution for the Oseen problem (O')) *Assume that $\boldsymbol{f} \in [\mathbf{X}_{r',p'}(\Omega)]'$ (see Lemma 1.1 and Remark 2.13), $h \in L^r(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ satisfying the compatibility condition (1.5) and $\boldsymbol{v} \in \mathbf{L}^s(\Omega)$ with (r, s) given by:*

$$s = \max\{3, p'\} \quad \text{if } p \neq \frac{3}{2} \quad \text{and } s > 3 \text{ if } p = \frac{3}{2}, \tag{2.67}$$

$$r \geq 1 \quad \textit{such that} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{s}. \tag{2.68}$$

We say that $(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ *is a* **very weak solution** *of* (O') *if the following equalities hold: For any* $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$ *and* $\chi \in W^{1,p'}(\Omega)$,

$$\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{\varphi} - \boldsymbol{v} \cdot \nabla \boldsymbol{\varphi}) \, d\boldsymbol{x} \quad - \quad \langle \pi, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_\Gamma,$$

$$\int_\Omega \boldsymbol{u} \cdot \nabla \chi \, d\boldsymbol{x} \quad = \quad - \int_\Omega h \, \chi \, d\boldsymbol{x} + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \chi \rangle_\Gamma,$$

where the dualities on Ω *and* Γ *are defined by:*

$$\langle \cdot, \cdot \rangle_\Omega = \langle \cdot, \cdot \rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)}, \; \langle \cdot, \cdot \rangle_\Gamma = \langle \cdot, \cdot \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)}, \tag{2.69}$$

and the space $\mathbf{Y}_{p'}(\Omega)$ *is characterised in two different ways in (1.4).*

Remark 2.13 *If* $p < \frac{3}{2}$, *then* $r = 1$ *and the hypothesis on* $\boldsymbol{f}$ *means that* $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$ *with* $\mathbb{F}_0 \in \mathbb{L}^1(\Omega)$ *and* $f_1 \in W^{-1,p}(\Omega)$. *In this case, we have:*

$$\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega = - \int_\Omega \mathbb{F}_0 : \nabla \boldsymbol{\varphi} \, d\boldsymbol{x} + \langle f_1, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)}, \quad \forall \boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega).$$

Theorem 2.14 (Very weak solution for (O')) *Assume that* $\boldsymbol{f} \in [\mathbf{X}_{r',p'}(\Omega)]'$, $h \in L^r(\Omega)$ *and* $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ *satisfy the compatibility condition (1.5) and* $\boldsymbol{v} \in \mathbf{L}^s(\Omega)$ *with* $\|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ *sufficiently small (in the sense of (2.55) if* $p > 6$ *and in the sense of (2.9) in the other case), and* (r, s) *given by (2.68) and (2.67). Then, the Oseen problem* (O') *has a unique solution* $(\boldsymbol{u}, \pi) \in \mathbf{T}_{p,r}(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ *verifying the following estimate:*

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + (1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)})^{-1} \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \quad &\leq \quad C \, \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big) \Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\big) \\ &\times \quad \Big(\|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \Big) \Big), \end{aligned} \tag{2.70}$$

if $1 < p \leq 6$ *and estimate (2.70) replacing* C *by* $C \, \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)$ *and* $\|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}$ *by* $\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}$ *if* $p > 6$.

Proof. Following the proof of Theorem 17 in [7], we have to prove two steps to conclude the statement of the Theorem:

(A) The solution $(\boldsymbol{u}, \pi)$ of (O') belongs to $\mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ and satisfies the two first equations of (O').

(A1) We first consider the case where $\boldsymbol{g} \cdot \boldsymbol{n}|_\Gamma = 0$ and $\int_\Omega h(\boldsymbol{x}) \, d\boldsymbol{x} = 0$.

We prove then that problem (O') is equivalent to the variational formulation: For a given $\boldsymbol{v} \in \mathbf{L}^s(\Omega)$ with s defined by (2.67), find $(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ such that: $\forall \boldsymbol{w} \in \mathbf{Y}_{p'}(\Omega)$,

$\forall \chi \in W^{1,p'}(\Omega)$

$$\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{w} - \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \nabla \chi)\, d\boldsymbol{x} - \langle \pi,\, \nabla \cdot \boldsymbol{w} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \int_\Omega h\, \chi\, d\boldsymbol{x} - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma . \tag{2.71}$$

Now, for any pair $(\mathbf{F}, \varphi) \in \mathbf{L}^{p'}(\Omega) \times [W_0^{1,p'}(\Omega) \cap L_0^{p'}(\Omega)]$, we have:

$$\begin{aligned} \left| \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \int_\Omega h\, \chi\, d\boldsymbol{x} - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma \right| &\leq C \left(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right) \\ &\times \left(\|\boldsymbol{w}\|_{\mathbf{W}^{2,p'}(\Omega)} + \|\chi\|_{W^{1,p'}} \right) \end{aligned} \tag{2.72}$$

where $(\boldsymbol{w}, \chi)$ is the solution of problem:

$$-\Delta \boldsymbol{w} - \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \nabla \chi = \mathbf{F}, \quad \nabla \cdot \boldsymbol{w} = \varphi \quad \text{in } \Omega, \qquad \boldsymbol{w} = \mathbf{0} \quad \text{on } \Gamma.$$

Observe that hypothesis of $\boldsymbol{v} \in \mathbf{L}^s(\Omega)$ with s defined in (2.67) is necessary in order to give a sense to the term:

$$\int_\Omega \boldsymbol{u} \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{w})\, d\boldsymbol{x} \tag{2.73}$$

in the case of $p = \frac{3}{2}$, where $\boldsymbol{u} \in \mathbf{L}^{3/2}(\Omega)$, $\nabla \boldsymbol{w} \in \mathbf{W}^{1,3}(\Omega) \hookrightarrow \mathbf{L}^b(\Omega)$ (for any $b \in (1, +\infty)$). In the cases $p \neq \frac{3}{2}$, the previous integral is always defined because $\boldsymbol{u} \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{w}) \in L^1(\Omega)$.

Taking into account that in any case $\boldsymbol{v} \in \mathbf{L}^3(\Omega)$ and using Theorem 2.5 (if $1 < p \leq 6$) and Theorem 2.10 (if $p > 6$), then the solution $(\boldsymbol{w}, \chi)$ belongs to $\mathbf{W}^{2,p'}(\Omega) \times W^{1,p'}(\Omega)$ and satisfies:

$$\|\boldsymbol{w}\|_{\mathbf{W}^{2,p'}(\Omega)} + \|\chi\|_{W^{1,p'}(\Omega)/\mathbb{R}} \leq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \left(\|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \|\varphi\|_{W^{1,p'}(\Omega)} \right)$$

if $p' \geq 6/5$ and for $1 < p' < 6/5$ the same estimate holds replacing C by $C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$.

Therefore the mapping:

$$(\mathbf{F}, \varphi) \longmapsto \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \int_\Omega h\, \chi\, d\boldsymbol{x} - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma \tag{2.74}$$

defines an element $(\boldsymbol{u}, \pi)$ of the dual space of $\mathbf{L}^{p'}(\Omega) \times [W_0^{1,p'} \cap L_0^{p'}(\Omega)]$, which is equal to $\mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$. Furthermore, $(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ and verifies (2.71) and the following estimate if $1 < p \leq 6$:

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} &+ (1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)})^{-1} \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \\ &\times \left(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right). \end{aligned} \tag{2.75}$$

In the case of $p > 6$, estimate (2.75) is also true but replacing C by $C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$. Thus, we have proved that estimates for $1 < p \leq 6$ and for $p > 6$ are true when $\boldsymbol{v} \in \mathbf{L}^s(\Omega)$ with s defined by (2.67).

(A2) Second, we suppose that $\displaystyle\int_\Omega h(\boldsymbol{x})\,d\boldsymbol{x} = \langle \mathbf{g}\cdot\boldsymbol{n}, 1\rangle_\Gamma$ and consider the Neumann problem: Find $\theta \in W^{1,p}(\Omega)/\mathbb{R}$ such that:

$$(N) \quad \Delta\theta = h \ \text{ in } \Omega, \ \ \frac{\partial\theta}{\partial\boldsymbol{n}} = \boldsymbol{g}\cdot\boldsymbol{n} \ \text{ on } \Gamma,$$

which has a unique solution $\theta \in W^{1,p}(\Omega)/\mathbb{R}$ and verifies the estimate:

$$\|\theta\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\,\left(\|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\cdot\boldsymbol{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right). \tag{2.76}$$

Set $\boldsymbol{u}_0 = \nabla\theta$. Using the precedent case, there exists a unique $(\boldsymbol{z},\pi) \in \mathbf{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$ solution of problem:

$$-\Delta\boldsymbol{z} + \nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{z}) + \nabla\pi = \boldsymbol{f} + \nabla h - \nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u}_0) \ \text{ and } \ \nabla\cdot\boldsymbol{z} = 0 \ \text{ in } \Omega, \ \boldsymbol{z} = \boldsymbol{g} - \boldsymbol{u}_0|_\Gamma \ \text{ on } \Gamma,$$

where the characterization given by Lemma 1.1 implies that ∇h and $\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u}_0)$ belong to $[\mathbf{X}_{r',p'}(\Omega)]'$ (because of $h \in L^r(\Omega) \hookrightarrow W^{-1,p}(\Omega)$ and $\boldsymbol{v}\otimes\boldsymbol{u} \in \mathbb{L}^r(\Omega)$ thanks to (2.68)). Thus, $\boldsymbol{f} + \nabla h - \nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u}_0)$ belongs to $[\mathbf{X}_{r',p'}(\Omega)]'$. Moreover, $\boldsymbol{g} - \boldsymbol{u}_0|_\Gamma$ satisfies the compatibility conditions for the precedent case. Hence, using (2.75) if $1 < p \leq 6$, $(\boldsymbol{z},\pi)$ satisfies:

$$\begin{aligned}
&\|\boldsymbol{z}\|_{\mathbf{L}^p(\Omega)} + (1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)})^{-1}\|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \\
&\quad\leq C\,\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|\boldsymbol{f} + \nabla h - \nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u}_0)\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|\boldsymbol{g} - \boldsymbol{u}_0\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right) \\
&\quad\leq C\,\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{W^{-1,p}(\Omega)} + \|\boldsymbol{v}\otimes\boldsymbol{u}_0\|_{\mathbb{L}^r(\Omega)} + \|\boldsymbol{g} - \boldsymbol{u}_0\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right) \\
&\quad\leq C\,\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)\left(\|h\|_{W^{-1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right)\right),
\end{aligned}$$

where we have used that $L^r(\Omega) \hookrightarrow W^{-1,p}(\Omega)$, which implies that $\|h\|_{W^{-1,p}(\Omega)} \leq C_8\,\|h\|_{L^r(\Omega)}$ for a certain constant C_8, the bound $\|\boldsymbol{v}\otimes\boldsymbol{u}_0\|_{\mathbb{L}^r(\Omega)} \leq \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\|\boldsymbol{u}_0\|_{\mathbf{L}^p(\Omega)}$ for (r,s) defined in (2.67)-(2.68), and estimate (2.76). In a similar manner, if $p > 6$ we use (2.75) replacing C by $C\,\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$ obtaining the estimate:

$$\begin{aligned}
\|\boldsymbol{z}\|_{\mathbf{L}^p(\Omega)} \ &+ \ (1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)})^{-1}\|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C\,\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2 \\
&\times \ \left(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|h\|_{W^{-1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right)\right).
\end{aligned}$$

Finally, the pair of functions $(\boldsymbol{u},\pi) = (\boldsymbol{z} + \boldsymbol{u}_0, \pi)$ is the required solution satisfying the required estimates.

(B) The trace of $\boldsymbol{u}$ satisfies $\boldsymbol{u} = \boldsymbol{g}$ on Γ and belongs to $\mathbf{W}^{-1/p,p}(\Gamma)$.

In order to obtain that $\boldsymbol{u} \in \mathbf{T}_{p,r}(\Omega)$, we need to prove that $\Delta\boldsymbol{u} \in [\mathbf{X}_{r',p'}(\Omega)]'$. From (2.68) and (2.67), it suffices to note that $\Delta\boldsymbol{u} = \nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u}) + \nabla\pi - \boldsymbol{f}$ and $\boldsymbol{v}\otimes\boldsymbol{u} \in \mathbb{L}^r(\Omega)$. Therefore, the tangential trace of $\boldsymbol{u}$ belongs to $\mathbf{W}^{-1/p,p}(\Gamma)$. In that way, as $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ and $\nabla \cdot \boldsymbol{u} \in L^r(\Omega)$, then $\boldsymbol{u} \cdot \boldsymbol{n}|_\Gamma \in W^{-1/p,p}(\Gamma)$, and the whole trace $\boldsymbol{u}|_\Gamma \in \mathbf{W}^{-1/p,p}(\Gamma)$ can be identified with $\boldsymbol{u}|_\Gamma = \boldsymbol{g}$.

□ ▮

Remark 2.15 *When $\boldsymbol{g} \cdot \boldsymbol{n}|_\Gamma = 0$ and $\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = 0$ (see case* **(A1)** *in the previous proof), estimates in Theorem 2.14 can be replaced by estimate (2.75) for $1 < p \leq 6$ and estimate (2.75) replacing C by $C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$ for $p > 6$. For both values of C, estimate (2.75) does not depend on the norm $\|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}$ but $\boldsymbol{v}$ must belong to the space $\mathbf{L}^s(\Omega)$ (with s defined by (2.67)) in order to give a sense to the term (2.73).*

Definition 2.16 (Very weak solution for the Oseen problem (O)**)** *Assume that $\boldsymbol{f} \in [\mathbf{X}_{r',p'}(\Omega)]'$ (see Lemma 1.1 and Remark 2.13), $h \in L^r(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ satisfying the compatibility condition (1.5) and $\boldsymbol{v} \in \mathbf{L}^s(\Omega)$ with (r, s) given by (2.68) and (2.67), and $\nabla \cdot \boldsymbol{v} \in L^t(\Omega)$ for t defined by:*

$$t = \max\left\{p', \frac{3}{2}\right\}. \tag{2.77}$$

We say that $(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ is a **very weak solution** *of (O) if the following equalities hold: For any $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$, with in addition $\boldsymbol{\varphi} \in \mathbf{L}^\infty(\Omega)$ if $p = 3$, and $\chi \in W^{1,p'}(\Omega)$,*

$$\begin{aligned}\int_\Omega \boldsymbol{u} \cdot (-\Delta\boldsymbol{\varphi} - \boldsymbol{v} \cdot \nabla\boldsymbol{\varphi} - (\nabla \cdot \boldsymbol{v})\, \boldsymbol{\varphi})\, d\boldsymbol{x} \quad &- \quad \langle \pi, \nabla \cdot \boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}}\rangle_\Gamma, \\ \int_\Omega \boldsymbol{u} \cdot \nabla\chi\, d\boldsymbol{x} \quad &= \quad -\int_\Omega h\, \chi\, d\boldsymbol{x} + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \chi\rangle_\Gamma,\end{aligned} \tag{2.78}$$

and where the dualities on Ω and Γ are defined by (2.69).

Theorem 2.17 (Very weak solution for (O)**)** *Let $\boldsymbol{f} \in [\mathbf{X}_{r',p'}(\Omega)]'$, $h \in L^r(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ satisfy the compatibility condition (1.5) and $\boldsymbol{v} \in \mathbf{L}^s(\Omega)$ with $\|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)}$ sufficiently small (in the sense of (2.36) if $p > 6$ and in the sense of (2.9) in the other case), and (r, s) given by (2.68) and (2.67). Assume also $\nabla \cdot \boldsymbol{v} \in L^t(\Omega)$ for t defined by (2.77). Then, the Oseen problem (O) has a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{T}_{p,r}(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ verifying the following estimate:*

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} \quad &+ \quad (1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)})^{-1} \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} + \|\nabla \cdot \boldsymbol{v}\|_{L^{3/2}(\Omega)}\right) \\ &\times\Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)} + \|\nabla \cdot \boldsymbol{v}\|_{L^t(\Omega)}\right)\Big(\|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\Big)\Big),\end{aligned} \tag{2.79}$$

if $1 < p \leq 6$ and estimate (2.79) replacing C by $C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$, $\|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}$ by $\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}$ and $\|\nabla \cdot \boldsymbol{v}\|_{\mathbf{L}^t(\Omega)}$ by $\|\nabla \cdot \boldsymbol{v}\|_{\mathbf{L}^{3/2}(\Omega)}$ if $p > 6$.

Proof. The proof follows a scheme similar to Theorem 2.14:

(A) The solution $(\boldsymbol{u},\pi)$ of (O) belongs to $\mathbf{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$ and satisfies the two first equations of (O).

(A1) Again we consider the case where $\boldsymbol{g}\cdot\boldsymbol{n}|_\Gamma=0$ and $\displaystyle\int_\Omega h(\boldsymbol{x})\,d\boldsymbol{x}=0$. Moreover, we start by considering $\nabla\cdot\boldsymbol{v}\in L^{\tilde q}(\Omega)$ for $\tilde q$ defined by:

$$\tilde q=\max\left\{p',\frac{3}{2}\right\},\quad \text{if } p\neq 3,\quad \text{and}\quad \tilde q>3/2\quad \text{if } p=3. \tag{2.80}$$

instead of $\nabla\cdot\boldsymbol{v}\in L^t(\Omega)$ with t defined in (2.77).

It remains to prove that problem (O) is equivalent to the variational formulation: Find $(\boldsymbol{u},\pi)\in\mathbf{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$ such that: $\forall\boldsymbol{w}\in\mathbf{Y}_{p'}(\Omega)$, $\forall\chi\in W^{1,p'}(\Omega)$

$$\begin{aligned}&\int_\Omega \boldsymbol{u}\cdot(-\Delta\boldsymbol{w}-\boldsymbol{v}\cdot\nabla\boldsymbol{w}-(\nabla\cdot\boldsymbol{v})\,\boldsymbol{w}+\nabla\chi)\,d\boldsymbol{x}-\langle\pi,\,\nabla\cdot\boldsymbol{w}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)}\\&\quad=\langle\boldsymbol{f},\boldsymbol{w}\rangle_\Omega-\int_\Omega h\,\chi\,d\boldsymbol{x}-\left\langle\boldsymbol{g}_\tau,\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{n}}\right\rangle_\Gamma.\end{aligned} \tag{2.81}$$

Now, for any pair $(\mathbf{F},\varphi)\in\mathbf{L}^{p'}(\Omega)\times[W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega)]$, inequality (2.72) is satisfied, but this time $(\boldsymbol{w},\chi)$ is the solution of a problem of type (O'):

$$-\Delta\boldsymbol{w}-\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{w})+\nabla\chi=\mathbf{F},\quad \nabla\cdot\boldsymbol{w}=\varphi\quad\text{in }\Omega,\qquad \boldsymbol{w}=\mathbf{0}\quad\text{on }\Gamma.$$

Observe that hypothesis of $\nabla\cdot\boldsymbol{v}\in\mathbf{L}^{\tilde q}(\Omega)$ with $\tilde q$ defined in (2.80) is necessary in order to give a sense to the term:

$$\int_\Omega \boldsymbol{u}\,(\nabla\cdot\boldsymbol{v})\cdot\boldsymbol{w}\,d\boldsymbol{x} \tag{2.82}$$

in the case of $p=3$, where $\boldsymbol{u}\in\mathbf{L}^3(\Omega)$, $\boldsymbol{w}\in\mathbf{W}^{2,3/2}(\Omega)\hookrightarrow\mathbf{W}^{1,3}(\Omega)\hookrightarrow\mathbf{L}^b(\Omega)$ (for any $b\in(1,+\infty)$). In the cases $p\neq 3$, the previous integral is always defined because $\boldsymbol{u}\,(\nabla\cdot\boldsymbol{v})\cdot\boldsymbol{w}\in L^1(\Omega)$.

Taking into account that in any case $\boldsymbol{v}\in\mathbf{L}^3(\Omega)$ and $\nabla\cdot\boldsymbol{v}\in L^{3/2}(\Omega)$ and using Theorem 2.4 and Theorem 2.11, then the solution $(\boldsymbol{w},\chi)$ belongs to $\mathbf{W}^{2,p'}(\Omega)\times W^{1,p'}(\Omega)/\mathbb{R}$, satisfying:

$$\|\boldsymbol{w}\|_{\mathbf{W}^{2,p'}(\Omega)}+\|\chi\|_{W^{1,p'}(\Omega)/\mathbb{R}}\le C\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\boldsymbol{v}\|_{L^{3/2}(\Omega)}\right)\left(\|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)}+\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\|\varphi\|_{W^{1,p'}(\Omega)}\right)$$

if $1<p\le 6$ and the same estimate holds for $p>6$ replacing C by $C\,\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$.

Therefore the mapping (2.74) defines an element $(\boldsymbol{u},\pi)$ of the dual space of $\mathbf{L}^{p'}(\Omega)\times[W_0^{1,p'}\cap L_0^{p'}(\Omega)]$, which is equal to $\mathbf{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$. Furthermore, $(\boldsymbol{u},\pi)\in\mathbf{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$ verifies (2.81) and following estimate if $1<p\le 6$:

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}+(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)})^{-1}\|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}}\ &\le\ C\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\boldsymbol{v}\|_{L^{3/2}(\Omega)}\right)\\&\quad\times\ \left(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'}+\|h\|_{L^r(\Omega)}+\|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right).\end{aligned} \tag{2.83}$$

The same estimate replacing C by $C\,\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$ holds if $p>6$. Thus, we have proved that estimates for $1<p\leq 6$ and for $p>6$ are true when $\nabla\cdot\boldsymbol{v}\in L^{\tilde{q}}(\Omega)$ with $\tilde{q}$ is defined by (2.80).

(A2) When the general case is treated, considering $h\neq 0$ or $\boldsymbol{g}\neq\mathbf{0}$ and satisfying the compatibility condition (1.5), we consider the Neumann Problem (N) whose unique solution $\theta\in W^{1,p}(\Omega)/\mathbb{R}$ and verifies estimate (2.76). Setting $\boldsymbol{u}_0=\nabla\theta$ and using the precedent case, there exists a unique $(\boldsymbol{z},\pi)\in\mathbf{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$ solution of problem:

$$-\Delta\boldsymbol{z}+\boldsymbol{v}\cdot\nabla z+\nabla\pi=\boldsymbol{f}+\nabla h-\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u}_0)+(\nabla\cdot\boldsymbol{v})\,\boldsymbol{u}_0\ \text{ and }\ \nabla\cdot\boldsymbol{z}=0\ \text{ in }\Omega,\ \boldsymbol{z}=\boldsymbol{g}-\boldsymbol{u}_0|_\Gamma\ \text{ on }\Gamma,$$

where the characterization given by Lemma 1.1 implies that ∇h and $\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u}_0)$ belong to $[\mathbf{X}_{r',p'}(\Omega)]'$. As we are proving immediately (see (2.84)), then $(\nabla\cdot\boldsymbol{v})\,\boldsymbol{u}_0$ belongs to $\mathbf{W}^{-1,r}(\Omega)$ and hence to $[\mathbf{X}_{r',p'}(\Omega)]'$. Moreover, $\boldsymbol{g}-\boldsymbol{u}_0|_\Gamma$ satisfies the compatibility conditions for the precedent case.

Hence, using (2.83) if $1<p\leq 6$, $(\boldsymbol{z},\pi)$ satisfies:

$$\begin{aligned}
&\|\boldsymbol{z}\|_{\mathbf{L}^p(\Omega)}+(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)})^{-1}\|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}}\\
&\quad\leq C\,\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\boldsymbol{v}\|_{L^{3/2}(\Omega)}\right)\\
&\quad\times\left(\|\boldsymbol{f}+\nabla h-\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u}_0)+(\nabla\cdot\boldsymbol{v})\,\boldsymbol{u}_0\|_{[\mathbf{X}_{r',p'}(\Omega)]'}+\|\boldsymbol{g}-\boldsymbol{u}_0\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right)\\
&\quad\leq C\,\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\boldsymbol{v}\|_{L^{3/2}(\Omega)}\right)\\
&\quad\times\left(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'}+\|h\|_{W^{-1,p}(\Omega)}+\|\boldsymbol{v}\otimes\boldsymbol{u}_0\|_{\mathbb{L}^r(\Omega)}+\|(\nabla\cdot\boldsymbol{v})\,\boldsymbol{u}_0\|_{[\mathbf{X}_{r',p'}(\Omega)]'}+\|\boldsymbol{g}-\boldsymbol{u}_0\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right)\\
&\quad\leq C\,\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\boldsymbol{v}\|_{L^{3/2}(\Omega)}\right)\\
&\quad\times\left(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'}+\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}+\|\nabla\cdot\boldsymbol{v}\|_{L^{\tilde{q}}(\Omega)}\right)\,\left(\|h\|_{W^{-1,p}(\Omega)}+\|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right)\right),
\end{aligned}$$

where, in addition to the estimates used in the proof of Theorem 2.14, we have used the bound $\|(\nabla\cdot\boldsymbol{v})\,\boldsymbol{u}_0\|_{[\mathbf{X}_{r',p'}(\Omega)]'}\leq\|\nabla\cdot\boldsymbol{v}\|_{L^t(\Omega)}\|\boldsymbol{u}_0\|_{\mathbf{L}^p(\Omega)}$ for $\widetilde{q}$ defined in (2.80) and estimate (2.76). In a similar manner, if $p>6$ we use (2.83), replacing C by $C\,\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$, obtaining the estimate:

$$\begin{aligned}
\|\boldsymbol{z}\|_{\mathbf{L}^p(\Omega)}\ &+\ (1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)})^{-1}\|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}}\leq C\,(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)})\,\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\boldsymbol{v}\|_{L^{3/2}(\Omega)}\right)\\
&\times\ \left(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'}+\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}+\|\nabla\cdot\boldsymbol{v}\|_{L^{\tilde{q}}(\Omega)}\right)\,\left(\|h\|_{W^{-1,p}(\Omega)}+\|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right)\right).
\end{aligned}$$

Finally, the pair of functions $(\boldsymbol{u},\pi)=(\boldsymbol{z}+\boldsymbol{u}_0,\pi)$ is the required solution satisfying the required estimates in the statement of the theorem for $1<p\leq 6$ and $p>6$.

(B) The trace of $\boldsymbol{u}$ satisfies $\boldsymbol{u}=\boldsymbol{g}$ on Γ and belongs to $\mathbf{W}^{-1/p,p}(\Gamma)$. As in Theorem 2.14 **(B)**, we need to prove that $\boldsymbol{u}\in\mathbf{T}_{p,r}(\Omega)$ and therefore its tangential trace belongs to $\mathbf{W}^{-1/p,p}(\Gamma)$.

It suffices to prove that $\Delta \boldsymbol{u} \in [\mathbf{X}_{r',p'}(\Omega)]'$ but now, the problem is that $\boldsymbol{v} \cdot \nabla \boldsymbol{u} \neq \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{u})$ and, in principle, we do not know if $\Delta \boldsymbol{u} = \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi - \boldsymbol{f}$ belongs to $[\mathbf{X}_{r',p'}(\Omega)]'$. However, if we prove that

$$(\nabla \cdot \boldsymbol{v})\, \boldsymbol{u} \in \mathbf{W}^{-1,r}(\Omega), \tag{2.84}$$

with r given by (2.68), from (1.3) we deduce that $\Delta \boldsymbol{u} \in [\mathbf{X}_{r',p'}(\Omega)]'$. Condition (2.84) is true because:

- **i)** Case of $p > 3/2$. First, if $p > 3$, then $\nabla \cdot \boldsymbol{v} \in L^{3/2}(\Omega)$ and therefore $(\nabla \cdot \boldsymbol{v})\, \boldsymbol{u} \in \mathbf{L}^{\frac{3p}{2p+3}}(\Omega)$ which is embedded in $\mathbf{W}^{-1,r}(\Omega)$ for $r = \frac{3p}{p+3}$. Second, if $p = 3$, then $\nabla \cdot \boldsymbol{v} \in L^{\widetilde{q}}(\Omega)$ for $\widetilde{q} > 3/2$ and therefore $(\nabla \cdot \boldsymbol{v})\, \boldsymbol{u} \in \mathbf{L}^{q}(\Omega)$ with $q > 1$ which is embedded in $\mathbf{W}^{-1,3/2}(\Omega)$. Third, if $\frac{3}{2} < p < 3$, then $\nabla \cdot \boldsymbol{v} \in L^{p'}(\Omega)$ and therefore $(\nabla \cdot \boldsymbol{v})\, \boldsymbol{u} \in \mathbf{L}^{1}(\Omega)$ which is embedded in $\mathbf{W}^{-1,r}(\Omega)$ for $r = \frac{3p}{p+3}$.

- **ii)** Case $p = \frac{3}{2}$: In this case $\nabla \cdot \boldsymbol{v} \in L^{3}(\Omega)$, and therefore $(\nabla \cdot \boldsymbol{v})\, \boldsymbol{u} \in \mathbf{L}^{1}(\Omega)$ which is embedded in $\mathbf{W}^{-1,r}(\Omega)$ for $r \in (1, 3/2)$.

- **iii)** Case $1 < p < \frac{3}{2}$: In this case $\nabla \cdot \boldsymbol{v} \in L^{p'}(\Omega)$ and therefore $(\nabla \cdot \boldsymbol{v})\, \boldsymbol{u} \in \mathbf{L}^{1}(\Omega)$ which is embedded in $\mathbf{W}^{-1,r}(\Omega)$ for $r = 1$.

In that way, as $\boldsymbol{u} \in \mathbf{L}^{p}(\Omega)$ and $\nabla \cdot \boldsymbol{u} \in L^{r}(\Omega)$, then $\boldsymbol{u} \cdot \boldsymbol{n}|_{\Gamma} \in W^{-1/p,p}(\Gamma)$, and the whole trace $\boldsymbol{u}|_{\Gamma} \in \mathbf{W}^{-1/p,p}(\Gamma)$ can be identified with $\boldsymbol{u}|_{\Gamma} = \boldsymbol{g}$.

(C) Now, we consider $\nabla \cdot \boldsymbol{v} \in L^{t}(\Omega)$ for t defined by (2.77), which only differs from the case of considering $\nabla \cdot \boldsymbol{v} \in L^{\tilde{q}}(\Omega)$ with $\tilde{q}$ defined by (2.80) when $p = 3$.

The case of $p = 3$**.** Suppose that $\nabla \cdot \boldsymbol{v} \in L^{3/2}(\Omega)$ and let $\boldsymbol{v}_k \in \mathcal{D}(\overline{\Omega})$ such that $\boldsymbol{v}_k \to \boldsymbol{v}$ in $\mathbf{L}^{3}(\Omega)$ and $\nabla \cdot \boldsymbol{v}_k \to \nabla \cdot \boldsymbol{v}$ in $L^{3/2}(\Omega)$. Thus, the very weak solution $(\boldsymbol{u}_k, \pi_k)$ for the Oseen problem:

$$(O_k) \quad -\Delta \boldsymbol{u}_k + \nabla \cdot (\boldsymbol{v}_k \otimes \boldsymbol{u}_k) - \boldsymbol{u}_k\, (\nabla \cdot \boldsymbol{v}_k) + \nabla \pi_k = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u}_k = h \quad \text{in } \Omega, \quad \boldsymbol{u}_k = \boldsymbol{g} \quad \text{on } \Gamma$$

belongs to $\mathbf{T}_{3,3/2}(\Omega) \times W^{-1,3}(\Omega)$ with the estimate:

$$\begin{aligned} \|\boldsymbol{u}_k\|_{\mathbf{L}^3(\Omega)} + (1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)})^{-1} \|\pi_k\|_{W^{-1,3}(\Omega)/\mathbb{R}} \;&\leq\; C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} + \|\nabla \cdot \boldsymbol{v}\|_{L^{3/2}(\Omega)}\right) \\ &\times \;\left(\|\boldsymbol{f}\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|h\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/3,3}(\Gamma)}\right), \end{aligned} \tag{2.85}$$

which thanks to (1.3) implies that $\Delta \boldsymbol{u}_k \in [\mathbf{X}_{3,3/2}(\Omega)]'$. We can deduce that $\boldsymbol{u}_k|_{\Gamma} = \boldsymbol{g}$.

From (2.85), we can deduce the following convergences:

$$\begin{array}{rclll} \boldsymbol{u}_k & \rightharpoonup & \boldsymbol{u} & \text{in } \mathbf{L}^3(\Omega), & \pi_k + C_k \rightharpoonup \pi \quad \text{in } W^{-1,3}(\Omega), \\ \Delta \boldsymbol{u}_k & \rightharpoonup & \Delta \boldsymbol{u} & \text{in } \mathbf{W}^{-2,3}(\Omega), & \boldsymbol{v}_k \otimes \boldsymbol{u}_k \rightharpoonup \boldsymbol{v} \otimes \boldsymbol{u} \quad \text{in } \mathbb{L}^{3/2}(\Omega), \\ (\nabla \cdot \boldsymbol{v}_k)\, \boldsymbol{u}_k & \rightharpoonup & (\nabla \cdot \boldsymbol{v})\, \boldsymbol{u} & \text{in } \mathbb{L}^1(\Omega), & \end{array}$$

and also that $\nabla \cdot \boldsymbol{u}_k = h \in L^{3/2}(\Omega)$ and $\boldsymbol{u}_k|_\Gamma = \boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$, for $(\boldsymbol{u}, \pi) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ being the solution of the limit problem (of Oseen type (O)), which can be written as:

$$-\Delta \boldsymbol{u} + \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{u}) - \boldsymbol{u}\,(\nabla \cdot \boldsymbol{v}) + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = h \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma.$$

As a consequence, estimate (2.83) for $1 < p \leq 6$ and estimate (2.83) replacing C by $C\,\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$ for $p > 6$ are also true when $\nabla \cdot \boldsymbol{v} \in L^t(\Omega)$ with t is defined by (2.77). □ ∎

Remark 2.18 *When* $\boldsymbol{g} \cdot \boldsymbol{n}|_\Gamma = 0$ *and* $\displaystyle\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = 0$ *(see case* **(A1)** *in the previous proof), estimates in Theorem 2.17 can be replaced by estimate (2.83) for* $1 < p \leq 6$ *and estimate (2.83) replacing* C *by* $C\,\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$ *for* $p > 6$*. For both values of* C*, estimate (2.83) does not depend on the norm* $\|\nabla \cdot \boldsymbol{v}\|_{\mathbf{L}^t(\Omega)}$ *but* $\nabla \cdot \boldsymbol{v}$ *must belong to the space* $L^t(\Omega)$ *(with* t *defined by (2.77)) in order to give a sense to the term (2.82).*

3 The Navier-Stokes problem

The results on the existence of very weak solution for the Navier-Stokes equations (NS) in [7] are true, but the proofs are correct only for the case of $h = 0$, that is, $\nabla \cdot \boldsymbol{u} = 0$. Here we prove the case of $h \neq 0$, that needs the results from Section 2. As in [7], we start proving the result for small data:

$$\|\boldsymbol{f}\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|h\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/3,3}(\Gamma)} << 1.$$

In this case, we slightly rewrite the notion of very weak solution (with respect to [7]) for the Navier-Stokes equations in order to take into account that $\nabla \cdot \boldsymbol{u} \neq 0$:

Definition 3.1 (Very weak solution for the Navier-Stokes problem) *Let* $\boldsymbol{f} \in [\mathbf{X}_{r',p'}(\Omega)]'$, $h \in L^r(\Omega)$ *and* $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ *satisfy the compatibility condition* (1.5)*. We say that* $(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ *is a* **very weak solution** *of* (NS) *if the following equalities hold: For any* $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$ *and* $\chi \in W^{1,p'}(\Omega)$,

$$\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{\varphi} - \boldsymbol{u} \cdot \nabla \boldsymbol{\varphi} - h\,\boldsymbol{\varphi})\; d\boldsymbol{x} \quad - \quad \langle \pi, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_\Gamma,$$

$$\int_\Omega \boldsymbol{u} \cdot \nabla \chi\, d\boldsymbol{x} \quad = \quad -\int_\Omega h\,\chi\, d\boldsymbol{x} + \langle (\boldsymbol{g} \cdot \boldsymbol{n}), \chi \rangle_\Gamma,$$

where the dualities on Ω *and* Γ *are defined in* (2.69).

As a consequence of the previous study, we look for giving a result of existence of a very weak solution:

Theorem 3.2 (Very weak solution for Navier-Stokes, $h \neq 0$, small data case) *Let $\boldsymbol{f} \in [\mathbf{X}_{3,3/2}(\Omega)]'$, $h \in L^{3/2}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$ verify (1.5) and $\|h\|_{W^{-1,3}(\Omega)} \leq \delta_0$ with δ_0 defined in (2.9). There exists a constant $\alpha > 0$ such that, if*

$$\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| \; h \; \|_{L^{3/2}(\Omega)} + \| \; \boldsymbol{g} \; \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leq \alpha, \tag{3.86}$$

then, there exists a very weak solution $(\boldsymbol{u}, \pi) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ of problem (NS) and the following estimates hold:

$$\|\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} \leq \widetilde{C} \left(\|\boldsymbol{f}\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|h\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/3,3}(\Gamma)} \right), \tag{3.87}$$

$$\|\pi\|_{W^{-1,3}/\mathbb{R}} \leq \widetilde{C} \, (1+\eta) \left(\|\boldsymbol{f}\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|h\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/3,3}(\Gamma)} \right), \tag{3.88}$$

where $\alpha = (16C^2)^{-1}$, $\widetilde{C} = C\,(1+\eta)^2$ with $C > 1$ is the constant given in (3.89) and η defined by (3.94).

Proof. We want to prove the existence of a very weak solution applying Banach's fixed point theorem. With this objective, we define a space over which we shall define an invariant operator. Then, we search for a fixed point for the application T defined as:

$$\begin{cases} T \; : \; \mathbf{B}_\eta \; \rightarrow \; \mathbf{B}_\eta \\ \qquad\quad \boldsymbol{v} \; \mapsto \; T\boldsymbol{v} = \boldsymbol{u} \end{cases}$$

where the neighborhood $\mathbf{B}_\eta$ is defined as:

$$\mathbf{B}_\eta \;=\; \{\boldsymbol{v} \in \mathbf{L}^3(\Omega);\, \nabla \cdot \boldsymbol{v} \in L^{3/2}(\Omega) \text{ with } \nabla \cdot \boldsymbol{v} = h \text{ and } \|\nabla \cdot \boldsymbol{v}\|_{W^{-1,3}(\Omega)} \leq \delta_0,\, \|\boldsymbol{v}\|_{\mathbf{B}_\eta} \leq \eta\}.$$

(where δ_0 is given in (2.9)) endowed with the topology given by the norm:

$$\|\boldsymbol{v}\|_{\mathbf{B}_\eta} = \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} + \|\nabla \cdot \boldsymbol{v}\|_{L^{3/2}(\Omega)}$$

and for $\eta > 0$ defined in (3.94). The operator T is defined as follows: for a given $\boldsymbol{v} \in \mathbf{B}_\eta$, its image $T\boldsymbol{v} = \boldsymbol{u}$ is the unique solution of the problem:

$$(O) \qquad \begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi \;=\; \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} \;=\; h \quad \text{in } \Omega, \\ \boldsymbol{u} \;=\; \boldsymbol{g} \quad \text{on } \Gamma. \end{cases}$$

From Theorem 2.17 for $p = 3$, $r = \frac{3}{2}$, $s = 3$ and $t = \frac{3}{2}$, the solution $(\boldsymbol{u}, \pi)$ of the Oseen problem (O) belongs to $\mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)/\mathbb{R}$, and satisfies estimate (2.79), that is:

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} \; &+ (1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)})^{-1} \|\pi\|_{W^{1-,3}(\Omega)} \leq C\Big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} + \|\nabla \cdot \boldsymbol{v}\|_{L^{3/2}(\Omega)}\Big) \\ &\times \left(\|\boldsymbol{f}\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \Big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} + \|\nabla \cdot \boldsymbol{v}\|_{L^{3/2}(\Omega)}\Big)\Big(\|\boldsymbol{h}\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/3,3}(\Gamma)}\Big) \right). \end{aligned} \tag{3.89}$$

Note that the operator T is well-defined in $\mathbf{B}_\eta$: for a given $\boldsymbol{v} \in \mathbf{B}_\eta$ and using (3.89), we obtain:

$$\|\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} \le C\ (1+\eta)\ (\beta_1 + (1+\eta)\,\beta_2)$$

where the constants β_1 and β_2 are defined as:

$$\beta_1 = \|\boldsymbol{f}\|_{[\mathbf{X}_{3,3/2}(\Omega)]'}, \quad \beta_2 = \|h\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/3,3}(\Gamma)} \quad \text{and} \quad \beta = \beta_1 + \beta_2. \tag{3.90}$$

In order to have that $T(\mathbf{B}_\eta) \subseteq \mathbf{B}_\eta$, we need to prove that $\boldsymbol{u} \in \mathbf{B}_\eta$. From (3.89), we know that:

$$\|\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} + \|\nabla\cdot\boldsymbol{u}\|_{\mathbf{L}^{3/2}(\Omega)} \le C\ (1+\eta)\ (\beta_1 + (1+\eta)\,\beta_2) + \|h\|_{L^{3/2}(\Omega)} \le C\ (1+\eta)\ (\beta_1 + (1+\eta)\,\beta_2) + \beta_2.$$

The inclusion $T(\mathbf{B}_\eta) \subseteq \mathbf{B}_\eta$ will be satisfied if the following inequality holds:

$$C\ (1+\eta)\ (\beta_1 + (1+\eta)\,\beta_2) + \beta_2 \le \eta. \tag{3.91}$$

It is clear that if β_1 and β_2 are sufficiently small, as we will see below, there exist some values of η satisfying (3.91). Further, we first prove that T is a contractive operator.

Now, for the contraction method we must prove: there exists $\theta \in]0,1[$ such that:

$$\|T\boldsymbol{v}_1 - T\boldsymbol{v}_2\|_{\mathbf{B}_\eta} = \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\mathbf{B}_\eta} \le \theta \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\mathbf{B}_\eta}.$$

Observe that for each $\boldsymbol{u}_i$, $i = 1, 2$, we have

$$\begin{cases} -\Delta \boldsymbol{u}_i + \boldsymbol{v}_i \cdot \nabla \boldsymbol{u}_i + \nabla \pi_i &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}_i &= h \quad \text{in } \Omega, \\ \boldsymbol{u}_i &= \boldsymbol{g} \quad \text{on } \Gamma, \end{cases}$$

with the estimates

$$\begin{aligned} \|\boldsymbol{u}_i\|_{\mathbf{L}^3(\Omega)} &+ (1 + \|\boldsymbol{v}_i\|_{\mathbf{L}^3(\Omega)})^{-1} \|\pi_i\|_{W^{1-,3}(\Omega)} \le C\Big(1 + \|\boldsymbol{v}_i\|_{\mathbf{L}^3(\Omega)} + \|\nabla\cdot\boldsymbol{v}_i\|_{L^{3/2}(\Omega)}\Big) \\ \times\Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} &+ \Big(1 + \|\boldsymbol{v}_i\|_{\mathbf{L}^3(\Omega)} + \|\nabla\cdot\boldsymbol{v}_i\|_{L^{3/2}(\Omega)}\Big)\Big(\|\boldsymbol{h}\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/3,3}(\Gamma)}\Big)\Big). \end{aligned} \tag{3.92}$$

However, in order to estimate the difference $\boldsymbol{u}_1 - \boldsymbol{u}_2$, we have to reason differently. We start with the problem verified by $(\boldsymbol{u}, \pi) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, \pi_1 - \pi_2)$, which is the following one:

$$-\Delta \boldsymbol{u} + \boldsymbol{v}_1 \cdot \nabla \boldsymbol{u} + \nabla \pi = -\nabla\cdot(\boldsymbol{v} \otimes \boldsymbol{u}_2) \quad \text{and} \quad \nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma,$$

where $\boldsymbol{u}_1 = T\boldsymbol{v}_1$, $\boldsymbol{u}_2 = T\boldsymbol{v}_2$ and $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$. Using the very weak estimates (3.92) made for the Oseen problem successively for $\boldsymbol{u}$ and for $\boldsymbol{u}_2$ (observe that $\nabla\cdot\boldsymbol{v} = 0$ and $\nabla\cdot\boldsymbol{u} = 0$), we obtain that:

$$\begin{aligned}
\|\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} &\leq C\left(1+\|\boldsymbol{v}_1\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\boldsymbol{v}_1\|_{L^{3/2}(\Omega)}\right)\|\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{u}_2)\|_{[\mathbf{X}_{3,3/2}(\Omega)]'}\\
&\leq C\left(1+\|\boldsymbol{v}_1\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\boldsymbol{v}_1\|_{L^{3/2}(\Omega)}\right)\|\boldsymbol{v}\otimes\boldsymbol{u}_2\|_{\mathbf{L}^{3/2}(\Omega)}\\
&\leq C\left(1+\|\boldsymbol{v}_1\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\boldsymbol{v}_1\|_{L^{3/2}(\Omega)}\right)\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\,\|\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}\\
&\leq C^2\left(1+\|\boldsymbol{v}_1\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\boldsymbol{v}_1\|_{L^{3/2}(\Omega)}\right)\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\left(1+\|\boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\boldsymbol{v}_2\|_{L^{3/2}(\Omega)}\right)\\
&\times\left\{\beta_1+\left(1+\|\boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\boldsymbol{v}_2\|_{L^{3/2}(\Omega)}\right)\beta_2\right\},
\end{aligned}$$

where the constants β_1 and β_2 are given by (3.90). Therefore, we have to prove that

$$C^2\,(1+\eta)^2\,\left(\beta_1+(1+\eta)\,\beta_2\right)<1. \tag{3.93}$$

For that, it suffices to suppose that $C>1$. Again, it is clear that if β_1 and β_2 are sufficiently small, there exist some values of η satisfying (3.93). We can choose, for instance:

$$\eta=(2C^2\,\beta)^{-1/3}-1\quad\text{with}\quad\beta<(16C^2)^{-1}, \tag{3.94}$$

which implies that $\eta>1$. With this choice, we have:

$$C^2\,(1+\eta)^2\,\left(\beta_1+(1+\eta)\,\beta_2\right)\leq C^2\,(1+\eta)^3\,\beta=\frac{1}{2},$$

which implies (3.93).

We are going to prove now that (3.91) is satisfied. Observe that from definition (3.94) of η, (3.91) is equivalent to prove that:

$$\frac{C}{(2\,C^2\,\beta)^{1/3}}\left(\beta_1+\frac{1}{(2C^2\beta)^{1/3}}\,\beta_2\right)+\beta_2\leq\frac{1}{(2C^2\beta)^{1/3}}-1$$

or equivalently

$$C\left(\beta_1+\frac{1}{(2C^2\beta)^{1/3}}\,\beta_2\right)+(\beta_2+1)\,(2C^2\beta)^{1/3}\leq 1.$$

Observe that, thanks to the smallness condition in (3.94) and $C>1$:

$$\begin{aligned}
C\left(\beta_1+\frac{1}{(2C^2\beta)^{1/3}}\,\beta_2\right)+(\beta_2+1)\,(2C^2\beta)^{1/3}&\leq C\,\beta\left(1+\frac{1}{(2C^2\beta)^{1/3}}\right)+(\beta_2+1)\,(2C^2\beta)^{1/3}\\
\leq C\beta+\frac{C^{1/3}\,\beta^{2/3}}{2^{1/3}}+(\beta_2+1)\,(2C^2\beta)^{1/3}&<\frac{1}{16C}+\frac{1}{8C}+\left(\frac{1}{16}+1\right)\frac{1}{2}<\frac{3}{16}+\frac{1}{32}+\frac{1}{2}<1,
\end{aligned}$$

which implies that (3.91) holds.

Therefore, thanks to the contraction of T we get that the unique fixed point $\bar{\boldsymbol{u}} \in \mathbf{L}^3(\Omega)$ satisfying (3.89) for $\boldsymbol{v} = \bar{\boldsymbol{u}}$, which implies:

$$\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)} \leq C\,\Big(1+\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\bar{\boldsymbol{u}}\|_{L^{3/2}(\Omega)}\Big)\,\Big(\beta_1+\Big(1+\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\bar{\boldsymbol{u}}\|_{L^{3/2}(\Omega)}\Big)\,\beta_2\Big).$$

Because of $\bar{\boldsymbol{u}} \in \mathbf{B}_\eta$, then:

$$\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)} \leq C\,(1+\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\bar{\boldsymbol{u}}\|_{L^{3/2}(\Omega)})^2\,\beta \leq C(1+\eta)^2\beta,$$

which implies:

$$\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)} \leq \widetilde{C}\,\beta$$

for $\widetilde{C} = C(1+\eta)^2$ and therefore (3.87) holds. Moreover, the equation $\nabla\cdot\bar{\boldsymbol{u}} = h$ implies that $\nabla\cdot\bar{\boldsymbol{u}} \in L^{3/2}(\Omega)$.

Concerning the estimate for the pressure $\bar{\pi}$, observe that $(\bar{\boldsymbol{u}}, \bar{\pi})$ is solution of an Oseen problem of type (O). Therefore, from (3.89) we deduce that:

$$\begin{aligned}\|\bar{\pi}\|_{W^{1-,3}(\Omega)} &\leq C\,\Big(1+\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)}\Big)\,\Big(1+\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\bar{\boldsymbol{u}}\|_{\mathbf{L}^{3/2}(\Omega)}\Big)\\ &\times\,\Big(\beta_1+\Big(1+\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)}+\|\nabla\cdot\bar{\boldsymbol{u}}\|_{\mathbf{L}^{3/2}(\Omega)}\Big)\,\beta_2\Big)\\ &\leq C\,\Big(1+\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)}\Big)\,(1+\eta)^2\,\beta \leq \widetilde{C}\,(1+\eta)\,\beta\end{aligned}$$

and thus, we arrive at (3.88). □ ▮

Remark 3.3 *As in Theorem 19 of [7], if the data are even small than in Theorem 3.2, then the uniqueness of very weak solution for the (NS) problem can be deduced.*

The proof of the following result can be taken from Corollary 9 in [7].

Corollary 3.4 *Let $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfy (1.5), (3.86) and*

$\boldsymbol{f} \in [\mathbf{X}_{r',p'}(\Omega)]'$, $\quad h \in L^r(\Omega)$ *such that* $\|h\|_{W^{-1,3}(\Omega)} \leq \delta_0$ *with* δ_0 *sufficiently small defined in (2.9),*

$$\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma),$$

with

$$\max\{r,3\} \leq p, \qquad \frac{1}{r} \leq \frac{1}{p}+\frac{1}{3}.$$

Then, the solution $(\boldsymbol{u}, \pi)$ given by Theorem 3.2 belongs to $\mathbf{L}^p(\Omega)\times W^{-1,p}(\Omega)$ with $\nabla\cdot\boldsymbol{u} \in L^r(\Omega)$ and $\|\nabla\cdot\boldsymbol{u}\|_{W^{-1,3}(\Omega)}$ sufficiently small.

Now, we introduce the result for arbitrary $\boldsymbol{f}$ only imposing smallness on:

$$\|h\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/3,3}(\Gamma)} << 1$$

The proof is similar to Theorem 20 in [7].

Theorem 3.5 (Very weak solution of Navier-Stokes, arbitrary forces) *Let* $\boldsymbol{f} \in [\mathbf{X}_{3,3/2}(\Omega)]'$, $h \in L^{3/2}(\Omega)$ *be such that* $\|h\|_{W^{-1,3}(\Omega)} \leq \delta_0$ *with* δ_0 *defined in (2.9) and* $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$ *satisfy the compatibility condition* (1.5). *There exists a constant* $\delta > 0$ *only depending on* Ω *such that if:*

$$\|h\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_j}| \leq \delta. \tag{3.95}$$

then the problem (NS) *has a very weak solution* $(\boldsymbol{u}, \pi) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$.

Proof. We decompose the problem into two: The first problem is to find $(\boldsymbol{v}_\varepsilon, \pi^1_\varepsilon)$ solution of the problem:

$$(NS_1) \begin{cases} -\Delta \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon + \nabla \pi^1_\varepsilon &= \boldsymbol{f} - \boldsymbol{f}_\varepsilon \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{v}_\varepsilon &= h - h_\varepsilon \quad \text{in } \Omega, \\ \boldsymbol{v}_\varepsilon &= \boldsymbol{g} - \boldsymbol{g}_\varepsilon \quad \text{on } \Gamma, \end{cases}$$

for $\boldsymbol{f}_\varepsilon \in \boldsymbol{\mathcal{D}}(\Omega)$, $h_\varepsilon \in \mathcal{D}(\overline{\Omega})$ and $\boldsymbol{g}_\varepsilon \in \mathcal{C}^\infty(\Gamma)$ for any $\varepsilon > 0$ sufficiently small satisfying the compatibility condition

$$\int_\Omega h_\varepsilon \, d\boldsymbol{x} = \langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1\rangle_\Gamma, \tag{3.96}$$

verifying

$$\|\boldsymbol{f} - \boldsymbol{f}_\varepsilon\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|h - h_\varepsilon\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g} - \boldsymbol{g}_\varepsilon\|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leq \varepsilon,$$

$$\|h_\varepsilon\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}| \leq 2 \left(\|h\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}| \right) \leq 2\,\delta,$$

for a δ that will be specified in (3.101). From the compatibility condition (1.5) and (3.96), applying Theorem 3.2 with $\varepsilon \leq \min\{\delta_0, \alpha\}$ (α defined in Theorem 3.2), then the solution $(\boldsymbol{v}_\varepsilon, \pi^1_\varepsilon)$ belongs to $\mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$. Moreover,

$$\|\boldsymbol{v}_\varepsilon\|_{\mathbf{L}^3(\Omega)} \leq C \left(\|\boldsymbol{f} - \boldsymbol{f}_\varepsilon\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|h - h_\varepsilon\|_{L^{3/2+\varepsilon}(\Omega)} + \|\boldsymbol{g} - \boldsymbol{g}_\varepsilon\|_{\mathbf{W}^{-1/3,3}(\Gamma)} \right) := \delta(\varepsilon). \tag{3.97}$$

The second problem is to find $(\boldsymbol{z}_\varepsilon, \pi^2_\varepsilon)$ solution of the problem:

$$(NS_2) \begin{cases} -\Delta \boldsymbol{z}_\varepsilon + \boldsymbol{z}_\varepsilon \cdot \nabla \boldsymbol{z}_\varepsilon + \boldsymbol{z}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{z}_\varepsilon + \nabla \pi^2_\varepsilon &= \boldsymbol{f}_\varepsilon \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{z}_\varepsilon &= h_\varepsilon \quad \text{in } \Omega, \\ \boldsymbol{z}_\varepsilon &= \boldsymbol{g}_\varepsilon \quad \text{on } \Gamma, \end{cases}$$

where $\boldsymbol{f}_\varepsilon \in \mathbf{H}^{-1}(\Omega),\ h_\varepsilon \in L^2(\Omega)$ and $\boldsymbol{g}_\varepsilon \in \mathbf{H}^{1/2}(\Gamma)$.

Let first $\theta^\varepsilon \in W^{2,3/2}(\Omega)$ be the unique solution of the problem:

$$\Delta\theta^\varepsilon = h_\varepsilon \ \text{ in } \Omega, \quad \theta^\varepsilon = 0 \ \text{ on } \Gamma, \tag{3.98}$$

which verifies (in particular) the estimate $\|\theta^\varepsilon\|_{W^{2,3/2}(\Omega)} \le C\,\|h_\varepsilon\|_{L^{3/2}(\Omega)}$. Multiplying the equation defined in Ω in (3.98) by 1 and integrating on Γ, we obtain:

$$\int_\Gamma \frac{\partial\theta^\varepsilon}{\partial\mathbf{n}}\,d\sigma = \int_\Omega h_\varepsilon\,d\boldsymbol{x}.$$

Using the Hopf's Lemma (see [15], page 610, Lemma IX.4.2 for instance), for any $\nu > 0$ there exists $\mathbf{Y}^\varepsilon \in \mathbf{H}^1(\Omega)$ the solution of the problem:

$$\nabla\cdot\mathbf{Y}^\varepsilon = 0 \quad \text{in } \Omega, \qquad \mathbf{Y}^\varepsilon = \boldsymbol{g}_\varepsilon - \nabla\theta^\varepsilon|_\Gamma \quad \text{on } \Gamma,$$

such that it verifies: for any $\boldsymbol{w} \in \mathbf{H}^1_0(\Omega)$,

$$\begin{aligned}\left|\int_\Omega (\boldsymbol{w}\cdot\nabla)\mathbf{Y}^\varepsilon\cdot\boldsymbol{w}\,d\boldsymbol{x}\right| &\le \left(\nu + C\sum_{i=1}^{i=I}|\langle(\boldsymbol{g}_\varepsilon - \nabla\theta^\varepsilon|_\Gamma)\cdot\boldsymbol{n},1\rangle_{\Gamma_i}|\right)\|\nabla\boldsymbol{w}\|^2_{\mathbf{L}^2(\Omega)}\\ &\le \left(\nu + C\sum_{i=1}^{i=I}|\langle\boldsymbol{g}_\varepsilon\cdot\boldsymbol{n},1\rangle_{\Gamma_i}| + C\,\|h_\varepsilon\|_{L^{3/2}(\Omega)}\right)\|\nabla\boldsymbol{w}\|^2_{\mathbf{L}^2(\Omega)}.\end{aligned} \tag{3.99}$$

Setting $\boldsymbol{y}_\varepsilon = \mathbf{Y}^\varepsilon + \nabla\theta^\varepsilon$, we have:

$$\nabla\cdot\boldsymbol{y}_\varepsilon = h_\varepsilon \text{ in } \Omega, \quad \boldsymbol{y}_\varepsilon = \boldsymbol{g}_\varepsilon \text{ on } \Gamma,$$

and the study of problem (NS_2) becomes the study of:

$$(\widetilde{NS_2})\begin{cases} -\Delta\boldsymbol{w}_\varepsilon + (\boldsymbol{v}_\varepsilon + \boldsymbol{w}_\varepsilon + \boldsymbol{y}_\varepsilon)\cdot\nabla\boldsymbol{w}_\varepsilon + \nabla\pi^2_\varepsilon\ + \boldsymbol{w}_\varepsilon\cdot\nabla\boldsymbol{y}_\varepsilon + \boldsymbol{w}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon = \mathbf{F}_\varepsilon & \text{in } \Omega,\\ \nabla\cdot\boldsymbol{w}_\varepsilon = 0 & \text{in } \Omega,\\ \boldsymbol{w}_\varepsilon = \mathbf{0} & \text{on } \Gamma,\end{cases}$$

where $\boldsymbol{w}_\varepsilon = \boldsymbol{z}_\varepsilon - \boldsymbol{y}_\varepsilon$ and $\mathbf{F}_\varepsilon = \boldsymbol{f}_\varepsilon + \Delta\boldsymbol{y}_\varepsilon - \boldsymbol{y}_\varepsilon\cdot\nabla\boldsymbol{y}_\varepsilon - \boldsymbol{y}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon - \boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{y}_\varepsilon \in \mathbf{H}^{-1}(\Omega)$. Indeed, note that:

$$\boldsymbol{y}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon = \nabla\cdot(\boldsymbol{y}_\varepsilon\otimes\boldsymbol{v}_\varepsilon) - (\nabla\cdot\boldsymbol{y}_\varepsilon)\,\boldsymbol{v}_\varepsilon = \nabla\cdot(\boldsymbol{y}_\varepsilon\otimes\boldsymbol{v}_\varepsilon) - h_\varepsilon\,\boldsymbol{v}_\varepsilon$$

and, since $\boldsymbol{y}_\varepsilon \in \mathbf{L}^6(\Omega)$, then $\boldsymbol{y}_\varepsilon\otimes\boldsymbol{v}_\varepsilon \in \mathbb{L}^2(\Omega)$, $h_\varepsilon\,\boldsymbol{v}_\varepsilon \in \mathbf{L}^{6/5}(\Omega) \hookrightarrow \mathbf{H}^{-1}(\Omega)$ and $\boldsymbol{y}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon \in \mathbf{H}^{-1}(\Omega)$. Additionally,

$$\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{y}_\varepsilon = \nabla\cdot(\boldsymbol{v}_\varepsilon\otimes\boldsymbol{y}_\varepsilon) - (\nabla\cdot\boldsymbol{v}_\varepsilon)\,\boldsymbol{y}_\varepsilon = \nabla\cdot(\boldsymbol{v}_\varepsilon\otimes\boldsymbol{y}_\varepsilon) - h_\varepsilon\,\boldsymbol{y}_\varepsilon,$$

where $\boldsymbol{v}_\varepsilon \otimes \boldsymbol{y}_\varepsilon \in \mathbb{L}^2(\Omega)$ and $h_\varepsilon\, \boldsymbol{y}_\varepsilon \in \mathbf{L}^{3/2}(\Omega) \hookrightarrow \mathbf{H}^{-1}(\Omega)$; and $\boldsymbol{y}_\varepsilon \otimes \boldsymbol{y}_\varepsilon \in \mathbb{L}^3(\Omega)$ together with $h_\varepsilon\, \boldsymbol{y}_\varepsilon \in \mathbf{L}^{3/2}(\Omega)$ imply that $\mathbf{F}_\varepsilon \in \mathbf{H}^{-1}(\Omega)$.

Taking $\boldsymbol{w}_\varepsilon$ as a test function in $(\widetilde{NS}_2)$, we obtain:

$$\|\nabla \boldsymbol{w}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)} + \int_\Omega (\boldsymbol{v}_\varepsilon + \boldsymbol{y}_\varepsilon)\cdot\nabla \boldsymbol{w}_\varepsilon \cdot \boldsymbol{w}_\varepsilon\, d\boldsymbol{x} + \int_\Omega \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{y}_\varepsilon \cdot \boldsymbol{w}_\varepsilon\, d\boldsymbol{x} + \int_\Omega \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon \cdot \boldsymbol{w}_\varepsilon\, d\boldsymbol{x} = \langle \mathbf{F}_\varepsilon, \boldsymbol{w}_\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega)\times \mathbf{H}^1_0(\Omega)}. \tag{3.100}$$

The first integral in (3.100) can be rewritten as:

$$\int_\Omega (\boldsymbol{v}_\varepsilon + \boldsymbol{y}_\varepsilon)\cdot\nabla \boldsymbol{w}_\varepsilon \cdot \boldsymbol{w}_\varepsilon\, d\boldsymbol{x} = -\frac{1}{2}\int_\Omega h\, |\boldsymbol{w}_\varepsilon|^2\, d\boldsymbol{x} \le \frac{1}{2}\|h\|_{L^{3/2}(\Omega)} \|\boldsymbol{w}_\varepsilon\|^2_{\mathbf{L}^6(\Omega)} \le \frac{C_1^2}{2}\, \|h\|_{L^{3/2}(\Omega)} \|\nabla \boldsymbol{w}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)}$$

for C_1 the product of the constant of the Sobolev embedding $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$ and the Poincaré constant associated to $\mathbf{H}^1_0(\Omega)$. The bound for the second integral in (3.100) can be made using that

$$\begin{aligned} \left| \int_\Omega (\boldsymbol{w}_\varepsilon \cdot \nabla)(\nabla \theta^\varepsilon) \cdot \boldsymbol{w}_\varepsilon\, d\boldsymbol{x} \right| &= -\left| \int_\Omega (\boldsymbol{w}_\varepsilon \cdot \nabla) \boldsymbol{w}_\varepsilon \cdot (\nabla \theta^\varepsilon)\, d\boldsymbol{x} \right| \\ &\le \|\boldsymbol{w}_\varepsilon\|_{\mathbf{L}^6(\Omega)} \|\nabla \theta^\varepsilon\|_{\mathbf{L}^3(\Omega)} \|\nabla \boldsymbol{w}_\varepsilon\|_{\mathbf{L}^2(\Omega)} \le C\, C_1\, \|h_\varepsilon\|_{L^{3/2}(\Omega)}\, \|\nabla \boldsymbol{w}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

and (3.99), obtaining:

$$\begin{aligned} \left| \int_\Omega \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{y}_\varepsilon \cdot \boldsymbol{w}_\varepsilon\, d\boldsymbol{x} \right| &\le \left(\nu + C \sum_{i=1}^{i=I} |\langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i}| + C\, C_1\, \|h_\varepsilon\|_{L^{3/2}(\Omega)} \right) \|\nabla \boldsymbol{w}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)} \\ &\le (\nu + 2\, C_1\, C\, \delta)\, \|\nabla \boldsymbol{w}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Finally, the bound for the third integral in (3.100) is

$$\begin{aligned} \left| \int_\Omega \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon \cdot \boldsymbol{w}_\varepsilon\, d\boldsymbol{x} \right| &= \left| - \int_\Omega \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{w}_\varepsilon \cdot \boldsymbol{v}_\varepsilon\, d\boldsymbol{x} \right| \le \|\boldsymbol{w}_\varepsilon\|_{\mathbf{L}^6(\Omega)} \|\boldsymbol{v}_\varepsilon\|_{\mathbf{L}^3(\Omega)} \|\nabla \boldsymbol{w}_\varepsilon\|_{\mathbf{L}^2(\Omega)} \\ &\le C_1\, \|\boldsymbol{v}_\varepsilon\|_{\mathbf{L}^3(\Omega)}\, \|\nabla \boldsymbol{w}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)} \le C_1\, \delta(\varepsilon)\, \|\nabla \boldsymbol{w}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)} \end{aligned}$$

with $\delta(\varepsilon)$ being given by (3.97). We choose ε, $\|h_\varepsilon\|_{L^{3/2}(\Omega)}$ and $\|\boldsymbol{g}_\varepsilon\|_{\mathbf{H}^{1/2}(\Gamma)}$ such that:

$$\left(\frac{C_1^2}{2}\, \|h\|_{L^{3/2}(\Omega)} + \nu + 2\, C_1\, C\, \delta + C_1\, \delta(\varepsilon) \right) \le 1/2. \tag{3.101}$$

Then, the classical theory for the problem $(\widetilde{NS}_2)$ implies the existence of a solution $(\boldsymbol{w}_\varepsilon, \pi^2_\varepsilon) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$. The pair $(\boldsymbol{u}, \pi) = (\boldsymbol{v}_\varepsilon + \boldsymbol{z}_\varepsilon, \pi^1_\varepsilon + \pi^2_\varepsilon)$ belonging to $\mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)/\mathbb{R}$ is then solution to problem (NS). □ ∎

The following result can be proved by adapting the proof of Theorem 21 in [7].

Theorem 3.6 *Let $(\boldsymbol{u}, \pi) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ be the solution given by Theorem 3.5. Then, the following regularity results hold:*

i) Suppose that

$$\boldsymbol{f} \in [\mathbf{X}_{r',p'}(\Omega)]', \;\; h \in L^r(\Omega) \quad \textit{and} \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$$

verify the compatibility condition (1.5) and there exists $\delta_3 > 0$ *such that* $\|h\|_{W^{-1,3}(\Omega)} \leq \delta_3$ *(*$\delta_3 = \delta_1$*, defined in (2.36), if* $p > 6$ *and* $\delta_3 = \delta_0$*, defined in (2.9), in the other case), with* $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ *and* $\max\{r, 3\} \leq p$*. Then* $(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$.

ii) Let $r \geq 3/2$ *and suppose that*

$$\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega), \;\; h \in L^r(\Omega) \; \textit{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma)$$

verify the compatibility condition (1.5) and there exists $\delta_4 > 0$ *such that* $\|h\|_{W^{-1,3}(\Omega)} \leq \delta_4$*, for* $\delta_4 = \delta_1$ *if* $r \geq 2$ *and* $\delta_4 = \delta_0$ *if* $r \in (1, 2)$*. Then* $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$.

iii) Let $1 < r < \infty$ *and suppose that*

$$\boldsymbol{f} \in \mathbf{L}^r(\Omega), \;\; h \in W^{1,r}(\Omega) \quad \textit{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/r,r}(\Gamma),$$

verify the compatibility condition (1.5) and there exists $\delta_5 > 0$ *such that* $\|h\|_{W^{-1,3}(\Omega)} \leq \delta_5$*, for* $\delta_5 = \delta_0$ *if* $r \geq 6/5$ *and* $\delta_5 = \delta_2$ *(defined in (2.55)) in the other case. Then* $(\boldsymbol{u}, \pi) \in \mathbf{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$.

Proof. Under the assumptions in i), ii) and iii), we have that $\boldsymbol{f} \in [\mathbf{X}_{3,3/2}(\Omega)]'$, $h \in L^{3/2}(\Omega)$ with $\|h\|_{W^{-1,3}(\Omega)}$ small enough and $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$.
i) Let $(\boldsymbol{u}, \pi) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ be the solution given by Theorem 3.5. Using Theorem 2.17 with $\boldsymbol{v} = \boldsymbol{u}$, there exists a unique $(\boldsymbol{w}, \chi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ satisfying $-\Delta \boldsymbol{w} + \boldsymbol{u} \cdot \nabla \boldsymbol{w} + \nabla \chi = \boldsymbol{f} = -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi$, $\nabla \cdot \boldsymbol{w} = h$ in Ω and $\boldsymbol{w} = \boldsymbol{g}$ on Γ. Setting $\boldsymbol{z} = \boldsymbol{w} - \boldsymbol{u}$ and $\theta = \chi - \pi$, that means that

$$-\Delta \boldsymbol{z} + \boldsymbol{u} \cdot \nabla \boldsymbol{z} + \nabla \theta = \mathbf{0}, \quad \nabla \cdot \boldsymbol{z} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{z} = \mathbf{0} \quad \text{on } \Gamma,$$

and thanks to Theorem 2.17 and uniqueness argument, we deduce that $\boldsymbol{z} = \nabla \theta = \mathbf{0}$ and then $\boldsymbol{w} = \boldsymbol{u}$ and $\chi = \pi + c$, with c constant. The point i) is proved.

ii) Let $r \geq 3/2$. Thanks to $(\boldsymbol{u}, \pi) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ and $h \in L^r(\Omega) \hookrightarrow W^{-1,3}(\Omega)$, Theorem 2.6 if $r \geq 2$, Theorem 2.8 if $r \in (1, 2)$ and the uniqueness argument given in i), we deduce that $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$.

iii) Let $1 < r < \infty$. Thanks to $(\boldsymbol{u}, \pi) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ and $h \in L^r(\Omega) \hookrightarrow W^{-1,3}(\Omega)$, Theorem 2.5 if $r \geq 6/5$ and Theorem 2.10 if $1 < r < 6/5$, and the uniqueness argument given in i), we deduce that $(\boldsymbol{u}, \pi) \in \mathbf{W}^{2,r}(\Omega) \times W^{2,r}(\Omega)$. We can also use a Stokes argument and point ii) to have the same conclusion. □ ∎

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