# Coefficient fields and scalar extension in positive characteristic

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**Abstract**

Let $k$ be a perfect field of positive characteristic, $k(t)_{per}$ the perfect closure of $k(t)$ and $A = k[[X_1, \ldots, X_n]]$. We show that for any maximal ideal $\mathfrak{n}$ of $A' = k(t)_{per} \otimes_k A$, the elements in $\widehat{A'_{\mathfrak{n}}}$ which are annihilated by the "Taylor" Hasse-Schmidt derivations with respect to the $X_i$ form a coefficient field of $\widehat{A'_{\mathfrak{n}}}$.

*Keywords:* Complete local ring; Coefficient field; Hasse-Schmidt derivation.

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## Introduction

Let $k$ be a perfect field, $k_{(\infty)} = k(t)_{per}$ the perfect closure of $k(t)$ and $A = k[[X_1, \ldots, X_n]]$.

If $k$ is of characteristic 0, then $k_{(\infty)} = k(t)$ and $A(t) = A \otimes_k k(t)$ is obviously noetherian. Actually, $A(t)$ is an $n$-dimensional regular non-local ring (see Example (2.3)) whose maximal ideals have the same height ($= n$). In [8] the second author proved that there is a uniform way to obtain a coefficient field in the completions $\widehat{(A(t))_{\mathfrak{n}}}$, for all maximal ideals $\mathfrak{n}$ in $A(t)$. Namely, the elements in $\widehat{(A(t))_{\mathfrak{n}}}$ which are annihilated by the partial derivatives $\frac{\partial}{\partial X_i}$ form a coefficient field of $\widehat{(A(t))_{\mathfrak{n}}}$.

In this paper, we generalize the above result to the positive characteristic case.

At first sight, in positive characteristic it seems natural to consider Hasse-Schmidt derivations instead of usual derivations (see [4, Theorem 3.17]), but Example (2.3) shows that the question is not so clear.

Consequently, in the characteristic $p > 0$ case we take the scalar extension $k \to k_{(\infty)}$ instead of $k \to k(t)$, but a new problem appears: it is not obvious that the ring $A_{(\infty)} = A \otimes_k k_{(\infty)}$ is noetherian. We have proved that result in [3].

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The main result in this paper says that, for every maximal ideal $\mathfrak{n}$ in $A_{(\infty)}$, the elements in $(\widehat{A_{(\infty)}})_{\mathfrak{n}}$ which are annihilated by the "Taylor" Hasse-Schmidt derivations with respect to the $X_i$ form a coefficient field of $(\widehat{A_{(\infty)}})_{\mathfrak{n}}$.

Let us now comment on the content of this paper.

In Section 1 we introduce our basic notations and recall some results, mainly from [3].

In Section 2 we prove our main result and give the (counter)Example (2.3).

In the Appendix we give a complete proof of the Normalization Lemma for power series rings over perfect fields, which is an important ingredient in the proof of Theorem (2.1) and that we have not found in the literature. Our proof closely follows the proof in [1], but the latter works only for infinite perfect fields.

# 1 Preliminaries and notations

All rings and algebras considered in this paper are assumed to be commutative with unit element. If $B$ is a ring, we shall denote by $\dim(B)$ its Krull dimension and by $\Omega(B)$ the set of its maximal ideals. We shall use the letters $K, L, k$ to denote fields and $\mathbb{F}_p$ to denote the finite field of $p$ elements, for a prime number $p$. If $\mathfrak{p} \in \operatorname{Spec}(B)$, we shall denote by $\operatorname{ht}(\mathfrak{p})$ the height of $\mathfrak{p}$. Remember that a ring $B$ is said to be *biequidimensional* if all its saturated chains of prime ideals have the same length.

If $B$ is an integral domain, we denote by $\operatorname{Qt}(B)$ its quotient field.

If $k$ is a ring and $B$ is a $k$-algebra, the set of all derivations (resp. of all Hasse-Schmidt derivations) of $B$ over $k$ (cf. [5] and [6], §27) will be denoted by $\operatorname{Der}_k(B)$ (resp. $\operatorname{HS}_k(B)$).

Now, we recall the notations and some results of [3] which are used in this paper.

For any $\mathbb{F}_p$–algebra $B$, we denote $B^\sharp := \bigcap_{e \geq 0} B^{p^e}$.

Let $k$ be a field of characteristic $p > 0$ and consider the field extension

$$k_{(\infty)} := \bigcup_{m \geq 0} k\left(t^{\frac{1}{p^m}}\right) \supset k(t).$$

If $k$ is perfect, $k_{(\infty)}$ coincides with the perfect closure of $k(t)$.

For each $k$-algebra $A$, we denote $A(t) := k(t) \otimes_k A$. For the sake of brevity, we will write $t_m = t^{\frac{1}{p^m}}$ and denote

$$A_{(m)} := A(t_m) := A \otimes_k k(t_m) = A(t) \otimes_{k(t)} k(t_m), \quad A_{[m]} := A[t_m],$$

$$A_{(\infty)} := A \otimes_k k_{(\infty)} = \bigcup_{m \geq 0} A_{(m)}, \quad A_{[\infty]} := \bigcup_{m \geq 0} A[t_m].$$

Each $A_{(m)}$ (resp. $A_{[m]}$) is a free module over $A(t)$ (resp. over $A[t]$) of rank $p^m$.

For each prime ideal $N$ of $A_{(\infty)}$ we denote $N_{[\infty]} := N \cap A_{[\infty]}$, $N_{[m]} := N \cap A_{[m]}$ and $N_{(m)} := N \cap A_{(m)}$. Similarly, if $P$ is a prime ideal of $A_{[\infty]}$ we denote $P_{[m]} := P \cap A_{[m]}$.

(1.1) We have the following properties [3, 4, 8]:

(i) $N = \bigcup_{m\geq 0} N_{(m)}$, $N_{[\infty]} = \bigcup_{m\geq 0} N_{[m]}$, (resp. $P = \bigcup_{m\geq 0} P_{[m]}$).

(ii) $N_{(n)} \cap A_{(m)} = N_{(m)}$ and $N_{[n]} \cap A_{[m]} = N_{[m]}$ for all $n \geq m$ (resp. $P_{[n]} \cap A_{[m]} = P_{[m]}$ for all $n \geq m$).

(iii) The following conditions are equivalent:

(a) $N$ is maximal (resp. $P$ is maximal).

(b) $N_{(m)}$ (resp. $P_{[m]}$) is maximal for some $m \geq 0$.

(c) $N_{(m)}$ (resp. $P_{[m]}$) is maximal for all $m \geq 0$.

(iv) $\mathrm{ht}(N) = \mathrm{ht}(N_{[\infty]}) = \mathrm{ht}(N_{(m)}) = \mathrm{ht}(N_{[m]})$ for all $m \geq 0$. Moreover, $\dim(A_{(\infty)}) = \dim(A_{(m)})$.

(v) [8, Proposition (1.4) and Theorem (1.6)] Let us assume that $A$ is noetherian and that for every maximal ideal $\mathfrak{m}$ of $A$, the residue field $A/\mathfrak{m}$ is algebraic over $k$. Then for every $m \geq 0$ we have $\dim(A_{(\infty)}) = \dim(A_{(m)}) = \dim(A(t))$. Moreover, if $A$ is biequidimensional, universally catenarian of Krull dimension $n$, then every maximal ideal of $A_{(\infty)}$ (or of $A_{(m)}$) has height $n$.

(vi) [3, Proposition 2.2] If $k$ is perfect and $B = k[[X_1, \ldots, X_n]]$, then $\mathrm{Qt}(B)^\sharp = k$.

(vii) [3, Proposition 3.4] If $k$ is perfect, $A$ is an integral $k$-algebra, $K = Qt(A)$ and $K^\sharp$ is algebraic over $k$, then any prime ideal $P \in \mathrm{Spec}(A_{[\infty]})$ with $P \cap k[t] = 0$ and $P \cap A = 0$ is the extended ideal of some $P_{[m_0]}$, $m_0 \geq 0$.

(viii) [3, Corollary 3.10] If $k$ is perfect, $A$ is noetherian and for every maximal ideal $\mathfrak{m}$ of $A$, the residue field $A/\mathfrak{m}$ is algebraic over $k$, then $A_{(\infty)}$ is also noetherian. In particular $k[[X_1, \ldots, X_n]]_{(\infty)}$ is noetherian.

(ix) [6, Theorem 30.6] Let $(R, \mathfrak{m})$ be an equicharacteristic $n$-dimensional regular local ring containing a quasi-coefficient field $k_0$, and $D_1, \ldots, D_n \in \mathrm{Der}_{k_0}(R)$, $a_1, \ldots, a_n \in R$ such that $D_i(a_j) = \delta_{ij}$. Then, $\mathrm{Der}_{k_0}(R)$ is a free $R$-module with basis $\{D_1, \ldots, D_n\}$.

(x) [4, Theorem 3.17] Let $(R, \mathfrak{m})$ be an equicharacteristic $n$-dimensional regular local ring containing a quasi-coefficient field $k_0$, and let $\underline{D}^1, \ldots, \underline{D}^n \in \mathrm{HS}_{k_0}(R)$ such that their degree 1 components $\{D_1^1, \ldots, D_1^n\}$ form a basis of $\mathrm{Der}_{k_0}(R)$. Let $\widehat{\underline{D}}^1, \ldots, \widehat{\underline{D}}^n$ be the extensions of $\underline{D}^1, \ldots, \underline{D}^n$ to $\widehat{R}$. Then, the set

$$\{a \in \widehat{R} \mid \widehat{D}_i^j(a) = 0 \quad \forall j = 1, \ldots, n,\ i \geq 1\}$$

is a coefficient field of $\widehat{R}$ (the only one containing $k_0$).

(1.2) *Taylor expansions* (cf. [7]).

Let $n \geq 1$ be an integer. We write $\mathbf{X} = (X_1, \ldots, X_n)$, $\mathbf{T} = (T_1, \ldots T_n)$, $\mathbf{X} + \mathbf{T} = (X_1 + T_1, \ldots, X_n + T_n)$ and, for $\alpha \in \mathbb{N}^n$, $\mathbf{X}^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$.

Let $A$ be the formal power series ring $k[[\mathbf{X}]]$ (or the polynomial ring $k[\mathbf{X}]$). For any $f(\mathbf{X}) = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha \mathbf{X}^\alpha \in A$ we define $\Delta^{(\alpha)}(f(\mathbf{X}))$ by: $f(\mathbf{X} + \mathbf{T}) = \sum_{\alpha \in \mathbb{N}^n} \Delta^{(\alpha)}(f(\mathbf{X}))\mathbf{T}^\alpha$. One has

$$\Delta^{(\alpha)}(f \cdot g) = \sum_{\beta + \sigma = \alpha} \Delta^{(\beta)}(f)\Delta^{(\sigma)}(g) \tag{1}$$

and $\alpha!\Delta^{(\alpha)} = (\frac{\partial}{\partial X_1})^{\alpha_1} \cdots (\frac{\partial}{\partial X_n})^{\alpha_n}$. For $i \in \mathbb{N}$, $1 \leq j \leq n$ and $\alpha = (0, \ldots, \underbrace{i}_{j}, \ldots, 0)$ we denote $\Delta_i^j = \Delta^{((0, \ldots, \underbrace{i}_{j}, \ldots, 0))}$. From (1) we obtain

$$\Delta_i^j(f \cdot g) = \sum_{r+s=i} \Delta_r^j(f)\Delta_s^j(g),$$

i.e. the sequences $\underline{\Delta}^j := (1_A, \Delta_1^j, \Delta_2^j, \ldots)$, $1 \leq j \leq n$, are Hasse-Schmidt derivations of $A$ (over $k$) (cf. [6], §27).

Now, let us recall the following basic well known result (cf. [2, Propositions 5.5.3 and 5.5.6]).

(1.3) Proposition. *Let $B$ be a noetherian ring, $P$ be a prime ideal of $B[t]$ and $\mathfrak{p} = P \cap B$. Then, one of the following conditions holds:*

- *(a) $P = \mathfrak{p}[t]$, $\operatorname{ht}(P) = \operatorname{ht}(\mathfrak{p})$ and $B[t]/P \simeq (A/\mathfrak{p})[t]$.*
- *(b) $P \supset \mathfrak{p}[t]$, $\operatorname{ht}(P) = \operatorname{ht}(\mathfrak{p}) + 1$ and $B[t]/P$ is an algebraic extension of $B/\mathfrak{p}$ (generated by $t \mod P$).*

## 2 Coefficients fields and the extension $k \to k_{(\infty)}$.

Let $k$ be a perfect field of characteristic $p > 0$ and $A$ a $k$-algebra. For every Hasse-Schmidt derivation $\underline{\mathfrak{D}} \in \mathrm{HS}_k(A)$, we also denote by $\underline{\mathfrak{D}} \in \mathrm{HS}_{k_{(\infty)}}(A_{(\infty)})$ the extended Hasse-Schmidt derivation. If $\mathfrak{n} \subset A_{(\infty)}$ is a maximal ideal, we denote by $\underline{\mathfrak{D}}_\mathfrak{n}$ and $\widehat{\underline{\mathfrak{D}}_\mathfrak{n}}$ the extended Hasse-Schmidt derivations to $(A_{(\infty)})_\mathfrak{n}$ and $\widehat{(A_{(\infty)})_\mathfrak{n}}$, respectively.

The following theorem generalizes Theorem 2.3 of [8] to the positive characteristic case.

(2.1) Theorem. *Let $k$ be a perfect field of positive characteristic $p > 0$, $A = k[[X_1, \ldots, X_n]]$ the power series ring and let us consider the Hasse-Schmidt derivations $\underline{\Delta}^j \in \mathrm{HS}_k(A)$, $j = 1, \ldots, n$, defined in (1.2). Then, for each maximal ideal $\mathfrak{n} \subset A_{(\infty)}$ the set*

$$K_0 = \left\{ a \in \widehat{(A_{(\infty)})_\mathfrak{n}} \mid \widehat{(\Delta_i^j)_\mathfrak{n}}(a) = 0 \quad \forall j = 1, \ldots, n; \ \forall i \geq 1 \right\}$$

*is a coefficient field of the complete local ring* $\widehat{(A_{(\infty)})_{\mathfrak{n}}}$.

Proof. We proceed in two steps, as in the proof of Theorem 2.3 of [8]: reduction to the case $n = 1$ and treatment of this case.

**Step 1: the reduction.** Let us write $P = \mathfrak{n} \cap A_{[\infty]}$, $\mathfrak{p} = \mathfrak{n} \cap A = P \cap A = P_{(m)} \cap A$. From (1.1) (iii), (iv) we know that the ideals $\mathfrak{n}_{(m)}$ are maximal and $\mathrm{ht}(\mathfrak{n}_{(m)}) = \mathrm{ht}(\mathfrak{n})$ for all $m \geq 0$. By Remark (1.8) of [8], there are only two possibilities for the prime ideal $\mathfrak{p}$:

(i) $\mathrm{ht}(\mathfrak{p}) = n$, and then $\mathfrak{p} = (X_1, \ldots, X_n)$ and $\mathfrak{n} = \mathfrak{p}^e$.

(ii) $\mathrm{ht}(\mathfrak{p}) = n - 1$.

In case (i), $k_{(\infty)}$ is a coefficient field of $(A_{(\infty)})_{\mathfrak{n}}$ as well as of its completion, and $\widehat{(\Delta_i^j)_{\mathfrak{n}}}(k_{(\infty)}) = 0$ for every $j = 1, \ldots, n$, $i \geq 1$. The theorem is then a consequence of (1.1) (ix), (x).

Let us suppose we are in case (ii). By Theorem (A.6) (Normalization Lemma) there exists a new set of variables $X'_1, \ldots, X'_n$ in $A$ such that

- $\mathfrak{p} \cap k[[X'_1]] = (0)$,
- $k[[X'_1]] \hookrightarrow A/\mathfrak{p}$ is a finite extension, and since $A/\mathfrak{p}$ is finitely generated over $k[[X'_1]]$, $A/\mathfrak{p}$ is a finite $k[[X'_1]]$–module,
- $k((X'_1)) \hookrightarrow \mathrm{Qt}(A/\mathfrak{p})$ is a separable finite extension.

Since the Hasse-Schmidt derivations of $A$ over $k$ with respect to the variables $X'_i$ can be expressed in terms of the $\underline{\Delta}^j$ ([4, Theorem 2.8]), we can suppose $X'_i = X_i$.

Let us write $K = A_{(\infty)}/\mathfrak{n} = \mathrm{Qt}\left(A_{[\infty]}/P\right)$, $R = A/\mathfrak{p}$, $A' = k[[X_1]]$, $\mathfrak{n}' = \mathfrak{n} \cap A'_{(\infty)}$, $P' = P \cap A'_{[\infty]} = \mathfrak{n}' \cap A'_{[\infty]}$ and $K' = A'_{(\infty)}/\mathfrak{n}' = \mathrm{Qt}\left(A'_{[\infty]}/P'\right)$.

We have $R_{[m]} = \dfrac{A_{[m]}}{\mathfrak{p}A_{[m]}}$, $R_{[\infty]} = \dfrac{A_{[\infty]}}{\mathfrak{p}A_{[\infty]}}$, $K = \bigcup_{m \geq 0} \dfrac{A_{(m)}}{\mathfrak{n}_{(m)}}$ and $K' = \bigcup_{m \geq 0} \dfrac{A'_{(m)}}{\mathfrak{n}'_{(m)}}$.

Let us consider the following commutative diagram of inclusion

$$\begin{array}{ccccc} & & A'[t]/P'_{[0]} & & \\ & \nearrow & & \searrow & \\ A' & & & & A[t]/P_{[0]}. \\ & \searrow & & \nearrow & \\ & & R = A/\mathfrak{p} & & \end{array}$$

The bottom inclusions are algebraic ($R$ is a finite $A'$-module and $P_{[0]} \cap A = \mathfrak{p}$), hence the top ones must be so. In particular $A'[t]/P'_{[0]}$ is algebraic over $A'$, which implies (Proposition (1.3)) that $P'_{[0]} \neq 0$, then $\mathfrak{n}'_{(0)} \neq 0$ and $\mathfrak{n}' \neq 0$. Therefore $\mathfrak{n}'$ is maximal since $\dim(A') = 1$.

Let us show that the inclusion $K' \subset K$ is separable algebraic. For that, it is enough to prove that the extensions

$$\frac{A'_{(m)}}{\mathfrak{n}'_{(m)}} \subset \frac{A_{(m)}}{\mathfrak{n}_{(m)}}$$

are finite and separable.

Let us write $L' = \mathrm{Qt}(A') = k((X_1))$, $L = \mathrm{Qt}(A/\mathfrak{p})$ and consider the following diagram of field extensions

$$\begin{array}{ccc} L' = \mathrm{Qt}(A') & \subset & \mathrm{Qt}\left(\dfrac{A'_{[m]}}{P'_{[m]}}\right) = \dfrac{A'_{(m)}}{\mathfrak{n}'_{(m)}} \\ \cap & & \cap \\ L = \mathrm{Qt}(R) & \subset & \mathrm{Qt}\left(\dfrac{A_{[m]}}{P_{[m]}}\right) = \dfrac{A_{(m)}}{\mathfrak{n}_{(m)}}. \end{array}$$

These extensions satisfy the following properties:

i) $L' \subset L$ is finite and separable. Hence, there is a primitive element $e$, $L = L'[e]$, whose minimal polynomial $f(X) \in L'[X]$ satisfies $f'(X) \neq 0$.

ii) By Proposition (1.3), the extensions $L \subset \mathrm{Qt}\left(\dfrac{A_{[m]}}{P_{[m]}}\right)$, $L' \subset \mathrm{Qt}\left(\dfrac{A'_{[m]}}{P'_{[m]}}\right)$ are finite and generated by the class $\overline{t}$ of $t$.

Therefore,

$$\frac{A_{(m)}}{\mathfrak{n}_{(m)}} = \mathrm{Qt}\left(\frac{A_{[m]}}{P_{[m]}}\right) = L[\overline{t}] = L'[e][\overline{t}] = \left(\mathrm{Qt}\left(\frac{A'_{[m]}}{P'_{[m]}}\right)\right)[e] = \left(\frac{A'_{(m)}}{\mathfrak{n}'_{(m)}}\right)[e]$$

and the extension

$$\frac{A'_{(m)}}{\mathfrak{n}'_{(m)}} \subset \frac{A_{(m)}}{\mathfrak{n}_{(m)}}$$

is finite and separable for all $m \geq 0$. Hence, $K' \subset K$ is separable algebraic.

Let us assume that the theorem is proved for $n = 1$. Then

$$K'_0 = \left\{ a \in \widehat{(A'_{(\infty)})_{\mathfrak{n}'}} \mid \widehat{(\Delta^1_i)_{\mathfrak{n}'}}(a) = 0 \quad \forall i \geq 1 \right\}$$

is a coefficient field of $\widehat{(A'_{(\infty)})_{\mathfrak{n}'}}$.

We can consider $K'_0$ as a subfield of $\widehat{(A_{(\infty)})_{\mathfrak{n}}}$ via the inclusion $\widehat{(A'_{(\infty)})_{\mathfrak{n}'}} \hookrightarrow \widehat{(A_{(\infty)})_{\mathfrak{n}}}$. Since $K'_0 \xrightarrow{\sim} K'$ and $K' \subset K$ is separable algebraic, we deduce that $K'_0$ is a quasi-coefficient field of $\widehat{(A_{(\infty)})_{\mathfrak{n}}}$.

It is clear that for all $a \in K'_0$

$$\widehat{(\Delta^j_i)_{\mathfrak{n}}}(a) = 0 \quad \forall j = 1, \ldots, n, \ \forall i \geq 1.$$

In particular, the $\widehat{(\underline{\Delta}^j)_{\mathfrak{n}}}$ are Hasse-Schmidt derivations over $K_0'$, and by (1.1) (ix), the $\{\Delta_1^1, \ldots, \Delta_1^n\}$ form a basis of $\mathrm{Der}_{K_0'}(\widehat{(A_{(\infty)})_{\mathfrak{n}}})$.

Now, by applying (1.1) (x), we obtain that

$$\left\{a \in \widehat{(A_{(\infty)})_{\mathfrak{n}}} \mid \widehat{(\Delta_i^j)_{\mathfrak{n}}}(a) = 0 \quad \forall j = 1, \ldots, n,\ \forall i \geq 1\right\}$$

is a coefficient field of $\widehat{(A_{(\infty)})_{\mathfrak{n}}}$ and the theorem is proved.

**Step 2: the case n=1.** Let us write $A = k[[X]]$, $L = \mathrm{Qt}(A) = k((X))$ and let $\mathfrak{n}$ be a maximal ideal of $A_{(\infty)} = A \otimes_k k_{(\infty)}$. Let us denote $P = \mathfrak{n} \cap A_{[\infty]}$. By (1.1) (iv), we know that

$$\mathrm{ht}(\mathfrak{n}) = \mathrm{ht}(\mathfrak{n}_{(m)}) = \mathrm{ht}(P_{[m]}) = \mathrm{ht}(P) = 1.$$

As in the first step, we focus on the case $\mathfrak{n} \cap A = (0)$ (and then $P \cap A = (0)$). Since each $A_{[m]} = A[t_m]$ is a unique factorization domain and each $P_{[m]}$ is a prime ideal of $A_{[m]}$ of height 1, $P_{[m]}$ is generated by an irreducible polynomial $F_m(t_m) \in A[t_m]$ of degree $d \geq 1$ and with some non-constant coefficient, since $P_{[m]} \cap k[t_m] = (0)$. By irreducibility, at least one of the coefficients of $F_m(t_m)$ must be a unit, so we may assume that it is 1.

Let us write $K = \frac{A_{(\infty)}}{\mathfrak{n}}$ and $K_m = \frac{A_{(m)}}{\mathfrak{n}_{(m)}}$. Since $A_{(\infty)}$ and $A_{(m)}$ are localizations of $A_{[\infty]}$ and $A_{[m]}$ respectively, it follows that

$$K = \frac{A_{(\infty)}}{\mathfrak{n}} = \mathrm{Qt}\left(\frac{A_{[\infty]}}{P}\right), \quad K_m = \frac{A_{(m)}}{\mathfrak{n}_{(m)}} = \mathrm{Qt}\left(\frac{A_{[m]}}{P_{[m]}}\right).$$

The minimal polynomial of $\theta_m := \left(t_m \mod P_{[m]}\right)$ over $L$ is $F_m(t_m)$. We have $K_m = L[\theta_m]$,

$$K = \bigcup_{m \geq 0} K_m = \bigcup_{m \geq 0} L[\theta_m] = L[\theta_0, \theta_1, \theta_2, \ldots],$$

where $\theta_m = \theta_{m+1}^p$, and the inclusion $k_{(\infty)} \hookrightarrow K$ is a $k$-morphism which sends each $t_m$ onto $\theta_m$.

By (1.1) (vi), it follows that $L^\sharp = k((X))^\sharp = k$, and we can apply (1.1) (vii) to conclude that there exists $m_0 \geq 0$ such that $P$ is the extended ideal of $P_{[m_0]} = (F_{m_0}(t_{m_0}))$. Then, $P$ (resp. $\mathfrak{n}$) is the ideal of $A_{[\infty]}$ (resp. of $A_{(\infty)}$) generated by $\mu = F_{m_0}(t_{m_0})$. Moreover, for every $j \geq 1$, $P_{[m_0+j]}$ is the extended ideal of $P_{[m_0]}$ and some of the coefficients of $\mu$ is not a $p$–th power. Hence, we can take

$$F_{m_0+j}(t_{m_0+j}) = F_{m_0}(t_{m_0}) = F_{m_0}(t_{m_0+j}^{p^j}), \quad j \geq 1.$$

Since $k_{(\infty)}$ is perfect, the field extension $k_{(\infty)} \subset K$ is separable and, by Cohen structure theorem, there exists a $k_{(\infty)}$–isomorphism

$$\varphi : \widehat{(A_{(\infty)})_{\mathfrak{n}}} \xrightarrow{\sim} K[[s]] \tag{2}$$

which induces the identity on residue fields and sends the regular parameter $\mu$ of $\widehat{(A_{(\infty)})_{\mathfrak{n}}}$ onto $s$. One has:

$$\begin{array}{rcl} \varphi(\mu) & = & s \\ \varphi(t_m) & = & \theta_m \\ \varphi(X) & = & X+\xi \quad \text{with} \quad \xi \in (s). \end{array}$$

Let us denote by

$$\underline{\Delta}^X = (1, \Delta_1^X, \Delta_2^X, \dots) \in \mathrm{HS}_k(k[[X]])$$

the Hasse-Schmidt derivation defined in (1.2) and let us assume, for the moment, that $\varphi$ satisfies the relation

$$\varphi(a(X)) = a(X+\xi) \subseteq k[[X,\xi]] \subseteq K[[\xi]] \subseteq K[[s]] \tag{3}$$

for all $a(X) \in A = k[[X]]$.
Then, writing $\mu = a_d(X) t_{m_0}^d + \cdots + a_0(X)$,

$$s = \varphi(\mu) = \varphi\left(\sum_{r=0}^{d} a_r(X) t_{m_0}^r\right) = \sum_{r=0}^{d} \varphi\left(a_r(X)\right)\theta_{m_0}^r = \sum_{r=0}^{d} a_r(X+\xi)\theta_{m_0}^r \overset{(1.2)}{=}$$
$$= \sum_{r=0}^{d}\left(\sum_{i=0}^{\infty} \Delta_i^X(a_r(X))\xi^i\right)\theta_{m_0}^r = \sum_{i=0}^{\infty}\left(\sum_{r=0}^{d} \Delta_i^X(a_r(X))\theta_{m_0}^r\right)\xi^i \in K[[\xi]],$$

and $\xi$ must be of order one in $s$. Hence, $\xi$ is a new variable in $K[[s]]$ and $K[[s]] = K[[\xi]]$.
Let us denote by $\underline{\Delta}'$ the unique extension of $\underline{\Delta}^X$ to $K[[s]]$ through

$$A \xrightarrow{\text{scalar ext.}} A \otimes_k k_{(\infty)} \xrightarrow{\text{local.}} (A_{(\infty)})_{\mathfrak{n}} \xrightarrow{\text{compl.}} \widehat{(A_{(\infty)})_{\mathfrak{n}}} \xrightarrow{\varphi\ \simeq} K[[s]],$$

which belongs to $\mathrm{HS}_{k_{(\infty)}}(K[[s]])$, and let us denote by

$$\underline{\Delta}^\xi = (1, \Delta_1^\xi, \Delta_2^\xi, \dots) \in \mathrm{HS}_K(K[[\xi]]) = \mathrm{HS}_K(K[[s]])$$

the Hasse-Schmidt derivation defined in (1.2), this time with respect to the variable $\xi$.

We will show that relation (3) implies that $\underline{\Delta}^\xi = \underline{\Delta}'$, i.e.

$$(\varphi \circ \Delta_i^X)(a) = (\Delta_i^\xi \circ \varphi)(a) \quad \forall i \geq 0, \forall a \in k[[X]], \tag{4}$$

and then

$$\varphi^{-1}(K) = \varphi^{-1}\left(\left\{c \in K[[s]] \mid \Delta_i^\xi(c) = 0,\ \forall i > 0\right\}\right) = \left\{a \in \widehat{(A_{(\infty)})_{\mathfrak{n}}} \mid \widehat{(\Delta_i^X)_{\mathfrak{n}}}(a) = 0,\ \forall i > 0\right\}$$

is a coefficient field of $\widehat{(A_{(\infty)})_{\mathfrak{n}}}$ and the step 2 would be finished.

Let $\varphi_0 : A = k[[X]] \to k[[X,\xi]]$ be the local $k$-homomorphism defined by $\varphi_0(X) = X+\xi$. Relation (3) says that $\varphi(a(X)) = \varphi_0(a(X))$ for all $a(X) \in A$.

Let $Y$ be a new variable and consider the local $k$-homomorphisms $\delta : k[[X]] \to k[[X,Y]]$, $\varepsilon : k[[X,\xi]] \to k[[X,\xi,Y]]$ and $\widetilde{\varphi_0} : k[[X,Y]] \to k[[X,\xi,Y]]$ defined by:

$$\delta(X) = X+Y, \quad \varepsilon(X) = X, \quad \varepsilon(\xi) = \xi + Y, \quad \widetilde{\varphi_0}(Y) = Y, \quad \widetilde{\varphi_0}(X) = X + \xi.$$

Let us also consider the local $K$-homomorphism $\Theta : K[[\xi]] \to K[[\xi, Y]]$ defined by $\Theta(\xi) = \xi + Y$. Then, the following diagram

$$\begin{array}{ccccc} k[[X]] & \xrightarrow{\varphi_0} & k[[X,\xi]] & \xrightarrow{\subset} & K[[\xi]] \\ {\scriptstyle\delta}\downarrow & & {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\Theta} \\ k[[X,Y]] & \xrightarrow{\widetilde{\varphi_0}} & k[[X,\xi,Y]] & \xrightarrow{\subset} & K[[\xi,Y]] \end{array}$$

is commutative and we have

$$\sum_{i=0}^{\infty} \Delta_i^{\xi}(\varphi(a))Y^i = \Theta(\varphi(a)) = \varepsilon(\varphi_0(a)) = \widetilde{\varphi_0}(\delta(a)) =$$
$$\widetilde{\varphi_0}\left(\sum_{i=0}^{\infty} \Delta_i^X(a)Y^i\right) = \sum_{i=0}^{\infty} \varphi_0(\Delta_i^X(a))Y^i = \sum_{i=0}^{\infty} \varphi(\Delta_i^X(a))Y^i$$

for all $a \in k[[X]]$. Therefore relation (4) is proved and $\underline{\Delta}^{\xi} = \underline{\Delta}'$.

The point now is to construct a $\varphi$ in (2) satisfying (3). We first find $\varphi(X) = X + \xi \in K[[s]]$, and for this we state and prove the following lemma which is a generalization of Lemma (2.3.3) of [8].

(2.2) Lemma. *There exists a unique $\xi \in K[[s]]$ such that $\xi(0) = 0$ of order 1 satisfying*

$$a_d(X+\xi)\theta_{m_0}^d + \cdots + a_0(X+\xi) = s.$$

Proof. The lemma is a consequence of the implicit function theorem. Let $G(s,\sigma) = a_d(X+\sigma)\theta_{m_0}^d + \cdots + a_0(X+\sigma) - s \in K[[s,\sigma]]$, with

$$G(0,0) = a_d(X)\theta_{m_0}^d + \cdots + a_0(X) = F_{m_0}(\theta_{m_0}) = 0.$$

We have to check that

$$\left(\frac{\partial G}{\partial \sigma}\right)|_{s=\sigma=0} = a_d'(X)\theta_{m_0}^d + \cdots + a_0'(X) \neq 0 \quad \text{in} \quad K.$$

Assume the contrary: then $a_d'(X)t_{m_0}^d + \cdots + a_0'(X)$ should be a multiple of $F_{m_0}(t_{m_0})$ in $k((X))[t_{m_0}]$ and there would be an $\alpha \in k((X))$ such that

$$a_r'(X) = \alpha(X)a_r(X) \quad \text{for every} \quad r = 0, 1, \ldots, d.$$

Since some of the coefficients $a_r$ is 1, we deduce that $\alpha(X) = 0$ and $a'_r(X) = 0$ for every $r = 0, 1, \dots, d$, and then there are $b_r(X) \in k[[X]]$ such that $a_r(X) = b_r(X^p)$. Since $k$ is perfect we conclude that $a_r(X) = b_r(X)^p$, contradicting the fact that some of the coefficients of $\mu$ is not a $p$-th power.

So $\left(\frac{\partial G}{\partial \sigma}\right)|_{s=\sigma=0} \neq 0$, and by the implicit function theorem, there is a unique $\xi \in K[[s]]$ such that $\xi(0) = 0$ and $G(s, \xi) = 0$. Then $\xi$ has order 1 since

$$\left(\frac{\partial \xi}{\partial s}\right)(0) = \left[\left(\frac{\partial G}{\partial \sigma}\right)(0,0)\right]^{-1} \neq 0.$$

Q.E.D.

Let us finish the proof of Theorem (2.1). Let $\xi \in K[[s]]$ be as in the Lemma (2.2) and let us consider the local $k$-homomorphism

$$\varphi_0 : \ A = k[[X]] \to k[[X, \xi]]$$

such that $\varphi_0(X) = X + \xi$. Let us call $\varphi : A \to K[[s]]$ the composition of $\varphi_0$ with the inclusion $k[[X, \xi]] \subset K[[\xi]] = K[[s]]$.

We extend $\varphi$ to $A_{(\infty)}$ by defining $\varphi(t_m) = \theta_m \in K_m \subseteq K$ and we obtain a $k_{(\infty)}$-homomorphism $\varphi : A_{(\infty)} \to K[[s]]$ satisfying (3) by construction and sending

$$\mu = F_{m_0}(t_{m_0}) = a_d(X)t_{m_0}^d + \cdots + a_0(X)$$

onto the element

$$a_d(X + \xi)\theta_{m_0}^d + \cdots + a_0(X + \xi) = s.$$

Therefore, the contraction of the maximal ideal $(s)$ by $\varphi$ must be $\mathfrak{n} = (\mu)$, and so we can extend $\varphi$, first to a local $k_{(\infty)}$-homomorphism $\varphi : (A_{(\infty)})_{\mathfrak{n}} \to K[[s]]$, and second, by completion, to $\varphi : \widehat{(A_{(\infty)})_{\mathfrak{n}}} \to K[[s]]$

Such a $\varphi$ induces the identity map on residue fields and sends the regular parameter $\mu = F_{m_0}(t_{m_0})$ onto $s$. Since both local rings are regular of dimension 1, we deduce that $\operatorname{gr}\varphi$ is an isomorphism, and since both rings are complete, we deduce that $\varphi : \widehat{(A_{(\infty)})_{\mathfrak{n}}} \to K[[s]]$ is a $k_{(\infty)}$-isomorphism satisfying (3) as desired. Q.E.D.

The following example shows that, in order to generalize Theorem (2.3) in [8] to the positive characteristic case, one has to consider the scalar extension $k \to k_{(\infty)}$ instead of $k \to k(t)$.

(2.3) Example. Let $k$ be a perfect field of characteristic $p > 0$, $A = k[[X]]$ and consider the maximal ideal $\mathfrak{n} = (X^p t - 1)$ in $A(t) = A \otimes_k k(t)$. Then, there is no coefficient field of $\widehat{(A(t))_{\mathfrak{n}}}$ on which the $\widehat{(\Delta_i^X)_{\mathfrak{n}}}$, $i > 0$, vanish.

Assume the contrary, i.e. there exists a coefficient field $K_0$ of $B := \widehat{(A(t))_{\mathfrak{n}}}$ such that $\widehat{(\Delta_i^X)_{\mathfrak{n}}}(K_0) = 0$ for all $i > 0$, i.e. $\widehat{(\underline{\Delta}^X)_{\mathfrak{n}}} \in \mathrm{HS}_{K_0}(B)$.

Since $\widehat{(\Delta_1^X)_{\mathfrak{n}}}(X) = 1$, $\widehat{(\Delta_1^X)_{\mathfrak{n}}}$ would be a basis of $\mathrm{Der}_{K_0}(B)$ by Theorem 30.6 of [6], and by Theorem 3.17 of [4] we would have the equality

$$K_0 = \left\{ a \in B \mid \widehat{(\Delta_i^X)_{\mathfrak{n}}}(a) = 0,\ \forall i > 0 \right\}.$$

In particular $k(t) \subset K_0$.

The residue field of $B$ is

$$K = \frac{A(t)}{\mathfrak{n}} = \mathrm{Qt}\left( \frac{A[t]}{(X^p t - 1)} \right) = k[[X]][X^{-p}] = k((X)),$$

where the inclusion $k(t) \hookrightarrow K$ sends $t$ to $X^{-p}$. Let $\tau : K_0 \xrightarrow{\sim} K$ be the $k(t)$-isomorphism induced by the inclusion $K_0 \subset B$.

By Cohen structure theorem, the inclusion $K_0 \subset B$ would be extended to an isomorphism $\psi : K_0[[s]] \xrightarrow{\sim} B$ such that $\psi(s) = X^p t - 1$ ($B$ is a one dimensional complete local noetherian local ring with parameter $X^p t - 1$) and the diagram

$$\begin{array}{ccc} K_0[[s]] & \xrightarrow[\sim]{\psi} & B = \widehat{(A(t))_{\mathfrak{n}}} \\ {\scriptstyle \text{res.}}\downarrow & & \downarrow{\scriptstyle \text{res.}} \\ K_0 & \xrightarrow[\sim]{\tau} & K \end{array}$$

is commutative.

Since $\tau^{-1}(X)$ is congruent to $X$ mod. the maximal ideal of $B$, we deduce that $\psi^{-1}(X)$ is congruent to $\tau^{-1}(X)$ mod. $s$, i.e. $\psi^{-1}(X) = \tau^{-1}(X) + \xi$, with $\xi \in (s)$.

On the other hand,

$$s = \psi^{-1}(X^p t - 1) = \psi^{-1}(X^p)\psi^{-1}(t) - 1 = \psi^{-1}(X)^p t - 1 = \left(\tau^{-1}(X) + \xi\right)^p t - 1 =$$

$$\left(\tau^{-1}(X)^p + \xi^p\right) t - 1 = \left(\tau^{-1}(X^p) + \xi^p\right) t - 1 = \left(t^{-1} + \xi^p\right) t - 1 = t\xi^p \in (s^p),$$

which is a contradiction.

## Appendix: The Normalization Lemma for power series rings over perfect fields

In this appendix we give a proof of the normalization lema for power series rings over an arbitrary perfect field of positive characteristic. Our proof is an adaptation of Abhyankar's proof [1], 23.7 and 24.5, which uses generic linear changes of coordinates and thus requieres the field $k$ to be infinite.

The following lemma is straightforward.

(A.1) LEMMA. *Let $L$ be a field of characteristic $p>0$, and let $L\subset K=L[\alpha_1,\dots,\alpha_n]$ a field extension with $\alpha_i^p\in L$ for $i=1,\dots,n$, and $[K:L]=p^e$. Then, there exist $\alpha_{i_1},\dots,\alpha_{i_e}$ such that $K=L[\alpha_{i_1},\dots,\alpha_{i_e}]$.*

A series $f(X_1,\dots,X_n)\in k[[X_1,\dots,X_n]]$ is said to be $X_n$*-distinguished* if $f(0,\dots,0,X_n)\neq 0$.

The following combinatorial lemma is classical.

(A.2) LEMMA. *Let $\sigma=(\sigma_1,\dots,\sigma_{n-1})\in(\mathbb{N}^*)^{n-1}$ and $L_\sigma:\mathbb{N}^n\to\mathbb{N}$ defined by $L_\sigma(\alpha)=\sigma_1\alpha_1+\dots+\sigma_{n-1}\alpha_{n-1}+\alpha_n$ for all $\alpha\in\mathbb{N}^n$. Then, for each finite subset $F\subset\mathbb{N}^n$, there exists a constant $C\geq 1$ such that the restriction $L_\sigma|_F$ is inyective for all $\sigma$ with $\sigma_1\geq\sigma_2 C$, $\sigma_2\geq\sigma_3 C$,..., $\sigma_{n-2}\geq\sigma_{n-1}C$, $\sigma_{n-1}\geq C$.*

PROOF. The proof is standard by a double induction on $n$ and $\sharp F$. Q.E.D.

(A.3) LEMMA. *Let $f(X_1,\dots,X_n)\in k[[\mathbf{X}]]$ be a non-zero and non-unit formal power series. Then for $\sigma_1\gg\sigma_2\gg\dots\gg\sigma_{n-1}\gg 0$, the series $f(X_1+X_n^{\sigma_1},\dots,X_{n-1}+X_n^{\sigma_{n-1}},X_n)$ is $X_n$-distinguished.*

PROOF. Let us write $f(X_1,\dots,X_n)=\sum_{\alpha\in\mathbb{N}^n} f_\alpha X_1,\dots,X_n{}^\alpha$ and consider the Newton's diagram

$$\mathcal{N}(f)=\{\alpha\in\mathbb{N}^n\mid f_\alpha\neq 0\}\neq\emptyset,\quad \underline{0}\notin\mathcal{N}(f).$$

Let $F\subset\mathcal{N}(f)$ be the finite set of minimal elements with respect to the usual partial ordering in $\mathbb{N}^n$. We have $\mathcal{N}(f)\subset F+\mathbb{N}^n$ .
By Lemma (A.2), we obtain that $L_\sigma|_F$ is inyective for $\sigma_1\gg\sigma_2\gg\dots\gg\sigma_{n-1}\gg 0$, and then the series

$$f(0+X_n^{\sigma_1},\dots,0+X_n^{\sigma_{n-1}},X_n)=\sum_{\alpha\in\mathcal{N}(f)} f_\alpha X_n^{L_\sigma(\alpha)}$$

has order $\min_{\alpha\in F}L_\sigma(\alpha)$ and is non zero. Q.E.D.

(A.4) PROPOSITION. *Let $\mathfrak{a}\subset A=k[[X_1,\dots,X_n]]$ be a proper ideal with $e=\dim(A/\mathfrak{a})$. Then there exists a change of coordinates of the form*

$$\begin{cases} Y_1 &= X_1+F_1(X_2^p,\dots,X_n^p)\\ Y_2 &= X_2+F_2(X_3^p,\dots,X_n^p)\\ \vdots & \vdots\quad \vdots\\ Y_{n-1} &= X_{n-1}+F_{n-1}(X_n^p)\\ Y_n &= X_n\end{cases}$$

*with $F_i\in\mathbb{F}_p[X_{i+1},\dots,X_n]$ for $i=1,\dots,n-1$, such that $\mathfrak{a}\cap k[[Y_1,\dots,Y_e]]=\{0\}$ and the extension $k[[Y_1,\dots,Y_e]]\hookrightarrow A/\mathfrak{a}$ is finite.*

PROOF. We proceed by induction on $n$.
For $n=1$: let $\mathfrak{a}$ a proper ideal of $A=k[[X_1]]$ of height 1. Then $\mathfrak{a}=(X_1^m)$ and

$$k \subset \frac{k[[X_1]]}{\mathfrak{a}} = k[\overline{X}_1]$$

is finite of rank $m$.

Suppose now the result is true for $n-1$, and let $\mathfrak{a}$ be a proper ideal of $A=k[[X_1,\dots,X_n]]$. Let us take a non-zero and non-unit formal power series $f(X_1,\dots,X_n)\in\mathfrak{a}$.

By the change

$$\begin{cases} Y_j = X_j - X_n^{\sigma_j},\ j=1,\dots,n-1 \\ Y_n = X_n, \end{cases}$$

with $\sigma_j=\dot{p}$, $\sigma_1 \gg \sigma_2 \gg \cdots \gg \sigma_{n-1} \gg 0$, and by Lemma (A.3), we deduce that the series

$$g(Y_1,\dots,Y_{n-1},Y_n) = f(Y_1+Y_n^{\sigma_1},\dots,Y_{n-1}+Y_n^{\sigma_{n-1}},Y_n) = f(X_1,\dots,X_n)$$

is $Y_n$-distinguished.

By Weierstrass preparation theorem we can write $g(Y_1,\dots,Y_{n-1},Y_n)=u\cdot H$, where $u$ is a unit and

$$H = Y_n^q + a_{q-1}(Y_1,\dots,Y_{n-1})Y_n^{q-1} + \cdots + a_0(Y_1,\dots,Y_{n-1}),$$

$q=\operatorname{ord}_{X_n}(f(X_n^{\sigma_1},\dots,X_n^{\sigma_{n-1}},X_n))\geq 1$ and $a_i(\underline{0})=0$.

Consequently $H\in\mathfrak{a}$ and the ring extension

$$\frac{k[[Y_1,\dots,Y_{n-1}]]}{\mathfrak{a}^c} \subseteq \frac{k[[Y_1,\dots,Y_n]]}{\mathfrak{a}} = k[[Y_1,\dots,Y_{n-1}]][Y_n]$$

is finite. The proposition follows by applying induction hypothesis to $\mathfrak{a}^c$. Q.E.D.

From now on $k$ will be a perfect field of characteristic $p>0$, $\mathfrak{p}$ a prime ideal in $A=k[[X_1,\dots,X_n]]$, $R=A/\mathfrak{p}$, $L=\operatorname{Qt}(A)=k((X_1,\dots,X_n))$ and $K=\operatorname{Qt}(R)$. Let us denote $e=\dim R$ and $\overline{a}\in R$ the class mod $\mathfrak{p}$ of any element $a\in A$.

The following proposition is an adaptation of (24.1) and (24.4) of [1], which uses Proposition (A.4) instead of (23.3) of loc. cit.

(A.5) PROPOSITION. *Under the above hypothesis, the relations*

$$K = K^p[\overline{X}_1,\dots,\overline{X}_n],\ [K:K^p]=p^e,$$

*hold and the set* $\left\{\overline{X}_1^{\sigma_1}\cdots\overline{X}_n^{\sigma_n} :\ 0\leq\sigma_i<p,\ i=1,\dots,n\right\}$ *is a system of generators of the extension* $K^p\subset K$. *Moreover, after a permutation of variables, we have* $K=K^p[\overline{X}_1,\dots,\overline{X}_e]$ *and* $\left\{\overline{X}_1^{\sigma_1}\cdots\overline{X}_e^{\sigma_e} :\ 0\leq\sigma_i<p,\ i=1,\dots,e\right\}$ *is a basis of* $K$ *as* $K^p$*-vector space.*

PROOF. Since $k$ is perfect, one has $A=A^p[X_1,\dots,X_n]$, $L=L^p[X_1,\dots,X_n]$ and

$$\{X_1^{\sigma_1}\cdots X_n^{\sigma_n} :\ 0\leq\sigma_1<p,\dots,0\leq\sigma_n<p\}$$

is basis of $L$ (resp. of $A$) as $L^p$-vector space (resp. as $A^p$-module). In particular $[L : L^p] = p^n$ and $A$ is a finite $A^p$-module.

Hence, $R = R^p[\overline{X}_1, \ldots, \overline{X}_n]$, $K = K^p[\overline{X}_1, \ldots, \overline{X}_n]$ and

$$\left\{\overline{X}_1^{\sigma_1} \cdots \overline{X}_n^{\sigma_n} : \ 0 \leq \sigma_i < p, \ i = 1, \ldots, n\right\}$$

is a system of generators of the extension $K^p \subset K$.

By Proposition (A.4) we obtain a finite ring extension $B = k[[Y_1, \ldots, Y_e]] \subset R$ and then $L_1 = \mathrm{Qt}(B) = k((Y_1, \ldots, Y_e)) \subset K$ is a finite field extension.

By using Frobenius morphism one proves that $[K : L_1] = [K^p : L_1^p]$, and from

$$[K : L_1][L_1 : L_1^p] = [K : L_1^p] = [K : K^p][K^p : L_1^p]$$

we deduce that $[K : K^p] = [L_1 : L_1^p] = p^e$.

Finally, by Lemma (A.1) we know that after a permutation of variables

$$\{\overline{X}_1^{\sigma_1} \cdots \overline{X}_e^{\sigma_e} : \ 0 \leq \sigma_i < p, \ i = 1, \ldots, e\},$$

is a basis of $K$ as $K^p$-vector space. Q.E.D.

(A.6) Theorem. (Normalization Lemma for power series ring over perfect fields in positive characteristics) *In the situation of Proposition (A.5), there exists a new set of variables* $Y_1, \ldots, Y_n \in A = k[[X_1, \ldots, X_n]]$ *such that*

*(1)* $\mathfrak{p} \cap k[[Y_1, \ldots, Y_e]] = \{0\}$.

*(2)* $B = k[[Y_1, \ldots, Y_e]] \hookrightarrow R = A/\mathfrak{p}$ *is a finite ring extension.*

*(3)* $L_1 = \mathrm{Qt}(B) \hookrightarrow K = \mathrm{Qt}(R)$ *is a separable finite extension.*

Proof. In view of Proposition (A.5), after a permutation of variables $X_i$ we get $K = K^p[\overline{X}_1, \ldots, \overline{X}_e]$ and $\left\{\overline{X}_1^{\sigma_1} \cdots \overline{X}_e^{\sigma_e} : \ 0 \leq \sigma_1 < p, \ldots, 0 \leq \sigma_e < p\right\}$ is basis of $K$ as $K^p$-vector space.

By Proposition (A.4), there is a new set of variables $Y_1, \ldots, Y_n$ in $k[[X_1, \ldots, X_n]]$ of the form

$$Y_j = X_j + F_j(X_{j+1}^p, \ldots, X_n^p), \quad 1 \leq j \leq n-1$$

and $Y_n = X_n$, with $F_j \in \mathbb{F}_p[X_{j+1}, \ldots, X_n]$, such that $\mathfrak{p} \cap k[[Y_1, \ldots, Y_e]] = \{0\}$ and the extension $B = k[[Y_1, \ldots, Y_e]] \hookrightarrow A/\mathfrak{p}$ is finite. Hence, $K$ is a finite field extension of $L_1 = \mathrm{Qt}(B)$.

Since

$$\overline{X}_1^{\sigma_1} \cdots \overline{X}_e^{\sigma_e} = \left(\overline{Y}_1 - F_1(\overline{X}_2^p, \ldots, \overline{X}_n^p)\right)^{\sigma_1} \cdots \left(\overline{Y}_e - F_e(\overline{X}_{e+1}^p, \ldots, \overline{X}_n^p)\right)^{\sigma_e},$$

$\overline{Y}_1, \ldots \overline{Y}_e \in L_1 = k((Y_1, \ldots, Y_e))$ and $F_j(\overline{X}_{j+1}^p, \ldots, \overline{X}_n^p) = F_j(\overline{X}_{j+1}, \ldots, \overline{X}_n)^p \in K^p$, we deduce that $\overline{X}_1^{\sigma_1} \cdots \overline{X}_e^{\sigma_e} \in K^p(L_1)$ and $K = K^p(L_1)$. Therefore $K$ is a separable finite extension of $L_1$(cf. [9], Theorem 8 on p. 69). Q.E.D.

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