Discrete solitons in optical BEC lattices. Effects of n-body interactions

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Abstract

In this poster we show some recent results concerning discrete solitons in strong optical lattices, which can be described by the Discrete Nonlinear Schrödinger equation. These results are related to a variation of this equation including saturable nonlinearity terms, a feature throughoutly studied in nonlinear optics. After presenting the derivation of the DNLS equation from the Gross-Pitaevskii equation in the presence of a strong optical lattice, we study the existence of thresholds in the quadratic norm of discrete solitons in the cubic DNLS, cubic-quintic DNLS and photorefractive DNLS. The second part of the poster is devoted to moving discrete solitons in the photorefractive DNLS equation. In the one hand, we study the existence of radiationless moving discrete solitons; on the other hand, we study the collisions of moving discrete solitons.

BECs in one-dimensional optical lattices

The dynamics of a BEC in an optical lattice satisfies the Gross-Pitaevskii equation (Dalfovo et al. 1999):

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Phi + [V_{\text{ext}} + g_0|\Phi^2|]\Phi$$

where $g_0 = 4\pi\hbar^2 a/m$, with a the s-wave scattering length, m the atomic mass and being V_{ext} the optical potential:

$$V_{\text{ext}}(\vec{r}) = V_L(y,z)\cos^2(2\pi x/\lambda)$$

where λ is the trap wavelength (the lattice space is $\lambda/2$). The trap depth at the center of the beam is V_0 that can be measured in units of $E_R = 2\pi^2\hbar^2/m\lambda^2$ (Cataliotti et al. 2001). If V_0 is much higher than the chemical potentials, a tight-binding approximation can be used (Trombettoni and Smerzi 2001). The order parameter Φ can be decomposed as a sum of wavefunctions located at each well of the periodic potential:

$$\Phi(\vec{r},t) = \sqrt{N_T}\sum_n \psi_n(t)\phi(\vec{r}-\vec{r}_n)$$

where $\psi_n = \sqrt{N_n/N_T}\exp i\theta_n(t)$, with N_n and θ_n being the number of particles and phase at the n-th well, and $N_T = \int \mathrm{d}\vec{r}|\Phi(\vec{r})|^2$ the total number of particles. With this ansatz, the GPE transforms into a Discrete Nonlinear Schrödinger (DNLS) Equation (Alfimov et al. 2002):

$$i\hbar\frac{\partial\psi_n}{\partial t} = -K(\psi_{n-1}+\psi_{n+1}) + (\epsilon_n + U|\psi_n|^2)\psi_n$$

where the tunneling rate is

$$K = -\int \mathrm{d}\vec{r}\left[\frac{\hbar^2}{2m}\nabla\phi_n\cdot\nabla\phi_{n+1} + \phi_n V_{\text{ext}}\phi_{n+1}\right]$$

the on-site energies are

$$\epsilon_n = \int \mathrm{d}\vec{r}\left[\frac{\hbar^2}{2m}(\nabla\phi_n)^2 + V_{\text{ext}}\phi_n^2\right]$$

and the nonlinear coefficient U supposed equal in each site is:

$$U = g_0 N_T \int \mathrm{d}\vec{r}\phi_n^4$$

Stationary solutions of the DNLS equation have the form:

$$\psi_n(t) = u_n\exp(-iEt/\hbar)$$

BECs in two-dimensional optical lattices

DNLS equation can also by derived for BECs confined in 2D optical lattices (Kalosakas et al. 2002). Starting from the single-mode boson-Hubbard Hamiltonian:

$$H = -K\sum_{<i,j>} b_i^\dagger b_j + U\sum_i b_i^\dagger b_i^\dagger b_i b_i$$

b_i ($b^\dagger$) is the bosonic annihilation (creation) operator at the i-th trap. $< i,j >$ denotes double summation over nearest neighbours.

A mean field approximation leads to the following DNLS equation for the wave function $\psi_{n,m}$ of the condensate at the n-th trap:

$$i\hbar\frac{\partial\psi_{n,m}}{\partial t} = -K\Delta_2\psi_{n,m} + U|\psi_{n,m}|^2\psi_{n,m}$$

with $\Delta\psi_{n,m} = \psi_{n+1,m} + \psi_{n-1,m} + \psi_{n+1,m} + \psi_{n,m-1} + \psi_{n,m+1}$ being the discrete Laplacian in two dimensions.

Thresholds for discrete solitons in the DNLS equation

We consider the DNLS equation in d dimensions:

$$i\frac{\partial\psi_n}{\partial t} + K(\Delta_d\psi_n - 2\psi_n) - U|\psi_n|^2\psi_n = 0$$

with n being a d-dimensional index. K is positive and the sign of U depends on the repulsive/attractive character of the BEC. For an attractive BEC, $U < 0$, and for a repulsive BEC, $U > 0$. Stationary states are given by:

$$\psi_n(t) = \mathrm{e}^{-i\Lambda t}\psi_n$$

with $U\Lambda > 0$. We have calculated thresholds in the total number of particles of the solitons, also known as the quadratic norm:

$$N_T = \sum_n |\psi_n|^2$$

In other words, the soliton norm must be above a threshold.

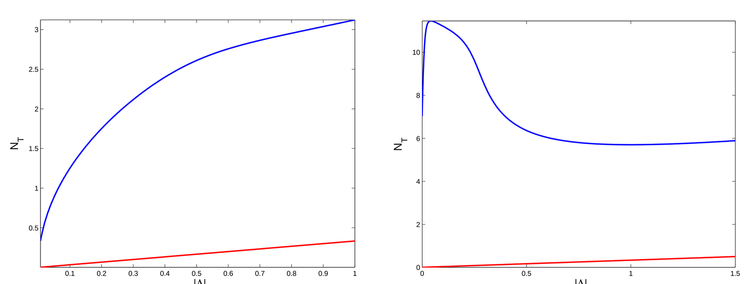
In (Cuevas et al. 2006), these thresholds are calculated using a fixed point argument. For $U > 0$ (Repulsive BEC):

$$\begin{cases} d = 1 & N_T \geq \frac{\Lambda-4K}{2U} \\ d \geq 2 & N_T \geq \frac{\Lambda-4dK}{U} \end{cases}$$

For $U < 0$ (attractive BEC):

$$N_T \geq \frac{\Lambda}{3U}$$

This result is independent of the dimension of the lattice and the coupling constant K. The following figure compares the thresholds analytically calculated with the real soliton norm



Thresholds (red line) and quadratic norm (blue line) for solitons in the cubic DNLS and $U = -1$. Left (right) panel corresponds to the one(two)-dimensional case.

Thresholds for discrete solitons in the cubic-quintic (CQ) DNLS equation

The dynamics of an attractive BEC loaded in an optical lattice can be approximately described by a CQ-DNLS equation if repulsive three-body interactions are taken into account (Michinel et al. 2002).

The CQ-DNLS can be written as:

$$i\frac{\partial\psi_n}{\partial t} + K(\Delta_d\psi_n - 2\psi_n) + U(1 - |\psi_n|^2 + |\psi_n|^4)\psi_n = 0$$

where $U < 0$ holds for attractive BECs and $U > 0$ holds for repulsive BECs. In (Cuevas et al. 2006) the threshold for the quadratic norm in the CQ-DNLS is also calculated by using a fixed point argument. This threshold is:

$$N_T \geq \frac{-3 + \sqrt{29 - 10(\Lambda/U)}}{10}$$

Thresholds for discrete solitons in the DNLS equation with saturable/photorefractive (PR) nonlinearity

Discrete solitons has recently been observed in one-dimensional waveguides arrays (Fleischer et al. 2003) and two-dimensional photonic lattices (Fleischer et al. 2003) made of photorefractive materials as SBN:75.

This kind of optical lattices can be described by the DNLS equation with a saturable nonlinearity term, which we call, for abbreviation, the PR-DNLS equation:

$$i\frac{\partial\psi_n}{\partial t} + K(\Delta_d\psi_n - 2\psi_n) + U\frac{1}{1+|\psi_n|^2}\psi_n = 0$$

Note that this equation can be transformed into the cubic DNLS or the CQ-DNLS equation by means of a Taylor series expansion of the nonlinear term.

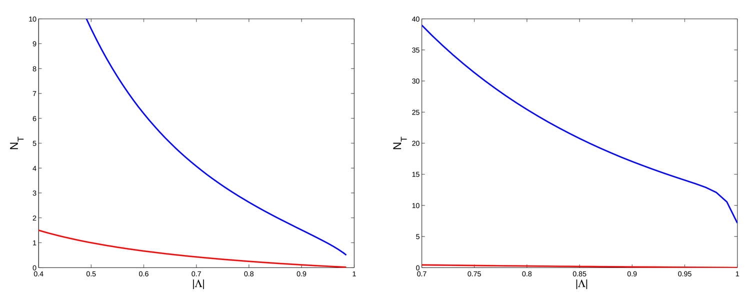
This equation has never been applied to BECs, but it might model the n-body interactions of a condensate.

The main feature of discrete solitons in the PR-DNLS equation is that their quadratic norm does not follows a monotone tendency when Λ is varied, contrary to the cubic DNLS equation where the norm grows when $|\Lambda|$ is increased. This is explained by the fact that, for small norms, the equation behaves as the cubic DNLS but exhibits saturation for higher norms due to the competition with repulsive higher order norms. This property has very important consequences, as we will see below.

Thresholds for discrete solitons in the PR-DNLS, with $U > 0$ and $\Lambda > 0$ are calculated in (Cuevas et al. 2006) by a variational technique and are given by:

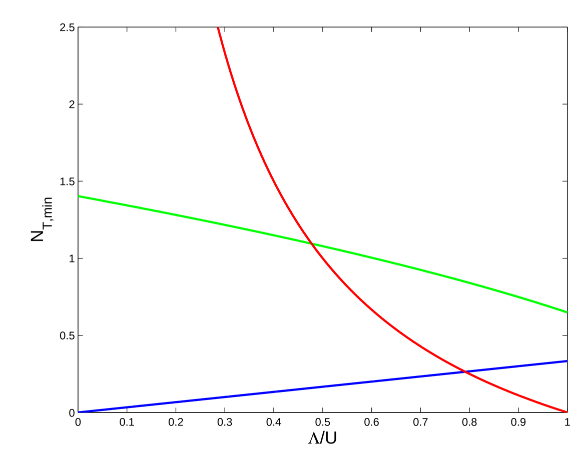
$$N_T \geq \frac{U}{\Lambda} - 1$$

The following figure compares the thresholds analytically calculated with the real soliton norm:



Thresholds (red line) and quadratic norm (blue line) for solitons in the cubic DNLS and $U = -1$. Left (right) panel corresponds to the one(two)-dimensional case.

Comparison between the thresholds in the cubic DNLS, CQ-DNLS and PR-DNLS

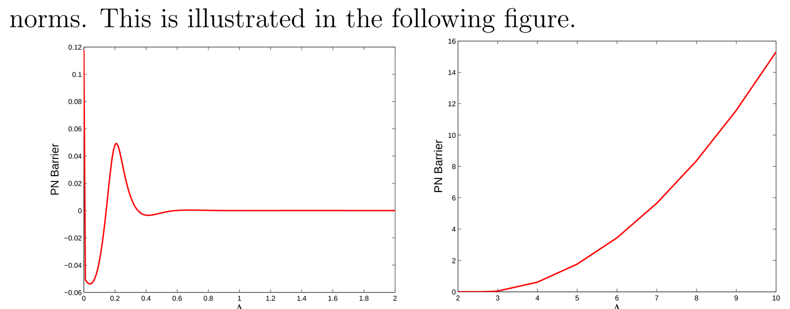


Quadratic norm thresholds for attractive condensates in the cubic DNLS (blue), CQ-DNLS (green) and PR-DNLS (red).

Radiationless travelling waves in the PR-DNLS equation

Contrary to continuum systems, which are translationally invariant, moving discrete solitons are strongly affected by the lattice. In particular, they always radiate linear waves while moving. This radiation is due to the existence of a Peierls-Nabarro (PN) potential (or barrier) whose origin relies in the discreteness of the lattice and whose consequence is the hindrance of the movement and, if this potential is deep enough, can make the movement impossible.

PR-DNLS lattices has the important feature that, for certain parameters range, the PN barrier (defined as the energy difference between a bond-centered and a site-centered soliton with the same norms) is small enough, and, in certain points, it vanishes. This effect is very different to the cubic DNLS where the PN potential has a monotonic dependence and has a high value except at small norms. This is illustrated in the following figure.



Peierls-Nabarro barrier for the PR-DNLS (left) and the cubic DNLS (right). $U = -1$ in both cases.

The figure shows the existence of points where the PN-barrier is zero. These points are *transparent* as the translational invariance of the lattice is restored (they also coincide with an exchange of stability between a site-centered and a bond-centered solution). The existence of these points suggests the possibility of finding travelling solutions without linear modes radiation.

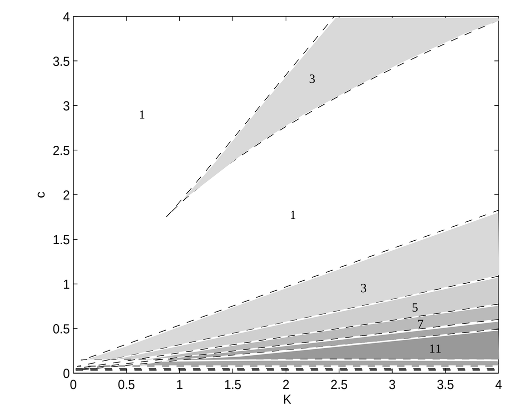
We consider travelling waves as localized moving solutions in the form (Melvin et al. 2006):

$$\psi_n(t) = F(n - ct)\exp(-i\Lambda t), \qquad F(n) = F(n \pm k), \qquad k \in \mathbb{Z}$$

where c is the velocity of the wave. These travelling waves can be considered as embedded solitons (an embedded soliton is, roughly speaking, a soliton dressed with several linear waves). The number of linear waves in the soliton coincides with the number of roots of the following equation in λ:

$$c\lambda + \Lambda - 1 = 4K\sin^2\left(\frac{\lambda}{2}\right)$$

The following figure shows the (c, K) parameter space together with the number of roots

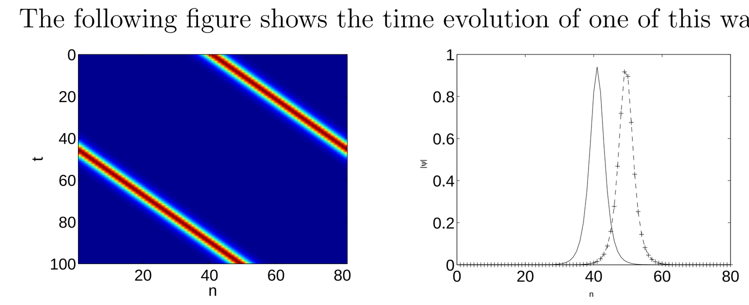


Values indicate the number of roots for $\Lambda = 0.5$. The shaded areas show where there is more than one branch of linear waves.

The only way of obtaining a radiationless travelling wave is that only one linear wave must be embedded. This linear wave is forced to have zero amplitude.

These travelling waves get undressed for high velocities, contrary to the expectation that the radiation is reduced at small velocities.

The following figure shows the time evolution of one of this waves:



Example of direct integration of the solution with $K = 0.911396$ and $c = 0.894153$. The space time contour plot of the solution modulus (left) and the modulus before (solid line) and after (dashed line) 100 time steps are shown.

A preliminary Floquet stability analysis shows that these radiationless exact moving solitons are orbitally linearly stable if c is high enough.

The non-existence of radiation for exact moving solitons has also been observed in the CQ-DNLS.

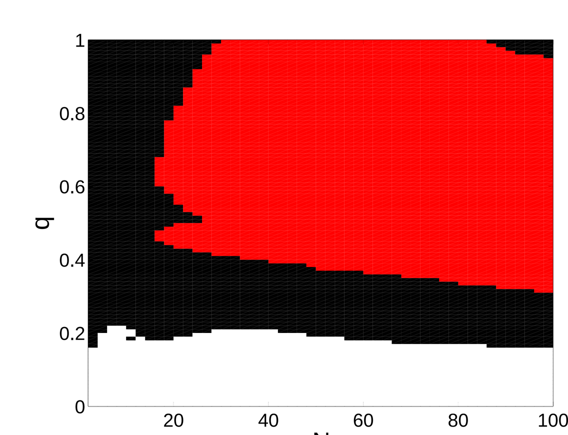
Moving discrete solitons collisions in the PR-DNLS equation

The scenario of binary collisions of two identical moving discrete solitons in the cubic DNLS equation, that was studied in (Papacharalampous et al. 2003) depends on several parameters as the velocity of the incident solitons, their relative phase or the collision point (i.e., on-site or inter-site collisions). Considering only in-phase incoming solitons, the scenario can be roughly simplified to the following one: for small velocities, the solitons merge and remains pinned, creating a breathing state; for high enough velocities, the solitons are reflected. The critical value separating both behaviours is very much higher for on-site collisions due to the PN barrier.

This behaviour is observed in the PR-DNLS equation for low-norm solitons. Thanks to the bounded PN-barrier of PR-DNLS solitons, high-norm moving solutions are allowed (contrary to the cubic DNLS case). When high-norm solutions collide, a new behaviour is observed: the so-called *breather creation* (Cuevas and Eilbeck 2006).

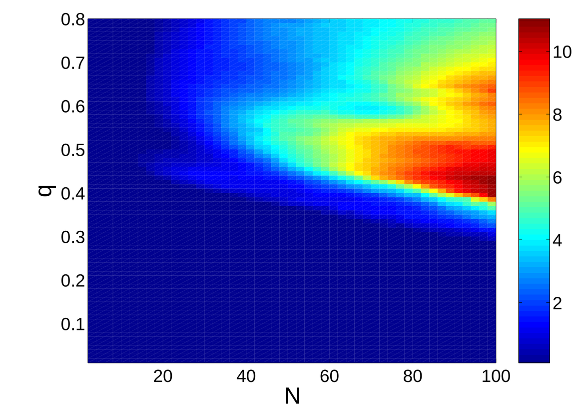
The breather creation consists in a partial trapping of the energy of the incoming solitons together with the reflection of the initial solitons. This effect was previously observed in the continuum PR-NLS equation (Królikowski et al. 2003) and in continuous CQ-NLS models (Cowan et al. 1986). We will focus on inter-site collisions although the results are quite similar for on-site collisions because of the small value of the PN barrier when $U = -1$, which is the value chosen in our analysis.

In the following figure, the parameters range at which each phenomenon takes place, is depicted:



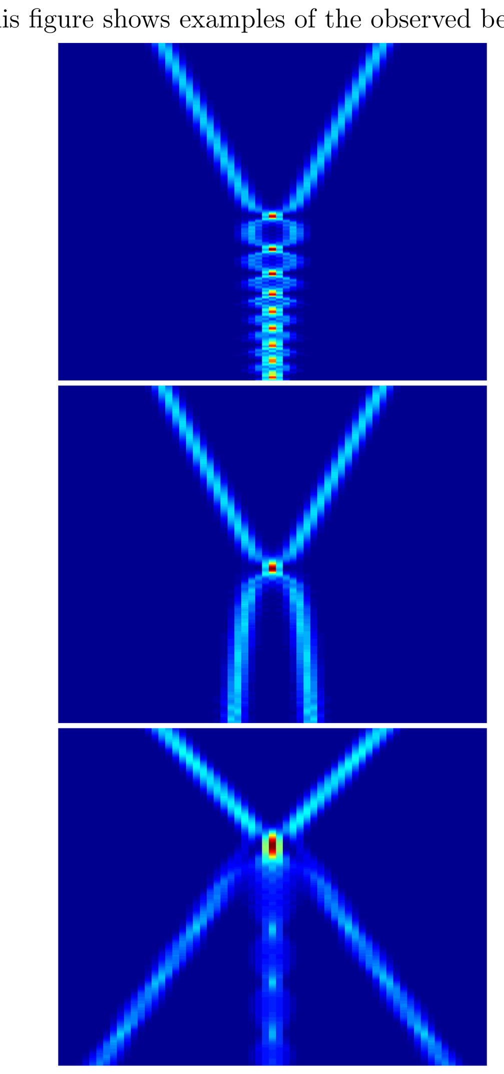
Different regimes observed in the binary collisions. The colours represent the following: white – merge; black – reflection; red – creation.

The trapped energy as a function of the velocity and the norm is shown in the following figure:



Minimum value of the amplitude at the collision point after the collision as a function of the norm and the momentum. We have supposed that the trapped energy in the merge regime is zero in order to clarify the figure.

Finally, this figure shows examples of the observed behaviours:



Energy density plot for the three behaviours observed in binary collisions in the PR-DNLS equation. From top to bottom: merge, reflection and creation.

Conclusions

We have described some novel phenomenon that appear in nonlinear optical lattices described by the DNLS equation with a saturable photorefractive nonlinearity. This phenomena seems to occur also for cubic-quintic nonlinearity.

Although the only kind of DNLS equation that derives from the Gross-Pitaevskii equation is the cubic one, taking into account n-body interaction terms can lead to modifications in the GPE that, at the same time, lead to the CQ and PR-DNLS. Thus, the phenomena observed for saturable nonlinearity could also be observed in Bose-Einstein condensates.

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