The Computation of the Logarithmic Cohomology for Plane Curves

Francisco Jesús Castro-Jiménez and Nobuki Takayama

November 28, 2007, Revised December 13, 2007

Abstract: We will give algorithms of computing bases of logarithmic cohomology groups for square-free polynomials in two variables.

1 Introduction

Let us denote by $R = \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ the polynomial ring, by $A_n = \mathbf{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ the Weyl algebra of order n over the complex numbers $\mathbb{C}$ and by $(\Omega_R^\bullet, d)$ the complex of polynomial (or regular) differential forms (i.e. the complex of differential forms with polynomial coefficients) where d is the exterior derivative.

The elements of A_n are called linear differential operators with polynomial coefficients. An element $P(x, \partial)$ in A_n can be written as a finite sum $P(x, \partial) = \sum_\alpha a_\alpha(x)\partial^\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $a_\alpha(x) \in R$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. Here ∂_i stands for the partial derivative $\frac{\partial}{\partial x_i}$.

For a non zero polynomial $f \in R$ we denote by R_f the ring of rational functions

$$R_f = \{\frac{g}{f^m} \,|\, g \in R, m \in \mathbb{N}\}$$

and by $(\Omega_f^\bullet, d) := (R_f \otimes_R \Omega_R^\bullet, d)$ the complex of rational differential forms with coefficients in R_f where d is the corresponding exterior derivative.

Let us denote by $Der_{\mathbb{C}}(R)$ the free R–module of polynomial vector fields (or equivalently of $\mathbb{C}$-linear derivations of R). Following K. Saito [17] we will denote by $Der_R(-\log f)$ the R–module of logarithmic vector fields with respect to f, i.e.

$$Der_R(-\log f) = \{\delta = \sum_{i=1}^n a_i(x)\partial_i \in Der_{\mathbb{C}}(R) \,|\, \delta(f) \in R \cdot f\}.$$

$Der(-\log f)$ is canonically isomorphic to the R–module $Syz_R(\partial_1(f), \ldots, \partial_n(f), f)$ of syzygies among $(\partial_1(f), \ldots, \partial_n(f), f)$. This isomorphism associates the logarithmic vector field $\delta = \sum_i a_i(x)\partial_i$ with the syzygy $(a_1(x), \ldots, a_n(x), -\frac{\delta(f)}{f})$. We will denote simply $Der(-\log f)$ if no confusion is possible.

If f is a non zero constant, then $Der(-\log f) = Der_{\mathbb{C}}(R)$. So we will assume from now that f is a non constant polynomial in R.

It is clear that

$$fDer_{\mathbb{C}}(R) \subset Der_R(-\log f) \subset Der_{\mathbb{C}}(R)$$

and then $Der(-\log f)$ has rank n as R–module. The R–module $Der_R(-\log f)$ does not depend on the polynomial f but only on the hypersurface $D = \mathcal{V}(f) := \{a \in \mathbb{C}^n \mid f(a) = 0\} \subset \mathbb{C}^n$.

Assume f is reduced (i.e. f is square-free). According to K.Saito [17] a rational differential p-form $\omega \in \Omega_f^p$ is said to be logarithmic with respect to f (or with respect to the hypersurface $D = \mathcal{V}(f) \subset \mathbb{C}^n$) if both $f\omega$ and $fd\omega$ are regular (i.e. $f\omega \in \Omega_R^p$ and $fd\omega \in \Omega_R^{p+1}$). We denote by $\Omega^p(\log f)$ the R–module of logarithmic differential p–forms with respect to f. K. Saito [17, Corollary 1.6] proved that $Der_R(-\log f)$ is a reflexive R–module whose dual is $\Omega^1(\log f)$. We denote by $(\Omega^\bullet(\log f), d)$ the complex

$$0 \longrightarrow \Omega^0(\log f) \xrightarrow{d} \Omega^1(\log f) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(\log f) \longrightarrow 0$$

which will be called the logarithmic de Rham complex and is also, for simple notation, denoted by $\Omega^\bullet(\log f)$ if no confusion arises.

Algorithms of computing dimensions and bases of the de Rham cohomology groups $H^i(\Omega_f^\bullet)$ are given by T.Oaku and N.Takayama [13], [15] and U.Walther [19]. Here, f is any non-zero polynomial in n-variables. The purpose of this paper is to give algorithms of computing dimensions and bases of the logarithmic de Rham cohomology groups $H^i(\Omega^\bullet(\log f))$ as $\mathbb{C}$-vector spaces in the case of two variables.

1.1 Logarithmic Comparison Theorem

The rings R and R_f have natural structures of left A_n–module where ∂_i acts on a polynomial g and on a rational function $\frac{g}{f^m}$ as the partial derivative with respect to x_i.

The de Rham complex of a left A_n–module M, denote by $DR(M)$, is by definition the complex of $\mathbb{C}$–vector spaces $(M \otimes_R \Omega_R^\bullet, \nabla^\bullet)$ where

$$\nabla^p : M \otimes_R \Omega_R^p \to M \otimes_R \Omega_R^{p+1}$$

is defined, for $p \geq 1$, by $\nabla^p(m \otimes \omega) = \nabla^0(m) \wedge \omega + m \otimes d\omega$ and $\nabla^0(m) = \sum_i \partial_i(m) \otimes dx_i$. Note that $am \otimes \omega = m \otimes a\omega$ for $m \in M$, $\omega \in \Omega^p$ and $a \in R$. The complexes $\Omega_f^\bullet$ and $DR(R_f)$ are naturally isomorphic.

For any non zero $f \in R$, the inclusion i_f is a natural morphism of complexes

$$i_f : \Omega^\bullet(\log f) \to \Omega_f^\bullet.$$

We say (see [3]) that f satisfies the (global) logarithmic Comparison Theorem if the morphism i_f is a quasi-isomorphism (i.e. if i_f induces an isomorphism $H^p(\Omega^\bullet(\log f)) \to H^p(\Omega_f^\bullet)$ for any p).

If $n = 2$, by [3, Cor. 2.7] and [2, Th. 1.3], i_f is a quasi-isomorphism if and only if f is a quasi-homogeneous polynomial.

1.2 The case $n = 2$. Bases for $Der_R(-\log f)$

If $n = 2$, any reflexive R–module is projective and then, by Quillen-Suslin theorem, this R–module is free. So, if $n = 2$, the R-module $Der_R(-\log f)$ is free of rank 2. In this case, we would like to compute a basis of $Der_R(\log f)$ by taking the polynomial $f = f(x, y)$ as input. By using the isomorphism

$$Der(-\log f) \simeq Sys_R(\partial_1(f), \partial_2(f), f)$$

and using Groebner basis computation, a system of generators of $Der_R(-\log f)$ can be calculated. Then we can apply Quillen-Suslin algorithm (as presented for example in [8] and implemented in [6]) to compute such a basis. Known Quillen-Suslin algorithms use Groebner bases computation. Nevertheless, in some cases, for a big family of polynomials $f(x_1, x_2)$ we will use an easier way to compute a basis of $Der(-\log f)$.

First of all, we can assume f to be a reduced polynomial since $Der(-\log f)$ depends only on the affine plane curve $D = \mathcal{V}(f) = \{(a_1, a_2) \in \mathbb{C}^2 \mid f(a_1, a_2) = 0\} \subset \mathbb{C}^2$.

Assume the plane curve $D = \mathcal{V}(f)$ is not smooth. The singular points of the plane curve $D = \mathcal{V}(f)$ (i.e. the affine algebraic set

$$Sing(D) := \mathcal{V}(f, f_1, f_2) = \{\underline{a} = (a_1, a_2) \in \mathbb{C}^2 \mid f(\underline{a}) = f_1(\underline{a}) = f_2(\underline{a}) = 0\})$$

–where $f_1 = \partial_1(f)$, $f_2 = \partial_2(f)$ – consists in a finite number of points (and it is not the empty set).

We will consider the affine plane $\mathbb{C}^2$ as a Zariski open subset of the projective plane $\mathbb{P}_2(\mathbb{C})$, the affine point (a_1, a_2) is mapped into the point with homogeneous coordinates $(1 : a_1 : a_2)$. Coordinates in $\mathbb{P}_2(\mathbb{C})$ will be denoted by $(x_0 : x_1 : x_2)$ and then the line at infinity is defined by $x_0 = 0$.

Let us denote $h = H(f), h_1 = H(f_1)$ and $h_2 = H(f_2)$ where $H(-)$ denotes dehomogenization with respect to the variable x_0. We will denote by $Z = \mathcal{V}_{\mathbb{P}}(h, h_1, h_2) \subset \mathbb{P}_2(\mathbb{C})$ (resp. $Z' = \mathcal{V}(h, h_1, h_2) \subset \mathbb{C}^3$) the projective algebraic set (resp. the affine algebraic set) defined by the polynomials h, h_1, h_2. The non-empty set Z (resp. Z') consists of a finite number of points in $\mathbb{P}_2(\mathbb{C})$ (resp. a finite number of straight lines in $\mathbb{C}^3$). Denote by $S = \mathbb{C}[x_0, x_1, x_2]$ the ring of polynomials graded by the degree of the polynomials. If $J = (h, h_1, h_2)$ denotes the ideal in S generated by h, h_1, h_2 then the quotient ring S/J has Krull dimension 1. Let us denote by S_+ the irrelevant ideal in S, i.e. the ideal generated by x_0, x_1, x_2.

Proposition 1.1 *The graded ring S/J is Cohen-Macaulay if and only if J is unmixed (i.e. S_+ is not an embedded prime associated with J).*

Proof: If S/J is Cohen-Macaulay then J is unmixed (see [9]). If J is unmixed then S_+ is not an embedded prime of J and then the set of non zero-divisors of S/J contains homogeneous elements of positive degree. That proves $depth(S/J) \geq 1$ but we also have $\mathrm{depth}(S/J) \leq \dim(S/J) = 1$. $\Box$

If S/J is Cohen-Macaulay then the projective dimension of S/J is 2 and J satisfies the Hilbert-Burch Theorem [5], i.e. there exists an exact sequence

$$0 \to S^2 \xrightarrow{\phi_2} S^3 \xrightarrow{\phi_1} J \to 0$$

where $\phi_1(g_0, g_1, g_2) = g_0 h + g_1 h_1 + g_2 h_2$ and ϕ_2 is defined by a syzygy matrix of ϕ_1. In particular, since $\ker(\phi_1) = Syz_S(h, h_1, h_2)$ is a graded free S–module of rank 2 we can compute $\{s^{(1)} = (s_{10}, s_{11}, s_{12}), s^{(2)} = (s_{20}, s_{21}, s_{22})\}$ a minimal system of generators and this system is in fact a basis of $\ker(\phi_1)$. By dehomogenization (i.e. by setting $x_0 = 1$), we obtain a system $\{s^{(1)}_{|x_0=1}, s^{(2)}_{|x_0=1}\}$ of generators of $Syz_R(f, f_1, f_2) \simeq Der_R(-\log f)$ and since this R–module is free of rank 2, this last system is in fact a basis.

If S/J is not Cohen-Macaulay we cannot apply, in general, the Hilbert-Burch theorem and the previous procedure fails to compute a basis of $Der_R(-\log f)$.

Example 1.2 (a) Consider the polynomial $f = (x^3 + y^4 + xy^3)(x^2 - y^2)$. With the notations as before (and writing $x_1 = x, x_2 = y, x_0 = t$) we can use `Macaulay 2` to prove that the corresponding S/J is Cohen-Macaulay and to compute a minimal system of generators of $Syz_S(h, h_1, h_2)$ and then a basis of $Der_R(-\log f)$.

```
Macaulay 2, version 0.9.2
--Copyright 1993-2001, D. R. Grayson and M. E. Stillman
--Singular-Factory 1.3b, copyright 1993-2001, G.-M. Greuel, et al.
--Singular-Libfac 0.3.2, copyright 1996-2001, M. Messollen

i1 : R=QQ[t,x,y];

i2 : f=(x^3+y^4+x*y^3)*(x^2-y^2);

i3 : f1=diff(x,f),f2=diff(y,f),h=homogenize(f,t),h1=homogenize(f1,t),h2=homogenize(f2,t);

i4 : Jf=ideal(h,h1,h2);

o4 : Ideal of R

i5 : pdim coker gens Jf

o5 = 2

i6 : Syzf=kernel matrix({{h1,h2,h}})

o6 = image {5} | x3+1/3x2y-4/3xy2 -tx2+4txy+3x2y+4xy2-y3 |
           {5} | 2/3x2y+1/3xy2-y3 tx2-txy+3ty2+2xy2+4y3  |
           {6} | -5x2-5/3xy+6y2   5tx-18ty-15xy-23y2     |

                                 3
o6 : R-module, submodule of R

i7 : mingens Syzf

o7 = {5} | x3+1/3x2y-4/3xy2 -tx2+4txy+3x2y+4xy2-y3 |
     {5} | 2/3x2y+1/3xy2-y3 tx2-txy+3ty2+2xy2+4y3  |
     {6} | -5x2-5/3xy+6y2   5tx-18ty-15xy-23y2     |
```

```
                 3       2
o7 : Matrix R  <--- R
```

Then the basis of $Der_R(-\log f)$ is

$\{(x^3+\frac{1}{3}x^2y-\frac{4}{3}xy^2)\partial_x+(\frac{2}{3}x^2y+\frac{1}{3}xy^2-y^3)\partial_y,\ (-x^2+4xy+3x^2y+4xy^2-y^3)\partial_x+(x^2-xy+3y^2+2xy^2+4y^3)\partial_y\}$

(b) Consider the polynomial $g=(x^3+y^4+xy^3)(x^2+y^2)$. With the notations as before (and writing $x_1=x, x_2=y, x_0=t$) we can use `Macaulay 2` to prove that the corresponding S/J is not Cohen-Macaulay and the minimal number of generators of $Syz_S(h,h_1,h_2)$ is 3. We can continue the last `Macaulay 2` session:

```
i8 : g=(x^3+y^4+x*y^3)*(x^2+y^2);

i9 : g1=diff(x,g),g2=diff(y,g),h=homogenize(g,t),h1=homogenize(g1,t),h2=homogenize(g2,t);

i10 : Jg=ideal(h,h1,h2);

i11 : pdim coker gens Jf

o11 = 3

i12 : Syzg=kernel matrix({{h1,h2,h}})

o12 =
image
{5} | tx2-5x3-4txy-20/3x2y-2xy2-5/3y3  x4+4/3x3y+x2y2+4/3xy3    tx3-tx2y+4x3y+4txy2+16/3x2y2+2xy3+4/3y4 |
{5} | tx2+txy-10/3x2y-3ty2-5xy2-1/3y3  2/3x3y+x2y2+2/3xy3+y4    -txy2+8/3x2y2+3ty3+4xy3+2/3y4           |
{6} | -5tx+25x2+18ty+100/3xy+11/3y2    -5x3-20/3x2y-13/3xy2-6y3 -5tx2+5txy-20x2y-18ty2-80/3xy2-16/3y3   |

                                 3
o12 : R-module, submodule of R

i13 : mingens Syzg

o13 =
{5} | tx2-5x3-4txy-20/3x2y-2xy2-5/3y3  x4+4/3x3y+x2y2+4/3xy3    tx3-tx2y+4x3y+4txy2+16/3x2y2+2xy3+4/3y4 |
{5} | tx2+txy-10/3x2y-3ty2-5xy2-1/3y3  2/3x3y+x2y2+2/3xy3+y4    -txy2+8/3x2y2+3ty3+4xy3+2/3y4           |
{6} | -5tx+25x2+18ty+100/3xy+11/3y2    -5x3-20/3x2y-13/3xy2-6y3 -5tx2+5txy-20x2y-18ty2-80/3xy2-16/3y3   |

                  3       3
o13 : Matrix R  <--- R
```

We will revisit this example in Example 4.1.

2 Logarithmic A_n–modules

Let us denote by $M^{\log f}$ the quotient A_n–module $M^{\log f}=\frac{A_n}{A_n Der_R(-\log f)}$. Moreover, we denote by $\widetilde{Der}_R(-\log f)$ the set

$$\widetilde{Der}_R(-\log f)=\{\delta+\frac{\delta(f)}{f}\mid \delta\in Der_R(-\log f)\}$$

and by $\widetilde{M}^{\log f}$ the quotient A_n–module

$$\widetilde{M}^{\log f} = \frac{A_n}{A_n \widetilde{Der}_R(-\log f)}.$$

As quoted in subsection 1.2, for $n = 2$ the R–module $Der(-\log f)$ (and hence $\Omega^1(\log f)$) is free of rank 2. Moreover, by [17, 1.8] there exists a R-basis $\{\delta_1, \delta_2\}$ of $Der(-\log f)$ satisfying $\det(A) = f$ where

$$\delta_i = a_{i1}\partial_1 + a_{i2}\partial_2, \ \ i = 1, 2$$

and A is the matrix (a_{ij}). Then the dual basis of $\{\delta_1, \delta_2\}$ is $\{\omega_1, \omega_2\}$ with

$$\omega_1 = \frac{1}{f}(a_{22}dx_1 - a_{21}dx_2) \ \ \omega_2 = \frac{1}{f}(-a_{12}dx_1 + a_{11}dx_2).$$

The R–module $\Omega^2(\log f)$ is free of rank 1 and $\omega_1 \wedge \omega_2$ is a basis of it. Moreover we have $\omega_1 \wedge \omega_2 = \frac{dx_1 \wedge dx_2}{f}$.

Proposition 2.1 *Let $f \in R = \mathbb{C}[x, y]$ be a non zero reduced polynomial. There exists a natural quasi-isomorphism*

$$\Omega^\bullet(\log f) \xrightarrow{\simeq} \mathbf{R}Hom_{A_2}(M^{\log f}, R)$$

where the last complex is the solution complex of $M^{\log f}$ with values in R.

This Proposition is proven in [1] in a more general setting using the notion of V_0–module. We will give here a direct proof to apply for our algorithm of computing logarithmic cohomology groups.
Proof: F.J. Calderón [1] defines the so called *logarithmic Spencer complex* associated with $M^{\log f}$. In our situation, once a basis $\{\delta_1, \delta_2\}$ is fixed in $Der(-\log f)$, this complex is nothing but

$$0 \to A \xrightarrow{\epsilon_2} A^2 \xrightarrow{\epsilon_1} A \to 0 \tag{1}$$

where A stands for A_2, the A-module morphism ϵ_1 is defined by $\epsilon_1(P_1, P_2) = P_1\delta_1 + P_2\delta_2$ (for $P_i \in A$) and ϵ_2 is defined by $\epsilon_2(Q) = Q(-\delta_2 - b_1, \delta_1 - b_2)$ for $Q \in A$ and the polynomials b_i being defined by the equality $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1 = b_1\delta_1 + b_2\delta_2$. In [1] it is proven that this complex is a A–free resolution of the module $M^{\log f}$. We will use this resolution to find a complex of $\mathbb{C}$–vector spaces representing the solution complex $\mathbf{R}Hom_A(M^{\log f}, R)$. Applying the functor $Hom_A(-, R)$ to the logarithmic Spencer complex and using the natural isomorphism $R \simeq Hom_A(A, R)$, we obtain the complex

$$0 \to R \xrightarrow{\epsilon_1^*} R^2 \xrightarrow{\epsilon_2^*} R \to 0$$

where $\epsilon_1^*(g) = (\delta_1(g), \delta_2(g))$ for $g \in R$ and $\epsilon_2^*(h_1, h_2) = \delta_1(h_2) - \delta_2(h_1) - b_1h_1 - b_2h_2$ for $h_i \in R$. There is a natural morphism of complexes

$$\begin{array}{ccccc} \Omega^0(\log f) = R & \stackrel{d}{\to} & \Omega^1(\log f) & \stackrel{d}{\to} & \Omega^2(\log f) \\ \eta_0 \downarrow & & \eta_1 \downarrow & & \eta_2 \downarrow \\ R & \stackrel{\epsilon_1^*}{\longrightarrow} & R^2 & \stackrel{\epsilon_2^*}{\longrightarrow} & R \end{array}$$

where $\eta_0 = id$, $\eta_1(h_1\omega_1 + h_2\omega_2) = (h_1, h_2)$ and $\eta_2(g\omega_1 \wedge \omega_2) = g$ for $h_1, h_2, g \in R$ and where $\{\omega_1, \omega_2\}$ is the dual basis in $\Omega^1(\log f)$ of the basis $\{\delta_1, \delta_2\}$ in $Der(-\log f)$. It is obvious that this morphism $\eta_\bullet$ of complexes of vector spaces is in fact an isomorphism of complexes. That proves the proposition. []

To each finitely generated left A_n–module M we associate the complex of finitely generated right A_n–modules $\mathbf{R}Hom_{A_n}(M, A_n)$. To this one we associate the complex of finitely generated left A_n–modules $Hom_R(\Omega^n_R, \mathbf{R}Hom_{A_n}(M, A_n))$ which is by definition the dual M^* of the left A_n–module M.

If M is holonomic (i.e. if the dimension of the characteristic variety of M is n) then it can be shown that $Ext^i_{A_n}(M, A_n) = 0$ for $i \neq n$ and then M^* is the left holonomic A_n–module $Hom_R(\Omega^n_R, Ext^n_{A_n}(M, A_n))$ (see e.g. [10, pag. 41]). Assume $Ext^n_{A_n}(M, A_n) = \frac{A_n}{J}$ for some right ideal $J \subset A_n$. Then $Hom_R(\Omega^n_R, A_n/J)$ is naturally isomorphic to the left A_n–module $\frac{A_n}{J^T}$ where J^T is the left ideal $J^T = \{P^T \mid P \in J\}$ and P^T is the formal adjoint of the operator P.

If N_1, N_2 are finitely generated left A_n–modules there exists a natural isomorphism of complexes

$$\mathbf{R}Hom_{A_n}(N_1, N_2) \to \mathbf{R}Hom_{A_n}(\mathbf{R}Hom_{A_n}(N_2, A_n), \mathbf{R}Hom_{A_n}(N_1, A_n))$$

and then a natural isomorphism

$$\mathbf{R}Hom_{A_n}(N_1, N_2) \to \mathbf{R}Hom_{A_n}(N_2^*, N_1^*).$$

In particular, if $N_2 = R = \mathbb{C}[x_1, \ldots, x_n]$ then there exists a natural isomorphism from $\mathbf{R}Hom_{A_n}(N_1, R)$ (i.e. the solution complex of N_1) to

$$\mathbf{R}Hom_{A_n}(R^*, N_1^*).$$

As the complex $\mathbf{R}Hom_{A_n}(R, A_n)$ is naturally isomorphic to Ω^n_R we can identify R and R^* and then we have a natural isomorphism

$$\mathbf{R}Hom_{A_n}(N_1, R) \stackrel{\simeq}{\to} \mathbf{R}Hom_{A_n}(R, N_1^*) \stackrel{\simeq}{\to} DR(N_1^*). \tag{2}$$

Proposition 2.2 *Let $f \in \mathbb{C}[x, y]$ be a non zero reduced polynomial. Then there exists a natural isomorphism*

$$(M^{\log f})^* \simeq \widetilde{M}^{\log f}.$$

Proof: This is one of the main results in [4]. We include here its proof for the sake of completeness. First of all, both A_2–modules $M^{\log f}$ and $\widetilde{M}^{\log f}$ are holonomic. That can be deduced from [1, Cor. 4.2.2] since the set of principal symbols $\{\sigma(\delta_1), \sigma(\delta_2)\}$ is a regular sequence in the polynomial ring $R[\xi_1, \xi_2]$ and then the Krull dimension of the quotient ring

$$R[\xi_1, \xi_2]/\langle\sigma(\delta_1), \sigma(\delta_2)\rangle$$

is 2. Then the characteristic variety of both A_2–modules $M^{\log f}$ and $\widetilde{M}^{\log f}$ has dimension 2 and the modules are holonomic.

We will use the logarithmic Spencer complex associated with $M^{\log f}$ (see the complex (1)) in order to compute $Ext^2_A(M, A)$ where $A = A_2$ and $M = M^{\log f}$. Applying the functor $Hom_{A_2}(-, A_2)$ to the complex (1) we get (by using the natural isomorphism $Hom_{A_2}(A_2, A_2) \simeq A_2$)

$$0 \longrightarrow A \xrightarrow{\overline{\epsilon_1}} A^2 \xrightarrow{\overline{\epsilon_2}} A \longrightarrow 0$$

where $\overline{\epsilon_1}(P) = (\delta_1 P, \delta_2 P)$ and $\overline{\epsilon_2}(P_1, P_2) = (-\delta_2 - b_1)P_1 + (\delta_1 - b_2)P_2$. Then we have

$$Ext^2_A(M, A) \simeq \frac{A}{(-\delta_2 - b_1, \delta_1 - b_2)A}.$$

So,

$$M^* \simeq \frac{A}{A((-\delta_2 - b_1)^T, (\delta_1 - b_2)^T)}.$$

Finally, $(-\delta_2 - b_1)^T = \delta_2 + \frac{\delta_2(f)}{f}$ and $(\delta_1 - b_2)^T = -\delta_1 - \frac{\delta_1(f)}{f}$ (see [4, Cor. 3.1]). $\square$

Theorem 2.3 *For any non zero reduced polynomial $f \in \mathbb{C}[x, y]$, the complexes $\Omega^\bullet(\log f)$ and $DR(\widetilde{M}^{\log f})$ are naturally quasi-isomorphic.*

As a consequence of this theorem and by [13], [15] and [19], the cohomology of the complex $\Omega^\bullet(\log f)$ can be computed starting with the given polynomial f, since a system of generators of the R-module $\widetilde{Der}_R(-\log f)$ can be computed using the R–syzygies of $(\partial_1(f), \partial_2(f), f)$.

Proof: Let us simply denote $R = \mathbb{C}[x, y]$, $A = A_2$, $M = M^{\log f}$, $\widetilde{M} = \widetilde{M}^{\log f}$.

By Proposition 2.1 there exists a natural isomorphism

$$\Omega^\bullet(\log f) \xrightarrow{\simeq} \mathbf{R}Hom_A(M, R)$$

and by equation (2) there exists a natural isomorphism

$$\mathbf{R}Hom_A(M, R) \xrightarrow{\simeq} DR(M^*).$$

By Proposition 2.2 we have $DR(M^*) \simeq DR(\widetilde{M})$.

We can give the explicit form of this quasi-isomorphism of complexes $\tau^\bullet$: $\Omega^\bullet(\log f) \to DR(\widetilde{M})$.

$\tau^0 : R \to \widetilde{M}$ is defined by $\tau^0(g) = \overline{gf}$ where $\overline{()}$ means the equivalent class in the ideal $A_2\widetilde{Der}_R(\log f)$.

$\tau^1 : \Omega^1(\log f) \to \widetilde{M} \otimes_R \Omega^1_R$ is defined by

$$\tau^1(c_1\omega_1 + c_2\omega_2) = \sum_i \overline{c_i} \otimes f\omega_i.$$

$\tau^2 : \Omega^2(\log f) \to \widetilde{M} \otimes_R \Omega^2_R$ is defined by $\tau^2(g\omega_1 \wedge \omega_2) = \overline{g} \otimes f\omega_1 \wedge \omega_2$. $\square$

3 Algorithm

Let us summarize our algorithm of computing logarithmic cohomology groups in the two dimensional case. Most tensor products $\otimes$ in the sequel are over A_2. If we omit the subscript A_2 for $\otimes$, it means that the tensor product is over A_2.

Algorithm 3.1
Input: a non zero reduced polynomial $f(x,y)$
Output: dimensions and bases of $H^i(\Omega^\bullet(\log f))$.

1. Compute a free basis $s = (s_0, s_1, s_2)$ and $t = (t_0, t_1, t_2)$ of the syzygy module of f, f_x, f_y over the polynomial ring $\mathbf{C}[x,y]$. This step can be performed by the following way.
   (a) Compute the minimal syzygy of $h(f)$, $h(f_x)$, $h(f_y)$. Here, $h(g)$ is the homogenization of g. If the number of generators is 2, then the dehomogenizations of these generators are s and t.
   (b) If we fail on the first step, apply an algorithm for the Quillen-Suslin theorem to obtain s and t (call the procedure Quillen-Suslin).
2. Define a left ideal in A_2 by
   $$I = A_2 \cdot \{-s_0 + s_1\partial_x + s_2\partial_y, -t_0 + t_1\partial_x + t_2\partial_y\}. \tag{3}$$
   Compute the dimensions and bases of the de Rham cohomology groups for $\widetilde{M} = A_2/I$ with the algorithm in [13], [15]. In other words, replace the A_2-module $\mathbf{C}[x,y,1/f]$ by A_2/I of (3) in the algorithm 1.2 in [13].
3. The bases of the previous step are given in $A_2/(\partial_x A_2 + \partial_y A_2) \otimes \widetilde{M}^\bullet$ where $\widetilde{M}^\bullet$ is $(1,1,-1,-1)$-adaptive free resolution of $\widetilde{M}$. Bases of de Rham cohomology groups in $\Omega^\bullet \otimes \widetilde{M} \simeq_{q.i.s} DR(\widetilde{M}) \simeq_{q.i.s} \Omega^\bullet(\log f)$ are determined by the transfer algorithm of U.Walther [19, Theorem 2.5 (Transfer Theorem)] and the correspondence τ^i given in our Theorem 2.3. Here, $\Omega^\bullet$ is the Koszul resolution of the right A_2-module $A_2/(\partial_x A_2 + \partial_y A_2)$.

In the first step, we should firstly try to find the minimal syzygy. Because, mostly it is faster than applying implementations and algorithms for the Quillen-Suslin theorem.

The following example will illustrate how our algorithm works.

Example 3.2 We consider the case of $f = xy(x-y)$. Two canonical generators of $I = \widetilde{Der}_R(\log f)$ are

$$\ell_1 = 3 + x\partial_x + y\partial_y, \ \ell_2 = -(2x-y) + (-x^2+xy)\partial_x$$

The associated canonical logarithmic forms are

$$\omega_1 = \frac{1}{f}x(x-y)dy, \ \omega_2 = \frac{1}{f}(-ydx + xdy)$$

Let us proceed on the step 2. We apply the procedure of computing the de Rham cohomology groups [13], [16] for A_2/I. The maximal integral root of the b function for $I = A_2 \cdot \{\ell_1, \ell_2\}$ with respect to the weight $(1,1,-1,-1)$ is 1. The dehomogenization of the $(1,1,-1,-1)$-minimal filtered free resolution of A_2/I is

$$A^\bullet : \qquad A_2[0] \xrightarrow{a^{-2}} A_2[1] \oplus A_2[0] \xrightarrow{a^{-1}} A_2[1] \tag{4}$$

where

$$\begin{array}{rcl} a^{-2}(c) & = & c(-\ell_2, \ell_1 - 1) \quad \text{for } c \in A_2 \\ a^{-1}(c,d) & = & (c,d)\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \quad \text{for } (c,d) \in A_2[1] \oplus A_2[0] \end{array}$$

Following [13, procedure 1.8], we truncate the complex $A_2/(\partial_x A_2 + \partial_y A_2) \otimes_{A_2} A^\bullet$ to the forms of $(1,1,-1,-1)$-degree at most 1 since the maximal integral root of the b-function is 1. The truncated complex is the following complex of finite dimensional vector spaces

$$\mathbf{C} \xrightarrow{\bar{a}^{-2}} (\mathbf{C} + \mathbf{C}x + \mathbf{C}y) \oplus \mathbf{C} \xrightarrow{\bar{a}^{-1}} (\mathbf{C} + \mathbf{C}x + \mathbf{C}y) \xrightarrow{\bar{a}^{0}} 0 \tag{5}$$

Here,

$$\begin{array}{rcl} \bar{a}^{-2}(1) & = & (-\ell_2, \ell_1 - 1) \bmod \partial_x A_2 + \partial_y A_2 \\ & = & (0,0) \\ \bar{a}^{-1}(a + bx + cy, d) & = & (a + bx + cy)\ell_1 + d\ell_2 \bmod \partial_x A_2 + \partial_y A_2 \\ & = & a \end{array}$$

Therefore, the cohomology groups $H^i(A_2/(\partial_x A_2 + \partial_y A_2) \otimes A^\bullet)$ are

$$\begin{array}{rcl} H^0(A_2/(\partial_x A_2 + \partial_y A_2) \otimes A^\bullet) & = & \mathrm{Ker}\, \bar{a}^{-2} = \mathbf{C} \\ H^1(A_2/(\partial_x A_2 + \partial_y A_2) \otimes A^\bullet) & = & \mathrm{Ker}\, \bar{a}^{-1}/\mathrm{Im}\, \bar{a}^{-2} = (\mathbf{C}x + \mathbf{C}y) \oplus \mathbf{C} \\ H^2(A_2/(\partial_x A_2 + \partial_y A_2) \otimes A^\bullet) & = & \mathrm{Ker}\, \bar{a}^{0}/\mathrm{Im}\, \bar{a}^{-1} = \mathbf{C}x + \mathbf{C}y \end{array}$$

Finally, we perform the step 3. Put $\widetilde{M} = A_2/I$. In order to give bases of the cohomology groups in $\widetilde{M} \otimes_R \Omega_R^i$, we apply the transfer theorem (algorithm) of Uli Walther [19].

We consider the following double complex (c.f., 2.4 of [19]).

$$
\begin{tikzcd}
\Omega(2)\otimes A_2 \arrow[r, "1\otimes a^{-2}"'] & \Omega(2)\otimes(A_2\oplus A_2) \arrow[r, "1\times a^{-1}"'] & \Omega(2)\otimes A_2 & \\
A_2\otimes A_2 \arrow[u] \arrow[r, "\alpha^{2,-2}"] & A_2\otimes(A_2\oplus A_2) \arrow[u] \arrow[r, "\alpha^{2,-1}"] & A_2\otimes A_2 \arrow[u] \arrow[r] & A_2\otimes\widetilde{M} \\
\begin{pmatrix} A_2 \\ \oplus \\ A_2 \end{pmatrix}\otimes A_2 \arrow[u, "\varepsilon^{1,-2}"] \arrow[r, "\alpha^{1,-2}"] & \begin{pmatrix} A_2 \\ \oplus \\ A_2 \end{pmatrix}\otimes(A_2\oplus A_2) \arrow[u, "\varepsilon^{1,-1}"] \arrow[r, "\alpha^{1,-1}"] & \begin{pmatrix} A_2 \\ \oplus \\ A_2 \end{pmatrix}\otimes A_2 \arrow[u, "\varepsilon^{1,0}"] \arrow[r] & \begin{pmatrix} A_2 \\ \oplus \\ A_2 \end{pmatrix}\otimes\widetilde{M} \arrow[u, "\varepsilon^{1,-2}"] \\
A_2\otimes A_2 \arrow[u, "\varepsilon^{0,-2}"] \arrow[r, "\alpha^{0,-2}"] & A_2\otimes(A_2\oplus A_2) \arrow[u, "\varepsilon^{0,-1}"] \arrow[r, "\alpha^{0,-1}"] & A_2\otimes A_2 \arrow[u, "\varepsilon^{0,0}"] \arrow[r] & A_2\otimes\widetilde{M} \arrow[u, "\varepsilon^{0,-2}"]
\end{tikzcd}
$$

Here we denote $A_2/(\partial_x A_2+\partial_y A_2)$ by $\Omega(2)$, which is isomorphic to Ω_R^2 as the right A_2-module. The vertical complex is constructed by the Koszul resolution of $\Omega(2)$ as the right module denoted by $\Omega^\bullet$. The horizontal complex is constructed by $A^\bullet$. Note that we have the following maps in the complex:

$$
\begin{aligned}
\varepsilon^{1,-2}((a,b)\otimes c) &= (-\partial_y a+\partial_x b)\otimes c \\
\varepsilon^{0,-2}(a\otimes c) &= (\partial_x a,\partial_y a)\otimes c
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^{1,-1}((a,b)\otimes(c,d)) &= (\partial_y a-\partial_x b)\otimes(c,d) \\
\varepsilon^{0,-1}(a\otimes(c,d)) &= (\partial_x a,\partial_y a)\otimes(c,d)
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^{1,0}((a,b)\otimes c) &= (-\partial_y a+\partial_x b)\otimes c \\
\varepsilon^{0,0}(a\otimes c) &= (\partial_x a,\partial_y a)\otimes c
\end{aligned}
$$

$$
\begin{aligned}
\alpha^{2,-2}(a\otimes c) &= a\otimes c(-\ell_2,\ell_1-1) \\
\alpha^{2,-1}(a\otimes(c,d)) &= a\otimes(c\ell_1+d\ell_2)
\end{aligned}
$$

$$
\begin{aligned}
\alpha^{1,-2}((a,b)\otimes c) &= (a,b)\otimes c(-\ell_2,\ell_1-1) \\
\alpha^{1,-1}((a,b)\otimes(c,d)) &= (a,b)\otimes(c\ell_1+d\ell_2)
\end{aligned}
$$

$$
\begin{aligned}
\alpha^{0,-2}(a\otimes c) &= a\otimes c(-\ell_2,\ell_1-1) \\
\alpha^{0,-1}(a\otimes(c,d)) &= a\otimes(c\ell_1+d\ell_2)
\end{aligned}
$$

The last vertical complex is quasi isomorphic to $DR(\widetilde{M})$. Let us compute transfers. Two cohomology classes x and y in $\operatorname{Ker}\bar{a}^0\subset\Omega(2)\otimes A_2$ are lifted to $1\otimes x$

and $1 \otimes y$ in $A_2 \otimes A_2$ respectively, and we push them to $A_2 \otimes \widetilde{M}$. It follows from the definition of τ^2, $x\omega_1 \wedge \omega_2$ and $y\omega_1 \wedge \omega_2$ is the basis of $H^2(\Omega(\log f)^\bullet)$.

Let us compute transfers of bases of $H^1(\Omega(2) \otimes A^\bullet)$. The cohomology class $1 \otimes (x, 0)$ in $\mathrm{Ker}\, \bar{a}^1$ are lifted to $1 \otimes (x, 0)$ in $A_2 \otimes (A_2 \oplus A_2)$. We have $\alpha^{2,-1}(1 \otimes (x, 0)) = 1 \otimes x\ell_1$. Solving $-\partial_y a + \partial_x b = x\ell_1$ in A_2, we obtain the preimage by $\varepsilon^{1,0}$; we have $\varepsilon^{1,0}((-xy, x^2) \otimes 1) = x\ell_1$. Push this element to $\begin{pmatrix} A_2 \\ \oplus \\ A_2 \end{pmatrix} \otimes \widetilde{M}$, we obtain $(xydx - x^2dy) \otimes 1$. Let us compute the preimage by τ^1. Solving $c_1 f\omega_1 + c_2 f\omega_2 = xydx - x^2dy$, we obtain $c_1 = 0, c_2 = -x$. Therefore, $1 \otimes (x, 0)$ stands for $-x\omega_2$. Analogously, $1 \otimes (y, 0)$ is transfered to $-y^2dx + xydy$ and stands for $-y\omega_2$ and $1 \otimes (0, 1)$ is transfered to $x(y - x)dy$ and stands for ω_1. In summary,

$$H^1(\Omega(\log f)^\bullet) = \mathbf{C}(-x)\omega_2 + \mathbf{C}(-y)\omega_2 + \mathbf{C}\omega_1.$$

Finally, we compute transfers of bases of $H^0(\Omega(2) \otimes A^\bullet)$. Since $\alpha^{2,-2}(1 \otimes 1) = 1 \otimes (-\ell_2, \ell_1 - 1)$, we firstly need to compute the preimage of this element by $\varepsilon^{1,-1}$. Since the projection of this element to $\Omega(2) \otimes (A_2 \oplus A_2)$ is zero, we have $-\ell_2 = \partial_x x(x - y)$ and $\ell_1 - 1 = \partial_x x + \partial_y y$. We decompose $1 \otimes (-\ell_2, \ell_1 - 1)$ as

$$\begin{aligned} 1 \otimes (-\ell_2, 0) + 1 \otimes (0, \ell_1 - 1) &= -\ell_2 \otimes (1, 0) + (\ell_1 - 1) \otimes (0, 1) \\ &= \partial_x x(x - y) \otimes (1, 0) + (\partial_x x + \partial_y y) \otimes (0, 1) \end{aligned}$$

Since $\varepsilon^{1,-1}$ is linear, this sum is equal to $\varepsilon^{1,-1}(c)$ where $c_1 = (0, -x(x - y)) \otimes (1, 0) + (y, -x) \otimes (0, 1)$. Since $\alpha^{1,-1}(c_1) = (y\ell_2, -x(x - y)\ell_1 - x\ell_2) \otimes 1 = (\partial_x xy(y - x), \partial_y xy(y - x)) \otimes 1$, the preimage of $\alpha^{1,-1}(c_1)$ by $\varepsilon^{0,0}$ is equal to $xy(y-x) \otimes 1 \in A_2 \otimes \widetilde{M}$. Therefore, the preimage of τ^0 is equal to -1 and hence $H^0(\Omega(\log f)^\bullet) = \mathbf{C}(-1)$. Although we have done this computation by hand, computation of transfers can be done by Gröbner basis computation. See [19] and the source code for `deRhamAll` of the Macaulay 2 package for D-modules [7].

Before presenting implementations and larger examples, we explain a bit about a procedure to find a preimage of τ^i in general. The transfer algorithm gives an element in $\Omega^i \otimes_{A_2} \widetilde{M}$ where $\Omega^\bullet$ is the Koszul resolution of $\Omega(2) \simeq \Omega^2_R$ as the right A_2-module. This element can be identified with a differential form with coefficients in $\tilde{M}$ and we need to find the preimage of it by τ^i which lies in $\Omega^i(\log f)$. This can be performed by the method of undetermined coefficients.

Consider the case of τ^1. Take an element $c_1\omega_1 + c_2\omega_2$ in $\Omega^1(\log f)$ where $c_i \in R$. We have seen in Theorem 2.3 that

$$\tau^1(c_1\omega_1 + c_2\omega_2) = f\bar{\omega}_1 \otimes_{A_2} \bar{c}_1 + f\bar{\omega}_2 \otimes_{A_2} \bar{c}_2 \ \in \begin{pmatrix} A_2 \\ \oplus \\ A_2 \end{pmatrix} \otimes_{A_2} \widetilde{M} \qquad (6)$$

Here, we identify $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes_{A_2} m_1$ with $m_1 \otimes_R dx$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes_{A_2} m_2$ with $m_2 \otimes_R dy$, $m_i \in \widetilde{M}$ (comparison theorem) and when $\omega_i = a_i dx + b_i dy$, we denote $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$ by

$\bar{\omega}_i$. As the output of the transfer algorithm, we are given an element $m_1 dx + m_2 dy$, $m_i \in \widetilde{M}$. We regard m_i as an element in A_2 in the sequel. We rewrite $f\omega_i$ as $f\omega_1 = Adx + Bdy$ and $f\omega_2 = Cdx + Ddy$. Assume I is generated by ℓ_1 and ℓ_2. Then, the definition of τ^1 (6) induces the following identity in A_2 by taking coefficients of dx and dy

$$Ac_1 + Cc_2 = m_1 + \sum_{j=1}^{2} d_1^j \ell_j + \partial_x e \quad (7)$$

$$Bc_1 + Dc_2 = m_2 + \sum_{j=1}^{2} d_2^j \ell_j + \partial_y e \quad (8)$$

where $c_i \in R$, $d_i^j, e \in A_2$ are unknown. Fix a degree bound m for these elements and determine these elements by the method of unknown coefficients. The identities (7) and (8) induce a system of linear equations over $\mathbf{C}$ for the coefficients. Increasing the degree bound and solving the system, we will be able to obtain c_1 and c_2 in finite steps by virtue of Theorem 2.3.

Consider the case of τ^2. Since our basis in $H^2(\Omega^\bullet \otimes \tilde{M})$ is given in terms of x and y and $f\omega_1 \wedge \omega_2 = dx \wedge dy$, we need no computation to find the preimage by τ^2.

Let us consider the case of τ^0. Let m be an output of the transfer algorithm. It lies in A_2 in general. Finding the preimage g of τ^0 can be done by solving $gf = m + \sum_{j=1}^{2} d_j \ell_j$ where $g \in R$ and $d_j \in A_2$.

4 Implementation and Examples

The second and third steps of Algorithm 3.1 can be performed with the help of the D-module package on Macaulay2; use the commands `DintegrationAll` to obtain the dimension of the cohomology groups, `DintegrationClasses` to obtain the bases of cohomology groups, and a modification of `DeRhamAll` to obtain the bases of cohomology groups in $\Omega^\bullet \otimes \widetilde{M}$. Unfortunately, this implementation has not installed an efficient algorithm of computing b-function by Noro [11] to get the truncated complex in [13], [15]. Then, only relatively small examples are feasible. The Example 4.1 is computed by our Macaulay2 program. The Example 4.2 is computed by our implementation on kan/k0 and Risa/Asir with an implementation of [11] (the transfer algorithm has not been implemented yet for kan/k0). This implementation also uses the minimal filtered resolution to reduce the size of complex of A_2-modules [16]. The program is contained in the OpenXM package with the name `logc2.k` (`http://www.openxm.org`). Our implementation does not contain that for the Quillen-Suslin theorem. We utilize the implementation by A.Fabianska on Maple when the step 1-(a) fails. We also note that computation of the preimage of τ^1 may become a bottleneck of computation.

Example 4.1 (Continued from Example 1.2 (b).) We will determine bases of $H^i(\Omega^\bullet(\log f))$ where $f = (x^3+y^4+xy^3)(x^2+y^2)$. We firstly use Fabianska's program for the Quillen-Suslin theorem to find the 2 free generators of the syzygies of f, f_x, f_y. The two rows of the following matrix S are the generators

$$S = \begin{pmatrix} S_{11} & (-23/6y+1/2)x^2+(y^3+y^2-2y)x-5/6y^3 & (1/3y+1/2)x^2+(-3y^2+1/2y)x+y^4+4/3y^3-3/2y^2 \\ S_{21} & -46/75x^3+(4/25y^2-2/25y)x^2-8/15y^2x & 4/75x^3-12/25yx^2+(4/25y^3-2/75y^2)x-2/5y^3 \end{pmatrix}$$

where $S_{11} = (115/6y-5/2)x-6y^3-43/6y^2+9y$, $S_{21} = 46/15x^2+(-24/25y^2+22/75y)x+12/5y^2$. Put $A = \begin{pmatrix} S_{12} & S_{13} \\ S_{22} & S_{23} \end{pmatrix}$. Then, $\det(A) = \frac{1}{3}f$. We put $\omega_1 = \frac{1}{f}(a_{22}dx - a_{21}dy)$ and $\omega_2 = \frac{1}{f}(-a_{12}dx + a_{11}dy)$. ($\sqrt{3}\omega_i$ agrees with the ω_i in Theorem 2.3.)

We apply the integration algorithm and the transfer algorithm for $\widetilde{M}$. We obtain the following result. (1) $H^0(DR(\widetilde{M}))$ is spanned by $1 \otimes f$ and then we have $H^0(\Omega^\bullet(\log f)) \simeq \mathbf{C}\cdot 1$. (2) $H^2(DR(\widetilde{M}))$ is spanned $1 \otimes a$ where a runs over

```
                    3          3        2      3      4
o9 = {{1}, {-x}, {y }, {-x*y }, {x*y }, {x y}, {y }}
```

(We have pasted the output of our Macaulay 2 program `trans.m2`.) Then, we have

$$H^2(\Omega^\bullet(\log f)) \simeq (\mathbf{C}\cdot 1 + \mathbf{C}\cdot(-x) + \cdots + \mathbf{C}\cdot y^4)\,\omega_1\wedge\omega_2$$

(3) $H^1(DR(\widetilde{M}))$ is spanned by 3 differential forms m_1dx+m_2dy where m_1, m_2 are elements in A_2, of which explicit expressions are a little lengthy. We solve the identities (7) and (8) to find c_1 and c_2. In other words, we need to compute preimages of m_1dx+m_2dy by τ^1. As we explained, this can be done by the method of undetermined coefficients degree by degree. We can find solutions when the degree of c_i, d_i^j, e with respect to x, y is 6 and that with respect to ∂_x, ∂_y is 0. Here is a basis of 3-dimensional vector space $H^1(\Omega^\bullet(\log f))$ obtained by this method.

$$-yx\omega_1 - 4/25x^2\omega_2$$
$$((215/28y-1101/280)x-367/56y^2)\omega_1 + (43/35x^2-367/350yx)\omega_2$$
$$((y-11/30)x-28/9y^3-13/6y^2+14/3y)\omega_1 + (4/25x^2+(-112/225y^2+2/5y)x+56/45y^2)\omega_2$$

All programs and session logs to find this answer is obtainable from
`http://www.math.kobe-u.ac.jp/OpenXM/Math/LogCohomology/2007-11/log-2007-11-22.txt`
The logarithmic comparison theorem does not hold for this example. In fact, the dimensions of the de Rham cohomology groups $H^i(\Omega_f^\bullet)$, $(i = 2,1,0)$ are $5,3,1$ respectively.

Example 4.2 We apply a part of our algorithm to compute the dimensions of the cohomology groups $H^i(\Omega^\bullet(\log f))$ for $f = x^p+y^q+xy^{q-1}$. Here is a table of p, q and the dimensions of H^2, H^1, H^0 and timing data.

p	q	Dimensions	Timing in seconds
10	11	(8,1,1)	3.5
10	12	(9,1,1)	4.6
10	13	(10,1,1)	6.9
10	14	(11,1,1)	9.4
10	20	(17,1,1)	55.0
10	21	(18,1,1)	86.8

The program is executed on a machine with 2G RAM and Pentium III (1G Hz).

The homogenization of f, f_x, f_y generates an ideal that is Cohen-Macaulay. These examples do not need to call the subprocedure Quillen-Suslin. However, the logarithmic comparison theorem does not hold for these examples. Computation of de Rham cohomology groups is not feasible by our implementation.

5 A Yet Another Algorithm

In the previous section, we have presented a general algorithm of computing a basis of the logarithmic cohomology groups for plane curves. However, this algorithm relies on algorithms for the Quillen-Suslin theorem and they are sometimes slow. We will present a yet another algorithm, which is free from the Quillen-Suslin theorem, but it works only for computing a basis of the middle dimensional cohomology group $H^2(\Omega^\bullet(\log f))$ under some conditions on f. This section can be read independently from other sections. For reader's convenience, we will also redefine some notations.

Before stating the main algorithm, we start with an introductory example, which explains the idea of our algorithm.

Put $K = \mathbf{C}$ and $L = (1-x)x\partial + 2x (= \theta_x - x(\theta_x - 2))$. We consider the problem of determining a basis of the K-vector space $K[x]/L \cdot K[x]$. Since L is a K-linear map and $K[x]$ is an infinite dimensional K-vector space, the quotient has the structure of a K-vector space. However, note that $L \cdot K[x]$ is not an ideal and we cannot use Gröbner basis to get a basis.

Let us act L on monomials; $L \cdot x^k = kx^k - (k-2)x^{k+1}$. For small k, they are $L \cdot 1 = 2x$, $L \cdot x = x + x^2$, $L \cdot x^2 = 2x^2$. Then $x^{k+1} \simeq \frac{k}{k-2} x^k$ modulo $L \cdot K[x]$. In particular, if $k \geq 3$, then the monomial x^{k+1} can be reduced to a lower order monomial modulo $L \cdot K[x]$. Hence, the set of monomials $1, x, x^2, x^3$ generates $K[x]/L \cdot K[x]$. More precisely, we can prove that it is isomorphic to $F_3/L \cdot F_2$. Where F_k is the set of polynomials of which degree is less than or equal to k. The monomials $1, x, x^2, x^3$ are not independent modulo $L \cdot K[x]$ and satisfies the relation above. Finally, we conclude that $K[x]/LK[x] \simeq K \cdot 1 + K \cdot x^3$.

Note that 3 is the magic number, which is characterized as follows. Put $L^* = -(1-x)x\partial - 1 + 4x$. $\mathrm{in}_{(1,-1)}(L^*) \cap K[-\partial x]$ is generated by $b(-\partial x)$ where $b(s) = s - 3$. The polynomial $b(s)$ is called the indicial polynomial (b-function) for integration. The magic number 3 is the root of $b(s) = 0$. We will call the method to bound a degree by a root of a b-function *b-function criterion.* T.Oaku firstly introduced the b-function criterion to compute restrictions and integrations of D-modules [12]. The topic of computing $K[x]/L\cdot K[x]$ by the b-function was also discussed in more detail in an expository book "D-modules and Computational Mathematics" (in Japanese) by T.Oaku.

Let f be a polynomial in two variables. Put

$$\Omega_f^k = k\text{-form with coefficients in } K[x,y,1/f]$$

As we have explained in the introduction, the k form $\omega \in \Omega_f^k$ is called *logarithmic* k-form iff both of $f\omega$ and $df \wedge \omega$ have polynomial coefficients. The space of logarithmic k-forms is denoted by $\Omega^k(\log f)$. The question we address in this section is the computation of $\frac{\Omega^2(\log f)}{d\,\Omega^1(\log f)}$. It is easy to see that $\Omega^2(\log f) = \dfrac{K[x,y]dx\wedge dy}{f}$. Let us determine all the logarithmic 1-forms. Let (p,q,r) a triple of polynomials such that

$$f_y p - f_x q + fr = 0 \quad \text{(syzygy equation)}. \tag{9}$$

Note that $(0,f,f_x)$, $(f,0,-f_y)$, $(f_x,f_y,0)$ are trivial solutions of the syzygy equation. For a solution (p,q,r) of the syzygy equation, $\omega = \frac{pdx+qdy}{f}$ belongs to $\Omega^1(\log f)$. Conversely, any logarithmic 1-form can be expressed in this way. In fact, the condition that $df\wedge\omega$ has a polynomial coefficient is equivalent to that $f_y p - f_x q$ is a multiple of f.

Put $\omega = \frac{pdx+qdy}{f}$. Let $e(x,y)$ be any polynomial. Then, $d(e\omega) = (Le)\frac{dx\wedge dy}{f}$ where

$$L = q\partial_x - p\partial_y + q_x - p_y + \frac{f_y p - f_x q}{f}$$

We denote the Weyl algebra A_2 by D for simplicity in the sequel. Suppose that L_i, $(i = 1,\ldots,m)$ stand for a set of generators of the solution space of the syzygy equation, which is a $K[x,y]$-module. Then $d\,\Omega^1(\log f) = \sum L_i K[x,y]dx\wedge dy/f$. Therefore, the computation of H^2 is nothing but the computation of $K[x,y]/\sum_{i=1}^m L_i \bullet K[x,y]$. Put $I^* = D\cdot\{L_1^*,\ldots,L_m^*\}$, which is a left D ideal. We denote by F_k the K-subvector space of D of which $(1,1,-1,-1)$-order is less than or equal to k [18, p.14, p.203]

Algorithm 5.1 $H^2(\Omega^\cdot(\log f))$.
Step 1. Find generators of the syzygy equation and obtain explicit expressions of L_i.
Step 2. Compute $(1,1,-1,-1)$-Gröbner basis (standard basis) of I. We denote the elements of the Gröbner basis by L^{i^*} (renaming).
Step 3. Find the monic generator $b(-\partial_x x - \partial_y y)$ of $\mathrm{in}_{(1,1,-1,-1)}(I) \cap K[-\partial_x x - \partial_y y]$.
Step 4. Let k_0 be the maximal non-negative root of $b(s)=0$. Then, return K-vector space basis $\{c_i\}$ of

$$F_{k_0}/\sum_i L_i \cdot F_{k_0 - \mathrm{ord}_{(1,1,-1,-1)}(L_i)}.$$

$\{c_i dx \wedge dy/f\}$ is a basis of H^2.

The steps 2, 3, 4 can also be done by computing $D/(I^* + \partial_x D + \partial_y D)$ (0-th integral module) where I^* is the formal adjoint of I. (As to details for the steps 2, 3, 4, see [14].)
Note: Although our discussion is independent from the discussions of the previous sections, the left ideal generated by L_i^* is nothing but $\widetilde{Der}_R(-\log f)$ and hence this algorithm and the Algorithm 3.1 are analogous for computing a basis of $H^2(\Omega^\bullet(f))$. We also note that finding bases for $H^i(\Omega^\bullet \otimes_{A_2} \widetilde{M})$ can be performed by applying the integration algorithm and the transfer algorithm for D/I^*. The Algorithm 3.1 relies on algorithms for the Quillen-Suslin theorem to find bases for $H^i(\Omega(\log f)^\bullet)$, $i = 1, 0$.

Theorem 5.2 *If* $\dim V(f, f_x, f_y) \leq 0$, $\dim V(f, f_x) \leq 1$, $\dim V(f, f_y) \leq 1$, *then the Algorithm 5.1 is correct.*

Proof: Let I be the left ideal in D generated by $L_1, \ldots, L_m$. We may assume that I contains $f\partial_x$, $f\partial_y$ and $f_y\partial_x - f_x\partial_y$. Therefore, the characteristic variety of I is contained in $V(f(x,y)\xi, f(x,y)\eta, f_y(x,y)\xi - f_x(x,y)\eta)$, of which dimension is less than or equal to 2 from the assumption. In fact, assume $(a,b) \in V(f, f_x, f_y)$. Then, ξ and η are free and then the dimension of the characteristic variety is less than or equal to 2. Assume $(a,b) \in V(f, f_x) \setminus V(f, f_x, f_y)$. Then, we have $f(a,b) = 0, f_x(a,b) = 0$ and $f_y(a,b) \neq 0$. Then, η is free and $\xi = 0$ and then the dimension of the characteristic variety is less than or equal to 2. The rest cases can be shown analogously. Therefore, D/I is a holonomic D-module and hence a non-trivial b exists ([18, Chapter 5, Theorem 5.1.2]). The rest of the correctness proof is analogous with that of the 0-th integration algorithm of D-modules [12], [18, Chapter 5; Theorems 5.2.6 and 5.5.1]. $\square$

Note: The algorithm works to get $H^n(\Omega^\bullet(\log f))$ in the n-variable case if $\dim V(f, f_{x_{i_1}}, \ldots, f_{x_{i_m}}) \leq n - m$ for all $m = 1, \ldots, n$ and all combinations $i_1, \ldots, i_m$. The algorithm and the correctness proof are analogous. In fact, since $f\xi_i$ and $(-1)^i f_{x_j}\xi_i - (-1)^j f_{x_i}\xi_j$, $(1 \leq i \neq j \leq n)$ are in the characteristic ideal for I and then the dimension of the characteristic variety is less than or equal to n by utilizing the condition.

Example 5.3 For $f = (x^3 + y^4 + xy^3)(x^2 + y^2)$, we have $\dim H^2(\Omega^\bullet(\log f)) = 7$ with our yet another algorithm 5.1. The execution time is 1.9s. We need to call the procedure Quillen-Suslin if we use the first algorithm.

We close this paper with a final note and the acknowledgement of this paper. We think that logarithmic differential forms give nice simple bases for some of hypergeometric integrals as pairings of twisted cycles and cocycles when the logarithmic comparison theorem holds for twisted de Rham complex. We hope that our result have applications to study hypergeometric integrals. The authors are grateful to A.Fabianska for helping us to compute free bases of syzygies by using her implementation for Quillen-Suslin's theorem.

References

[1] F. J. Calderón-Moreno, Logarithmic differential operators and logarithmic de Rham complexes relative to a free divisor, Annales Scientifiques de lÉcole Normale Supérieure (4) **32** (1999), 701–714.

[2] F. J. Calderón-Moreno, D. Mond, L. Narváez-Macarro and F. J. Castro-Jiménez, Logarithmic Cohomology of the Complement of a Plane Curve, Commentarii Mathematici Helvetici **77** (2002), 24–38.

[3] F. J. Castro-Jiménez, L. Narváez-Macarro and D. Mond, Cohomology of the complement of a free divisor, Transactions of the American Mathematical Society **348** (1996), 3037–3049.

[4] F. J. Castro-Jiménez and J. M. Ucha, Explicit comparison theorems for $\mathcal{D}$-modules, Journal of Symbolic Computation **32** (2001), 677–685.

[5] D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry*, Springer, New York, (1995).

[6] A. Fabianska, QuillenSuslin package, `http://wwwb.math.rwth-aachen.de/QuillenSuslin/`

[7] A. Leykin, H. Tsai, D-module package for Macaulay2, 1999–2007. `http://www.math.uiuc.edu/Macaulay2`

[8] A. Logar and B. Sturmfels, Algorithms for the Quillen-Suslin Theorem, Journal of Algebra **145**, (1992), 231-239.

[9] H. Matsumura, *Commutative Ring Theory,* Cambridge University Press, (1986).

[10] Mebkhout, Z. *Le formalisme des six opérations de Grothendieck pour les $\mathcal{D}_X$-modules cohérents.* Travaux en Cours, 35. Hermann, Paris, 1989.

[11] M. Noro, An Efficient Modular Algorithm for Computing the Global B-Function, Mathematical Software, Proceedings of the first international congress of mathematical software, Beijing, Edited by A. M. Cohen, X. S. Gao, N. Takayama, World Scientific, (2002), 147–157.

[12] T. Oaku, Algorithms for b-functions, restrictions, and algebraic local cohomology groups of D-modules. Advances in Applied Mathematics **19** (1997), 61–105,

[13] T. Oaku and N. Takayama, An algorithm for de Rham cohomology groups of the complement of an affine variety via D-module computation, Journal of Pure and Applied Algebra, **139** (1999), 201–233.

[14] T. Oaku, N. Takayama, H. Tsai, Polynomial and rational solutions of holonomic systems. Journal of Pure and Applied Algebra **164** (2001), 199–220.

[15] T. Oaku and N. Takayama, Algorithms for D-modules—restriction, tensor product, localization, and local cohomology groups, Journal of Pure and Applied Algebra **156** (2001), 267–308.

[16] T. Oaku and N. Takayama, Minimal Free Resolutions of Homogenized D-modules, Journal of Symbolic Computation, **32** (2001), 575–592.

[17] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields, Journal of Faculty of Science, University of Tokyo. Section IA. **27** (1980), 265–291.

[18] M.Saito, B.Sturmfels, N.Takayama, *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics, 6. Springer-Verlag, Berlin, 2000. viii+254 pp.

[19] U. Walther, Computing the cup product structure for complements of complex affine varieties, Journal of Pure Applied Algebra **164** (2001), 247–273.