



ANNALES

DE

L'INSTITUT FOURIER

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Tome 59, n° 7 (2009), p. 2979-3014.

http://aif.cedram.org/item?id=AIF_2009__59_7_2979_0

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HASSE-SCHMIDT DERIVATIONS, DIVIDED POWERS AND DIFFERENTIAL SMOOTHNESS

by Luis NARVÁEZ MACARRO (*)

ABSTRACT. — Let k be a commutative ring, A a commutative k -algebra and D the filtered ring of k -linear differential operators of A . We prove that: (1) The graded ring $\text{gr } D$ admits a canonical embedding θ into the graded dual of the symmetric algebra of the module $\Omega_{A/k}$ of differentials of A over k , which has a canonical divided power structure. (2) There is a canonical morphism ϑ from the divided power algebra of the module of k -linear Hasse-Schmidt integrable derivations of A to $\text{gr } D$. (3) Morphisms θ and ϑ fit into a canonical commutative diagram.

RÉSUMÉ. — Soit k un anneau commutatif, A une k -algèbre commutative et D l'anneau filtré des opérateurs différentiels k -linéaires de A . Nous montrons que : (1) l'anneau gradué $\text{gr } D$ admet un plongement canonique θ dans le dual gradué de l'algèbre symétrique du module $\Omega_{A/k}$ des différentielles de A sur k , qui a une structure canonique de puissances divisées. (2) Il existe un morphisme canonique ϑ de l'algèbre des puissances divisées du module des dérivations k -linéaires et intégrables dans le sens de Hasse-Schmidt de A vers $\text{gr } D$. (3) Les morphismes θ et ϑ forment partie d'un diagramme commutatif canonique.

Introduction

In the case of a polynomial ring $A = k[x_1, \dots, x_n]$ or a power series ring $A = k[[x_1, \dots, x_n]]$ with coefficients in some ring k , it is well known that the k -linear differential operators $\Delta^{(\alpha)} : A \rightarrow A$, $\alpha \in \mathbb{N}^n$, given by Taylor's development

$$F(x_1 + T_1, \dots, x_n + T_n) = \sum_{\alpha \in \mathbb{N}^n} \Delta^{(\alpha)}(F) T^\alpha, \quad \forall F \in A,$$

Keywords: Derivation, integrable derivation, differential operator, divided powers structure.

Math. classification: 13N15, 13N10.

(*) Partially supported by MTM2007-66929 and FEDER.

form a basis of the ring of k -linear differential operators $\text{Diff}_{A/k}$ regarded as left (or right) A -module. More precisely, any k -linear differential operator $P : A \rightarrow A$ or order $\leq d$ can be uniquely written as

$$P = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq d}} a_\alpha \Delta^{(\alpha)}, \quad a_\alpha \in A, \quad \text{with} \quad a_\alpha = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (-1)^{|\beta|} x^\beta P(x^{\alpha-\beta}),$$

where $\beta \leq \alpha$ stands for the usual partial ordering: $\beta_i \leq \alpha_i$ for all $i = 1, \dots, n$.

For any $i = 1, \dots, n$ and any integer $m \geq 0$ let us write $\Delta_m^{(i)} = \Delta^{(0, \dots, \overset{(i)}{m}, \dots, 0)}$. In particular $\Delta_1^{(i)} = \frac{\partial}{\partial x_i}$. The $\Delta^{(\alpha)}$ satisfy the following easy and well known rules:

- (a) $\Delta^{(\alpha)}(x^\beta) = \begin{cases} \binom{\beta}{\alpha} x^{\beta-\alpha} & \text{if } \beta \geq \alpha \\ 0 & \text{if } \beta \not\geq \alpha. \end{cases}$
- (b) $\Delta^{(\alpha)} \circ \Delta^{(\beta)} = \Delta^{(\beta)} \circ \Delta^{(\alpha)} = \binom{\alpha+\beta}{\alpha} \Delta^{(\alpha+\beta)}$.
- (c) $\Delta^{(\alpha)} = \Delta_{\alpha_1}^{(1)} \circ \dots \circ \Delta_{\alpha_n}^{(n)}$.

Let us write $\text{Diff}_{A/k}^{(d)}$, $d \geq 0$, for the A -module of k -linear differential operators of order $\leq d$ and let us consider the graded ring

$$\text{gr Diff}_{A/k} = \bigoplus_{d \geq 0} \text{Diff}_{A/k}^{(d)} / \text{Diff}_{A/k}^{(d-1)} \quad (\text{where } \text{Diff}_{A/k}^{(-1)} = 0),$$

which is commutative. Let us also write $\sigma^{(\alpha)}$ (resp. $\sigma_m^{(i)}$) for the class (or symbol) of $\Delta^{(\alpha)}$ (resp. of $\Delta_m^{(i)}$) in $\text{gr}^d \text{Diff}_{A/k} = \text{Diff}_{A/k}^{(d)} / \text{Diff}_{A/k}^{(d-1)}$, with $d = |\alpha|$ (resp. with $d = m$). From the above properties, the following ones hold:

- (1) The family $\{\sigma^{(\alpha)}, |\alpha| = d\}$ is a basis of the A -module $\text{gr}^d \text{Diff}_{A/k}$,
- (2) $\sigma^{(\alpha)} \sigma^{(\beta)} = \binom{\alpha+\beta}{\alpha} \sigma^{(\alpha+\beta)}$,
- (3) $\sigma^{(\alpha)} = \sigma_{\alpha_1}^{(1)} \dots \sigma_{\alpha_n}^{(n)}$.

So, there is an isomorphism of (commutative) graded A -algebras between the algebra of divided powers $\Gamma_A(\xi_1, \dots, \xi_n)$ of the free A -module with basis ξ_1, \dots, ξ_n ([11, 12]) and the graded ring $\text{gr Diff}_{A/k}$ sending ξ_i to $\sigma_i^{(1)}$. Let us call this isomorphism $\vartheta_0 : \Gamma_A(\xi_1, \dots, \xi_n) \xrightarrow{\sim} \text{gr Diff}_{A/k}$. In particular, the ring $\text{gr Diff}_{A/k}$ has a divided power structure (in the sense of [12] and [2]).

On the other hand, there is a canonical homomorphism of graded A -algebras $\tau : \text{Sym}_A \text{Der}_k(A) \rightarrow \text{gr Diff}_{A/k}$ (which in fact always exist for any k -algebra A and not only for polynomial or power series rings), which

is an isomorphism if $\mathbb{Q} \subset A$. Furthermore, if $\mathbb{Q} \subset A$, then the symmetric algebra $\text{Sym}_A \text{Der}_k(A)$ coincides with the algebra of divided powers $\Gamma_A \text{Der}_k(A)$ and the isomorphism ϑ_0 coincides with τ , once the basis $\{\xi_1 = \frac{\partial}{\partial x_1}, \dots, \xi_n = \frac{\partial}{\partial x_n}\}$ of the A -module $\text{Der}_k(A)$ is chosen.

If we do not assume anymore that $\mathbb{Q} \subset A$, it is still possible to define an isomorphism $\vartheta : \Gamma_A \text{Der}_k(A) \xrightarrow{\sim} \text{gr Diff}_{A/k}$ by using the coordinates x_1, \dots, x_n of A and the above basis of $\text{Der}_k(A)$. It turns out that ϑ is independent of the basis choice and it extends the canonical homomorphism τ through the canonical map from the symmetric algebra to the algebra of divided powers.

The following natural questions appear:

- (Q-1) Can we canonically define a divided power structure on $\text{gr Diff}_{A/k}$ for an arbitrary k -algebra A ?
- (Q-2) Can we canonically define a homomorphism of graded A -algebras $\vartheta : \Gamma_A \text{Der}_k(A) \rightarrow \text{gr Diff}_{A/k}$ which becomes an isomorphism under convenient smoothness hypotheses, for instance when $A = k[x_1, \dots, x_n]$ or $A = k[[x_1, \dots, x_n]]$?

A positive answer to (Q-1) would imply, of course, a positive answer to (Q-2).

The aim of this paper is to explore the above questions. Our main results are the following: for any commutative ring k and any commutative k -algebra A , the following properties hold:

- (A-1) There is a canonical embedding θ of $\text{gr Diff}_{A/k}$ into the graded dual of the symmetric algebra of the module of differentials $\Omega_{A/k}$, $(\text{Sym } \Omega_{A/k})_{\text{gr}}^*$, which carries a canonical divided power structure by general reasons. Moreover, θ is given by:

$$\theta(\sigma_d(P)) \left(\prod_{i=1}^d dx_i \right) = [[\dots [[P, x_d], x_{d-1}], \dots, x_2], x_1]$$

for each $P \in \text{Diff}_{A/k}^{(d)}$ and for any $x_1, \dots, x_d \in A$.

- (A-2) There is a submodule $\text{IDer}_k(A) \subset \text{Der}_k(A)$ (the elements of $\text{IDer}_k(A)$ are the “integrable” derivations in the sense of Hasse-Schmidt) and a canonical homomorphism of graded A -algebras $\vartheta : \Gamma_A \text{IDer}_k(A) \rightarrow \text{gr Diff}_{A/k}$. When $\mathbb{Q} \subset A$, we have $\text{IDer}_k(A) = \text{Der}_k(A)$ and morphism ϑ coincides with the canonical morphism $\tau : \text{Sym}_A \text{Der}_k(A) \rightarrow \text{gr Diff}_{A/k}$.

(A-3) There is a canonical commutative diagram

$$\begin{array}{ccc}
 \text{gr Diff}_{A/k} & \xrightarrow{\theta} & (\text{Sym } \Omega_{A/k})_{\text{gr}}^* \\
 \uparrow \vartheta & & \uparrow \\
 \Gamma \text{IDer}_k(A) & \xrightarrow{\text{nat.}} & \Gamma \text{Der}_k(A).
 \end{array}$$

Our results are strongly based on the notions of Hasse–Schmidt derivation and of integrable derivation. In fact, our starting point was the observation that the symbols of the components of any Hasse–Schmidt derivation only depend on its component of degree 1 (see proposition 2.6).

Any k -derivation of A is integrable in two relatively “orthogonal” situations:

- In characteristic 0, *i.e.*, when $\mathbb{Q} \subset A$.
- When A is a smooth k -algebra.

So, the property that any derivation is integrable seems to be an interesting step in understanding singularities in positive or unequal characteristics.

Let us now comment on the content of this paper.

In section 1 we review the basic notions used throughout the paper: Hasse–Schmidt derivations, integrable derivations, rings of differential operators, exponential type series, algebras of divided powers and divided power structures.

Section 2 contains the main results of this paper: the construction of the embedding $\theta : \text{gr Diff}_{A/k} \hookrightarrow (\text{Sym } \Omega_{A/k})_{\text{gr}}^*$, the construction of the morphism $\vartheta : \Gamma \text{IDer}_k(A) \rightarrow \text{gr Diff}_{A/k}$ and the commutative diagram relating θ and ϑ . As a consequence we obtain a relationship between the differential smoothness of A/k , in the sense of [6], 16.10, and the behavior of θ and ϑ , and a proof of the following general result: If $\text{IDer}_k(A) = \text{Der}_k(A)$ and $\text{Der}_k(A)$ is a projective A -module of finite rank, then the canonical map $\vartheta : \Gamma_A \text{IDer}_k(A) = \Gamma_A \text{Der}_k(A) \rightarrow \text{gr Diff}_{A/k}$ is an isomorphism. In particular, if $\mathbb{Q} \subset A$ and $\text{Der}_k(A)$ is a projective A -module of finite rank, then the canonical map $\tau : \text{Sym}_A \text{Der}_k(A) \rightarrow \text{gr Diff}_{A/k}$ is an isomorphism, generalizing proposition 4 in [1].

Section 3 contains logarithmic versions of the preceding notions and their use for explicit computations. We give an example illustrating the problem of deciding whether a derivation is integrable or not.

I would like to thank Dan Laksov for his comments on a earlier version of this paper.

1. Notations and preliminaries

All rings and algebras considered in this paper are assumed to be commutative with unit element. For any family $x = \{x_i\}_{i \in I}$ of elements in a ring and for any finite subset $L \subset I$, we denote $x_L = \prod_{i \in L} x_i$. For any integer $n \geq 1$ we will denote $[n] = \{1, \dots, n\}$ and $[0] = \emptyset$.

Let k be a ring, A a k -algebra and M an A -module. We denote by $\text{Der}_k(A, M)$ the A -module of k -linear derivations from A to M . If $M = A$, we will write as usual $\text{Der}_k(A) = \text{Der}_k(A, A)$.

1.1. Hasse-Schmidt derivations

In this section, $k \xrightarrow{f} A \xrightarrow{g} B$ will be ring homomorphisms. For each integer $m \geq 0$ we set $B_m = B[t]/(t^{m+1})$ and for $m = \infty$, $B_\infty = B[[t]]$. We can view B_m as a k -algebra in a natural way (for $m \leq \infty$).

A *Hasse-Schmidt derivation* (over k) ([7]; see also [10], § 27, and [13], [14] for more recent references) of length $m \geq 1$ (resp. of length ∞) from A to B , is a sequence $D = (D_0, D_1, \dots, D_m)$ (resp. $D = (D_0, D_1, \dots)$) of k -linear maps $D_i : A \rightarrow B$, satisfying the conditions:

$$D_0 = g, \quad D_i(xy) = \sum_{r+s=i} D_r(x)D_s(y)$$

for all $x, y \in A$ and for all $i = 1, \dots, m$ (resp. for all $i \geq 0$). In particular, the component D_1 is a k -derivation from A to B . Moreover, D_i vanishes on $f(k)$ for all $i > 0$. When $A = B$ and $g = \text{Id}_A$, we simply say that D is a Hasse-Schmidt derivation of A (over k). We write $\text{HS}_k(A, B; m)$ for the set of all Hasse-Schmidt derivations (over k) of length m from A to B , $\text{HS}_k(A, B) = \text{HS}_k(A, B; \infty)$, $\text{HS}_k(A; m) = \text{HS}_k(A, A; m)$ and $\text{HS}_k(A) = \text{HS}_k(A, A; \infty)$.

It is clear that the map

$$(1.1) \quad (D_0, D_1) \in \text{HS}_k(A, B; 1) \mapsto D_1 \in \text{Der}_k(A, B)$$

is a bijection.

For any $b \in B$ and any $D \in \text{HS}_k(A, B; m)$, the sequence D' defined by $D'_0 = g$ and $D'_r = b^r D_r$ for $r > 0$ is again a Hasse-Schmidt derivation over k of the length m from A to B , which will be denoted by $b \bullet D$.

Any Hasse-Schmidt derivation $D \in \text{HS}_k(A, B; m)$ is determined by the k -algebra homomorphism $\Phi : A \rightarrow B_m$ defined by $\Phi(x) = \sum_{i=0}^m D_i(x)t^i$ and satisfying $\Phi(x) \equiv g(x) \pmod{t}$. When $B = A$ and $g = \text{Id}_A$ the k -algebra

homomorphism Φ can be uniquely extended to a k -algebra automorphism $\tilde{\Phi} : A_m \rightarrow A_m$ with $\tilde{\Phi}(t) = t$:

$$\tilde{\Phi} \left(\sum_{i=0}^m a_i t^i \right) = \sum_{i=0}^m \Phi(a_i) t^i.$$

So, we have a bijection between $\text{HS}_k(A; m)$ and the subgroup of $\text{Aut}_{k\text{-alg}}(A_m)$ consisting of the automorphisms $\tilde{\Phi}$ satisfying $\tilde{\Phi}(a) \equiv a \pmod t$ for all $a \in A$ and $\tilde{\Phi}(t) = t$. In particular, $\text{HS}_k(A; m)$ inherits a canonical group structure which is explicitly given by $D \circ D' = D''$ with $D''_n = \sum_{i+j=n} D_i \circ D'_j$, the identity element of $\text{HS}_k(A; m)$ being $(\text{Id}_A, 0, 0, \dots)$.

It is clear that in the case $A = B$, $g = \text{Id}_A$ and $m = 1$, the map (1.1) is an isomorphism of groups and so $\text{HS}_k(A; 1)$ is abelian.

For $1 \leq m \leq q \leq \infty$, let us denote by $\tau_{qm} : \text{HS}_k(A, B; q) \rightarrow \text{HS}_k(A, B; m)$ the m -truncation map defined as $\tau_{qm}(D) = (D_0, D_1, \dots, D_m)$.

Since any $D \in \text{HS}_k(A, B)$ is determined by its finite truncations, we have:

$$\text{HS}_k(A, B) = \varprojlim \text{HS}_k(A, B; m).$$

Let us note that $\tau_{qm}(b \bullet D) = b \bullet \tau_{qm}(D)$.

When $A = B$, the truncation maps τ_{qm} are group homomorphisms and the projective limit above can be taken in the category of groups. In the case $m = 1$, since $\text{HS}_k(A; 1) \equiv \text{Der}_k(A)$, we can think on τ_{q1} as a group homomorphism $\tau_{q1} : \text{HS}_k(A; q) \rightarrow \text{Der}_k(A)$ satisfying $\tau_{q1}(a \bullet D) = a \tau_{q1}(D)$.

DEFINITION 1.1. — (cf. [9]) We say that a k -derivation $\delta : A \rightarrow A$ is q -integrable (resp. integrable) (over k) if there is a Hasse–Schmidt derivation $D \in \text{HS}_k(A; q)$ of length q (resp. $D \in \text{HS}_k(A)$) such that $D_1 = \delta$. In such a case we say that D is a q -integral (resp. an integral) of δ . The set of q -integrable (resp. integrable) k -derivations of A is denoted by $\text{IDer}_k(A; q)$ (resp. $\text{IDer}_k(A)$).

It is clear that $\text{IDer}_k(A; q)$ and $\text{IDer}_k(A)$ are submodules of the A -module $\text{Der}_k(A)$. We have exact sequences of groups

$$(1.2) \quad 1 \rightarrow \ker \tau_{q1} \rightarrow \text{HS}_k(A; q) \rightarrow \text{IDer}_k(A; q) \rightarrow 0,$$

$$\text{Der}_k(A) = \text{IDer}_k(A; 1) \supset \text{IDer}_k(A; 2) \supset \text{IDer}_k(A; 3) \supset \dots,$$

$$\text{IDer}_k(A) \subset \bigcap_{q \in \mathbb{N}} \text{IDer}_k(A; q).$$

More generally, we say that a Hasse–Schmidt derivation $D' \in \text{HS}_k(A; m)$ of length m is q -integrable (over k) if there is a Hasse–Schmidt derivation

$D \in \text{HS}_k(A; q)$ of length q such that $\tau_{qm}(D) = D'$. In such a case we say that D is a q -integral of D' . We say that D' is integrable if it is ∞ -integrable.

Example 1.2. — Let $q \geq 1$ be an integer. If $q!$ is invertible in A , then any k -derivation δ of A is q -integrable: we can take $D \in \text{HS}_k(A; q)$ defined by $D_i = \frac{\delta^i}{i!}$ for $i = 0, \dots, q$, and $\tau_{q1}(D) = \delta$. In the case $q = \infty$, if $\mathbb{Q} \subset A$, one proves in a similar way that any k -derivation of A is integrable.

PROPOSITION 1.3. — *Let us assume that A is a 0-smooth k -algebra. Then any k -derivation of A is integrable.*

Proof. — It is enough to prove that, for each $m \geq 1$, the map $\tau_{m+1,m} : \text{HS}_k(A; m + 1) \rightarrow \text{HS}_k(A; m)$ is surjective. Let $D \in \text{HS}_k(A; m)$ and let $\Phi : A \rightarrow A_m = A[[t]]/(t^{m+1})$ be the corresponding homomorphism of k -algebras. Since A is 0-smooth over k (cf. [10], p. 193), we obtain a commutative diagram

$$\begin{array}{ccc}
 k & \longrightarrow & A_{m+1} \\
 f \downarrow & \nearrow \Phi' & \downarrow \text{projection} \\
 A & \xrightarrow{\Phi} & A_m
 \end{array}$$

and $D = \tau_{m+1,m}(D')$, where $D' \in \text{HS}_k(A; m + 1)$ is the Hasse-Schmidt derivation corresponding to Φ' . □

The following proposition answers a natural question.

PROPOSITION 1.4. — *Assume that $\text{Der}_k(A)$ is a finitely generated A -module and that $\text{Der}_k(A) = \text{IDer}_k(A)$. Then, for each $m \geq 1$, any Hasse-Schmidt derivation $D' \in \text{HS}_k(A; m)$ is $(m + 1)$ -integrable, and a fortiori it is integrable.*

Proof. — The proof is a consequence of [4], § 2. Let $\delta^1, \dots, \delta^n$ be a system of generators of the A -module $\text{Der}_k(A)$ and let $D^d \in \text{HS}_k(A)$ be an integral of δ^d . From theorem 2.8 in *loc. cit.* there exist $C_{ld} \in A$, $1 \leq d \leq n$, $1 \leq l \leq m$, such that

$$(1.3) \quad D'_i = \sum_{m=1}^i \left(\sum_{\substack{|\lambda|=i \\ |\mu|=m \\ \lambda \succeq \mu}} \prod_{d=1}^n \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d} \right) D_\mu$$

for all $i = 1, \dots, m$, where we write $\lambda \succeq \mu$ for $\lambda_d \geq \mu_d, d = 1, \dots, n$, and if $\mu_d = 0$ then $\lambda_d = 0$,

$$D_\mu = D_{\mu_1}^1 \circ \dots \circ D_{\mu_n}^n \quad \text{and} \quad \sum_{\substack{l \in \mathbb{N}_+^d \\ |l| = \lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d} = 1 \quad \text{if} \quad \mu_d = \lambda_d = 0.$$

Let us take arbitrary elements $C_{m+1,d} \in A$ (for instance $C_{m+1,d} = 0$) for $d = 1, \dots, n$ and let D'_{m+1} be defined by the equation (1.3) for $i = m + 1$. The sequence $(D'_0 = \text{Id}_A, D'_1, \dots, D'_m, D'_{m+1})$ is a Hasse–Schmidt derivation of A of length $m + 1$ and so D' is $(m + 1)$ -integrable. \square

Remark 1.5. — (a) Assuming that $\text{Der}_k(A)$ is a free A -module and $\delta^1, \dots, \delta^n$ is a basis in the proposition above, the sequence $C_{ld} \in A, 1 \leq d \leq n, 1 \leq l \leq m$, is uniquely determined and the choice $C_{m+1,d} = 0$ gives a “canonical” integral of D' .

(b) Formula (1.3) can be generalize to the case where $\text{Der}_k(A)$ is not necessarily finitely generated and $\{\delta^d\}$ is an arbitrary system of generators of $\text{Der}_k(A)$. In such a case one has that for each $l = 1, \dots, m$, the constants C_{ld} vanish except for a finite set of d 's.

In example 3.6 we will see that if A is a “normal crossing” k -algebra, then any k -derivation of A is integrable.

1.2. Rings of differential operators

A general reference for the notions and results in this section is [6], § 16, 16.8.

Let $k \xrightarrow{f} A \xrightarrow{g} B$ be ring homomorphisms and let E, F be two A -modules. The k -module $\text{Hom}_k(E, F)$ has a natural structure of $(A; A)$ -bimodule:

$$\begin{aligned} (a, h) \in A \times \text{Hom}_k(E, F) &\mapsto ah := [e \in E \mapsto (ah)(e) = ah(e) \in F], \\ (h, a) \in \text{Hom}_k(E, F) \times A &\mapsto ha := [e \in E \mapsto (ha)(e) = h(ae) \in F]. \end{aligned}$$

For $h \in \text{Hom}_k(E, F)$ and $a \in A$ let us write $[h, a] := ha - ah$. For any $c \in k$ one has $[h, c] = 0$.

For all $i \geq 0$, we inductively define the subsets $\text{Diff}_{A/k}^{(i)}(E, F) \subseteq \text{Hom}_k(E, F)$ in the following way:

$$\begin{aligned} \text{Diff}_{A/k}^{(0)}(E, F) &:= \text{Hom}_A(E, F), \\ \text{Diff}_{A/k}^{(i+1)}(E, F) &:= \{\varphi \in \text{Hom}_k(E, F) \mid [\varphi, a] \in \text{Diff}_{A/k}^{(i)}(E, F), \quad \forall a \in A\}. \end{aligned}$$

The elements of $\text{Diff}_{A/k}(E, F) := \bigcup_{i \geq 0} \text{Diff}_{A/k}^{(i)}(E, F)$ (resp. of $\text{Diff}_{A/k}^{(i)}(E, F)$) are called *k-linear differential operators* (resp. *k-linear differential operators of order $\leq i$*) from E to F .

The family $\{\text{Diff}_{A/k}^{(i)}(E, F)\}_{i \geq 0}$ is an increasing sequence of (A, A) -bimodules of $\text{Hom}_k(E, F)$, and if G is a third A -module, then

$$\text{Diff}_{A/k}^{(i)}(F, G) \circ \text{Diff}_{A/k}^{(j)}(E, F) \subset \text{Diff}_{A/k}^{(i+j)}(E, G), \quad \forall i, j \geq 0,$$

and so $\text{Diff}_{A/k}(F, G) \circ \text{Diff}_{A/k}(E, F) \subset \text{Diff}_{A/k}(E, G)$.

From the definition of Hasse-Schmidt derivations we know that for any $D \in \text{HS}_k(A, B; m)$ and any $a \in A$, the following equality holds:

$$(1.4) \quad [D_r, a] = \sum_{i=0}^{r-1} D_{r-i}(a)D_i, \quad \forall r > 0.$$

The proof of the following proposition proceeds easily by induction from (1.4).

PROPOSITION 1.6. — *For each Hasse-Schmidt derivation $D \in \text{HS}_k(A, B; m)$ and each $i = 0, \dots, m$, D_i is a k -linear differential operator from A to B of order $\leq i$, i.e., $D_i \in \text{Diff}_{A/k}^{(i)}(A, B)$.*

In the case $E = F$, $\text{Diff}_{A/k}(E, E)$ is a subring of $\text{End}_k(E)$. When $E = A$, one has a canonical decomposition $\text{Diff}_{A/k}^{(1)}(A, F) \simeq F \oplus \text{Der}_k(A, F)$ given by

$$P \in \text{Diff}_{A/k}^{(1)}(A, F) \mapsto (P(1), P - P(1)) \in F \oplus \text{Der}_k(A, F),$$

$$(f, \delta) \in F \oplus \text{Der}_k(A, F) \mapsto f + \delta \in \text{Diff}_{A/k}^{(1)}(A, F),$$

which fits into a commutative diagram

$$(1.5) \quad \begin{array}{ccc} F & \xrightarrow{\sim} & \text{Diff}_{A/k}^{(0)}(A, F) \\ \downarrow & & \downarrow \\ F \oplus \text{Der}_k(A, F) & \xrightarrow{\sim} & \text{Diff}_{A/k}^{(1)}(A, F). \end{array}$$

The ring $\text{Diff}_{A/k}(A, A)$ will be simply denoted by $\text{Diff}_{A/k}$. It is filtered by the $F^i \text{Diff}_{A/k} := \text{Diff}_{A/k}^{(i)}(A, A)$, $i \geq 0$. For any $P \in F^i \text{Diff}_{A/k}$, $Q \in F^j \text{Diff}_{A/k}$ one easily sees (by induction on $i+j$) that $[P, Q] \in F^{i+j-1} \text{Diff}_{A/k}$ and so the associated graded ring $\text{gr} \text{Diff}_{A/k}$ is commutative. From (1.5) we have a canonical isomorphism of A -modules $\text{Der}_k(A) \xrightarrow{\sim} \text{gr}^1 \text{Diff}_{A/k}$ and so a canonical map of commutative graded A -algebras

$$(1.6) \quad \tau_{A/k} : \text{Sym} \text{Der}_k(A) \rightarrow \text{gr} \text{Diff}_{A/k},$$

which is an isomorphism in degrees 0 and 1 (actually, in degree 0 it is the identity map of A).

Let us denote by $\sigma_r(P)$ the class in $\text{gr}^r \text{Diff}_{A/k} = \text{Diff}_{A/k}^{(r)} / \text{Diff}_{A/k}^{(r-1)}$ of a $P \in \text{Diff}_{A/k}^{(r)}$.

DEFINITION 1.7. — *The ring $\text{gr Diff}_{A/k}$ is endowed with a canonical homogeneous k -bilinear map $\{-, -\} : \text{gr Diff}_{A/k} \times \text{gr Diff}_{A/k} \rightarrow \text{gr Diff}_{A/k}$, called Poisson bracket, which is defined on homogeneous elements by*

$$\{\sigma_r(P), \sigma_s(Q)\} = \sigma_{r+s-1}([P, Q]), \quad \forall P \in \text{Diff}_{A/k}^{(r)}, \forall Q \in \text{Diff}_{A/k}^{(s)}.$$

It is a Lie bracket and a k -derivation on each component.

The notion of differential operator is linearized through the algebras of principal parts. Namely, let us consider the k -algebra $\mathcal{P}_{A/k} = A \otimes_k A$, the epimorphism of k -algebras $\pi : \mathcal{P}_{A/k} \rightarrow A$ and the homomorphisms of k -algebras $\mu_1 : A \rightarrow \mathcal{P}_{A/k}$, $\mu_2 : A \rightarrow \mathcal{P}_{A/k}$ defined by $\pi(a \otimes b) = ab$, $\mu_1(a) = a \otimes 1$, $\mu_2(a) = 1 \otimes a$, which endow $\mathcal{P}_{A/k}$ with a “left” and a “right” A -algebra structure.

Let us denote by $I_{A/k} = \ker \pi$. The ring $\mathcal{P}_{A/k}^n := \mathcal{P}_{A/k} / I_{A/k}^{n+1}$ is called the algebra of principal parts of order n of A over k , and is also endowed with a left and a right A -algebra structure. For each A -module E , let us denote $\mathcal{P}_{A/k}^n(E) = \mathcal{P}_{A/k}^n \otimes_A E$, where the tensor product is taken with respect to the right A -module structure on $\mathcal{P}_{A/k}^n$. The module $\mathcal{P}_{A/k}^n(E)$ will be always considered as a A -module through the left A -module structure on $\mathcal{P}_{A/k}^n$. Let us denote by $d_{A/k,E}^n : E \rightarrow \mathcal{P}_{A/k}^n(E)$ the k -linear map given by $d_{A/k,E}^n(e) = (\overline{1 \otimes 1}) \otimes e$.

The module of differentials of A over k is $\Omega_{A/k} := I_{A/k} / I_{A/k}^2$, on which the induced left and right A -module structures coincide. Moreover, the exact sequence of left A -modules $0 \rightarrow \Omega_{A/k} \rightarrow \mathcal{P}_{A/k}^1 \rightarrow A \rightarrow 0$ splits and we have a canonical decomposition $\mathcal{P}_{A/k}^1 = A \oplus \Omega_{A/k}$. So, the map $d_{A/k,A}^1 : A \rightarrow \mathcal{P}_{A/k}^1$ induces the differential $d_{A/k} : A \rightarrow \Omega_{A/k}$ defined by $d_{A/k}(a) = (1 \otimes a - a \otimes 1) + I_{A/k}^2$. The main facts are the following:

- (a) The map $d_{A/k} : A \rightarrow \Omega_{A/k}$ is a k -derivation and the map

$$h \in \text{Hom}_A(\Omega_{A/k}, F) \mapsto h \circ d_{A/k} \in \text{Der}_k(A, F)$$

is an isomorphism of A -modules.

- (b) The map $d_{A/k,E}^n : E \rightarrow \mathcal{P}_{A/k}^n(E)$ is a k -linear differential operator of order $\leq n$ and the map

$$h \in \text{Hom}_A(\mathcal{P}_{A/k}^n(E), F) \mapsto h \circ d_{A/k,E}^n \in \text{Diff}_{A/k}^{(n)}(E, F)$$

is an isomorphism of A -bimodules.

1.3. Exponential type series and divided powers

General references for the notions and results in this section are [11, 12] and [2].

Let B be an A -algebra and let $m \geq 1$ be an integer or $m = \infty$. The substitution $t \mapsto t + t'$ gives rise to a homomorphism of A -algebras

$$R(t) \in B_m = B[[t]]/(t^{m+1}) \mapsto R(t + t') \in B[[t, t']]/(t, t')^{m+1}.$$

DEFINITION 1.8. — *An element $R = R(t) = \sum_{i=0}^m R_i t^i$ in $B_m = B[[t]]/(t^{m+1})$ is said to be of exponential type if $R_0 = 1$ and $R(t + t') = R(t)R(t')$, or equivalently, if*

$$\binom{i + j}{i} R_{i+j} = R_i R_j, \quad \text{whenever } i + j < m + 1.$$

The set of elements in B_m of exponential type will be denoted by $\mathcal{E}_m(B)$. The set $\mathcal{E}_\infty(B)$ will be simply denoted by $\mathcal{E}(B)$.

The set $\mathcal{E}_m(B)$ is a subgroup of the group of units of B_m and the external operation

$$\left(a, \sum_{i=0}^m R_i t^i \right) \in B \times \mathcal{E}_m(B) \mapsto \sum_{i=0}^m R_i (at)^i = \sum_{i=0}^m R_i a^i t^i \in \mathcal{E}_m(B)$$

defines a natural B -module structure on $\mathcal{E}_m(B)$. It is clear that $\mathcal{E}_1(B)$ is canonically isomorphic to B .

Let C be another A -algebra. For each $m \geq 1$, any A -algebra map $h : B \rightarrow C$ induces obvious A -linear maps $\mathcal{E}_m(h) : \mathcal{E}_m(B) \rightarrow \mathcal{E}_m(C)$. In this way we obtain functors \mathcal{E}_m from the category of A -algebras to the category of A -modules. For $1 \leq m \leq q \leq \infty$, the projections $B_q \rightarrow B_m$ induce truncation natural transformations $\mathcal{E}_q \rightarrow \mathcal{E}_m$. The following result is proven in [11] in the case $m = \infty$. The proof for any integer $m \geq 1$ is completely similar.

PROPOSITION 1.9. — *For each A -module M and each $m \geq 1$ there is an universal pair $(\Gamma_m M, \gamma_m)$, where $\Gamma_m M$ is an A -algebra and $\gamma_m : M \rightarrow \mathcal{E}_m(\Gamma_m M)$ is an A -linear map, satisfying the following universal property: for any A -algebra B and any A -linear map $H : M \rightarrow \mathcal{E}_m(B)$ there is a unique morphism of A -algebras $h : \Gamma_m M \rightarrow B$ such that $H = \mathcal{E}_m(h) \circ \gamma_m$, or equivalently, the map*

$$h \in \text{Hom}_{A\text{-alg}}(\Gamma_m M, B) \mapsto \mathcal{E}_m(h) \circ \gamma_m \in \text{Hom}_A(M, \mathcal{E}_m(B))$$

is bijective.

The pair $(\Gamma_m M, \gamma_m)$ is unique up to a unique isomorphism. The A -algebra $\Gamma_m M$ is called the algebra of m -divided powers of M and it is canonically \mathbb{N} -graded with $\Gamma_m^0 M = A$, $\Gamma_m^1 M = M$. In the case $m = \infty$, $(\Gamma_\infty M, \gamma_\infty)$ is simply denoted by $(\Gamma M, \gamma)$ and it is called the algebra of divided powers of M .

In this way Γ_m becomes a functor from the category of A -modules to the category of (\mathbb{N} -graded) A -algebras, which is left adjoint to \mathcal{E}_m . For $1 \leq m \leq q \leq \infty$ the truncations $\mathcal{E}_q \rightarrow \mathcal{E}_m$ induce natural transformations $\Gamma_m \rightarrow \Gamma_q$.

For any A -module M and any integer $m \geq 1$ there is a canonical morphism of graded A -algebras $\text{Sym } M \rightarrow \Gamma_m M$, which is an isomorphism provided that $m!$ is invertible in A . In particular, $\text{Sym } M \simeq \Gamma_1 M$. In the case $m = \infty$ there is also a canonical morphism of graded A -algebras $\text{Sym } M \rightarrow \Gamma M$, which is an isomorphism provided that $\mathbb{Q} \subset A$.

The algebra ΓM has another important structure which we will recall for the ease of the reader.

DEFINITION 1.10. — ([12], [2], § 3) Let $I \subset B$ be an ideal. A divided power structure (or a system of divided powers) on I is a collection of maps $\varrho_i : I \rightarrow B$, $i \geq 0$, such that for all $x, y \in I$, $\lambda \in B$:

- (1) $\varrho_0(x) = 1$, $\varrho_1(x) = x$ and $\varrho_i(x) \in I$ for all $i \geq 1$.
- (2) $\varrho_k(x + y) = \sum_{i+j=k} \varrho_i(x)\varrho_j(y)$.
- (3) $\varrho_k(\lambda x) = \lambda^k \varrho_k(x)$.
- (4) $\varrho_i(x)\varrho_j(x) = \binom{i+j}{i} \varrho_{i+j}(x)$.
- (5) $\varrho_i(\varrho_j(x)) = \frac{(ij)!}{i!(j!)^i} \varrho_{ij}(x)$.

A such object $(B, I, \{\varrho_i\})$ is called a P.D. ring. A P.D. A -algebra is a P.D. ring which is also an A -algebra.

Morphisms between P.D. rings (or P.D. A -algebras) are defined in the obvious way.

Let us define $\gamma_i^0 : M \rightarrow \Gamma M$, $i \geq 0$, by $x \in M \mapsto \gamma(x) = \sum_{i=0}^\infty \gamma_i^0(x)t^i \in \mathcal{E}(\Gamma M)$. We have $\gamma_i^0(M) \subset \Gamma^i M$. Let us write $\Gamma^+ M$ for the ideal of ΓM generated by homogeneous elements of strictly positive degree, and let us note that $\gamma_1^0 : M \rightarrow \Gamma^+ M$ is an A -linear map. The following result is proved in [12] (see also [2], App. A).

THEOREM 1.11. — Under the above hypotheses, the following properties hold:

- (1) The $\{\gamma_i^0\}$ extend to a unique divided power structure on $\Gamma^+ M$, denoted by $\{\gamma_i\}$.

- (2) *The P.D. A-algebra $(\Gamma M, \Gamma^+ M, \{\gamma_i\})$ and the linear map $\gamma_1^0 : M \rightarrow \Gamma^+ M$ have the following universal property: If $(B, J, \{\varrho_i\})$ is a P.D. A-algebra and $\psi : M \rightarrow J$ is a A-linear map there is a unique morphism of P.D. A-algebras $\tilde{\psi} : (\Gamma M, \Gamma^+ M, \{\gamma_i\}) \rightarrow (B, J, \{\varrho_i\})$ such that $\tilde{\psi} \circ \gamma_1^0 = \psi$.*

Apart from the canonical morphism $\text{Sym } M \rightarrow \Gamma M$, there is another way to relate symmetric algebras with algebras of divided powers (see for instance [8] and [3], A2.4). Given an A-module M, the symmetric algebra $\text{Sym } M$ has a coproduct given by the homomorphism of graded A-algebras

$$\Delta : \text{Sym } M \rightarrow \text{Sym } M \otimes_A \text{Sym } M (\simeq \text{Sym}(M \oplus M))$$

induced by the diagonal map $M \rightarrow M \oplus M: \Delta(m) = m \otimes 1 + 1 \otimes m$ for any $m \in M$. Let us consider the *graded dual* of $\text{Sym } M$ as

$$(\text{Sym } M)_{\text{gr}}^* := \bigoplus_{i=0}^{\infty} (\text{Sym}^i M)^*,$$

where $(\text{Sym}^i M)^*$ is the dual A-module $\text{Hom}_A(\text{Sym}^i M, A)$. It is well known that $(\text{Sym } M)_{\text{gr}}^*$ becomes a (commutative) graded A-algebra by defining the *shuffle product* through the transposed map of Δ . Explicitly, the shuffle product of $u \in (\text{Sym}^i M)^*$ and $v \in (\text{Sym}^j M)^*$ is $u \star v \in (\text{Sym}^{i+j} M)^*$ given by

$$(1.7) \quad (u \star v) \left(\prod_{l=1}^{i+j} x_l \right) = \dots = \sum_{\substack{L \subset [i+j] \\ \#L=i}} u(x_L)v(x_{L'})$$

for any $x_1, \dots, x_{i+j} \in M$, where $L' = [i+j] \setminus L$.

For any element $w \in M^*$ and any integer $i > 0$, let $\zeta_i(w) \in (\text{Sym}^i M)^*$ be the linear form defined by

$$\zeta_i(w) \left(\prod_{l=1}^i x_l \right) = \prod_{l=1}^i \langle x_l, w \rangle, \quad \forall x_1, \dots, x_i \in M.$$

For $i = 0$ let us define $\zeta_0(w) = 1 \in A = (\text{Sym}^0 M)^*$. The element $\zeta(w) = \sum_{i=0}^{\infty} \zeta_i(w)t^i$ in $(\text{Sym } M)_{\text{gr}}^*[[t]]$ is of exponential type and the map

$$(1.8) \quad \zeta : w \in M^* \mapsto \zeta(w) \in \mathcal{E} \left((\text{Sym } M)_{\text{gr}}^* \right)$$

is A-linear. So, it induces a canonical homomorphism of graded A-algebras

$$(1.9) \quad \phi : \Gamma M^* \rightarrow (\text{Sym } M)_{\text{gr}}^*.$$

The homomorphism ϕ is an isomorphism if M is a projective module of finite rank (cf. [2], prop. A10). In fact we have the following more general result.

PROPOSITION 1.12. — *The above homomorphism ϕ is an isomorphism if M^* is a projective module of finite rank.*

Proof. — The proposition is a consequence of the fact that, if M^* is a projective module of finite rank, then the canonical homomorphism of graded A -algebras

$$(\text{Sym } M^{**})_{\text{gr}}^* \rightarrow (\text{Sym } M)_{\text{gr}}^*$$

is an isomorphism, or equivalently, for any $r \geq 1$ the canonical A -linear map $(\text{Sym}^r M^{**})^* \rightarrow (\text{Sym}^r M)^*$ is an isomorphism. The case $r = 1$ is clear.

We have canonical isomorphisms

$$(M^{\otimes r})^* \simeq \text{Hom}_A(M^{\otimes(r-1)}, M^*) \simeq (M^{\otimes(r-1)})^* \otimes_A M^*,$$

where the last one comes from the hypothesis on M^* , and so we find by induction on r that $((M^{**})^{\otimes r})^* \simeq (M^{\otimes r})^*$.

For any A -module N , let us consider the natural right exact sequences of A -modules

$$(N^{\otimes r})^{r-1} \xrightarrow{H} N^{\otimes r} \rightarrow \text{Sym}^r N \rightarrow 0, \quad r \geq 2,$$

where $H(t_1, \dots, t_{r-1}) = \sum H_i(t_i)$ and

$$H_i(n_1 \otimes \dots \otimes n_r) = n_1 \otimes \dots \otimes n_{i-1} \otimes (n_i \otimes n_{i+1} - n_{i+1} \otimes n_i) \otimes \dots \otimes n_r.$$

By taking A -duals we obtain natural left exact sequences

$$0 \rightarrow (\text{Sym}^r N)^* \rightarrow (N^{\otimes r})^* \xrightarrow{H^*} \left((N^{\otimes r})^{r-1} \right)^*, \quad r \geq 2.$$

By considering the cases $N = M$ and $N = M^{**}$ and the natural isomorphisms $((M^{**})^{\otimes r})^* \simeq (M^{\otimes r})^*$, we deduce that $(\text{Sym}^r M^{**})^* \simeq (\text{Sym}^r M)^*$. □

In fact, it is possible to define a canonical divided power structure on the A -algebra $(\text{Sym } M)_{\text{gr}}^*$, or more precisely, on the ideal generated by homogeneous elements of strictly positive degree. The case where M is free is treated in [3], proposition–definition A2.6. We will briefly sketch the general case.

Let us write for simplicity $B = (\text{Sym } M)_{\text{gr}}^*$ and $B^+ = \bigoplus_{d \geq 1} B^d \subset B$. We need to define a collection of maps $\varrho_i : B^+ \rightarrow B$, $i \geq 0$, satisfying the properties in definition 1.10. It is enough to define the restrictions $\varrho_{i,d} : B^d \rightarrow B^{id}$, $d \geq 1$.

Let us denote by $\mathcal{P}(i, d)$ the set of (unordered) partitions of $[di] = \{1, \dots, di\}$ formed by i subsets with d elements each one, i.e., an element $\mathcal{L} \in \mathcal{P}(i, d)$ is a subset $\mathcal{L} \subset \mathcal{P}([di])$ with $\#\mathcal{L} = i$, $\#L = d$ for all $L \in \mathcal{L}$ and $L \cap L' = \emptyset$ whenever $L, L' \in \mathcal{L}$ and $L \neq L'$.

Let us also denote by $\tilde{\mathcal{P}}(i, d)$ the set of ordered partitions of $[di] = \{1, \dots, di\}$ formed by i subsets with d elements each one, i.e., an element $\underline{L} \in \tilde{\mathcal{P}}(i, d)$ is $\underline{L} = (L_1, \dots, L_i)$ with $\mathcal{L}(L) := \{L_1, \dots, L_i\} \in \mathcal{P}(i, d)$.

The map $\underline{L} \in \tilde{\mathcal{P}}(i, d) \mapsto \mathcal{L}(L) \in \mathcal{P}(i, d)$ is clearly the quotient map by the action of the symmetric group \mathfrak{S}_i on $\tilde{\mathcal{P}}(i, d)$.

Given an element $u \in B^d = (\text{Sym}^d M)^*$, we define $\varrho_{i,d}(u) \in (\text{Sym}^{di} M)^*$ by

$$\varrho_{i,d}(u) \left(\prod_{l=1}^{di} x_l \right) = \sum_{\mathcal{L} \in \mathcal{P}(i,d)} \prod_{L \in \mathcal{L}} u(x_L).$$

Let us note that if $u_1, \dots, u_i \in (\text{Sym}^d M)^*$, then

$$(u_1 \star \dots \star u_i) \left(\prod_{l=1}^{di} x_l \right) = \sum_{\underline{L} \in \tilde{\mathcal{P}}(i,d)} \prod_{j=1}^i u_j(x_{L_j})$$

and so $u^{\star i} = i! \varrho_{i,d}(u)$.

The proof of the following proposition is left up to the reader.

PROPOSITION 1.13. — *The maps $\varrho_{i,d} : (\text{Sym}^d M)^* \rightarrow (\text{Sym}^{id} M)^*$ defined above extend uniquely to a system of divided powers on $B^+ = \bigoplus_{d>0} (\text{Sym}^d M)^*$, $\varrho_i : B^+ \rightarrow (\text{Sym} M)_{\text{gr}}^*$, $i \geq 0$.*

The preceding proposition joint with the universal property of theorem 1.11 gives another way to construct the canonical homomorphism (1.9).

In the case where A is a k -algebra and $M = \Omega_{A/k}$, the homomorphism (1.9) has an interesting (and obvious) interpretation in terms of multi-derivations.

DEFINITION 1.14. — *Let M be an A -module and $r \geq 1$ an integer. A k -multiderivation from A^r to M is a k -multilinear map $h : A^r \rightarrow M$ such that for any $i = 1, \dots, r$ and any $a_j \in A$ with $j \neq i$, the map*

$$x \in A \mapsto h(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_r) \in M$$

is a k -derivation. We say that a k -multiderivation $h : A^r \rightarrow M$ is symmetric if it is so as multilinear map.

Let us denote by $\text{Der}_k^r(A, M)$ (resp. $\text{SDer}_k^r(A, M)$) the set of k -multiderivations (resp. symmetric k -multiderivations) from A^r to M . If $M = A$ we

will write $\text{Der}_k^r(A)$ (resp. $\text{SDer}_k^r(A)$) instead of $\text{Der}_k^r(A, A)$ (resp. instead of $\text{SDer}_k^r(A, M)$). It is clear that $\text{Der}_k^r(A, M)$ is an A -module and that $\text{SDer}_k^r(A, M)$ is a submodule of $\text{Der}_k^r(A, M)$.

PROPOSITION 1.15. — (a) For each A -linear map $\tilde{h} : \Omega_{A/k}^{\otimes r} \rightarrow M$, the map $h : A^r \rightarrow M$ defined by

$$h(x_1, \dots, x_r) := \tilde{h}(dx_1 \otimes \dots \otimes dx_r)$$

is a k -multiderivation and the map

$$\tilde{h} \in \text{Hom}_A(\Omega_{A/k}^{\otimes r}, M) \mapsto h \in \text{Der}_k^r(A, M)$$

is an isomorphism of A -modules.

(b) For each A -linear map $\tilde{h} : \text{Sym}^r \Omega_{A/k} \rightarrow M$, the map $h : A^r \rightarrow M$ defined by

$$h(x_1, \dots, x_r) := \tilde{h}(dx_1 \cdots dx_r)$$

is a symmetric k -multiderivation and the map

$$\tilde{h} \in \text{Hom}_A(\text{Sym}^r \Omega_{A/k}, M) \mapsto h \in \text{SDer}_k^r(A, M)$$

is an isomorphism of A -modules.

Proof. — Part (a) is clear for $r = 1$. For $r \geq 2$ we proceed inductively by using the obvious A -linear isomorphism

$$\text{Der}_k^r(A, M) \simeq \text{Der}_k(A, \text{Der}_k^{r-1}(A, M)).$$

Part (b) is a straightforward consequence of part (a) and the fact that symmetric maps $\tilde{h} : \Omega_{A/k}^{\otimes r} \rightarrow M$ are characterized by factoring through $\text{Sym}^r \Omega_{A/k}$. □

The A -module of symmetric k -multiderivations of A is by definition the graded A -module

$$\text{SDer}_k^\bullet(A) = \bigoplus_{r=0}^{\infty} \text{SDer}_k^r(A).$$

From the above proposition, there is a natural graded A -linear isomorphism $(\text{Sym} \Omega_{A/k})_{\text{gr}}^* \simeq \text{SDer}_k^\bullet(A)$ and we can transfer the shuffle product from the left side to the right side in the following way: given $h : A^n \rightarrow A$, $h' : A^m \rightarrow A$ symmetric k -multiderivations, their shuffle product $h \star h' : A^{n+m} \rightarrow A$ is defined by (cf. [5], 2)

$$(h \star h')(x_1, \dots, x_{n+m}) = \sum_{\substack{L \subset [n+m] \\ \#L=n}} h(x_{L_1}, \dots, x_{L_n}) h'(x_{L'_1}, \dots, x_{L'_m}),$$

where $L' = [n+m] \setminus L$ and M_i stands for the i^{th} element of $M \subset [n+m]$ with respect to the induced ordering.

For example, if $\delta, \delta' : A \rightarrow A$ are k -derivations, then

$$(\delta \star \delta')(x_1, x_2) = \delta(x_1)\delta'(x_2) + \delta(x_2)\delta'(x_1).$$

In that way $\text{SDer}_k^\bullet(A)$ is a graded commutative A -algebra canonically isomorphic to $(\text{Sym } \Omega_{A/k})_{\text{gr}}^*$.

Remark 1.16. — In the case $M = \Omega_{A/k}$, the homomorphism (1.9) can be interpreted as $\phi : \Gamma \text{Der}_k(A) \rightarrow \text{SDer}_k^\bullet(A)$ determined by the A -linear map $\zeta : \text{Der}_k(A) \rightarrow \mathcal{E}(\text{SDer}_k^\bullet(A))$ defined by

$$(1.10) \quad \zeta(\delta) = \sum_{r=0}^{\infty} \zeta_r(\delta)t^r, \quad \delta \in \text{Der}_k(A),$$

where $\zeta_r(\delta) \in \text{SDer}_k^r(A)$ is given by

$$\zeta_r(\delta)(x_1, \dots, x_r) = \prod_{i=1}^r \delta(x_i), \quad \forall x_1, \dots, x_r \in A.$$

So, $\phi : \Gamma \text{Der}_k(A) \rightarrow \text{SDer}_k^\bullet(A)$ is an isomorphism if $\text{Der}_k(A)$ is a projective module of finite rank.

Remark 1.17. — We can define a unique homogeneous “Poisson bracket” on $\text{SDer}_k^\bullet(A)$ extending the usual Lie bracket on $\text{SDer}_k^1(A) = \text{Der}_k(A)$ and such that $\{h, a\}(x_1, \dots, x_{r-1}) = h(x_1, \dots, x_{r-1}, a)$ for any $h \in \text{SDer}_k^r(A)$ and any $a \in A = \text{SDer}_k^0(A)$. Namely, given $h \in \text{SDer}_k^r(A), h' \in \text{SDer}_k^s(A)$ we define $\{h, h'\} \in \text{SDer}_k^{r+s-1}(A)$ in the following way:

$$\begin{aligned} \{h, h'\}(x_1, \dots, x_{r+s-1}) &= \sum_{\substack{L \subset [r+s-1] \\ \#L=s}} h(x_{L'_1}, \dots, x_{L'_{r-1}}, h'(x_{L_1}, \dots, x_{L_s})) \\ &\quad - \sum_{\substack{M \subset [r+s-1] \\ \#M=r}} h'(x_{M'_1}, \dots, x_{M'_{s-1}}, h(x_{M_1}, \dots, x_{M_r})), \end{aligned}$$

where $L' = [r + s - 1] \setminus L, M' = [r + s - 1] \setminus M$. For instance, if $\delta \in \text{Der}_k(A) = \text{SDer}_k^1(A)$ then

$$\begin{aligned} \{h, \delta\}(x_1, \dots, x_r) &= h(\delta(x_1), \dots, x_r) + \dots + h(x_1, \dots, \delta(x_r)) \\ &\quad - \delta(h(x_1, \dots, x_r)). \end{aligned}$$

In that way $\text{SDer}_k^\bullet(A)$ becomes a Poisson algebra over k , but this structure will not be used in this paper.

2. Main Results

2.1. The embedding $\theta_{A/k} : \text{gr Diff}_{A/k} \hookrightarrow (\text{Sym } \Omega_{A/k})_{\text{gr}}^*$

Let A be a fixed k -algebra. For the sake of simplicity, we will omit the subscript “ A/k ” everywhere: $\mathcal{P}^* = \mathcal{P}_{A/k}^*, I = I_{A/k}, d^* : A \rightarrow \mathcal{P}^*, \Omega = \Omega_{A/k}(= I/I^2), d : A \rightarrow \Omega$ the universal differential, $\text{Diff}^* = \text{Diff}_{A/k}^*$, etc. Let us denote by

$$(2.1) \quad v : \text{Sym } \Omega = \text{Sym } I/I^2 \rightarrow \text{gr}_I \mathcal{P} = \bigoplus_{i=0}^{\infty} I^i/I^{i+1}$$

the canonical epimorphism of graded A -algebras.

Let $n \geq 0$ be an integer. We know that the map $h \in \text{Hom}_A(\mathcal{P}^n, A) \mapsto h \circ d^n \in \text{Diff}^{(n)}$ is an isomorphism of A -bimodules. For each $P \in \text{Diff}^{(n)}$ denote by $\tilde{P} : \mathcal{P}^n \rightarrow A$ the unique (left) A -linear map such that $P = \tilde{P} \circ d^n$. Since $\text{gr}_I^n \mathcal{P} = I^n/I^{n+1} \hookrightarrow \mathcal{P}/I^{n+1} = \mathcal{P}^n$, we can consider the map $\lambda_n^0 : \text{Diff}^{(n)} \rightarrow \text{Hom}_A(\text{gr}_I^n \mathcal{P}, A)$ defined by $\lambda_n^0(P) = \tilde{P}|_{\text{gr}_I^n \mathcal{P}}$, which is obviously A -linear. By looking at the exact sequence

$$(2.2) \quad 0 \rightarrow \text{gr}_I^n \mathcal{P} \rightarrow \mathcal{P}^n \rightarrow \mathcal{P}^{n-1} = \mathcal{P}/I^n \rightarrow 0$$

we see that a $P \in \text{Diff}^{(n)}$ belongs to the kernel of λ_n^0 if and only if \tilde{P} vanishes on $\text{gr}_I^n \mathcal{P}$, i.e., if \tilde{P} factorizes through \mathcal{P}^{n-1} , or equivalently, $P \in \text{Diff}^{(n-1)}$. So, we obtain an injective linear map

$$(2.3) \quad \lambda_n : \text{gr}^n \text{Diff} \hookrightarrow \text{Hom}_A(\text{gr}_I^n \mathcal{P}, A).$$

By composing with the transposed map of the homogeneous component of degree n of v , we obtain an injective A -linear map

$$\theta_n = v_n^* \circ \lambda_n : \text{gr}^n \text{Diff} \hookrightarrow \text{Hom}_A(\text{Sym}^n \Omega, A).$$

For $n = 0$ we have $\text{gr}^0 \text{Diff} = A, \text{Hom}_A(\text{Sym}^0 \Omega, A) = \text{Hom}_A(A, A) = A$ and $\theta_0 = \text{Id}_A$, and for $n = 1, \text{gr}^1 \text{Diff} = \text{Der}_k(A), \text{Hom}_A(\text{Sym}^1 \Omega, A) = \text{Hom}_A(\Omega, A) = \text{Der}_k(A)$ and $\theta_1 = \text{Id}_{\text{Der}_k(A)}$.

PROPOSITION 2.1. — *Let $n \geq 1$ be an integer, $P \in \text{Diff}^{(n)}$ and $x_1, \dots, x_n \in A$. With the above notations, the following equalities hold*

$$\begin{aligned} \theta_n(\sigma_n(P))(dx_1 \cdots dx_n) &= \sum_{L \subset [n]} (-1)^{\#L} x_L P(x_{[n] \setminus L}) \\ &= [[\cdots [P, x_n], x_{n-1}], \dots, x_2], x_1]. \end{aligned}$$

Proof. — For the first equality, let $\tilde{P} : \mathcal{P}^n \rightarrow A$ be the unique (left) A -linear map such that $P = \tilde{P} \circ d^n$. We have

$$\begin{aligned} \theta_n(\sigma_n(P))(dx_1 \cdots dx_n) &= \lambda_n(\sigma_n(P))(v_n(dx_1 \cdots dx_n)) \\ &= \lambda_n(\sigma_n(P)) \left(\prod_{i=1}^n (1 \otimes x_i - x_i \otimes 1) + I^{n+1} \right) \\ &= \tilde{P} \left(\prod_{i=1}^n (1 \otimes x_i - x_i \otimes 1) + I^{n+1} \right) = \tilde{P} \left(\sum_{L \subset [n]} (-1)^{\#L} x_L \otimes x_{L'} + I^{n+1} \right) \\ &= \sum_{L \subset [n]} (-1)^{\#L} \tilde{P}(x_L d^n(x_{L'})) = \sum_{L \subset [n]} (-1)^{\#L} x_L P(x_{L'}), \end{aligned}$$

with $L' = [n] \setminus L$.

For the second equality,

$$\begin{aligned} \theta_n(\sigma_n(P))(dx_1 \cdots dx_n) &= \sum_{L \subset [n]} (-1)^{\#L} x_L P(x_{L'}) \\ &= \sum_{\substack{L \subset [n] \\ n \notin L}} (-1)^{\#L} x_L P(x_{[n] \setminus L}) + \sum_{\substack{L \subset [n] \\ n \in L}} (-1)^{\#L} x_L P(x_{[n] \setminus L}) \\ &= \sum_{L \subset [n-1]} (-1)^{\#L} x_L P(x_n x_{[n-1] \setminus L}) - \sum_{K \subset [n-1]} (-1)^{\#K} x_K x_n P(x_{[n-1] \setminus K}) \\ &= \sum_{L \subset [n-1]} (-1)^{\#L} x_L [P, x_n](x_{[n-1] \setminus L}) \end{aligned}$$

and so

$$(2.4) \quad \theta_n(\sigma_n(P))(dx_1 \cdots dx_n) = \theta_{n-1}(\sigma_{n-1}([P, x_n]))(dx_1 \cdots dx_{n-1}).$$

By iterating (2.4) we obtain

$$\theta_n(\sigma_n(P))(dx_1 \cdots dx_n) = [[\cdots [[P, x_n], x_{n-1}], \dots, x_2], x_1].$$

□

THEOREM 2.2. — *The A -linear map*

$$\bigoplus_{n \geq 0} \theta_n : \text{gr Diff} \rightarrow (\text{Sym } \Omega)_{\text{gr}}^*$$

is a homomorphism of graded A -algebras.

Proof. — We need to prove that $\theta_{n+m}(\sigma_n(P)\sigma_m(Q)) = \theta_n(\sigma_n(P)) \star \theta_m(\sigma_m(Q))$ for all integers $n, m \geq 0$ and all $P \in \text{Diff}^{(n)}$ and $Q \in \text{Diff}^{(m)}$. We proceed by induction on $n + m$. For $n + m = 0$, the result is clear since $\theta_0 = \text{Id}_A$.

Let us assume that $\theta_{r+s}(\sigma_r(P')\sigma_s(Q')) = \theta_r(\sigma_r(P')) \star \theta_s(\sigma_s(Q'))$ for all integers $r, s \geq 0$, $r + s < n + m$ and all $P' \in \text{Diff}^{(r)}$ and $Q' \in \text{Diff}^{(s)}$, and take $P \in \text{Diff}^{(n)}$, $Q \in \text{Diff}^{(m)}$, $x_1, \dots, x_{n+m} \in A$. From the definition of the shuffle product (see (1.7)), and writing $x' = x_{n+m}$, $(dx)_L = \prod_{i \in L} dx_i$, $L' = [n + m] \setminus L$, $N'' = [n + m - 1] \setminus N$, we have:

$$\begin{aligned} (\theta_n(\sigma_n(P)) \star \theta_m(\sigma_m(Q)))(dx_1 \cdots dx_{n+m}) &= \sum_{\substack{L \subset [n+m] \\ \#L=n}} \theta_n(\sigma_n(P))((dx)_L) \theta_m(\sigma_m(Q))((dx)_{L'}) \\ &= \sum_{\substack{L \subset [n+m] \\ \#L=n}} \left(\sum_{K \subset L} (-1)^{\#K} x_K P(x_{L \setminus K}) \right) \left(\sum_{M \subset L'} (-1)^{\#M} x_M Q(x_{L' \setminus M}) \right) \\ &= \sum_{\substack{L \subset [n+m] \\ \#L=n \\ K \subset L, M \subset L'}} (-1)^{\#(K \sqcup M)} x_{K \sqcup M} P(x_{L \setminus K}) Q(x_{L' \setminus M}) \\ &= \sum_{N \subset [n+m]} (-1)^{\#N} x_N \sum_{\substack{L \subset [n+m] \\ \#L=n}} P(x_{L \cap N'}) Q(x_{L' \cap N'}) \\ &= \left(\sum_{\substack{N \subset [n+m] \\ n+m \notin N}} (-1)^{\#N} x_N \sum_{\substack{L \subset [n+m] \\ \#L=n}} (\dots) \right) + \left(\sum_{\substack{N \subset [n+m] \\ n+m \in N}} (-1)^{\#N} x_N \sum_{\substack{L \subset [n+m] \\ \#L=n}} (\dots) \right). \end{aligned}$$

For the first summand, since $N \subset [n + m]$ with $n + m \notin N$, we have $N \subset [n + m - 1]$, $N' = N'' \cup \{n + m\}$ and

$$\begin{aligned} \sum_{\substack{L \subset [n+m] \\ \#L=n}} (\dots) &= \sum_{\substack{L \subset [n+m] \\ \#L=n}} P(x_{L \cap N'}) Q(x_{L' \cap N'}) \\ &= \left(\sum_{\substack{L \subset [n+m] \\ \#L=n \\ n+m \notin L}} P(x_{L \cap N''}) Q(x' x_{L' \cap N''}) \right) + \left(\sum_{\substack{L \subset [n+m] \\ \#L=n \\ n+m \in L}} P(x' x_{L \cap N''}) Q(x_{L' \cap N''}) \right) \\ &= \left(\underbrace{\sum_{\substack{L \subset [n+m-1] \\ \#L=n}} P(x_{L \cap N''}) Q(x' x_{L' \cap N''})}_{A_N} \right) + \left(\underbrace{\sum_{\substack{K \subset [n+m-1] \\ \#K=n-1}} P(x' x_{K \cap N''}) Q(x_{K'' \cap N''})}_{B_N} \right). \end{aligned}$$

For the second summand, since $N \subset [n + m]$ with $n + m \in N$, we have $N = H \sqcup \{n + m\}$ with $H \subset [n + m - 1]$, $x_N = x_H x'$, $N' = H''$ and

$$\begin{aligned} \sum_{\substack{LC[n+m] \\ \#L=n}} (\dots) &= \sum_{\substack{LC[n+m] \\ \#L=n}} P(x_{L \cap N'}) Q(x_{L' \cap N'}) \\ &= \left(\sum_{\substack{LC[n+m] \\ \#L=n \\ n+m \notin L}} P(x_{L \cap H''}) Q(x_{L' \cap H''}) \right) + \left(\sum_{\substack{LC[n+m] \\ \#L=n \\ n+m \in L}} P(x_{L \cap H''}) Q(x_{L' \cap H''}) \right) \\ &= \underbrace{\left(\sum_{\substack{LC[n+m-1] \\ \#L=n}} P(x_{L \cap H''}) Q(x_{L'' \cap H''}) \right)}_{C_H} + \underbrace{\left(\sum_{\substack{K \subset [n+m-1] \\ \#K=n-1}} P(x_{K \cap H''}) Q(x_{K'' \cap H''}) \right)}_{D_H}. \end{aligned}$$

Putting all together

$$\begin{aligned} &(\theta_n(\sigma_n(P)) \star \theta_m(\sigma_m(Q)))(dx_1 \cdots dx_{n+m}) = \\ &\dots\dots\dots \\ &\left(\sum_{\substack{N \subset [n+m] \\ n+m \notin N}} (-1)^{\#N} x_N \sum_{\substack{LC[n+m] \\ \#L=n}} (\dots) \right) + \left(\sum_{\substack{N \subset [n+m] \\ n+m \in N}} (-1)^{\#N} x_N \sum_{\substack{LC[n+m] \\ \#L=n}} (\dots) \right) \\ &= \left(\sum_{N \subset [n+m-1]} (-1)^{\#N} x_N (A_N + B_N) \right) - \left(\sum_{H \subset [n+m-1]} (-1)^{\#H} x_H x' (C_H + D_H) \right) \\ &= \sum_{N \subset [n+m-1]} (-1)^{\#N} x_N (B_N - x' C_N) + \sum_{N \subset [n+m-1]} (-1)^{\#N} x_N (A_N - x' D_N) \\ &= \sum_{N \subset [n+m-1]} (-1)^{\#N} \sum_{\substack{K \subset [n+m-1] \\ \#K=n-1}} [P, x'] (x_{K \cap N''}) Q(x_{K'' \cap N''}) \\ &\quad + \sum_{N \subset [n+m-1]} (-1)^{\#N} \sum_{\substack{LC[n+m-1] \\ \#L=n}} P(x_{L \cap N''}) [Q, x'] (x_{L'' \cap N''}) \\ &= (\theta_{n-1}(\sigma_{n-1}([P, x'])) \star \theta_m(\sigma_m(Q)))(dx_1 \cdots dx_{n+m-1}) \\ &\quad + (\theta_n(\sigma_n(P)) \star \theta_{m-1}(\sigma_{m-1}([Q, x'])))(dx_1 \cdots dx_{n+m-1}). \end{aligned}$$

On the other hand, from the induction hypothesis

$$\begin{aligned}
 &\theta_{n+m-1}(\sigma_{n+m-1}([P \circ Q, x'])) \\
 &= \theta_{n+m-1}(\sigma_{n+m-1}(P \circ [Q, x'])) + \theta_{n+m-1}(\sigma_{n+m-1}([P, x'] \circ Q)) \\
 &= \theta_{n+m-1}(\sigma_n(P)\sigma_{m-1}([Q, x'])) + \theta_{n+m-1}(\sigma_{n-1}([P, x'])\sigma_m(Q)) \\
 &= \theta_n(\sigma_n(P)) \star \theta_{m-1}(\sigma_{m-1}([Q, x'])) + \theta_{n-1}(\sigma_{n-1}([P, x'])) \star \theta_m(\sigma_m(Q))
 \end{aligned}$$

and from proposition 2.1 we conclude that

$$\begin{aligned}
 &(\theta_n(\sigma_n(P)) \star \theta_m(\sigma_m(Q)))(dx_1 \cdots dx_{n+m}) = \\
 &\dots\dots\dots \\
 &(\theta_{n-1}(\sigma_{n-1}([P, x'])) \star \theta_m(\sigma_m(Q)))(dx_1 \cdots dx_{n+m-1}) \\
 &+ (\theta_n(\sigma_n(P)) \star \theta_{m-1}(\sigma_{m-1}([Q, x'])))(dx_1 \cdots dx_{n+m-1}) \\
 &= (\theta_{n+m-1}(\sigma_{n+m-1}([P \circ Q, x_{n+m}])))(dx_1 \cdots dx_{n+m-1}) \\
 &= \theta_{n+m}(\sigma_{n+m}(P \circ Q))(dx_1 \cdots dx_{n+m}) \\
 &= \theta_{n+m}(\sigma_n(P) \sigma_m(Q))(dx_1 \cdots dx_{n+m})
 \end{aligned}$$

and so $\theta_{n+m}(\sigma_n(P) \sigma_m(Q)) = \theta_n(\sigma_n(P)) \star \theta_m(\sigma_m(Q))$. □

We will denote

$$(2.5) \quad \theta_{A/k} = \bigoplus_{n \geq 0} \theta_n : \text{gr Diff}_{A/k} \hookrightarrow (\text{Sym } \Omega_{A/k})_{\text{gr}}^*$$

the homomorphism of theorem 2.2. Let us recall (see proposition 1.13) that $(\text{Sym } \Omega_{A/k})_{\text{gr}}^*$ has a canonical divided power structure.

Remark 2.3. — By using the Poisson bracket (see def. 1.7), proposition 1.15 and (2.4), the morphism $\theta_{A/k}$ can be interpreted as a homomorphism of graded A -algebras $\theta_{A/k} : \text{gr Diff}_{A/k} \rightarrow \text{SDer}_k^\bullet(A)$ given by

$$\theta_{A/k}(F)(x_1, \dots, x_n) = \{ \{ \cdots \{ \{ F, x_n \}, x_{n-1} \}, \cdots, x_2 \}, x_1 \}$$

for all $F \in \text{gr Diff}_{A/k}^{(n)}$ and all $x_1, \dots, x_n \in A$. One can see that $\theta_{A/k}$ is compatible with the Poisson bracket in $\text{SDer}_k^\bullet(A)$ described in remark 1.17.

2.2. The total symbol of a Hasse-Schmidt derivation

Let A be a fixed k -algebra. In this section, we will see how the diagram

$$\begin{array}{ccc} \text{gr Diff}_{A/k} & \xrightarrow{\theta_{A/k}} & (\text{Sym } \Omega_{A/k})_{\text{gr}}^* \\ & & \uparrow \phi \\ \Gamma \text{IDer}_k(A) & \xrightarrow{\text{nat.}} & \Gamma \text{Der}_k(A) \end{array}$$

can be completed up to a commutative diagram by defining a homomorphism of graded A -algebras $\vartheta_{A/k} : \Gamma \text{IDer}_k(A) \rightarrow \text{gr Diff}_{A/k}$.

DEFINITION 2.4. — For any Hasse-Schmidt derivation $D \in \text{HS}_k(A; m)$ we define its total symbol by

$$\Sigma_m(D) = \sum_{i=0}^m \sigma_i(D_i)t^i \in (\text{gr Diff}_{A/k})_m = (\text{gr Diff}_{A/k})[[t]]/(t^{m+1}).$$

It is clear that $\Sigma_m(D)$ is a unit and that the total symbol map Σ_m is a group homomorphism from $\text{HS}_k(A; m)$ to the multiplicative group of units of $(\text{gr Diff}_{A/k})_m$. In fact we have a more precise result.

PROPOSITION 2.5. — For any $D \in \text{HS}_k(A; m)$, the total symbol $\Sigma_m(D)$ is of exponential type in $(\text{gr Diff}_{A/k})_m$ and for any $a \in A$ we have $\Sigma_m(a \bullet D) = a \Sigma_m(D)$.

Proof. — The equality $\Sigma_m(a \bullet D) = a \Sigma_m(D)$ is clear. To prove the equality

$$\binom{r+s}{r} \sigma_{r+s}(D_{r+s}) = \sigma_r(D_r) \sigma_s(D_s), \quad \forall r, s \geq 0, r+s < m+1,$$

we need to prove that $\binom{r+s}{r} D_{r+s} - D_r \circ D_s \in \text{Diff}_{A/k}^{(r+s-1)}$ (Hasse-Schmidt derivations for which the equality $\binom{r+s}{r} D_{r+s} = D_r \circ D_s$ holds are called *iterative* [10], §27). We proceed by induction on $r+s$. For $r = s = 0$ the result is clear. Let us assume that $D_i \circ D_j - \binom{i+j}{i} D_{i+j} \in \text{Diff}_{A/k}^{(i+j-1)}$ for $i+j < r+s$.

Let us write $P = D_r \circ D_s - \binom{r+s}{r} D_{r+s}$. For each $a \in A$ we have

$$\begin{aligned}
 [P, a] &= D_r \circ [D_s, a] + [D_r, a] \circ D_s - \binom{r+s}{r} [D_{r+s}, a] \\
 &= D_r \circ \sum_{i=0}^{s-1} D_{s-i}(a) D_i + \sum_{j=0}^{r-1} D_{r-j}(a) D_j \circ D_s \\
 &\quad - \binom{r+s}{r} \sum_{k=0}^{r+s-1} D_{r+s-k}(a) D_k \\
 &= \sum_{\substack{0 \leq i \leq s-1 \\ 0 \leq q \leq r}} D_{r-q}(D_{s-i}(a)) D_q \circ D_i + \sum_{j=0}^{r-1} D_{r-j}(a) D_j \circ D_s \\
 &\quad - \binom{r+s}{r} \sum_{k=0}^{r+s-1} D_{r+s-k}(a) D_k.
 \end{aligned}$$

The only summands of possible order $r + s - 1$ in the above expression are those corresponding to $i = s - 1, q = r, j = r - 1$ y $k = r + s - 1$ and their sum

$$\begin{aligned}
 &D_1(a) D_r \circ D_{s-1} + D_1(a) D_{r-1} \circ D_s - \binom{r+s}{r} D_1(a) D_{r+s-1} \\
 &= D_1(a) \left[D_r \circ D_{s-1} + D_{r-1} \circ D_s - \binom{r+s}{r} D_{r+s-1} \right] \\
 &= D_1(a) \left[D_r \circ D_{s-1} + D_{r-1} \circ D_s - \left[\binom{r+s-1}{s-1} + \binom{r+s-1}{s} \right] D_{r+s-1} \right]
 \end{aligned}$$

has order $\leq n+m-2$ by the induction hypothesis. Hence, $[P, a] \in \text{Diff}_{A/k}^{(r+s-2)}$ for all $a \in A$ and so $P \in \text{Diff}_{A/k}^{(r+s-1)}$. □

Total symbol maps $\Sigma_m : \text{HS}_k(A; m) \rightarrow \mathcal{E}_m(\text{gr Diff}_{A/k})$ turn out to be group homomorphisms and for $1 \leq m \leq q \leq \infty$ the following diagram is commutative:

$$\begin{array}{ccc}
 \text{HS}_k(A; q) & \xrightarrow{\Sigma_q} & \mathcal{E}_q(\text{gr Diff}_{A/k}) \\
 \tau_{qm} \downarrow & & \downarrow \text{truncation} \\
 \text{HS}_k(A; m) & \xrightarrow{\Sigma_m} & \mathcal{E}_m(\text{gr Diff}_{A/k}).
 \end{array}$$

PROPOSITION 2.6. — *The total symbol map Σ_m vanishes on $\ker \tau_{m1}$ (here “vanishes” means that the restriction of Σ_m to $\ker \tau_{m1}$ is constant equal to 1, since the target of Σ_m is the group of units of $(\text{gr Diff}_{A/k})_m$).*

Proof. — For any $D \in \ker \tau_{m_1}$ we have $D_1 = 0$, and so $D_1 \in F^0 \text{Diff}_{A/k}$ and $\sigma_1(D_1) = 0$. From (1.4) we deduce inductively that $D_i \in F^{i-1} \text{Diff}_{A/k}$, and so $\sigma_i(D_i) = 0$, for all $i > 0$ and $\Sigma_m(D) = 1$. \square

COROLLARY 2.7. — *The total symbol map $\Sigma_m : \text{HS}_k(A; m) \rightarrow \mathcal{E}_m(\text{gr Diff}_{A/k})$ induces an A -linear map $\chi_m : \text{IDer}_k(A; m) \rightarrow \mathcal{E}_m(\text{gr Diff}_{A/k})$.*

Proof. — The corollary is a consequence of the above proposition, the exact sequence (1.2) and the fact that $\Sigma_m(a \bullet D) = a \Sigma_m(D)$. \square

It is clear that, for $1 \leq m \leq q \leq \infty$, the following diagram is commutative:

$$\begin{CD} \text{IDer}_k(A; q) @>\chi_q>> \mathcal{E}_q(\text{gr Diff}_{A/k}) \\ @VVV @VV\text{truncation}V \\ \text{IDer}_k(A; m) @>\chi_m>> \mathcal{E}_m(\text{gr Diff}_{A/k}). \end{CD}$$

From the universal property of the algebras of divided powers (see proposition 1.9), we obtain canonical homomorphisms of graded A -algebras

$$(2.6) \quad \vartheta_{A/k,m} : \Gamma_m \text{IDer}_k(A; m) \rightarrow \text{gr Diff}_{A/k}.$$

In the case $m = \infty$, $\vartheta_{A/k,\infty}$ will be simply denoted by $\vartheta_{A/k} : \Gamma \text{IDer}_k(A) \rightarrow \text{gr Diff}_{A/k}$.

It is clear that for each m the following diagram is commutative:

$$(2.7) \quad \begin{CD} \Gamma_m \text{IDer}_k(A; m) @>\vartheta_{A/k,m}>> \text{gr Diff}_{A/k} \\ @. @. \\ \text{Sym IDer}_k(A; m) @>\text{nat.}>> \text{Sym Der}_k(A). \end{CD}$$

$\begin{array}{ccc} \uparrow \text{nat.} & & \uparrow \tau_{A/k} \\ \text{Sym IDer}_k(A; m) & & \text{Sym Der}_k(A) \end{array}$

THEOREM 2.8. — *With the above notations, the following diagram of graded A -algebras*

$$\begin{CD} \text{gr Diff}_{A/k} @>\theta_{A/k}>> (\text{Sym } \Omega_{A/k})_{\text{gr}}^* \\ @. @. \\ \Gamma \text{IDer}_k(A) @>\text{nat.}>> \Gamma \text{Der}_k(A) \end{CD}$$

$\begin{array}{ccc} \uparrow \vartheta_{A/k} & & \uparrow \phi \\ \Gamma \text{IDer}_k(A) & & \Gamma \text{Der}_k(A) \end{array}$

is commutative.

Proof. — For simplicity, we will omit the subscript “ A/k ”. By the universal property of the algebra of divided powers (see prop. 1.9), it is enough

to prove the commutativity of the following diagram of A -modules:

$$\begin{array}{ccc}
 \mathcal{E}(\text{gr Diff}) & \xrightarrow{\mathcal{E}(\theta)} & \mathcal{E}\left(\left(\text{Sym } \Omega_{\text{gr}}^*\right)\right) \\
 \uparrow \chi & & \uparrow \zeta \\
 \text{IDer}_k(A) & \xrightarrow{\text{inc.}} & \text{Der}_k(A),
 \end{array}$$

where ζ and χ have been defined in (1.8) and corollary 2.7 respectively.

Let $\delta \in \text{IDer}_k(A)$ be an integrable derivation, $D \in \text{HS}_k(A)$ with $\delta = D_1$ and $x_1, \dots, x_n \in A$. We have $\chi(\delta) = \sum_{n=0}^\infty \sigma_n(D_n)t^n$, $(\mathcal{E}(\theta) \circ \chi)(\delta) = \sum_{n=0}^\infty \theta_n(\sigma_n(D_n))t^n$ and

$$\theta_n(\sigma_n(D_n))(dx_1 \cdots dx_n) = [\cdots [[D_n, x_n], x_{n-1}], \dots, x_1]$$

(see proposition 2.1). On the other hand, $\zeta(\delta) = \sum_{n=0}^\infty \zeta_n(\delta)t^n$ with

$$\zeta_n(\delta)(dx_1 \cdots dx_n) = \langle dx_1, \delta \rangle \cdots \langle dx_n, \delta \rangle = \delta(x_1) \cdots \delta(x_n).$$

So, the theorem is a consequence of lemma 2.9. □

LEMMA 2.9. — *Let $D \in \text{HS}_k(A; m)$ be a Hasse–Schmidt derivation of length m . Then, for any integer $n = 1, \dots, m$ and all $x_1, \dots, x_n \in A$ we have:*

$$[\cdots [[D_n, x_n], x_{n-1}], \dots, x_1] = D_1(x_1) \cdots D_1(x_n).$$

Proof. — From the equality $[D_n, x_n] = \sum_{i=0}^{n-1} D_{n-i}(x_n)D_i$ (see (1.4)) and the fact that for each $i = 0, \dots, n - 2$, D_i is a differential operator of order $\leq i$ and hence $[\cdots [D_i, x_{n-1}], \dots, x_1] = 0$, we deduce

$$[\cdots [[D_n, x_n], x_{n-1}], \dots, x_1] = [\cdots [D_1(x_n)D_{n-1}, x_{n-1}], \dots, x_1].$$

We conclude by induction on n . □

Given a family $\mathbf{D} = \{D^i\}_{1 \leq i \leq n}$ of Hasse–Schmidt derivations of A over k , let us write $\mathbf{D}_\alpha = D_{\alpha_1}^1 \circ \cdots \circ D_{\alpha_n}^n$ for each $\alpha \in \mathbb{N}^n$. It is clear that

$$\mathbf{D}_\alpha(ab) = \sum_{\sigma+\rho=\alpha} \mathbf{D}_\sigma(a)\mathbf{D}_\rho(b), \quad \forall a, b \in A.$$

PROPOSITION 2.10. — *Assume that the map $\vartheta: \Gamma \text{IDer}_k(A) \rightarrow \text{gr Diff}_{A/k}$ is surjective (and so $\text{Der}_k(A) = \text{IDer}_k(A)$) and that $\underline{\delta} = \{D_1^1, \dots, D_1^n\}$ is a system of generators of $\text{IDer}_k(A) = \text{Der}_k(A)$. Then, any k -linear differential operator $P: A \rightarrow A$ of order $\leq d$ can be written as*

$$P = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq d}} a_\alpha \mathbf{D}_\alpha, \quad a_\alpha \in A.$$

Proof. — Let us denote by $\gamma : \text{IDer}_k(A) \rightarrow \mathcal{E}(\Gamma \text{IDer}_k(A))$ the canonical map and $\gamma(\varepsilon) = \sum_{j=0}^\infty \gamma_j(\varepsilon)t^j$. From the definition (2.6) of ϑ we have $\vartheta(\gamma_j(D_1^i)) = \sigma_j(D_1^i)$. We know that the homogenous part of degree d of the algebra of divided powers of $\text{IDer}_k(A)$ is generated by the system $\gamma_\alpha(\underline{\delta}) := \prod_{i=1}^n \gamma_{\alpha_i}(D_1^i)$ with $|\alpha| = d$, and so, since ϑ is surjective, the system $\sigma_d(\mathbf{D}_\alpha) = \vartheta(\gamma_\alpha(\underline{\delta}))$, $|\alpha| = d$, generates the homogeneous part of degree d of $\text{gr Diff}_{A/k}$. The proof of proposition goes then by induction on d , the case $d = 1$ being obvious. \square

2.3. Relationship with differential smoothness

PROPOSITION 2.11. — *If the homomorphism of graded A -algebras*

$$\theta_{A/k} : \text{gr Diff}_{A/k} \hookrightarrow (\text{Sym } \Omega_{A/k})_{\text{gr}}^* \equiv \text{SDer}_k^\bullet(A)$$

is surjective (and so an isomorphism), then $\text{IDer}_k(A) = \text{Der}_k(A)$.

Proof. — For simplicity, we will omit the subscript “ A/k ”.

Let $\delta \in \text{Der}_k(A)$ be a derivation. We will show by induction on $n \geq 0$ that there are $D_n \in \text{Diff}^{(n)}$ such that $D_0 = \text{Id}_A$, $D_1 = \delta$ and

$$(2.8) \quad [D_m, a] = \sum_{i=0}^{m-1} D_{m-i}(a)D_i, \quad \forall a \in A$$

for all $m \geq 1$. The case $n = 1$ is obvious.

Assume that there are $D_m \in \text{Diff}^{(m)}$, $m = 1, \dots, n - 1$, with $n \geq 2$, satisfying the equality (2.8). In other words, $(D_0, D_1, \dots, D_{n-1})$ is a Hasse-Schmidt derivation of length $n - 1$ with $D_1 = \delta$. From lemma 2.9 we know that

$$[\dots [[D_m, x_m], x_{m-1}], \dots, x_1] = \prod_{i=1}^m \delta(x_i), \quad \forall m = 1, \dots, n-1, \forall x_1, \dots, x_m \in A.$$

Since θ_n is an isomorphism, there is a $P^{(1)} \in \text{Diff}^{(n)}$, unique modulo $\text{Diff}^{(n-1)}$, such that (see proposition 2.1)

$$[\dots [[P^{(1)}, x_1], x_2], \dots, x_n] = \prod_{i=1}^n \delta(x_i), \quad \forall x_1, \dots, x_n \in A.$$

Therefore,

$$\begin{aligned} & [\dots [[P^{(1)}, x_1] - \delta(x_1)D_{n-1}, x_2], \dots, x_n] \\ &= [\dots [[P^{(1)}, x_1], x_2], \dots, x_n] - \delta(x_1)[\dots [D_{n-1}, x_2], \dots, x_n] \\ &= \prod_{i=1}^n \delta(x_i) - \delta(x_1) \prod_{i=2}^n \delta(x_i) = 0 \end{aligned}$$

for all $x_1, \dots, x_n \in A$, and so $[P^{(1)}, x_1] - \delta(x_1)D_{n-1} \in \text{Diff}^{(n-2)}$, and also

$$[P^{(1)}, x_1] - \sum_{i=1}^{n-1} D_{n-i}(x_1)D_i \in \text{Diff}^{(n-2)} \quad \forall x_1 \in A.$$

Assume that for any integer r with $1 \leq r \leq n - 2$ we have found $P^{(r)} \in \text{Diff}^{(n)}$ such that

$$[P^{(r)}, x_1] - \sum_{i=1}^{n-1} D_{n-i}(x_1)D_i \in \text{Diff}^{(n-r-1)}$$

for all $x_1 \in A$, and let us write

$$R(x_1) := [P^{(r)}, x_1] - \sum_{i=1}^{n-1} D_{n-i}(x_1)D_i \in \text{Diff}^{(n-r-1)}.$$

Let $h : A^{n-r} \rightarrow A$ be the k -multilinear map defined by

$$\begin{aligned} h(x_1, \dots, x_{n-r}) &= [\dots [R(x_1), x_2], \dots, x_{n-r}] = [\dots [R(x_1), x_2], \dots, x_{n-r}](1) \\ &= [\dots [[P^{(r)}, x_1], x_2], \dots, x_{n-r}](1) \\ &\quad - \sum_{i=n-r-1}^{n-1} D_{n-i}(x_1)[\dots [D_i, x_2], \dots, x_{n-r}](1). \end{aligned}$$

From lemma 2.12, we deduce that the second summand above is equal to

$$\sum_{i=n-r-1}^{n-1} D_{n-i}(x_1) \left(\sum_{\substack{\alpha \in \mathbb{N}^{n-r-1} \\ |\alpha|=i \\ \alpha_l > 0}} \prod_{l=1}^{n-r-1} D_{\alpha_l}(x_{l+1}) \right) = \sum_{\substack{\beta \in \mathbb{N}^{n-r} \\ |\beta|=n \\ \beta_i > 0}} \prod_{l=1}^{n-r} D_{\beta_l}(x_l)$$

and so it is symmetric in the variables x_1, \dots, x_{n-r} . From lemma 2.13 we conclude that h is symmetric. On the other hand, it is clear that, for $x_1, \dots, x_{n-r-1} \in A$ fixed, the map

$$x_{n-r} \in A \mapsto h(x_1, \dots, x_{n-r-1}, x_{n-r}) \in A$$

is a k -derivation. So, h is a symmetric k -multiderivation,

Since θ_{n-r} is an isomorphism, there is a $Q \in \text{Diff}^{(n-r)}$, unique modulo $\text{Diff}^{(n-r-1)}$, such that

$$h(x_1, \dots, x_{n-r}) = [\dots [Q, x_1], \dots, x_{n-r}], \quad \forall x_1, \dots, x_{n-r} \in A.$$

Taking $P^{(r+1)} := P^{(r)} - Q \in \text{Diff}^{(n)}$ and

$$R'(x_1) := [P^{(r+1)}, x_1] - \sum_{i=1}^{n-1} D_{n-i}(x_1)D_i = R(x_1) - [Q, x_1] \in \text{Diff}^{(n-r-1)},$$

we have

$$[\dots [R'(x_1), x_2], \dots, x_{n-r}] = \dots = h(x_1, \dots, x_{n-r}) - [\dots [Q, x_1], \dots, x_{n-r}] = 0$$

for all $x_1, \dots, x_{n-r} \in A$, and so

$$[P^{(r+1)}, x_1] - \sum_{i=1}^{n-1} D_{n-i}(x_1)D_i \in \text{Diff}^{(n-(r+1)-1)}, \quad \forall x_1 \in A.$$

After a finite number of steps, we find a $P^{(n-1)} \in \text{Diff}^{(n)}$ such that

$$S(x_1) := [P^{(n-1)}, x_1] - \sum_{i=1}^{n-1} D_{n-i}(x_1)D_i \in \text{Diff}^{(0)} = A, \quad \forall x_1 \in A.$$

To conclude, we define $D_n = P^{(n-1)} - P^{(n-1)}(1)$ and we have

$$\begin{aligned} [D_n, x_1] &= [P^{(n-1)}, x_1] = \sum_{i=1}^{n-1} D_{n-i}(x_1)D_i + S(x_1) \\ &= \sum_{i=1}^{n-1} D_{n-i}(x_1)D_i + S(x_1)(1) = \sum_{i=1}^{n-1} D_{n-i}(x_1)D_i + [P^{(n-1)}, x_1](1) \\ &= \sum_{i=1}^{n-1} D_{n-i}(x_1)D_i + P^{(n-1)}(x_1) - x_1 P^{(n-1)}(1) \\ &= \sum_{i=0}^{n-1} D_{n-i}(x_1)D_i, \quad \forall x_1 \in A. \end{aligned}$$

It is clear that the sequence $\{D_n\}_{n \geq 0}$ defined in that way is a Hasse-Schmidt derivation with $D_1 = \delta$ and so δ is integrable. □

The following lemma generalizes the equality 1.4 and its proof goes by induction on k .

LEMMA 2.12. — *For any Hasse-Schmidt derivation $D \in \text{HS}_k(A; m)$ of length m , for any integer $k = 1, \dots, m$ and for any $x_1, \dots, x_k \in A$ the*

following equality holds

$$[\cdots [D_m, x_1], \dots, x_k] = \sum_{j=0}^{m-k} \left(\sum_{\substack{\alpha \in \mathbb{N}^k \\ |\alpha|=m-j \\ \alpha_i > 0}} \prod_{i=1}^k D_{\alpha_i}(x_i) \right) D_j.$$

The proof of the following lemma is clear.

LEMMA 2.13. — For any k -linear endomorphism $P : A \rightarrow A$ the map

$$(x_1, \dots, x_d) \in A^d \mapsto [\cdots [[P, x_1], x_2], \dots, x_d] \in \text{End}_k(A)$$

is symmetric.

THEOREM 2.14. — Assume that $\text{Der}_k(A)$ is a projective A -module of finite rank. The following properties are equivalent:

- (a) The homomorphism of graded A -algebras

$$\theta_{A/k} : \text{gr Diff}_{A/k} \rightarrow (\text{Sym } \Omega_{A/k})_{\text{gr}}^* \equiv \text{SDer}_k^*(A)$$

is an isomorphism.

- (b) The homomorphism of graded A -algebras

$$\vartheta_{A/k} : \Gamma \text{IDer}_k(A) \rightarrow \text{gr Diff}_{A/k}$$

is an isomorphism.

- (c) $\text{IDer}_k(A) = \text{Der}_k(A)$.

Proof. — For simplicity, we will omit the subscript “ A/k ”. From the hypothesis and proposition 1.12 we know that $\phi : \Gamma \text{Der}_k(A) \rightarrow (\text{Sym } \Omega)_{\text{gr}}^* \equiv \text{SDer}_k^*(A)$ is an isomorphism.

(a) \Rightarrow (b) From proposition 2.11, $\text{IDer}_k(A) = \text{Der}_k(A)$ and we conclude by applying theorem 2.8.

(b) \Rightarrow (c) It is clear since the degree 1 component of ϑ is the inclusion of $\text{IDer}_k(A)$ in $\text{gr}^1 \text{Diff} = \text{Der}_k(A)$.

(c) \Rightarrow (a) It is a consequence of 2.8 and the fact that θ is injective. □

COROLLARY 2.15. — Assume that $\text{IDer}_k(A) = \text{Der}_k(A)$ and that $\text{Der}_k(A)$ is a free A -module of finite rank with basis $\underline{d} = \{D_1^1, \dots, D_1^n\}$, and $D^i \in \text{HS}_k(A)$. Then, by using the notations in proposition 2.10, any k -linear differential operator $P : A \rightarrow A$ of order $\leq d$ can be uniquely written as

$$P = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq d}} a_\alpha \mathbf{D}_\alpha, \quad a_\alpha \in A.$$

Proof. — From theorem 2.14 we know that $\vartheta : \Gamma \text{IDer}_k(A) \rightarrow \text{gr Diff}_{A/k}$ is an isomorphism. After proposition 2.10, we only need to prove the uniqueness of the coefficients a_α , but this comes easily by induction on d . \square

The following result relates properties (a), (b), (c) in the theorem 2.14 with *differential smoothness*, as defined in [6], 16.10.

COROLLARY 2.16. — *Assume that $\Omega_{A/k}$ is a projective A -module of finite rank and that A is differentially smooth over k , i.e., the homomorphism of graded A -algebras (see (2.1))*

$$v_{A/k} : \text{Sym } \Omega_{A/k} \rightarrow \text{gr}_{I_{A/k}} \mathcal{P}_{A/k}$$

is an isomorphism. Then, the equivalent properties (a), (b), (c) of theorem 2.14 hold.

Proof. — For simplicity, we will omit the subscript “ A/k ”.

Since $v_n : \text{Sym}^n \Omega \xrightarrow{\sim} \text{gr}_I^n \mathcal{P}$ is an isomorphism of A -modules, each $\text{gr}_I^n \mathcal{P}$ is a projective A -module of finite rank and each \mathcal{P}^n it is so. Hence, by applying the functor $\text{Hom}_A(-, A)$ to the exact sequence (2.2) we obtain again an exact sequence

$$0 \rightarrow \text{Diff}^{(n-1)} \rightarrow \text{Diff}^{(n)} \rightarrow \text{Hom}_A(\text{gr}_I^{n-1} \mathcal{P}, A) \rightarrow 0,$$

and the map λ_n defined in (2.3) is an isomorphism. So $\theta_n = v_n^* \circ \lambda_n$ is also an isomorphism for all $n \geq 0$. \square

In the characteristic zero case (i.e., $\mathbb{Q} \subset A$), we have the following result.

COROLLARY 2.17. — *Assume that $\mathbb{Q} \subset A$ and that $\text{Der}_k(A)$ is a projective A -module of finite rank. Then, the canonical map (1.6)*

$$\tau_{A/k} : \text{Sym Der}_k(A) \rightarrow \text{gr Diff}_{A/k}$$

is an isomorphism.

Proof. — Since $\mathbb{Q} \subset A$, we have $\text{IDer}_k(A) = \text{Der}_k(A)$ and so, by theorem 2.14, we deduce that $\vartheta_{A/k} : \Gamma \text{IDer}_k(A) \rightarrow \text{gr Diff}_{A/k}$ is an isomorphism. On the other hand, the hypothesis $\mathbb{Q} \subset A$ implies that the canonical map $\text{Sym Der}_k(A) \rightarrow \Gamma \text{Der}_k(A)$ is an isomorphism. We conclude by looking at diagram (2.7). \square

3. Examples and questions

In this section we will assume that our k -algebra A is a quotient of the ambient k -algebra $R = k[x_1, \dots, x_n]$ or $R = k[[x_1, \dots, x_n]]$ by an ideal

J. Let us denote by $\pi : R \rightarrow A = R/J$ the natural projection and by $\Delta^\alpha : R \rightarrow R, \alpha \in \mathbb{N}^n$, Taylor's k -linear differential operators. The following properties hold:

$$\begin{aligned} \Delta^{(\alpha)}(x^\beta) &= \begin{cases} \binom{\beta}{\alpha} x^{\beta-\alpha} & \text{if } \beta \geq \alpha \\ 0 & \text{if } \beta \not\geq \alpha. \end{cases} \\ \Delta^{(\alpha)} \circ \Delta^{(\beta)} &= \Delta^{(\beta)} \circ \Delta^{(\alpha)} = \binom{\alpha+\beta}{\alpha} \Delta^{(\alpha+\beta)}. \\ \{\Delta^{(\alpha)}\}_{|\alpha| \leq d} &\text{ is a basis of } \text{Diff}_{R/k}^{(d)} \text{ as left (or right) } R\text{-module, for any } d \geq 0. \end{aligned}$$

For any $i = 1, \dots, n$ and any integer $e \geq 0$ let us write $\Delta_e^{(i)} = \Delta^{(0, \dots, \overset{(i)}{e}, \dots, 0)}$. In particular $\Delta_1^{(i)} = \frac{\partial}{\partial x_i}$ and $\Delta^{(i)} := (\text{Id}, \Delta_1^{(i)}, \Delta_2^{(i)}, \Delta_3^{(i)}, \dots) \in \text{HS}_k(R)$.

3.1. Logarithmic objects

DEFINITION 3.1. — A k -linear derivation $\delta : R \rightarrow R$ will be called J -logarithmic if $\delta(J) \subset J$. The set of k -linear derivations of R which are J -logarithmic will be denoted by $\text{Der}_k(\log J)$.

It is clear that $\text{Der}_k(\log J)$ is a R -submodule of $\text{Der}_k(R)$, and that any $\delta \in \text{Der}_k(\log J)$ gives rise to a unique $\bar{\delta} \in \text{Der}_k(A)$ satisfying $\bar{\delta} \circ \pi = \pi \circ \delta$. Moreover, the sequence of R -modules

$$0 \rightarrow J \text{Der}_k(R) \xrightarrow{\text{incl.}} \text{Der}_k(\log J) \xrightarrow{\delta \mapsto \bar{\delta}} \text{Der}_k(A) \rightarrow 0$$

is exact.

DEFINITION 3.2. — A Hasse–Schmidt derivation $D \in \text{HS}_k(R; m)$ is called J -logarithmic if $D_i(J) \subset J$ for any $i = 0, \dots, m$. The set of Hasse–Schmidt derivations of R over k of length m which are J -logarithmic will be denoted by $\text{HS}_k(\log J; m)$. When $m = \infty$ it will be simply denoted by $\text{HS}_k(\log J)$.

It is clear that $\text{HS}_k(\log J; m)$ is a subgroup of $\text{HS}_k(R; m)$, and that $a \bullet D$ is J -logarithmic whenever D is J -logarithmic.

For each integer $m \geq 1$ let us call $\pi_m : R_m = R[[t]]/(t^{m+1}) \rightarrow A_m = A[[t]]/(t^{m+1})$ the ring epimorphism induced by π . Any $D \in \text{HS}_k(\log J; m)$ gives rise to a unique $\bar{D} \in \text{HS}_k(A; m)$ such that $\bar{D}_i \circ \pi = \pi \circ D_i$ for all $i = 0, \dots, m$. Moreover, if $\Phi : R \rightarrow R_m$ is the k -algebra homomorphism determined by D , then the k -algebra homomorphism $\bar{\Phi} : A \rightarrow A_m$ determined by \bar{D} is characterized by $\bar{\Phi} \circ \pi = \pi_m \circ \Phi$, and the map

$$D \in \text{HS}_k(\log J; m) \mapsto \bar{D} \in \text{HS}_k(A; m)$$

is a surjective homomorphism of groups.

On the other hand, a $D \in \text{HS}_k(R; m)$ is J -logarithmic if and only if its corresponding k -algebra homomorphism $\Phi : R \rightarrow R_m$ satisfies $\Phi(J) \subset JR_m$.

DEFINITION 3.3. — We say that a J -logarithmic k -linear derivation $\delta : R \rightarrow R$ is J -logarithmically m -integrable if there is a $D \in \text{HS}_k(\log J; m)$ such that $D_1 = \delta$. The set of J -logarithmic k -linear derivations of R which are J -logarithmically m -integrable will be denoted by $\text{IDer}_k(\log J; m)$. When $m = \infty$ it will be simply denoted by $\text{IDer}_k(\log J)$.

It is clear that $\text{IDer}_k(\log J; m)$ is a R -submodule of $\text{Der}_k(\log J)$ and that

$$\text{Der}_k(\log J) = \text{IDer}_k(\log J; 1) \supset \text{IDer}_k(\log J; 2) \supset \text{IDer}_k(\log J; 3) \supset \dots$$

PROPOSITION 3.4. — Let $\varepsilon : A \rightarrow A$ be a k -linear derivation. The following properties are equivalent:

- (a) ε is m -integrable.
- (b) Any $\delta \in \text{Der}_k(\log J)$ with $\bar{\delta} = \varepsilon$ is J -logarithmically m -integrable.
- (c) There is a $\delta \in \text{Der}_k(\log J)$ with $\bar{\delta} = \varepsilon$ which is J -logarithmically m -integrable.

Proof. — The only implication to prove is (a) \Rightarrow (b): Let $E \in \text{HS}_k(A; m)$ be a m -integral of ε and let δ be a J -logarithmic k -derivation of R with $\bar{\delta} = \varepsilon$. There is a $D \in \text{HS}_k(\log J; m)$ such that $\bar{D} = E$. We have $\bar{D}_1 = E_1 = \varepsilon = \bar{\delta}$ and so $\overline{\delta - D_1} = 0$, i.e., there are $a_1, \dots, a_n \in J$ such that $\delta - D_1 = \sum a_i \Delta_1^{(i)}$. The Hasse-Schmidt derivation

$$E' = \tau_{\infty m} \left((a_1 \bullet \Delta^{(1)}) \circ \dots \circ (a_n \bullet \Delta^{(n)}) \right)$$

is obviously a J -logarithmic m -integral of $\delta - D_1$ and so $D \circ E'$ is a J -logarithmic m -integral of δ . □

COROLLARY 3.5. — The following properties are equivalent:

- (a) $\text{IDer}_k(A; m) = \text{Der}_k(A)$.
- (b) $\text{IDer}_k(\log J; m) = \text{Der}_k(\log J)$.

Proof. — The proof is a straightforward consequence of the preceding proposition. □

Example 3.6. — Let us write $F = \prod_{i=1}^m x_i$ and $J = (F) \subset R$. The R -module $\text{IDer}_k(\log J)$ is generated by $\{x_1 \Delta_1^{(1)}, \dots, x_m \Delta_1^{(m)}, \Delta_1^{(m+1)}, \dots,$

$\Delta_1^{(n)}\}$, and any of these J -logarithmic derivations are integrable J -logarithmically, since $\Delta^{(j)}, x_i \bullet \Delta^{(i)} \in \text{HS}_k(\log J)$ for $i = 1, \dots, m$ and $j = m + 1, \dots, n$. In particular $\text{IDer}_k(\log J) = \text{Der}_k(\log J)$ and $\text{IDer}_k(R/J) = \text{Der}_k(R/J)$.

Example 3.7. — Let k be a ring of characteristic 2, $R = k[x_1, x_2, x_3]$, $F = x_1^2 + x_2^3 + x_3^2$ and $J = (F)$. Let us consider the k -derivation $\delta = x_2^2 \frac{\partial}{\partial x_3} = x_2^2 \Delta^{(0,0,1)} \in \text{Der}_k(A)$. Since $\delta(F) = 0$, δ is J -logarithmic. The J -logarithmic Hasse–Schmidt derivation (of length 1) $(\text{Id}, D_1 = \delta)$ is determined by the homomorphism of k -algebras $\Phi_1 : R \rightarrow R[[t]]/(t^2)$ given by $\Phi(x_1) = x_1, \Phi_1(x_2) = x_2, \Phi_1(x_3) = x_3 + x_2^2 t$. Let us consider the lifting $\Phi_4 : R \rightarrow R[[t]]/(t^5)$ of Φ_1 given by $\Phi_4(x_1) = x_1, \Phi_4(x_2) = x_2 + x_2^2 t^2 + x_3^2 t^4, \Phi_4(x_3) = x_3 + x_2^2 t$. Since $\Phi_4(F) = F$, the Hasse–Schmidt derivation D corresponding to Φ_4 is J -logarithmic and it is explicitly given by $D = (\text{Id}, D_1 = \delta, D_2, D_3, D_4)$, with

$$D_2 = x_2^4 \Delta^{(0,0,2)} + x_2^2 \Delta^{(0,1,0)}, \quad D_3 = x_2^6 \Delta^{(0,0,3)} + x_2^4 \Delta^{(0,1,1)},$$

$$D_4 = x_2^8 \Delta^{(0,0,4)} + x_2^6 \Delta^{(0,1,2)} + x_2^4 \Delta^{(0,2,0)} + x_2^3 \Delta^{(0,1,0)}.$$

Let us consider now the Hasse–Schmidt derivation $D'' = (\text{Id}, 0, \Delta^{(1,0,0)}, 0) \in \text{HS}_k(R; 3)$ and $D' = \tau_{43}(D) \circ D'' = (\text{Id}, D'_1, D'_2, D'_3)$ with $D'_1 = \delta, D'_2 = D_2 + \Delta^{(1,0,0)} = x_2^4 \Delta^{(0,0,2)} + x_2^2 \Delta^{(0,1,0)} + \Delta^{(1,0,0)}, D'_3 = D_3 + D_1 \circ \Delta^{(1,0,0)} = x_2^6 \Delta^{(0,0,3)} + x_2^4 \Delta^{(0,1,1)} + x_2^2 \Delta^{(1,0,1)}$. It is clear that D'' is J -logarithmic, and so D' is a J -logarithmic 3-integral of δ .

Let $\widetilde{D}'' = (\text{Id}, 0, \Delta^{(1,0,0)}, 0, G_4)$ be a Hasse–Schmidt derivation of length 4 integrating D'' . It is clear that $(\text{Id}, \Delta^{(1,0,0)}, G_4) \in \text{HS}_k(R; 2)$, and the symbol of G_4 must be the same as the symbol of $\Delta^{(2,0,0)}$ (see corollary 2.7), i.e., $G_4 = \Delta^{(2,0,0)} + \delta'$, with $\delta' \in \text{Der}_k(R)$. So, $G_4(F) = 1 + \delta'(F) = 1 + 3\delta'(x_2)x_2^2 \notin (F)$ and \widetilde{D}'' is never J -logarithmic. We deduce that there is no $\widetilde{D}' \in \text{HS}_k(\log J; 4)$ such that $\tau_{43}(\widetilde{D}') = D'$.

3.2. Questions

Let $D = (\text{Id}, D_1, \dots, D_m)$ be a J -logarithmic Hasse–Schmidt derivation (over k) of length m of R and let $\Phi : R \rightarrow R_m = R[[t]]/(t^{m+1})$ be the homomorphism of k -algebras determined by D . Since $R = k[x]$ or $R = k[[x]]$, Φ can be canonically lifted to an homomorphism of k -algebras $\widetilde{\Phi} : R \rightarrow R_m = R[[t]]/(t^{m+2})$ and so we obtain a canonical $(m + 1)$ -integral $\widetilde{D} = (\text{Id}, D_1, \dots, D_m, D_{m+1}) \in \text{HS}_k(R; m+1)$ of D (this is a particular case of remark 1.5). If $\widetilde{D}' = (\text{Id}, D_1, \dots, D_m, D'_{m+1})$ is another $(m + 1)$ -integral

of D , then $D'_{m+1} - D_{m+1} \in \text{Der}_k(R)$. In particular, the existence of a such \tilde{D}' which is J -logarithmic is equivalent to the existence of a derivation $\delta \in \text{Der}_k(R)$ such that $(D_{m+1} + \delta)(J) \subset J$. For instance, when $J = (F)$, the above property is equivalent to the fact that $D_{m+1}(F) \in (F'_{x_1}, \dots, F'_{x_n}, F)$, that can be tested easily at least when k is a “computable” ring.

However, example 3.7 shows that there are (J -logarithmically) m -integrable derivations admitting (J -logarithmic) $(m-1)$ -integrals which are not (J -logarithmically) m -integrable, and so (J -logarithmic) m -integrability of a (J -logarithmic) derivation cannot be tested step by step.

QUESTION 3.8. — *Find an algorithm to decide whether a J -logarithmic derivation is J -logarithmically m -integrable or not, for $m \geq 3$.*

QUESTION 3.9. — *Find an algorithm to compute a system of generators of $\text{IDer}_k(\log J; m)$, for $m \geq 2$.*

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Manuscrit reçu le 3 février 2009,
accepté le 6 mars 2009.

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