

A q -analog of the Racah polynomials and the q -algebra $SU_q(2)^*$

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Abstract

We study some q -analogues of the Racah polynomials and some of their applications in the theory of representation of quantum algebras.

1 Introduction

In the paper [6] an orthogonal polynomial family that generalizes the Racah coefficients or $6j$ -symbols was introduced: the so-called Racah and q -Racah polynomials. These polynomials were in the top of the so-called Askey Scheme (see e.g. [14]) that contains all classical families of hypergeometric orthogonal polynomials. Some years later the same authors [7] introduced the celebrated Askey-Wilson polynomials. One of the important properties of these polynomials is that from them one can obtain all known families of hypergeometric polynomials and q -polynomials as particular cases or as limit cases (for a review on this see the nice survey [14]). The main tool in these two works was the hypergeometric and basic series, respectively. On the other hand, the authors of [21] (see also [20, Russian Edition]) considered the q -polynomials as the solution of a second order difference equation of hypergeometric-type on the non-linear lattice $x(s) = c_1q^s + c_2q^{-s} + c_3$. In particular, they show that the solution of the hypergeometric-type equation can be expressed as certain basic series and, in such a way, they recovered the results by Askey & Wilson.

The interest of such polynomials increase after the appearance of the q -algebras and quantum groups [9, 10, 12, 16, 24]. However, from the first attempts to build the q -analog of the Wigner-Racah formalism for the simplest quantum algebra $U_q(su(2))$ [13] (see also [1, 4, 17]) becomes clear that for obtaining the q -polynomials intimately connected with the q -analogues of the Racah and Clebsch Gordan coefficients, i.e., a q -analogue of the Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ and the dual Hahn polynomials $w_n^c(x(s), a, b)_q$, respectively, it is better to use a different lattice—in fact the q -Racah polynomials $R_n^{\beta,\gamma}(x(s), N, \delta)_q$ introduced in [7] (see also [14]) were defined on the lattice $x(s) = q^{-s} + \delta q^{-N}q^s$ that depends not only of the variable s but also on the parameters of the polynomials—, namely,

$$x(s) = [s]_q[s + 1]_q, \quad (1)$$

that only depends on s , where by $[s]_q$ we denote the q -numbers (in its symmetric form)

$$[s]_q = \frac{q^{s/2} - q^{-s/2}}{q^{1/2} - q^{-1/2}}, \quad \forall s \in \mathbb{C}. \quad (2)$$

With this choice the q -Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ are proportional to the q -Racah coefficients (or $6j$ -symbols) of the quantum algebra $U_q(su(2))$. A very nice and simple approach to $6j$ -symbols has been recently developed in [23].

Moreover, this connection gives the possibility to a deeper study of the Wigner-Racah formalism (or the q -analogue of the quantum theory of angular momentum [25, 26, 27, 28]) for the quantum algebras $U_q(su(2))$ and $U_q(su(1, 1))$ using the powerful and well-known theory of orthogonal polynomials on non-uniform lattices. On the other hand, using the q -analogue of the quantum theory of angular momentum [25, 26, 27, 28] we can obtain several results for the q -polynomials, some of which are non trivial from the point of view of the theory of orthogonal polynomials (see e.g. the nice surveys [15, 29]). In fact, in the present paper we present a detailed study of some q -analogues of the Racah polynomials on the lattice (1): the $u_n^{\alpha,\beta}(x(s), a, b)_q$ and the $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ as well as their connection with the q -Racah coefficients (or $6j$ -symbols) of the quantum algebra $U_q(su(2))$ in order to establish which properties of the polynomials correspond to the $6j$ -symbols and vice versa.

The structure of the paper is as follows: In section 2 we present some general results from the theory of orthogonal polynomials on the non-uniform lattices taken from [3, 20]. In Section 2.1 a detailed discussion of the Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ is presented, whereas in Section 2.2 the $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ are considered. In particular, a relation between these families is established. In section 3 the comparative analysis of such families and the $6j$ -symbols of the quantum algebra $U_q(su(2))$ is developed which gives, on one hand, some information about the Racah coefficients and, on the other hand, allow us to give a group-theoretical interpretation of the Racah polynomials on the lattice (1). Finally, some comments and remarks about q -Racah polynomials and the quantum algebra $U_q(su(3))$ are included.

2 Some general properties of q -polynomials

We will start with some general properties of orthogonal hypergeometric polynomials on the non-uniform lattices [8, 20].

The hypergeometric polynomials are the polynomial solutions $P_n(x(s))_q$ of the second order linear difference equation of hypergeometric-type on the non-uniform lattice $x(s)$ (SODE)

$$\sigma(s) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \frac{\nabla y(s)}{\nabla x(s)} + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda y(s) = 0, \quad x(s) = c_1[q^s + q^{-s-\mu}] + c_3, \quad q^\mu = \frac{c_1}{c_2}, \quad (3)$$

$$\nabla f(s) = f(s) - f(s-1), \quad \Delta f(s) = f(s+1) - f(s),$$

or, equivalently

$$A_s y(s+1) + B_s y(s) + C_s y(s-1) + \lambda y(s) = 0, \quad (4)$$

where

$$A_s = \frac{\sigma(s) + \tau(s)\Delta x(s - \frac{1}{2})}{\Delta x(s)\Delta x(s - \frac{1}{2})}, \quad C_s = \frac{\sigma(s)}{\nabla x(s)\Delta x(s - \frac{1}{2})}, \quad B_s = -(A_s + C_s).$$

Notice that $x(s) = x(-s - \mu)$.

In the following we will use the following notations¹ $P_n(s)_q := P_n(x(s))_q$ and $\sigma(-s - \mu) = \sigma(s) + \tau(s)\Delta x(s - \frac{1}{2})$. With this notation the Eq. (3) becomes

$$\sigma(-s - \mu) \frac{\Delta P_n(s)_q}{\Delta x(s)} - \sigma(s) \frac{\nabla P_n(s)_q}{\nabla x(s)} + \lambda_n \Delta x(s - \frac{1}{2}) P_n(s)_q = 0. \quad (5)$$

The polynomial solutions $P_n(s)_q$ of (3) can be obtained by the following Rodrigues-type formula [20, 22]

$$P_n(s)_q = \frac{B_n}{\rho(s)} \nabla^{(n)} \rho_n(s), \quad \nabla^{(n)} := \frac{\nabla}{\nabla x_1(s)} \frac{\nabla}{\nabla x_2(s)} \cdots \frac{\nabla}{\nabla x_n(s)}, \quad (6)$$

where $x_m(s) = x(s + \frac{m}{2})$,

$$\rho_n(s) = \rho(s+n) \prod_{m=1}^n \sigma(s+m), \quad (7)$$

¹In the exponential lattice $x(s) = c_1 q^{\pm s} + c_3$, so $\mu = \pm\infty$, therefore instead of using $\sigma(-s - \mu)$ one should use the equivalent function $\sigma(s) + \tau(s)\Delta x(s - \frac{1}{2})$.

and $\rho(s)$ is a solution of the Pearson-type equation $\Delta[\sigma(s)\rho(s)] = \tau(s)\rho(s)\Delta x(s-1/2)$, or equivalently,

$$\frac{\rho(s+1)}{\rho(s)} = \frac{\sigma(s) + \tau(s)\Delta x(s-\frac{1}{2})}{\sigma(s+1)} = \frac{\sigma(-s-\mu)}{\sigma(s+1)}. \quad (8)$$

Let us point out that the function ρ_n satisfy the equation $\Delta[\sigma(s)\rho_n(s)] = \tau_n(s)\rho_n(s)\Delta x_n(s-1/2)$, where $\tau_n(s)$ is given by

$$\tau_n(s) = \frac{\sigma(s+n) + \tau(s+n)\Delta x(s+n-\frac{1}{2}) - \sigma(s)}{\Delta x_{n-1}(s)} = \frac{\sigma(-s-n-\mu) - \sigma(s)}{\Delta x_n(s-\frac{1}{2})} = \tau'_n x_n(s) + \tau_n(0). \quad (9)$$

being

$$\tau'_n = -\frac{\lambda_{2n+1}}{[2n+1]_q}, \quad \tau_n(0) = \frac{\sigma(-s_n^* - n - \mu) - \sigma(s_n^*)}{x_n(s_n^* + \frac{1}{2}) - x_n(s_n^* - \frac{1}{2})},$$

where s_n^* is the zero of the function $x_n(s)$, i.e., $x_n(s_n^*) = 0$.

From (6) follows an explicit formula for the polynomials P_n [20, Eq.(3.2.30)]

$$P_n(s)_q = B_n \sum_{m=0}^n \frac{[n]_q! (-1)^{m+n} \nabla x(s+m-\frac{n-1}{2}) \rho_n(s-n+m)}{[m]_q! [n-m]_q! \prod_{l=0}^n \nabla x(s+\frac{m-l+1}{2}) \rho(s)}, \quad (10)$$

where $[n]_q$ denotes the *symmetric q-numbers* (2) and the q -factorials are given by

$$[0]_q! := 1, \quad [n]_q! := [1]_q [2]_q \cdots [n]_q, \quad n \in \mathbb{N}.$$

It can be shown [8, 20, 22] that the most general polynomial solution of the q -hypergeometric equation (3) corresponds to

$$\sigma(s) = A \prod_{i=1}^4 [s-s_i]_q = C q^{-2s} \prod_{i=1}^4 (q^s - q^{s_i}), \quad A \cdot C \neq 0 \quad (11)$$

and has the form [22, Eq. (49a), page 240]

$$P_n(s)_q = D_n {}_4\phi_3 \left(\begin{matrix} q^{-n}, q^{2\mu+n-1+\sum_{i=1}^4 s_i}, q^{s_1-s}, q^{s_1+s+\mu} \\ q^{s_1+s_2+\mu}, q^{s_1+s_3+\mu}, q^{s_1+s_4+\mu} \end{matrix} ; q, q \right), \quad (12)$$

where the normalizing factor D_n is given by ($\varkappa_q := q^{1/2} - q^{-1/2}$)

$$D_n = B_n \left(\frac{-A}{c_1 q^\mu \varkappa_q^5} \right)^n q^{-\frac{n}{2}(3s_1+s_2+s_3+s_4+\frac{3(n-1)}{2})} (q^{s_1+s_2+\mu}; q)_n (q^{s_1+s_3+\mu}; q)_n (q^{s_1+s_4+\mu}; q)_n.$$

The basic hypergeometric series ${}_r\phi_p$ are defined by [14]

$${}_r\phi_p \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_p \end{matrix} ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_p; q)_k} \frac{z^k}{(q; q)_k} \left[(-1)^k q^{\frac{k}{2}(k-1)} \right]^{p-r+1},$$

where $(a; q)_k = \prod_{m=0}^{k-1} (1 - aq^m)$, is the q -analogue of the Pochhammer symbol.

In this paper we will deal with orthogonal q -polynomials and functions. It can be proven [20], by using the difference equation of hypergeometric-type (3), that if the boundary conditions $\sigma(s)\rho(s)x^k(s-1/2)|_{s=a,b} = 0$, for all $k \geq 0$, holds, then the polynomials $P_n(s)_q$ are orthogonal with respect to the weight function ρ , i.e.,

$$\sum_{s=a}^{b-1} P_n(s)_q P_m(s)_q \rho(s) \Delta x(s-1/2) = \delta_{nm} d_n^2, \quad s = a, a+1, \dots, b-1. \quad (13)$$

The squared norm in (13) is [20, Eq. (3.7.15)]

$$d_n^2 = (-1)^n A_{n,n} B_n^2 \sum_{s=a}^{b-n-1} \rho_n(s) \Delta x_n(s-1/2), \quad (14)$$

where [20, page 66]

$$A_{n,k} = \frac{[n]_q!}{[n-k]_q!} \prod_{m=0}^{k-1} \left(-\frac{\lambda_{n+m}}{[n+m]_q} \right). \quad (15)$$

A simple consequence of the orthogonality is the three-term recurrence relation (TTRR)

$$x(s)P_n(s)_q = \alpha_n P_{n+1}(s)_q + \beta_n P_n(s)_q + \gamma_n P_{n-1}(s)_q, \quad (16)$$

where α_n , β_n and γ_n are given by

$$\alpha_n = \frac{a_n}{a_{n+1}}, \quad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \quad \gamma_n = \frac{a_{n-1}}{a_n} \frac{d_n^2}{d_{n-1}^2}, \quad (17)$$

being a_n and b_n the first and second coefficients in the power expansion of P_n , i.e., $P_n(s)_q = a_n x^n(s) + b_n x^{n-1}(s) + \dots$. Substituting $s = a$ in (16) we find

$$\beta_n = \frac{x(a)P_n(a)_q - \alpha_n P_{n+1}(a)_q - \gamma_n P_{n-1}(a)_q}{P_n(a)_q}, \quad (18)$$

which is an alternative way for finding the coefficient β_n . Also we can use the expression [3, page 148]

$$\beta_n = \frac{[n]_q \tau_{n-1}(0)}{\tau'_{n-1}} - \frac{[n+1]_q \tau_n(0)}{\tau'_n} + c_3([n]_q + 1 - [n+1]_q).$$

To compute α_n (and β_n) we need the following formulas (see e.g. [3, page 147])

$$a_n = \frac{B_n A_{n,n}}{[n]_q!}, \quad \frac{b_n}{a_n} = \frac{[n]_q \tau_{n-1}(0)}{\tau'_{n-1}} + c_3([n]_q - n). \quad (19)$$

The explicit expression of λ_n is [22, Eq. (52) page 232]

$$\begin{aligned} \lambda_n &= -\frac{Aq^\mu}{c_1^2(q^{1/2} - q^{-1/2})^4} [n]_q [s_1 + s_2 + s_3 + s_4 + 2\mu + n - 1]_q \\ &= -\frac{Cq^{-n+1/2}}{c_1^2(q^{1/2} - q^{-1/2})^2} (1 - q^n) (1 - q^{s_1+s_2+s_3+s_4+2\mu+n-1}), \end{aligned} \quad (20)$$

which can be obtained equating the largest powers of q^s in (5).

From the Rodrigues formula [19, 3, §5.6] follows that

$$\frac{\Delta P_n(s - \frac{1}{2})_q}{\Delta x(s - \frac{1}{2})} = \frac{-\lambda_n B_n}{\tilde{B}_{n-1}} \tilde{P}_{n-1}(s)_q, \quad (21)$$

where \tilde{P}_{n-1} denotes the polynomial orthogonal with respect to the weight function $\tilde{\rho}(s) = \rho_1(s - \frac{1}{2})$. On the other hand, rewriting (3) as

$$\left(\sigma(s) \frac{\nabla}{\nabla x_1(s)} + \tau(s)I \right) \frac{\Delta}{\Delta x(s)} P_n(s)_q = -\lambda_n P_n(s)_q,$$

it can be substituted by the following two first-order difference equations

$$\frac{\Delta}{\Delta x(s)} P_n(s)_q = Q(s), \quad \left(\sigma(s) \frac{\nabla}{\nabla x_1(s)} + \tau(s)I \right) Q(s) = -\lambda_n P_n(s)_q. \quad (22)$$

Using the fact that $\frac{\Delta}{\Delta x(s)} P_n(s)_q$ is a polynomial of degree $n - 1$ on $x(s + 1/2)$ (see [20, §3.1]) it follows that

$$\frac{\Delta}{\Delta x(s)} P_n(s)_q = C_n Q_{n-1}(s + \frac{1}{2}),$$

where C_n is a normalizing constant. Comparison with (21) implies that $Q(s)$ is the polynomial \tilde{P}_{n-1} orthogonal with respect to the function $\rho_1(s - \frac{1}{2})$ and $C_n = -\lambda_n B_n / \tilde{B}_{n-1}$. Therefore, the second expression in (22) becomes

$$P_n(s)_q = \frac{B_n}{\tilde{B}_{n-1}} \left(\sigma(s) \frac{\nabla}{\nabla x_1(s)} + \tau(s)I \right) \tilde{P}_{n-1}(s + \frac{1}{2})_q. \quad (23)$$

The q -polynomials satisfy the following differentiation-type formula [19, 3, §5.6.1]

$$\sigma(s) \frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q \tau'_n} \left[\tau_n(s) P_n(s)_q - \frac{B_n}{B_{n+1}} P_{n+1}(s)_q \right]. \quad (24)$$

Then, using the explicit expression for the coefficient α_n , we find

$$\sigma(s) \frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau'_n} P_n(s)_q - \frac{\alpha_n \lambda_{2n}}{[2n]_q} P_{n+1}(s)_q. \quad (25)$$

From the above equation using the identity $\Delta \frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\Delta P_n(s)_q}{\Delta x(s)} - \frac{\nabla P_n(s)_q}{\nabla x(s)}$ as well as the SODE (5) we find

$$\sigma(-s - \mu) \frac{\Delta P_n(s)_q}{\Delta x(s)} = \frac{\lambda_n}{[n]_q \tau'_n} \left[\left(\tau_n(s) - [n]_q \tau'_n \Delta x(s - \frac{1}{2}) \right) P_n(s)_q - \frac{B_n}{B_{n+1}} P_{n+1}(s)_q \right]. \quad (26)$$

To conclude this section we will introduce the following notation by Nikiforov and Uvarov [20, 22]. First we define another q -analog of the Pochhammer symbols [20, Eq. (3.11.1)]

$$(a|q)_k = \prod_{m=0}^{k-1} [a+m]_q = \frac{\tilde{\Gamma}_q(a+k)}{\tilde{\Gamma}_q(a)} = (-1)^k (q^a; q)_k (q^{1/2} - q^{-1/2})^{-k} q^{-\frac{k}{4}(k-1) - \frac{kq}{2}}, \quad (27)$$

where $\tilde{\Gamma}_q(x)$ is the q -analog of the Γ function introduced in [20, Eq. (3.2.24)], and related to the classical q -Gamma function Γ_q by formula

$$\tilde{\Gamma}_q(s) = q^{-\frac{(s-1)(s-2)}{4}} \Gamma_q(s) = q^{-\frac{(s-1)(s-2)}{4}} (1-q)^{1-s} \frac{(q; q)_\infty}{(q^s; q)_\infty}, \quad 0 < q < 1.$$

Next we define the q -hypergeometric function ${}_rF_p(\cdot|q, z)$

$${}_rF_p \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_p \end{matrix} \middle| q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1|q)_k (a_2|q)_k \cdots (a_r|q)_k}{(b_1|q)_k (b_2|q)_k \cdots (b_p|q)_k} \frac{z^k}{(1|q)_k} \left[\varkappa_q^{-k} q^{\frac{1}{4}k(k-1)} \right]^{p-r+1}, \quad (28)$$

where, as before, $\varkappa_q = q^{1/2} - q^{-1/2}$, and $(a|q)_k$ are given by (27). Notice that

$$\lim_{q \rightarrow 1} {}_rF_p \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_p \end{matrix} \middle| q, z \varkappa_q^{p-r+1} \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_p)_k} \frac{z^k}{k!} = {}_rF_p \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_p \end{matrix} \middle| z \right),$$

and

$${}_{p+1}F_p \left(\begin{matrix} a_1, a_2, \dots, a_{p+1} \\ b_1, b_2, \dots, b_p \end{matrix} \middle| q, t \right) \Big|_{t=t_0} = {}_{p+1}\mathcal{F}_p \left(\begin{matrix} q^{a_1}, q^{a_2}, \dots, q^{a_{p+1}} \\ q^{b_1}, q^{b_2}, \dots, q^{b_p} \end{matrix} \middle| q, z \right), \quad (29)$$

where $t_0 = z q^{\frac{1}{2}(\sum_{i=1}^{p+1} a_i - \sum_{i=1}^p b_i - 1)}$.

Using the above notation the polynomial solutions of (3) is [22, Eq. (49), page 232]

$$P_n(s)_q = B_n \left(\frac{A}{c_1 q^{-\frac{\mu}{2}} \varkappa_q^2} \right)^n (s_1 + s_2 + \mu|q)_n (s_1 + s_3 + \mu|q)_n \times \\ (s_1 + s_4 + \mu|q)_n {}_4F_3 \left(\begin{matrix} -n, 2\mu + n - 1 + \sum_{i=1}^4 s_i, s_1 - s, s_1 + s + \mu \\ s_1 + s_2 + \mu, s_1 + s_3 + \mu, s_1 + s_4 + \mu \end{matrix} \middle| q, 1 \right). \quad (30)$$

2.1 The q -Racah polynomials

Here we will consider the q -Racah polynomials $u_n^{\alpha, \beta}(x(s), a, b)_q$ on the lattice $x(s) = [s]_q [s+1]_q$ introduced in [2, 17, 20]. For this lattice one has

$$c_1 = q^{\frac{1}{2}} \varkappa_q^{-2}, \quad \mu = 1, \quad c_3 = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \varkappa_q^{-2}. \quad (31)$$

Let chose σ in (11) as

$$\sigma(s) = -\frac{q^{-2s}}{\varkappa_q^4 q^{\frac{\alpha+\beta}{2}}} (q^s - q^a)(q^s - q^{-b})(q^s - q^{\beta-a})(q^s - q^{b+\alpha}) = [s-a]_q [s+b]_q [s+a-\beta]_q [b+\alpha-s]_q,$$

i.e., $s_1 = a, s_2 = -b, s_3 = \beta - a, s_4 = b + \alpha, C = -q^{-\frac{1}{2}(\alpha+\beta)} \varkappa_q^{-4}, A = -1$, and let $B_n = (-1)^n / [n]_q!$. Here, as before, $\varkappa_q = q^{1/2} - q^{-1/2}$. Now from (20) we find

$$\lambda_n = q^{-\frac{1}{2}(\alpha+\beta+2n+1)} \varkappa_q^{-2} (1 - q^n)(1 - q^{\alpha+\beta+n+1}) = [n]_q [n + \alpha + \beta + 1]_q.$$

To obtain $\tau_n(s)$ we use (9). In this case $x_n(s) = [s + n/2]_q [s + n/2 + 1]_q$, then, choosing $s_n^* = -n/2$, we get

$$\tau_n(s) = \tau_n'(s) x_n(s) + \tau_n(0), \quad \tau_n' = -[2n + \alpha + \beta + 2]_q, \quad \tau_n(0) = \sigma(-n/2 - 1) - \sigma(-n/2). \quad (32)$$

Taking into account that $\tau(s) = \tau_0(s)$, we obtain the corresponding function $\tau(s)$

$$\tau(s) = -[2 + \alpha + \beta]_q x(s) + \sigma(-1) - \sigma(0).$$

2.1.1 The orthogonality and the norm d_n^2

A solution of the Pearson-type difference equation (8) is

$$\rho(s) = \frac{\tilde{\Gamma}_q(s+a+1)\tilde{\Gamma}_q(s-a+\beta+1)\tilde{\Gamma}_q(s+\alpha+b+1)\tilde{\Gamma}_q(b+\alpha-s)}{\tilde{\Gamma}_q(s-a+1)\tilde{\Gamma}_q(s+b+1)\tilde{\Gamma}_q(s+a-\beta+1)\tilde{\Gamma}_q(b-s)}.$$

Since $\sigma(a)\rho(a) = \sigma(b)\rho(b) = 0$, then the q -Racah polynomials satisfy the orthogonality relation

$$\sum_{s=a}^{b-1} u_n^{\alpha,\beta}(x(s), a, b)_q u_m^{\alpha,\beta}(x(s), a, b)_q \rho(s) [2s+1]_q = 0, \quad n \neq m,$$

with the restrictions $-\frac{1}{2} < a \leq b-1, \alpha > -1, -1 < \beta < 2a+1$. Let us now compute the square of the norm d_n^2 . From (7) and (15) follow

$$\rho_n(s) = \frac{\tilde{\Gamma}_q(s+n+a+1)\tilde{\Gamma}_q(s+n-a+\beta+1)\tilde{\Gamma}_q(s+n+\alpha+b+1)\tilde{\Gamma}_q(b+\alpha-s)}{\tilde{\Gamma}_q(s-a+1)\tilde{\Gamma}_q(s+b+1)\tilde{\Gamma}_q(s+a-\beta+1)\tilde{\Gamma}_q(b-s-n)},$$

$$A_{n,n} = [n]_q! (-1)^n \frac{\tilde{\Gamma}_q(\alpha+\beta+2n+1)}{\tilde{\Gamma}_q(\alpha+\beta+n+1)} \Rightarrow \Lambda_n := (-1)^n A_{n,n} B_n^2 = \frac{\tilde{\Gamma}_q(\alpha+\beta+2n+1)}{[n]_q! \tilde{\Gamma}_q(\alpha+\beta+n+1)}.$$

Taking into account that $\nabla x_{n+1}(s) = [2s+n+1]_q$, using (14), and the identity

$$\tilde{\Gamma}_q(A-s) = \frac{\tilde{\Gamma}_q(A)(-1)^s}{(1-A|q)_s}, \quad (33)$$

we have

$$\begin{aligned} d_n^2 &= \Lambda_n \sum_{s=a}^{b-n-1} \frac{\tilde{\Gamma}_q(s+n+a+1)\tilde{\Gamma}_q(s+n-a+\beta+1)\tilde{\Gamma}_q(s+n+\alpha+b+1)\tilde{\Gamma}_q(b+\alpha-s)}{\tilde{\Gamma}_q(s-a+1)\tilde{\Gamma}_q(s+b+1)\tilde{\Gamma}_q(s+a-\beta+1)\tilde{\Gamma}_q(b-s-n)} [2s+n+1]_q^{-1} \\ &= \Lambda_n \sum_{s=0}^{b-a-n-1} \frac{\tilde{\Gamma}_q(s+n+2a+1)\tilde{\Gamma}_q(s+n+\beta+1)\tilde{\Gamma}_q(s+n+\alpha+b+a+1)\tilde{\Gamma}_q(b-a+\alpha-s)}{\tilde{\Gamma}_q(s+1)\tilde{\Gamma}_q(s+b+a+1)\tilde{\Gamma}_q(s+2a-\beta+1)\tilde{\Gamma}_q(b-a-s-n)} [2s+2a+n+1]_q^{-1} \\ &= \frac{\tilde{\Gamma}_q(\alpha+\beta+2n+1)\tilde{\Gamma}_q(2a+n+1)\tilde{\Gamma}_q(n+\beta+1)\tilde{\Gamma}_q(a+b+n+\alpha+1)\tilde{\Gamma}_q(b+\alpha-a)}{[n]_q! \tilde{\Gamma}_q(\alpha+\beta+n+1)\tilde{\Gamma}_q(a+b+1)\tilde{\Gamma}_q(2a-\beta+1)\tilde{\Gamma}_q(b-a-n)} \times \\ &\quad \sum_{s=0}^{b-a-n-1} \frac{(n+2a+1, n+\beta+1, n+a+\alpha+b+1, 1-b+a+n|q)_s}{(1, a+b+1, 2a-\beta+1, 1-b+a-\alpha|q)_s} [2s+2a+n+1]_q. \end{aligned}$$

In the following we denote by S_n the sum in the last expression. If we now use that $(a|q)_n = (-1)^n (q^a; q)_n q^{-\frac{n}{2}(n+2a-1)} \varkappa_q^{-n}$, as well as the identity

$$[2s + 2a + n + 1]_q = q^{-s} [2a + n + 1]_q \frac{(q^{a+\frac{n+1}{2}+1}; q)(-q^{a+\frac{n+1}{2}+1}; q)_s}{(q^{a+\frac{n+1}{2}}; q)(-q^{a+\frac{n+1}{2}}; q)},$$

we obtain

$$\begin{aligned} S_n &= \sum_{s=0}^{b-a-n-1} \frac{(q^{2a+n+1}, q^{n+\beta+1}, q^{n+a+b+1}, q^{1-b+a+n}, q^{\frac{1}{2}(2a+n+3)}, -q^{\frac{1}{2}(2a+n+3)}; q)_s}{(q, q^{a+b+1}, q^{2a-\beta+1}, q^{1-b-\alpha+a}, q^{\frac{1}{2}(2a+n+1)}, -q^{\frac{1}{2}(2a+n+1)}; q)_s [2a+n+1]_q^{-1}} q^{-s(1+2n+\beta+\alpha)} \\ &= [2a+n+1]_q {}_6\phi_5 \left(\begin{matrix} q^{2a+n+1}, q^{n+\beta+1}, q^{n+a+b+1}, q^{1-b+a+n}, q^{\frac{1}{2}(2a+n+3)}, -q^{\frac{1}{2}(2a+n+3)} \\ q^{a+b+1}, q^{2a-\beta+1}, q^{1-b-\alpha+a}, q^{\frac{1}{2}(2a+n+1)}, -q^{\frac{1}{2}(2a+n+1)} \end{matrix} \middle| q, q^{-1-2n-\beta-\alpha} \right). \end{aligned}$$

But the above ${}_6\phi_5$ series is a very-well-poised ${}_6\phi_5$ basic series and therefore by using the summation formula [11, Eq. (II.21) page 238]

$${}_6\varphi_5 \left(\begin{matrix} a, qa^{1/2}, -qa^{1/2}, b, c, q^{-k} \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq^{k+1} \end{matrix} \middle| q, \frac{aq^{k+1}}{bc} \right) = \frac{(aq, aq/bc; q)_k}{(aq/b, aq/c; q)_k},$$

with $k = b - a - n - 1$, $a = q^{2a+n+1}$, $b = q^{n+\beta+1}$, $c = q^{n+a+\alpha+b+1}$, we obtain

$$\begin{aligned} S_n &= [2a+n+1]_q \frac{(q^{2a+n+2}, q^{-n+a-b-\alpha-\beta}; q)_{b-a-n-1}}{(q^{2a-\beta+1}, q^{a-b-\alpha+1}; q)_{b-a-n-1}} \\ &= [2a+n+1]_q \frac{(2a+n+2|q)_{b-a-n-1} (-n+a-b-\alpha-\beta|q)_{b-a-n-1}}{(2a-\beta+1|q)_{b-a-n-1} (a-b-\alpha+1|q)_{b-a-n-1}}. \end{aligned}$$

Finally, using (33) and (27)

$$S_n = [2a+n+1]_q \frac{\tilde{\Gamma}_q(a+b+1)\tilde{\Gamma}_q(2a-\beta+1)\tilde{\Gamma}_q(b-a+\alpha+\beta+n+1)\tilde{\Gamma}_q(\alpha+n+1)}{\tilde{\Gamma}_q(n+2a+2)\tilde{\Gamma}_q(b+a-\beta-n)\tilde{\Gamma}_q(\alpha+\beta+2n+2)\tilde{\Gamma}_q(b-a+\alpha)},$$

thus

$$\begin{aligned} d_n^2 &= \frac{\tilde{\Gamma}_q(\alpha+\beta+2n+1)\tilde{\Gamma}_q(2a+n+1)\tilde{\Gamma}_q(n+\beta+1)\tilde{\Gamma}_q(a+b+n+\alpha+1)\tilde{\Gamma}_q(b+\alpha-a)}{[n]_q! \tilde{\Gamma}_q(\alpha+\beta+n+1)\tilde{\Gamma}_q(a+b+1)\tilde{\Gamma}_q(2a-\beta+1)\tilde{\Gamma}_q(b-a-n)} S_n \\ &= \frac{\tilde{\Gamma}_q(\alpha+n+1)\tilde{\Gamma}_q(\beta+n+1)\tilde{\Gamma}_q(b-a+\alpha+\beta+n+1)\tilde{\Gamma}_q(a+b+\alpha+n+1)}{[\alpha+\beta+2n+1]_q \tilde{\Gamma}_q(n+1)\tilde{\Gamma}_q(\alpha+\beta+n+1)\tilde{\Gamma}_q(b-a-n)\tilde{\Gamma}_q(a+b-\beta-n)}. \end{aligned}$$

2.1.2 The hypergeometric representation

From formula (12) and (30) the following two equivalent hypergeometric representations hold

$$\begin{aligned} u_n^{\alpha,\beta}(x(s), a, b)_q &= \frac{q^{-\frac{n}{2}(2a+\alpha+\beta+n+1)} (q^{a-b+1}; q)_n (q^{\beta+1}; q)_n (q^{a+b+\alpha+1}; q)_n}{\varkappa_q^{2n}(q; q)_n} \times \\ &4\varphi_3 \left(\begin{matrix} q^{-n}, q^{\alpha+\beta+n+1}, q^{a-s}, q^{a+s+1} \\ q^{a-b+1}, q^{\beta+1}, q^{a+b+\alpha+1} \end{matrix} \middle| q, q \right), \end{aligned} \quad (34)$$

and

$$\begin{aligned} u_n^{\alpha,\beta}(x(s), a, b)_q &= \frac{(a-b+1|q)_n (\beta+1|q)_n (a+b+\alpha+1|q)_n}{[n]_q!} \times \\ &4F_3 \left(\begin{matrix} -n, \alpha+\beta+n+1, a-s, a+s+1 \\ a-b+1, \beta+1, a+b+\alpha+1 \end{matrix} \middle| q, 1 \right). \end{aligned} \quad (35)$$

Using the Sears transformation formula [11, Eq. (III.15)] we obtain the equivalent formulas

$$\begin{aligned} u_n^{\alpha,\beta}(x(s), a, b)_q &= \frac{q^{-\frac{n}{2}(-2b+\alpha+\beta+n+1)} (q^{a-b+1}; q)_n (q^{\alpha+1}; q)_n (q^{\beta-a-b+1}; q)_n}{\varkappa_q^{2n}(q; q)_n} \times \\ &4\varphi_3 \left(\begin{matrix} q^{-n}, q^{\alpha+\beta+n+1}, q^{-b-s}, q^{-b+s+1} \\ q^{a-b+1}, q^{\alpha+1}, q^{-a-b+\beta+1} \end{matrix} \middle| q, q \right), \end{aligned} \quad (36)$$

and

$$u_n^{\alpha,\beta}(x(s), a, b)_q = \frac{(a-b+1|q)_n(\alpha+1|q)_n(-a-b+\beta+1|q)_n}{[n]_q!} \times {}_4F_3 \left(\begin{matrix} -n, \alpha+\beta+n+1, -b-s, -b+s+1 \\ a-b+1, \alpha+1, -a-b+\beta+1 \end{matrix} \middle| q, 1 \right). \quad (37)$$

Remark: From the above formulas follow that the polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ are multiples of the standard q -Racah polynomials $R_n(\mu(q^{b+s}); q^\alpha, q^\beta, q^{a-b}, q^{-a-b}|q)$.

From the above hypergeometric representations also follow the values

$$\begin{aligned} u_n^{\alpha,\beta}(x(a), a, b)_q &= \frac{(a-b+1|q)_n(\beta+1|q)_n(a+b+\alpha+1|q)_n}{[n]_q!} \\ &= \frac{(q^{a-b+1}; q)_n(q^{\beta+1}; q)_n(q^{a+b+\alpha+1}; q)_n}{q^{\frac{n}{2}(2\alpha+\alpha+\beta+n+1)} \varkappa_q^{2n}(q; q)_n}, \end{aligned} \quad (38)$$

$$\begin{aligned} u_n^{\alpha,\beta}(x(b-1), a, b)_q &= \frac{(a-b+1|q)_n(\alpha+1|q)_n(-a-b+\beta+1|q)_n}{[n]_q!} \\ &= \frac{(q^{a-b+1}; q)_n(q^{\alpha+1}; q)_n(q^{\beta-a-b+1}; q)_n}{q^{\frac{n}{2}(-2b+\alpha+\beta+n+1)} \varkappa_q^{2n}(q; q)_n}. \end{aligned} \quad (39)$$

The formula (10) leads to the following explicit formula²

$$\begin{aligned} u_n^{\alpha,\beta}(x(s), a, b)_q &= \frac{\tilde{\Gamma}_q(s-a+1)\tilde{\Gamma}_q(s+b+1)\tilde{\Gamma}_q(s+a-\beta+1)\tilde{\Gamma}_q(b-s)}{\tilde{\Gamma}_q(s+a+1)\tilde{\Gamma}_q(s-a+\beta+1)\tilde{\Gamma}_q(s+\alpha+b+1)\tilde{\Gamma}_q(b+\alpha-s)} \times \\ &\quad \sum_{k=0}^n \frac{(-1)^k [2s+2k-n+1]_q \tilde{\Gamma}_q(s+k+a+1)\tilde{\Gamma}_q(2s+k-n+1)}{\tilde{\Gamma}_q(k+1)\tilde{\Gamma}_q(n-k+1)\tilde{\Gamma}_q(2s+k+2)\tilde{\Gamma}_q(s-n+k-a+1)} \times \\ &\quad \frac{\tilde{\Gamma}_q(s+k-a+\beta+1)\tilde{\Gamma}_q(s+k+\alpha+b+1)\tilde{\Gamma}_q(b+\alpha-s+n-k)}{\tilde{\Gamma}_q(s-n+k+b+1)\tilde{\Gamma}_q(s-n+k+a-\beta+1)\tilde{\Gamma}_q(b-s-k)}, \end{aligned} \quad (40)$$

from where follows

$$\begin{aligned} u_n^{\alpha,\beta}(x(a), a, b)_q &= \frac{(-1)^n \tilde{\Gamma}_q(b-a)\tilde{\Gamma}_q(\beta+n+1)\tilde{\Gamma}_q(b+a+\alpha+n+1)}{[n]! \tilde{\Gamma}_q(b-a-n)\tilde{\Gamma}_q(\beta+1)\tilde{\Gamma}_q(b+a+\alpha+1)}, \\ u_n^{\alpha,\beta}(x(b-1), a, b)_q &= \frac{\tilde{\Gamma}_q(b-a)\tilde{\Gamma}_q(\alpha+n+1)\tilde{\Gamma}_q(b+a-\beta)}{[n]! \tilde{\Gamma}_q(b-a-n)\tilde{\Gamma}_q(\alpha+1)\tilde{\Gamma}_q(b+a-\beta-n)}, \end{aligned} \quad (41)$$

that coincide with the values (38) and (39) obtained before.

From the hypergeometric representation the following symmetry property follows

$$u_n^{\alpha,\beta}(x(s), a, b)_q = u_n^{-b-a+\beta, b+a+\alpha}(x(s), a, b)_q.$$

Finally, notice that from (34) (or (36)) follows that $u_n^{\alpha,\beta}(x(s), a, b)_q$ is a polynomial of degree n on $x(s) = [s]_q[s+1]_q$. In fact,

$$(q^{a-s}; q)_k (q^{a+s+1}; q)_k = (-1)^k q^{k(a+\frac{k+1}{2})} \prod_{l=0}^{k-1} \left(\frac{x(s) - c_3}{c_1} - q^{-\frac{1}{2}}(q^{a+l+\frac{1}{2}} + q^{-a-l-\frac{1}{2}}) \right),$$

where c_1 and c_3 are given in (31).

2.1.3 Three-term recurrence relation and differentiation formulas

To derive the coefficients of the TTRR (16) we use (17) and (18). Using (19) and (17), we obtain

$$a_n = \frac{\tilde{\Gamma}_q(\alpha+\beta+2n+1)}{[n]_q! \tilde{\Gamma}_q(\alpha+\beta+n+1)}, \quad \alpha_n = \frac{[n+1]_q [\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+1]_q [\alpha+\beta+2n+2]_q}.$$

²Obviously the formulas (34) and (36) also give equivalent explicit formulas.

Table 1: Main data of the q -Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$

$P_n(s)$	$u_n^{\alpha,\beta}(x(s), a, b)_q, \quad x(s) = [s]_q[s+1]_q$
(a, b)	$[a, b-1]$
$\rho(s)$	$\frac{\tilde{\Gamma}_q(s+a+1)\tilde{\Gamma}_q(s-a+\beta+1)\tilde{\Gamma}_q(s+\alpha+b+1)\tilde{\Gamma}_q(b+\alpha-s)}{\tilde{\Gamma}_q(s-a+1)\tilde{\Gamma}_q(s+b+1)\tilde{\Gamma}_q(s+a-\beta+1)\tilde{\Gamma}_q(b-s)}$ $-\frac{1}{2} < a \leq b-1, \alpha > -1, -1 < \beta < 2a+1$
$\sigma(s)$	$[s-a]_q[s+b]_q[s+a-\beta]_q[b+\alpha-s]_q$
$\sigma(-s-1)$	$[s+a+1]_q[b-s-1]_q[s-a+\beta+1]_q[b+\alpha+s+1]_q$
$\tau(s)$	$[\alpha+1]_q[a]_q[a-\beta]_q + [\beta+1]_q[b]_q[b+\alpha]_q - [\alpha+1]_q[\beta+1]_q - [\alpha+\beta+2]_q x(s)$
$\tau_n(s)$	$-[\alpha+\beta+2n+2]_q x(s + \frac{n}{2}) + [a + \frac{n}{2} + 1]_q [b - \frac{n}{2} - 1]_q [\beta + \frac{n}{2} + 1 - a]_q [b + \alpha + \frac{n}{2} + 1]_q$ $- [a + \frac{n}{2}]_q [b - \frac{n}{2}]_q [\beta + \frac{n}{2} - a]_q [b + \alpha + \frac{n}{2}]_q$
λ_n	$[n]_q[\alpha+\beta+n+1]_q$
B_n	$\frac{(-1)^n}{[n]_q!}$
d_n^2	$\frac{\tilde{\Gamma}_q(\alpha+n+1)\tilde{\Gamma}_q(\beta+n+1)\tilde{\Gamma}_q(b-a+\alpha+\beta+n+1)\tilde{\Gamma}_q(a+b+\alpha+n+1)}{[\alpha+\beta+2n+1]_q\tilde{\Gamma}_q(n+1)\tilde{\Gamma}_q(\alpha+\beta+n+1)\tilde{\Gamma}_q(b-a-n)\tilde{\Gamma}_q(a+b-\beta-n)}$
$\rho_n(s)$	$\frac{\tilde{\Gamma}_q(s+n+a+1)\tilde{\Gamma}_q(s+n-a+\beta+1)\tilde{\Gamma}_q(s+n+\alpha+b+1)\tilde{\Gamma}_q(b+\alpha-s)}{\tilde{\Gamma}_q(s-a+1)\tilde{\Gamma}_q(s+b+1)\tilde{\Gamma}_q(s+a-\beta+1)\tilde{\Gamma}_q(b-s-n)}$
a_n	$\frac{\tilde{\Gamma}_q[\alpha+\beta+2n+1]_q}{[n]_q!\tilde{\Gamma}_q[\alpha+\beta+n+1]_q}$
α_n	$\frac{[n+1]_q[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+1]_q[\alpha+\beta+2n+2]_q}$
β_n	$[a]_q[a+1]_q - \frac{[\alpha+\beta+n+1]_q[a-b+n+1]_q[\beta+n+1]_q[a+b+\alpha+n+1]_q}{[\alpha+\beta+2n+1]_q[\alpha+\beta+2n+2]_q}$ $+ \frac{[\alpha+n]_q[b-a+\alpha+\beta+n]_q[a+b-\beta-n]_q[n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}$
γ_n	$\frac{[a+b+\alpha+n]_q[a+b-\beta-n]_q[\alpha+n]_q[\beta+n]_q[b-a+\alpha+\beta+n]_q[b-a-n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}$

To find γ_n we use (17)

$$\gamma_n = \frac{[a+b+\alpha+n]_q[a+b-\beta-n]_q[\alpha+n]_q[\beta+n]_q[b-a+\alpha+\beta+n]_q[b-a-n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}.$$

To compute β_n we use (18)

$$\begin{aligned} \beta_n &= x(a) - \alpha_n \frac{u_{n+1}^{\alpha,\beta}(x(a), a, b)_q}{u_n^{\alpha,\beta}(x(a), a, b)_q} - \gamma_n \frac{u_{n-1}^{\alpha,\beta}(x(a), a, b)_q}{u_n^{\alpha,\beta}(x(a), a, b)_q} \\ &= [a]_q[a+1]_q - \frac{[\alpha+\beta+n+1]_q[a-b+n+1]_q[\beta+n+1]_q[a+b+\alpha+n+1]_q}{[\alpha+\beta+2n+1]_q[\alpha+\beta+2n+2]_q} \\ &\quad + \frac{[\alpha+n]_q[b-a+\alpha+\beta+n]_q[a+b-\beta-n]_q[n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}. \end{aligned}$$

The differentiation formulas (21) and (23) yield

$$\frac{\Delta u_n^{\alpha,\beta}(x(s), a, b)_q}{\Delta x(s)} = [\alpha+\beta+n+1]_q u_{n-1}^{\alpha+1, \beta+1}(x(s + \frac{1}{2}), a + \frac{1}{2}, b - \frac{1}{2})_q, \quad (42)$$

$$\begin{aligned}
-[n]_q[2s+1]_q u_n^{\alpha,\beta}(x(s), a, b)_q &= \sigma(-s-1) u_{n-1}^{\alpha+1, \beta+1}(x(s+\frac{1}{2}), a+\frac{1}{2}, b-\frac{1}{2})_q \\
&\quad - \sigma(s) u_{n-1}^{\alpha+1, \beta+1}(x(s-\frac{1}{2}), a+\frac{1}{2}, b-\frac{1}{2})_q,
\end{aligned} \tag{43}$$

respectively. Finally, formulas (24) (or (25)) and (26) lead to the *differentiation formulas*

$$\sigma(s) \frac{\nabla u_n^{\alpha,\beta}(x(s), a, b)_q}{[2s]_q} = -\frac{[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+2]_q} \left[\tau_n(s) u_n^{\alpha,\beta}(x(s), a, b)_q + [n+1]_q u_{n+1}^{\alpha,\beta}(x(s), a, b)_q \right], \tag{44}$$

$$\begin{aligned}
\sigma(-s-1) \frac{\Delta u_n^{\alpha,\beta}(x(s), a, b)_q}{[2s+2]_q} &= -\frac{[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+2]_q} \times \\
&\quad \left[(\tau_n(s) + [n]_q[\alpha+\beta+2n+2]_q[2s+1]_q) u_n^{\alpha,\beta}(x(s), a, b)_q + [n+1]_q u_{n+1}^{\alpha,\beta}(x(s), a, b)_q \right],
\end{aligned} \tag{45}$$

where $\tau_n(s)$ is given in (32).

2.1.4 The duality of the Racah polynomials

In this section we will discuss the duality property of the q -Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$. We will follow [20, pages 38-39]. First of all, notice that the orthogonal relation (13) for the Racah polynomials can be written in the form

$$\sum_{t=0}^{N-1} C_{tn} C_{tm} = \delta_{n,m}, \quad C_{tn} = \frac{u_n^{\alpha,\beta}(x(t+a), a, b)_q \sqrt{\rho(t+a) \Delta x(t+a-1/2)}}{d_n}, \quad N = b-a,$$

where $\rho(s)$ and d_n are the weight function and the norm of the q -Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$, respectively. The above relation can be understood as the orthogonality property of the orthogonal matrix $C = (C_{tn})_{t,n=0}^{N-1}$ by its first index. If we now use the orthogonality of C by the second index we get

$$\sum_{n=0}^{N-1} C_{tn} C_{t'n} = \delta_{t,t'}, \quad N = b-a,$$

that leads to the dual orthogonality relation for the q -Racah polynomials

$$\sum_{n=0}^{N-1} u_n^{\alpha,\beta}(x(s), a, b)_q u_n^{\alpha,\beta}(x(s'), a, b)_q \frac{1}{d_n^2} = \frac{1}{\rho(s) \Delta x(s-1/2)} \delta_{s,s'}. \tag{46}$$

The next step is to identify the functions $u_n^{\alpha,\beta}(x(s), a, b)_q$ as polynomials on some lattice $x(n)$. Before starting let us mention that from the representation (34) and the identity

$$(q^{-n}; q)_k (q^{\alpha+\beta+n+1}; q)_k = \prod_{l=0}^{k-1} \left(1 + q^{\alpha+\beta+2l+1} - q^{\frac{\alpha+\beta+1}{2}+l} \left(\varkappa_q^2 x(t) + q^{\frac{1}{2}} + q^{-\frac{1}{2}} \right) \right),$$

where $x(t) = [t]_q [t+1]_q = [n + \frac{\alpha+\beta}{2}]_q [n + \frac{\alpha+\beta}{2} + 1]_q$, follows that $u_n^{\alpha,\beta}(x(s), a, b)_q$ also constitutes a polynomial of degree $s-a$ (for $s = a, a+1, \dots, b-a-1$) on $x(t)$ with $t = n + \frac{\alpha+\beta}{2}$.

Let us now define the polynomials —compare with the definition of the Racah polynomials (35)—

$$u_k^{\alpha',\beta'}(x(t), a', b')_q = \frac{(-1)^k \widetilde{\Gamma}_q(b'-a') \widetilde{\Gamma}_q(\beta'+k+1) \widetilde{\Gamma}_q(b'+a'+\alpha'+k+1)}{[k]! \widetilde{\Gamma}_q(b'-a'-k) \widetilde{\Gamma}_q(\beta'+1) \widetilde{\Gamma}_q(b'+a'+\alpha'+1)} \times \tag{47}$$

$${}_4F_3 \left(\begin{matrix} -k, \alpha'+\beta'+k+1, a'-t, a'+t+1 \\ a'-b'+1, \beta'+1, a'+b'+\alpha'+1 \end{matrix} \middle| q, 1 \right),$$

where

$$k = s - a, \quad t = n + \frac{\alpha + \beta}{2}, \quad a' = \frac{\alpha + \beta}{2}, \quad b' = b - a + \frac{\alpha + \beta}{2}, \quad \alpha' = 2a - \beta, \quad \beta' = \beta. \quad (48)$$

Obviously they are polynomials of degree $k = s - a$ on the lattice $x(t)$ that satisfy the orthogonality property

$$\sum_{t=a'}^{b'-1} u_k^{\alpha', \beta'}(x(t), a', b')_q u_m^{\alpha', \beta'}(x(t), a', b')_q \rho'(t) \Delta x(t - 1/2) = (d'_k)^2 \delta_{k,m}, \quad (49)$$

where $\rho'(t)$ and d'_k are the weight function ρ and the norm d_n given in table 1 with the corresponding change $a, b, \alpha, \beta, s, n$ by $a', b', \alpha', \beta', t, k$.

Furthermore, with the above choice (48) of the parameters of $u_k^{\alpha', \beta'}(x(t), a', b')_q$, the hypergeometric function ${}_4F_3$ in (47) coincides with the function ${}_4F_3$ in (35) and therefore the following relation between the polynomials $u_k^{\alpha', \beta'}(x(t), a', b')_q$ and $u_n^{\alpha, \beta}(x(s), a, b)_q$ holds

$$u_k^{\alpha', \beta'}(x(t), a', b')_q = \mathcal{A}(\alpha, \beta, a, b, n, s) u_n^{\alpha, \beta}(x(s), a, b)_q, \quad (50)$$

where

$$\mathcal{A}(\alpha, \beta, a, b, n, s) = \frac{(-1)^{s-a+n} \tilde{\Gamma}_q(b-a-n) \tilde{\Gamma}_q(s-a+\beta+1) \tilde{\Gamma}_q(b+\alpha+s+1) \tilde{\Gamma}_q(n+1)}{\tilde{\Gamma}_q(b-s) \tilde{\Gamma}_q(n+\beta+1) \tilde{\Gamma}_q(b+a+\alpha+n+1) \tilde{\Gamma}_q(s-a+1)}.$$

If we now substitute (50) in (49) and make the change (48), then (49) becomes into relation (46), i.e., the polynomial set $u_k^{\alpha', \beta'}(x(t), a', b')_q$ defined by (47) (or (50)) is the dual set associated to the Racah polynomials $u_n^{\alpha, \beta}(x(s), a, b)_q$.

To conclude this study, let us show that the TTRR (16) of the polynomials $u_k^{\alpha', \beta'}(x(t), a', b')_q$ is the SODE (4) of the polynomials $u_n^{\alpha, \beta}(x(s), a, b)_q$ whereas the SODE (4) of the $u_k^{\alpha', \beta'}(x(t), a', b')_q$ becomes into the TTRR (16) of $u_n^{\alpha, \beta}(x(s), a, b)_q$ and vice versa.

Let denote by $\varsigma(t)$ the σ function of the polynomial $u_k^{\alpha', \beta'}$, then

$$\varsigma(t) = [t - a']_q [t + b']_q [t + a' - \beta']_q [b' + \alpha' - t]_q = [n]_q [n + b - a + \alpha + \beta]_q [n + \alpha]_q [b + a - n - \beta]_q,$$

and therefore,

$$\varsigma(-t - 1) = [\alpha + \beta + n + 1]_q [b + a + \alpha + n + 1]_q [b - a - n - 1]_q [n + \beta + 1]_q, \\ \lambda_k = [k]_q [\alpha' + \beta' + k + 1]_q = [s - a]_q [s + a + 1]_q.$$

For the coefficients α'_k, β'_k and γ'_k , of the TTRR for the polynomials $u_k^{\alpha', \beta'}$ we have

$$\alpha'_k = \frac{[k + 1]_q [\alpha' + \beta' + k + 1]_q}{[\alpha' + \beta' + 2k + 1]_q [\alpha' + \beta' + 2k + 2]_q} = \frac{[s - a + 1]_q [s + a + 1]_q}{[2s + 1]_q [2s + 2]_q}, \\ \gamma'_k = \frac{[b + \alpha + s]_q [b + \alpha - s]_q [s + a - \beta]_q [s - a + \beta]_q [b + s]_q [b - s]_q}{[2s + 1]_q [2s]_q},$$

and

$$\beta'_k = [n + \frac{\alpha + \beta}{2}]_q [n + \frac{\alpha + \beta}{2} + 1]_q + \frac{\sigma(-s - 1)}{[2s + 1]_q [2s + 2]_q} + \frac{\sigma(s)}{[2s + 1]_q [2s]_q}.$$

Also we have $\Delta x(t) = [2t + 2]_q = [2n + \alpha + \beta + 2]_q$ and $x(s) = [s]_q [s + 1]_q = [k + a]_q [k + a + 1]_q$.

Let show that the SODE of the Racah polynomials $u_n^{\alpha, \beta}(x(s), a, b)_q$ is the TTRR of the $u_k^{\alpha', \beta'}(x(t), a', b')_q$ polynomials. First, we substitute the relation (50) in the SODE (4) of the polynomials $u_n^{\alpha, \beta}(x(s), a, b)_q$ and use that $u_n^{\alpha, \beta}(x(s \pm 1), a, b)_q$ is proportional to $u_{k \pm 1}^{\alpha', \beta'}(x(t), a', b')_q$ (see (50)). After some simplification, and using the last formulas we obtain

$$\alpha'_k u_{k+1}^{\alpha', \beta'}(x(t), a', b')_q + \left(\beta'_k - [n]_q [\alpha + \beta + n + 1]_q - \left[\frac{\alpha + \beta}{2} \right]_q \left[\frac{\alpha + \beta}{2} + 1 \right]_q \right) u_k^{\alpha', \beta'}(x(t), a', b')_q \\ + \gamma'_k u_{k-1}^{\alpha', \beta'}(x(t), a', b')_q = 0,$$

but

$$[n]_q [\alpha + \beta + n + 1]_q + \left[\frac{\alpha + \beta}{2} \right]_q \left[\frac{\alpha + \beta}{2} + 1 \right]_q = [n + \frac{\alpha + \beta}{2}]_q [n + \frac{\alpha + \beta}{2} + 1]_q = x(t),$$

i.e., we obtain the TTRR for the polynomials $u_k^{\alpha',\beta'}(x(t), a', b')_q$.

If we now substitute (50) in the TTRR (16) for the Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$, and use that $u_{n\pm 1}^{\alpha,\beta}(x(s), a, b)_q \sim u_k^{\alpha',\beta'}(x(t \pm 1), a', b')_q$, then we obtain the SODE

$$\begin{aligned} & \frac{\varsigma(-t-1)}{\Delta x(t)\Delta x(t-\frac{1}{2})} u_k^{\alpha',\beta'}(x(t+1), a', b')_q + \frac{\varsigma(t)}{\nabla x(t)\Delta x(t-\frac{1}{2})} u_k^{\alpha',\beta'}(x(t-1), a', b')_q \\ & - \left[\frac{\varsigma(-t-1)}{\Delta x(t)\Delta x(t-\frac{1}{2})} + \frac{\varsigma(t)}{\nabla x(t)\Delta x(t-\frac{1}{2})} + [a]_q[a+1]_q - [k+a]_q[k+a+1]_q \right] u_k^{\alpha',\beta'}(x(t), a', b')_q = 0. \end{aligned}$$

That is the SODE (4) of the $u_k^{\alpha',\beta'}(x(t), a', b')_q$ since

$$[a]_q[a+1]_q - [k+a]_q[k+a+1]_q = -[k]_q[k+2a+1]_q = -[k]_q[k+\alpha'+\beta'+1]_q = -\lambda_k.$$

2.2 The q -Racah polynomials $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$

There is another possibility to define the q -Racah polynomials as it is suggested in [17, 20]. It corresponds to the function

$$\sigma(s) = [s-a]_q[s+b]_q[s-a+\beta]_q[b+\alpha+s]_q,$$

i.e., $A=1$, $s_1=a$, $s_2=-b$, $s_3=a-\beta$, $s_4=-b-\alpha$. With this choice we obtain a new family of polynomials $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ that is orthogonal with respect to the weight function

$$\rho(s) = \frac{\tilde{\Gamma}_q(s+a+1)\tilde{\Gamma}_q(s+a-\beta+1)}{\tilde{\Gamma}_q(s+\alpha+b+1)\tilde{\Gamma}_q(b+\alpha-s)\tilde{\Gamma}_q(s-a+1)\tilde{\Gamma}_q(s+b+1)\tilde{\Gamma}_q(s-a+\beta+1)\tilde{\Gamma}_q(b-s)}.$$

All their characteristics can be obtained exactly in the same way as before. Moreover, they can be also obtained from the corresponding characteristics of the polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ by changing $\alpha \rightarrow -2b-\alpha$, $\beta \rightarrow 2a-\beta$ —and using the properties of the functions $\tilde{\Gamma}_q(s)$, $\Gamma_q(s)$, $(a|q)_n$ and $(a; q)_n$. We will resume the main data of the polynomials $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ in table 2.

2.2.1 The hypergeometric representation

For the $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ polynomials we have the following hypergeometric representation

$$\begin{aligned} \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q &= \frac{q^{-\frac{n}{2}(4a-2b-\alpha-\beta+n+1)}(q^{a-b+1}; q)_n(q^{2a-\beta+1}; q)_n(q^{a-b-\alpha+1}; q)_n}{\varkappa_q^{2n}(q; q)_n} \times \\ & 4\varphi_3 \left(\begin{matrix} q^{-n}, q^{2a-2b-\alpha-\beta+n+1}, q^{a-s}, q^{a+s+1} \\ q^{a-b+1}, q^{2a-\beta+1}, q^{a-b-\alpha+1} \end{matrix} \middle| q, q \right), \end{aligned} \quad (51)$$

or, in terms of the q -hypergeometric series (28),

$$\begin{aligned} \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q &= \frac{(a-b+1|q)_n(2a-\beta+1|q)_n(a-b-\alpha+1|q)_n}{[n]_q!} \times \\ & 4F_3 \left(\begin{matrix} -n, 2a-2b-\alpha-\beta+n+1, a-s, a+s+1 \\ a-b+1, 2a-\beta+1, a-b-\alpha+1 \end{matrix} \middle| q, 1 \right). \end{aligned} \quad (52)$$

Using the Sears transformation formula [11, Eq. (III.15)] we obtain the equivalent representation formulas

$$\begin{aligned} \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q &= \frac{q^{-\frac{n}{2}(2a-4b-\alpha-\beta+n+1)}(q^{a-b+1}; q)_n(q^{-2b-\alpha+1}; q)_n(q^{-\beta+a-b+1}; q)_n}{\varkappa_q^{2n}(q; q)_n} \times \\ & 4\varphi_3 \left(\begin{matrix} q^{-n}, q^{2a-2b-\alpha-\beta+n+1}, q^{-b-s}, q^{-b+s+1} \\ q^{a-b+1}, q^{-2b-\alpha+1}, q^{a-b-\beta+1} \end{matrix} \middle| q, q \right), \end{aligned} \quad (53)$$

and

$$\begin{aligned} \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q &= \frac{(a-b+1|q)_n(-2b-\alpha+1|q)_n(a-b-\beta+1|q)_n}{[n]_q!} \times \\ & 4F_3 \left(\begin{matrix} -n, 2a-2b-\alpha-\beta+n+1, -b-s, -b+s+1 \\ a-b+1, -2b-\alpha+1, a-b-\beta+1 \end{matrix} \middle| q, 1 \right). \end{aligned} \quad (54)$$

Table 2: Main data of the q -Racah polynomials $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$

$P_n(s)$	$\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q, \quad x(s) = [s]_q[s+1]_q$
(a, b)	$[a, b-1]$
$\rho(s)$	$\frac{\tilde{\Gamma}_q(s+a+1)\tilde{\Gamma}_q(s+a-\beta+1)}{\tilde{\Gamma}_q(s+\alpha+b+1)\tilde{\Gamma}_q(b+\alpha-s)\tilde{\Gamma}_q(s-a+1)\tilde{\Gamma}_q(s+b+1)\tilde{\Gamma}_q(s-a+\beta+1)\tilde{\Gamma}_q(b-s)}$ $-\frac{1}{2} < a \leq b-1, \alpha > -1, -1 < \beta < 2a+1$
$\sigma(s)$	$[s-a]_q[s+b]_q[s-a+\beta]_q[b+\alpha+s]_q$
$\sigma(-s-1)$	$[s+a+1]_q[b-s-1]_q[s+a-\beta+1]_q[b+\alpha-s-1]_q$
$\tau(s)$	$[2a-\beta+1]_q[b]_q[b+\alpha]_q - [2b+\alpha-1]_q[a]_q[a-\beta]_q - [2b+\alpha-1]_q[2a-\beta+1]_q$ $-[2b-2a+\alpha+\beta-2]_q x(s)$
$\tau_n(s)$	$-[2b-2a+\alpha+\beta-2n-2]_q x(s+\frac{n}{2}) + [a+\frac{n}{2}+1]_q[b-\frac{n}{2}-1]_q[a+\frac{n}{2}+1-\beta]_q[b-\frac{n}{2}+\alpha-1]_q$ $-[a+\frac{n}{2}]_q[b-\frac{n}{2}]_q[a+\frac{n}{2}-\beta]_q[b-\frac{n}{2}+\alpha]_q$
λ_n	$[n]_q[2b-2a+\alpha+\beta-n-1]_q$
B_n	$\frac{1}{[n]_q!}$
d_n^2	$\frac{\tilde{\Gamma}_q(2a+n-\beta+1)\tilde{\Gamma}_q(2b-2a+\alpha+\beta-n)[2b-2a-2n-1+\alpha+\beta]_q^{-1}}{\tilde{\Gamma}_q(n+1)\tilde{\Gamma}_q(b-a-n)\tilde{\Gamma}_q(b-a-n+\alpha)\tilde{\Gamma}_q(b-a+\beta-n)\tilde{\Gamma}_q(2b+\alpha-n)\tilde{\Gamma}_q(b-a+\alpha+\beta-n)}$
$\rho_n(s)$	$\frac{\tilde{\Gamma}_q(s+a+n+1)\tilde{\Gamma}_q(s+a+n-\beta+1)}{\tilde{\Gamma}_q(s+\alpha+b+1)\tilde{\Gamma}_q(b+\alpha-s-n)\tilde{\Gamma}_q(s-a+1)\tilde{\Gamma}_q(s+b+1)\tilde{\Gamma}_q(s-a+\beta+1)\tilde{\Gamma}_q(b-s-n)}$
a_n	$\frac{(-1)^n \tilde{\Gamma}_q[2b-2a+\alpha+\beta-n]_q}{[n]_q! \tilde{\Gamma}_q[2b-2a+\alpha+\beta-2n]_q}$
α_n	$-\frac{[n+1]_q[2b-2a+\alpha+\beta-n-1]_q}{[2b-2a+\alpha+\beta-2n-1]_q[2b-2a+\alpha+\beta-2n-2]_q}$
β_n	$[a]_q[a+1]_q + \frac{[2b-2a+\alpha+\beta-n-1]_q[a-b+n+1]_q[2a-\beta+n+1]_q[a-b-\alpha+n+1]_q}{[2b-2a+\alpha+\beta-2n-1]_q[2b-2a+\alpha+\beta-2n-2]_q}$ $+ \frac{[2b+\alpha-n]_q[b-a+\alpha+\beta-n]_q[b-a+\beta-n]_q[n]_q}{[2b-2a+\alpha+\beta-2n-1]_q[2b-2a+\alpha+\beta-2n]_q}$
γ_n	$-\frac{[2a-\beta+n]_q[b-a-n]_q[b-a-n+\alpha]_q[b-a-n+\beta]_q[2b+\alpha-n]_q[b-a+\alpha+\beta-n]_q}{[2b-2a+\alpha+\beta-2n-1]_q[2b-2a+\alpha+\beta-2n]_q}$

Remark: From the above formulas follow that the polynomials $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ are multiples of the standard q -Racah polynomials $R_n(\mu(q^{a-s}); q^{a-b-\alpha}, q^{a-b-\beta}, q^{a-b}, q^{a+b}|q)$.

Moreover, from the above hypergeometric representations follow the values

$$\begin{aligned} \tilde{u}_n^{\alpha,\beta}(x(a), a, b)_q &= \frac{(a-b+1|q)_n(2a-\beta+1|q)_n(a-b-\alpha+1|q)_n}{[n]_q!} \\ &= \frac{(q^{a-b+1}; q)_n(q^{2a-\beta+1}; q)_n(q^{a-b-\alpha+1}; q)_n}{q^{\frac{n}{2}(4a-2b-\alpha-\beta+n+1)} \mathcal{Z}_q^{2n}(q; q)_n}, \end{aligned} \quad (55)$$

$$\begin{aligned} \tilde{u}_n^{\alpha,\beta}(x(b-1), a, b)_q &= \frac{(a-b+1|q)_n(-2b-\alpha+1|q)_n(a-b-\beta+1|q)_n}{[n]_q!} \\ &= \frac{(q^{a-b+1}; q)_n(q^{-2b-\alpha+1}; q)_n(q^{-\beta+a-b+1}; q)_n}{q^{\frac{n}{2}(2a-4b-\alpha-\beta+n+1)} \mathcal{Z}_q^{2n}(q; q)_n}. \end{aligned} \quad (56)$$

Using (10) we obtain an explicit formula³

$$\begin{aligned} \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q &= \frac{\tilde{\Gamma}_q(s-a+1)\tilde{\Gamma}_q(s+b+1)\tilde{\Gamma}_q(s-a+\beta+1)\tilde{\Gamma}_q(b-s)\tilde{\Gamma}_q(s+\alpha+b+1)}{\tilde{\Gamma}_q(s+a+1)\tilde{\Gamma}_q(s+a-\beta+1)} \times \\ &\tilde{\Gamma}_q(b+\alpha-s) \sum_{k=0}^n \frac{(-1)^{k+n} [2s+2k-n+1]_q \tilde{\Gamma}_q(s+k+a+1) \tilde{\Gamma}_q(2s+k-n+1)}{\tilde{\Gamma}_q(k+1) \tilde{\Gamma}_q(n-k+1) \tilde{\Gamma}_q(2s+k+2) \tilde{\Gamma}_q(s-n+k-a+1) \tilde{\Gamma}_q(b-s-k)} \times \\ &\frac{\tilde{\Gamma}_q(s+k+a-\beta+1)}{\tilde{\Gamma}_q(s+k-n+\alpha+b+1) \tilde{\Gamma}_q(b+\alpha-s-k) \tilde{\Gamma}_q(s-n+k+b+1) \tilde{\Gamma}_q(s-n+k-a+\beta+1)}. \end{aligned} \quad (57)$$

From this expression follows that

$$\begin{aligned} \tilde{u}_n^{\alpha,\beta}(x(a), a, b)_q &= \frac{\tilde{\Gamma}_q(b-a)\tilde{\Gamma}_q(2a-\beta+n+1)\tilde{\Gamma}_q(b-a+\alpha)}{[n]! \tilde{\Gamma}_q(b-a-n)\tilde{\Gamma}_q(2a-\beta+1)\tilde{\Gamma}_q(b-a+\alpha-n)}, \\ \tilde{u}_n^{\alpha,\beta}(x(b-1), a, b)_q &= \frac{(-1)^n \tilde{\Gamma}_q(b-a)\tilde{\Gamma}_q(2b+\alpha)\tilde{\Gamma}_q(b-a+\beta)}{[n]! \tilde{\Gamma}_q(b-a-n)\tilde{\Gamma}_q(2b+\alpha-n)\tilde{\Gamma}_q(b-a+\beta-n)}, \end{aligned} \quad (58)$$

that are in agreement with the values (55) and (56) obtained before.

From the hypergeometric representation follows the symmetry property

$$\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q = \tilde{u}_n^{-b-a+\beta, b+a+\alpha}(x(s), a, b)_q.$$

2.2.2 The differentiation formulas

Next we use the differentiation formulas (21) and (23) to obtain

$$\frac{\Delta \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q}{\Delta x(s)} = -[2b-2a+\alpha+\beta-n-1]_q \tilde{u}_{n-1}^{\alpha,\beta}(x(s+\frac{1}{2}), a+\frac{1}{2}, b-\frac{1}{2})_q, \quad (59)$$

$$\begin{aligned} [n]_q [2s+1]_q \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q &= \sigma(-s-1) \tilde{u}_{n-1}^{\alpha,\beta}(x(s+\frac{1}{2}), a+\frac{1}{2}, b-\frac{1}{2})_q \\ &\quad - \sigma(s) \tilde{u}_{n-1}^{\alpha,\beta}(x(s-\frac{1}{2}), a+\frac{1}{2}, b-\frac{1}{2})_q, \end{aligned} \quad (60)$$

respectively. Finally, the formulas (24) (or (25)) and (26) lead to the following *differentiation formulas*

$$\sigma(s) \frac{\nabla \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q}{[2s]_q} = \frac{[2b-2a+\alpha+\beta-n-1]_q}{[2b-2a+\alpha+\beta-2n-2]_q} \left[\tau_n(s) \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q - [n+1]_q \tilde{u}_{n+1}^{\alpha,\beta}(x(s), a, b)_q \right], \quad (61)$$

$$\begin{aligned} \sigma(-s-1) \frac{\Delta \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q}{[2s+2]_q} &= -\frac{[2b-2a+\alpha+\beta-n-1]_q}{[2b-2a+\alpha+\beta-2n-2]_q} \times \\ &\left[(\tau_n(s) + [n]_q [2b-2a+\alpha+\beta-2n-2]_q [2s+1]_q) \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q - [n+1]_q \tilde{u}_{n+1}^{\alpha,\beta}(x(s), a, b)_q \right], \end{aligned} \quad (62)$$

respectively, where $\tau_n(s)$ is given in table 2.

2.3 The dual set to $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$

To obtain the dual set to $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ we use the same method as in the previous section. We start from the orthogonality relation (13) for the $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ polynomials defined by (54) and write the dual relation

$$\sum_{n=0}^{N-1} \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q \tilde{u}_n^{\alpha,\beta}(x(s'), a, b)_q \frac{1}{d_n^2} = \frac{1}{\rho(s) \Delta x(s-1/2)} \delta_{s,s'}, \quad N = b-a, \quad (63)$$

³Obviously the formulas (51–54) also give two equivalent explicit formulas.

where ρ and d_n^2 are the weight function and the norm of the $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ given in table 2. Furthermore, from (54) follows that the functions $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ are polynomials of degree $k = b - s - 1$ on the lattice $x(t) = [t]_q[t+1]_q$ where $t = b - a - n + \frac{\alpha+\beta}{2} - 1$ (the proof is similar to the one presented in section 2.1.4 and we will omit it here). To identify the dual set let us define the new set

$$\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q = \frac{(-1)^k \tilde{\Gamma}_q(b' - a') \tilde{\Gamma}_q(b' - a' + \beta') \tilde{\Gamma}_q(2b' + \alpha')}{[k]! \tilde{\Gamma}_q(b' - a' - k) \tilde{\Gamma}_q(b' - a' + \beta' - k) \tilde{\Gamma}_q(2b' + \alpha' - k)} \times {}_4F_3 \left(\begin{matrix} -k, 2a' - 2b' - \alpha' - \beta' + k + 1, -b' - t, -b' + t + 1 \\ a' - b' + 1, -2b' - \alpha' + 1, a' - b' - \beta' + 1 \end{matrix} \middle| q, 1 \right), \quad (64)$$

where

$$k = b - s - 1, \quad t = b - a - n + \frac{\alpha + \beta}{2} - 1, \quad a' = \frac{\alpha + \beta}{2}, \quad b' = b - a + \frac{\alpha + \beta}{2}, \quad \alpha' = 2a - \beta, \quad \beta' = \beta. \quad (65)$$

Obviously they satisfy the following orthogonality relation

$$\sum_{t=a'}^{b'-1} \tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q \tilde{u}_m^{\alpha',\beta'}(x(t), a', b')_q \rho'(t) \Delta x(t - 1/2) = (d'_k)^2 \delta_{k,m}, \quad (66)$$

where now $\rho'(t)$ and d'_k are the weight function ρ and the norm d_n , respectively, given in table 2 with the corresponding change of the parameters $a, b, \alpha, \beta, n, s$ by $a', b', \alpha', \beta', k, t$ (65).

Furthermore, with the above definition (65) for the parameters of $\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q$, the hypergeometric function ${}_4F_3$ in (64) coincides with the function ${}_4F_3$ in (54) and therefore the following relation between the polynomials $\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ and $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ holds

$$\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q = \tilde{\mathcal{A}}(\alpha, \beta, a, b, n, s) \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q, \quad (67)$$

where

$$\tilde{\mathcal{A}}(\alpha, \beta, a, b, n, s) = \frac{(-1)^{b-s-1-n} \tilde{\Gamma}_q(b-a-n) \tilde{\Gamma}_q(2b+\alpha-n) \tilde{\Gamma}_q(b-a+\beta-n) \tilde{\Gamma}_q(n+1)}{\tilde{\Gamma}_q(b-s) \tilde{\Gamma}_q(s-a+\beta+1) \tilde{\Gamma}_q(s+b+\alpha+1) \tilde{\Gamma}_q(s-a+1)}.$$

To prove that the polynomials $\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ are the dual set to $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ it is sufficient to substitute (67) in (66) and do the change (65) that transforms (66) into (46).

Let also mention that, as in the case of the q -Racah polynomials, the TTRR (16) of the polynomials $\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ is the SODE (4) of the polynomials $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ whereas the SODE (4) of the $\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ becomes into the TTRR (16) of $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ and vice versa.

To conclude this section let us point out that there exist a simple relation connecting both polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ and $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ (see (87) from below). We will establish it at the end of the next section.

3 Connection with the $6j$ -symbols of the q -algebra $SU_q(2)$

3.1 $6j$ -symbols of the quantum algebra $SU_q(2)$

It is known (see e.g. [26] and references therein) that the Racah coefficients $U_q(j_1 j_2 j_3; j_{12} j_{23})$ are used for the transition from the coupling scheme of three angular momenta j_1, j_2, j_3

$$|j_1 j_2(j_{12}), j_3 : jm\rangle = \sum_{m_1, m_2, m_3, m_{12}} \langle j_1 m_1 j_2 m_2 | j_{12} m_{12} \rangle \langle j_{12} m_{12} j_3 m_3 | jm \rangle |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle,$$

to the following ones

$$|j_1 j_2 j_3(j_{23}) : jm\rangle = \sum_{m_1, m_2, m_3, m_{23}} \langle j_2 m_2 j_3 m_3 | j_{23} m_{23} \rangle \langle j_1 m_1 j_{23} m_{23} | jm \rangle |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle,$$

where $\langle j_a m_a j_b m_b | j_{ab} m_{ab} \rangle$ denotes the Clebsh-Gordon Coefficients of the quantum algebra $su_q(2)$. In fact we have that the recoupling is given by

$$|j_1 j_2(j_{12}), j_3 : jm\rangle = \sum_{j_{23}} U_q(j_1 j_2 j j_3; j_{12} j_{23}) |j_1 j_2 j_3(j_{23}) : jm\rangle.$$

The Racah coefficients U define an unitary matrix, i.e., they satisfy the orthogonality relations

$$\sum_{j_{23}} U_q(j_1 j_2 j j_3; j_{12} j_{23}) U_q(j_1 j_2 j j_3; j'_{12} j_{23}) = \delta_{j_{12}, j'_{12}}, \quad (68)$$

$$\sum_{j_{12}} U_q(j_1 j_2 j j_3; j_{12} j_{23}) U_q(j_1 j_2 j j_3; j_{12} j'_{23}) = \delta_{j_{23}, j'_{23}}. \quad (69)$$

Usually instead of the Racah coefficients is more convenient to use the $6j$ -symbols defined by

$$U_q(j_1 j_2 j j_3; j_{12} j_{23}) = (-1)^{j_1+j_2+j_3+j} \sqrt{[2j_{12}+1]_q [2j_{23}+1]_q} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q.$$

The $6j$ -symbols have the following symmetry property

$$\left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q = \left\{ \begin{matrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12} \end{matrix} \right\}_q. \quad (70)$$

Here and without lost of generality we will suppose that $j_1 \geq j_2$ and $j_3 \geq j_2$, then for the moments j_{23} and j_{12} we have the intervals

$$j_3 - j_2 \leq j_{23} \leq j_2 + j_3, \quad j_1 - j_2 \leq j_{12} \leq j_1 + j_2,$$

respectively. Now, in order to avoid any other restrictions on these two momenta (caused by the so called triangle inequalities for the $6j$ -symbols) we will assume that the following restrictions hold

$$|j - j_3| \leq \min(j_{12}) = j_1 - j_2, \quad |j - j_1| \leq \min(j_{23}) = j_3 - j_2.$$

3.2 $6j$ -symbols and the q -Racah polynomials $u_n^{\alpha, \beta}(x(s), a, b)_q$

Now we are ready to establish the connection of $6j$ -symbols with the q -Racah polynomials. We fix the variable s as $s = j_{23}$ that runs on the interval $a \leq s \leq b - 1$ where $a = j_3 - j_2$, $b = j_2 + j_3 + 1$. Let us put

$$(-1)^{j_1+j_{23}+j} \sqrt{[2j_{12}+1]_q} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q = \sqrt{\frac{\rho(s)}{d_n^2}} u_n^{\alpha, \beta}(x(s), a, b)_q, \quad (71)$$

where $\rho(s)$ and d_n are the weight function and the norm, respectively, of the q -Racah polynomials on the lattice (1) $u_n^{\alpha, \beta}(x(s), a, b)_q$, and $n = j_{12} - j_1 + j_2$, $\alpha = j_1 - j_2 - j_3 + j \geq 0$, $\beta = j_1 - j_2 + j_3 - j \geq 0$.⁴

To verify the above relation we use the recurrence relation [27, Eq. (5.17)]

$$\begin{aligned} & [2]_q [2j_{23} + 2]_q A_q^- \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} - 1 \end{matrix} \right\}_q - \\ & \left(([2j_{23}]_q [2j_1 + 2]_q - [2]_q [j - j_{23} + j_1 + 1]_q [j + j_{23} - j_1]_q) \times \right. \\ & \left. ([2j_2]_q [2j_{23} + 2]_q - [2]_q [j_3 - j_2 + j_{23} + 1]_q [j_3 + j_2 - j_{23}]_q) - \right. \\ & \left. ([2j_2]_q [2j_1 + 2]_q - [2]_q [j_{12} - j_2 + j_1 + 1]_q [j_{12} + j_2 - j_1]_q) [2j_{23} + 2]_q [2j_{23}]_q \right) \times \\ & [2j_{23} + 1]_q \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q + [2]_q [2j_{23}]_q A_q^+ \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} + 1 \end{matrix} \right\}_q = 0, \end{aligned} \quad (72)$$

⁴Notice that this is equivalent to the following setting

$$\begin{aligned} j_1 &= (b - a - 1 + \alpha + \beta)/2, & j_2 &= (b - a - 1)/2, & j_3 &= (a + b - 1)/2, \\ j_{12} &= (2n + \alpha + \beta)/2, & j_{23} &= s, & j &= (a + b - 1 + \alpha - \beta)/2. \end{aligned}$$

where

$$\begin{aligned}
A_q^- &= \sqrt{[j+j_{23}+j_1+1]_q[j+j_{23}-j_1]_q[j-j_{23}+j_1+1]_q[j_{23}-j+j_1]_q} \times \\
&\quad \sqrt{[j_2+j_3+j_{23}+1]_q[j_2+j_3-j_{23}+1]_q[j_3-j_2+j_{23}]_q[j_2-j_3+j_{23}]_q}, \\
A_q^+ &= \sqrt{[j+j_{23}+j_1+2]_q[j+j_{23}-j_1+1]_q[j-j_{23}+j_1]_q[j_{23}-j+j_1+1]_q} \times \\
&\quad \sqrt{[j_2+j_3+j_{23}+2]_q[j_2+j_3-j_{23}]_q[j_3-j_2+j_{23}+1]_q[j_2-j_3+j_{23}+1]_q}.
\end{aligned} \tag{73}$$

Notice that

$$A_q^- = \sqrt{\sigma(j_{23})\sigma(-j_{23})}, \quad A_q^+ = \sqrt{\sigma(j_{23}+1)\sigma(-j_{23}-1)},$$

where

$$\begin{aligned}
\sigma(j_{23}) &= [j_{23}-j_3+j_2]_q[j_{23}+j_2+j_3+1]_q[j_{23}-j_1+j]_q[j+j_1-j_{23}+1]_q, \\
\sigma(-j_{23}-1) &= [j_{23}+j_3-j_2+1]_q[j_2+j_3-j_{23}]_q[j_{23}+j_1-j+1]_q[j+j_1+j_{23}+2]_q.
\end{aligned}$$

Substituting (71) in (72) and simplifying the obtained expression we get

$$\begin{aligned}
&[2s]_q\sigma(-s-1)u_n^{\alpha,\beta}(x(s+1), a, b)_q + [2s+2]_q\sigma(s)u_n^{\alpha,\beta}(x(s-1), a, b)_q + \\
&\left(\lambda_n[2s]_q[2s+1]_q[2s+2]_q - [2s]_q\sigma(-s-1) - [2s+2]_q\sigma(s)\right)u_n^{\alpha,\beta}(x(s), a, b)_q = 0,
\end{aligned}$$

which is the difference equation for the q -Racah polynomials (4). Since $u_0^{\alpha,\beta}(x(s), a, b)_q = 1$, (71) leads to

$$\begin{aligned}
&(-1)^{j_1+j_{23}+j} \sqrt{[2j_1-2j_2+1]_q} \left\{ \begin{matrix} j_1 & j_2 & j_1-j_2 \\ j_3 & j & j_{23} \end{matrix} \right\}_q = \sqrt{\frac{\rho(s)}{d_0^2}} \Rightarrow \\
&\left\{ \begin{matrix} j_1 & j_2 & j_1-j_2 \\ j_3 & j & j_{23} \end{matrix} \right\}_q := \left\{ \begin{matrix} j_1 & j_2 & j_1-j_2 \\ j_3 & j & s \end{matrix} \right\}_q \\
&= (-1)^{j+j_1+s} \sqrt{\frac{[j_1+j+s+1]_q! [j_1+j-s]_q! [j_1-j+s]_q! [j_3-j_2+s]_q!}{[j-j_1+s]_q! [j_3+j_2-s]_q! [j_2-j_3+s]_q! [j_2+j_3+s+1]_q!}} \times \\
&\sqrt{\frac{[2j_1-2j_2]_q! [2j_2]_q! [j_2+j_3+j-j_1]_q!}{[2j_1+1]_q! [j_1+j_3-j_2-j]_q! [j_1-j_3-j_2+j]_q! [j_1+j_3-j_2+j+1]_q!}}.
\end{aligned}$$

Furthermore, substituting the values $s = a$ and $s = b - 1$ in (71) and using (41) we find

$$\begin{aligned}
&\left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_3-j_2 \end{matrix} \right\}_q = (-1)^{j_{12}+j_3+j} \times \\
&\sqrt{\frac{[j_{12}+j_3-j]_q! [2j_2]_q! [j_{12}+j_3+j+1]_q! [2j_3-2j_2]_q! [j_2-j_1+j_{12}]_q!}{[j_1-j_2+j_3-j]_q! [j_1+j_2-j_{12}]_q! [j_1-j_2+j_3+j+1]_q!}} \times \\
&\sqrt{\frac{[j_1+j_2-j_3+j]_q! [j_1-j_2+j_{12}]_q! [j_3-j_{12}+j]_q!}{[2j_3+1]_q! [j_3-j_1-j_2+j]_q! [j_{12}-j_3+j]_q! [j_1+j_2+j_{12}+1]_q!}}
\end{aligned} \tag{74}$$

and

$$\begin{aligned}
&\left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_2+j_3 \end{matrix} \right\}_q = (-1)^{j_1+j_2+j_3+j} \times \\
&\sqrt{\frac{[2j_2]_q! [j_{12}-j_3+j]_q! [j_2-j_1+j_3+j]_q! [2j_3]_q! [j_1+j_2+j_3-j]_q!}{[j_1+j_2-j_{12}]_q! [j_1-j_2-j_3+j]_q! [j_3-j_{12}+j]_q!}} \times \\
&\sqrt{\frac{[j_2-j_1+j_{12}]_q! [j_1-j_2+j_{12}]_q! [j_1+j_2+j_3+j+1]_q!}{[2j_2+2j_3+1]_q! [j_{12}+j_3-j]_q! [j_1+j_2+j_{12}+1]_q! [j_{12}+j_3+j+1]_q!}},
\end{aligned} \tag{75}$$

that are in agreement with the results in [26].

The relation (71) allows us to obtain several recurrence relations for the $6j$ -symbols of the quantum algebra $SU_q(2)$ by using the properties of the q -Racah polynomials. So, the TTRR (16) gives

$$\begin{aligned}
& [2j_{12}]_q \tilde{A}_q^+ \left\{ \begin{matrix} j_1 & j_2 & j_{12} + 1 \\ j_3 & j & j_{23} \end{matrix} \right\}_q + [2j_{12} + 2]_q \tilde{A}_q^- \left\{ \begin{matrix} j_1 & j_2 & j_{12} - 1 \\ j_3 & j & j_{23} \end{matrix} \right\}_q \\
& - \left([2j_{12}]_q [2j_{12} + 1]_q [2j_{12} + 2]_q ([j_{23}]_q [j_{23} + 1]_q - [j_3 - j_2]_q [j_3 - j_2 + 1]_q) + [2j_{12}]_q \times \right. \\
& [j_1 - j_2 + j_{12} + 1]_q [j_{12} - j_1 - j_2]_q [j_{12} + j_3 - j + 1]_q [j_{12} + j_3 + j + 2]_q - [2j_{12} + 2]_q \times \\
& \left. [j_{12} - j_3 + j]_q [j_1 + j_2 + j_{12} + 1]_q [j_3 - j_{12} + j + 1]_q [j_2 - j_1 + j_{12}]_q \right) \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q = 0,
\end{aligned} \tag{76}$$

where

$$\begin{aligned}
\tilde{A}_q^- &= \frac{\sqrt{[j_2 - j_1 + j_{12}]_q [j_1 - j_2 + j_{12}]_q [j_{12} - j_3 + j]_q [j_{12} + j_3 - j]_q [j_1 + j_2 + j_{12} + 1]_q} \times}{\sqrt{[j_{12} + j_3 + j + 1]_q [j_1 + j_2 - j_{12} + 1]_q [j_3 - j_{12} + j + 1]_q}} \\
\tilde{A}_q^+ &= \frac{\sqrt{[j_2 - j_1 + j_{12} + 1]_q [j_1 - j_2 + j_{12} + 1]_q [j_{12} - j_3 + j + 1]_q [j_{12} + j_3 - j + 1]_q} \times}{\sqrt{[j_1 + j_2 + j_{12} + 2]_q [j_{12} + j_3 + j + 2]_q [j_1 + j_2 - j_{12}]_q [j_3 - j_{12} + j]_q}}.
\end{aligned} \tag{77}$$

The expressions (42) and (43) yield

$$\begin{aligned}
& \sqrt{\sigma(j_{23} + 1)} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} + 1 \end{matrix} \right\}_q + \sqrt{\sigma(-j_{23} - 1)} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q \\
& = [2j_{23} + 2]_q \sqrt{[j_2 - j_1 + j_{12}]_q [j_1 - j_2 + j_{12} + 1]_q} \left\{ \begin{matrix} j_1 + \frac{1}{2} & j_2 - \frac{1}{2} & j_{12} \\ j_3 & j & j_{23} + \frac{1}{2} \end{matrix} \right\}_q,
\end{aligned} \tag{78}$$

and

$$\begin{aligned}
& \sqrt{\sigma(-j_{23} - 1)} \left\{ \begin{matrix} j_1 + \frac{1}{2} & j_2 - \frac{1}{2} & j_{12} \\ j_3 & j & j_{23} + \frac{1}{2} \end{matrix} \right\}_q + \sqrt{\sigma(j_{23})} \left\{ \begin{matrix} j_1 + \frac{1}{2} & j_2 - \frac{1}{2} & j_{12} \\ j_3 & j & j_{23} - \frac{1}{2} \end{matrix} \right\}_q \\
& = [2j_{23} + 1]_q \sqrt{[j_{12} - j_1 + j_2]_q [j_{12} + j_1 - j_2 + 1]_q} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q,
\end{aligned} \tag{79}$$

respectively, whereas the differentiation formulas (44)–(45) give

$$\begin{aligned}
& [2j_{12} + 2]_q A_q^- \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} - 1 \end{matrix} \right\}_q + [2j_{23}]_q \tilde{A}_q^+ \left\{ \begin{matrix} j_1 & j_2 & j_{12} + 1 \\ j_3 & j & j_{23} \end{matrix} \right\}_q + \\
& \left(\sigma(j_{23}) [2j_{12} + 2]_q + [j_1 - j_2 + j_{12} + 1]_q [2j_{23}]_q \Lambda(j_{12}, j_{23}, j_1, j_2) \right) \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q = 0
\end{aligned} \tag{80}$$

and

$$\begin{aligned}
& [2j_{12} + 2]_q A_q^+ \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} + 1 \end{matrix} \right\}_q - [2j_{23} + 2]_q \tilde{A}_q^+ \left\{ \begin{matrix} j_1 & j_2 & j_{12} + 1 \\ j_3 & j & j_{23} \end{matrix} \right\}_q + \\
& \left([2j_{12} + 2]_q \sigma(-j_{23} - 1) - [2j_{23} + 2]_q [j_1 - j_2 + j_{12} + 1]_q (\Lambda(j_{12}, j_{23}, j_1, j_2) + \right. \\
& \left. [j_{12} - j_1 + j_2]_q [2j_{12} + 2]_q [2j_{23} + 1]_q) \right) \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q = 0,
\end{aligned} \tag{81}$$

respectively, where A_q^\pm are given by (73), \tilde{A}_q^\pm by (77) and

$$\begin{aligned}
\Lambda(j_{12}, j_{23}, j_1, j_2) &= \sigma \left(\frac{-j_{12} + j_1 - j_2}{2} - 1 \right) - \sigma \left(\frac{-j_{12} + j_1 - j_2}{2} \right) - \\
& [2j_{12} + 2]_q \left[j_{23} + \frac{j_{12} - j_1 + j_2}{2} \right]_q \left[j_{23} + \frac{j_{12} - j_1 + j_2}{2} + 1 \right]_q.
\end{aligned}$$

Using the hypergeometric representations (35) and (37) we obtain the representation of the $6j$ -symbols in terms of the q -hypergeometric function⁵ (28)

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = (-1)^{j_{12}+j_{23}+j_2+j} \frac{[2j_2]_q!}{[j_1-j_2+j_3-j]_q! [j_1-j_2+j_3+j+1]_q!} \\ \sqrt{\frac{[j_1+j+j_{23}+1]_q! [j_1+j-j_{23}]_q! [j_1-j+j_{23}]_q! [j_3-j_2+j_{23}]_q!}{[j-j_1+j_{23}]_q! [j_3+j_2-j_{23}]_q! [j_2-j_3+j_{23}]_q! [j_2+j_3+j_{23}+1]_q!}} \times \\ \sqrt{\frac{[j_{12}-j_1+j_2]_q! [j_{12}+j_1-j_2]_q! [j_3+j-j_{12}]_q! [j_3+j_{12}-j]_q! [j_3+j_{12}+j+1]_q!}{[j_{12}-j_3+j]_q! [j_1+j_2+j_{12}+1]_q! [j_1+j_2-j_{12}]_q!}} \times \\ {}_4F_3 \left(\begin{array}{c} j_1-j_2-j_{12}, j_1-j_2+j_{12}+1, j_3-j_2-j_{23}, j_{23}+j_3-j_2+1 \\ -2j_2, j_1-j_2+j_3-j+1, j_1-j_2+j_3+j+2 \end{array} \middle| q, 1 \right),$$

and

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = (-1)^{j_1+j_{23}+j} \frac{[2j_2]_q! [j_2+j_3-j_1+j]_q!}{[j_1-j_2-j_3+j]_q!} \\ \sqrt{\frac{[j_1+j+j_{23}+1]_q! [j_1+j-j_{23}]_q! [j_1-j+j_{23}]_q! [j_3-j_2+j_{23}]_q!}{[j-j_1+j_{23}]_q! [j_3+j_2-j_{23}]_q! [j_2-j_3+j_{23}]_q! [j_2+j_3+j_{23}+1]_q!}} \times \\ \sqrt{\frac{[j_{12}-j_1+j_2]_q! [j_{12}+j_1-j_2]_q! [j_{12}-j_3+j]_q!}{[j_1+j_2+j_{12}+1]_q! [j_1+j_2-j_{12}]_q! [j_3+j-j_{12}]_q! [j_{12}+j_3-j]_q! [j_3+j_{12}+j+1]_q!}} \times \\ {}_4F_3 \left(\begin{array}{c} j_1-j_2-j_{12}, j_1-j_2+j_{12}+1, -j_3-j_2+j_{23}, -j_{23}-j_3-j_2-1 \\ -2j_2, j_1-j_2-j_3+j+1, j_1-j_2-j_3-j \end{array} \middle| q, 1 \right).$$

Notice that from the above representations the values (74) and (75) immediately follows. Notice also that the above formulas give two alternative explicit formulas for computing the $6j$ -symbols. A third explicit formula follows from (40)

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = \sqrt{\frac{[j_{23}+j_2-j_3]_q! [j_{23}+j_2+j_3+1]_q! [j_{23}+j-j_1]_q! [j_2+j_3-j_{23}]_q!}{[j_{23}+j_3-j_2]_q! [j_{23}+j_1-j]_q! [j_{23}+j_1+j+1]_q! [j_1+j-j_{23}]_q!}} \times \\ \sqrt{\frac{[j_{12}-j_1+j_2]_q! [j_1-j_2+j_{12}]_q! [j_1+j_2-j_{12}]_q! [j_3-j_{12}+j]_q!}{[j_{12}-j_3+j]_q! [j_{12}+j_3-j]_q! [j_1+j_2+j_{12}+1]_q! [j_{12}+j_3+j]_q!}} \times \\ \sum_{k=0}^{j_{12}-j_1+j_2} \frac{(-1)^{k+j_1+j_{23}+j} [2k+j_1-j_2-j_{12}+2j_{23}+1]_q! [k+j_{23}+j_3-j_2]_q!}{[k]_q! [j_{12}-j_1+j_2-k]_q! [2j_3+1+k]_q! [k+j_{23}+j_1-j_{12}-j_3]_q!} \times \\ \frac{[2j_{23}+k-j_{12}+j_1-j_2]_q! [k+j_{23}+j_1-j]_q! [k+j_{23}+j_1+j+1]_q! [j_1+j-j_{23}-k]_q!}{[k+j_{23}+j_1-j_{12}+j_3+1]_q! [k+j_{23}+j-j_2-j_{12}]_q! [j_2+j_3-j_{23}+1-k]_q!}$$

To conclude this section let us point out that the orthogonality relations (68) and (69) lead to the orthogonality relations for the Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ (35) and their *duals* $u_k^{\alpha',\beta'}(x(t), a', b')_q$, respectively, and also that the relation (50) between q -Racah and dual q -Racah corresponds to the symmetry property (70).

3.3 $6j$ -symbols and the alternative q -Racah polynomials $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$

In this section we will provide the same comparative analysis but for the alternative q -Racah polynomials $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$. We again choose $s = j_{23}$ that runs on the interval $[a, b-1]$, $a = j_3-j_2$, $b = j_2+j_3+1$. In this case the connection is given by formula

$$(-1)^{j_{12}+j_3+j} \sqrt{[2j_{12}+1]_q} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = \sqrt{\frac{\rho(s)}{d_n^2}} \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q, \quad (82)$$

where $\rho(s)$ and d_n are the weight function and the norm, respectively, of the alternative q -Racah polynomials $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ (see Section 2.2) on the lattice (1), and $n = j_1+j_2-j_{12}$, $\alpha = j_1-j_2-j_3+j \geq 0$, $\beta = j_1-j_2+j_3-j \geq 0$.

⁵To obtain the representation in terms of the basic hypergeometric series it is sufficient to use the relation (29).

Using the above relations we see that the SODE (4) for the $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ polynomials becomes into the recurrence relation (72) as well as the TTRR (16) becomes into the recurrence relation (76). Evaluating (82) in $s = j_{23} = j_3 - j_2$ and $s = j_{23} = j_2 + j_3 + 1$ and using (58) we recover the values (74) and (75), respectively. If we now put $n = 0$, i.e., $j_{12} = j_1 + j_2$ we obtain the value

$$\begin{aligned} & \left\{ \begin{matrix} j_1 & j_2 & j_1 + j_2 \\ j_3 & j & j_{23} \end{matrix} \right\}_q := \left\{ \begin{matrix} j_1 & j_2 & j_1 + j_2 \\ j_3 & j & s \end{matrix} \right\}_q \\ & = (-1)^{j_1 + j_2 + j_3 + j} \sqrt{\frac{[2j_1]_q! [2j_2]_q! [j_1 + j_2 + j_3 + j + 1]_q! [j_1 + j_2 - j_3 + j]_q!}{[2j_1 + 2j_2 + 1]_q! [-j_1 - j_2 + j_3 + j]_q! [j_2 + j_3 + s + 1]_q!}} \times \\ & \sqrt{\frac{[s - j_1 + j]_q! [s - j_2 + j_3]_q!}{[j_1 + j - s]_q! [j_1 - j + s]_q! [j_1 + j + s + 1]_q! [j_2 + j_3 - s]_q! [j_2 - j_3 + s]_q!}}. \end{aligned}$$

The expressions (59) and (60) yield

$$\begin{aligned} & \sqrt{\varsigma(j_{23} + 1)} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} + 1 \end{matrix} \right\}_q - \sqrt{\varsigma(-j_{23} - 1)} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q \\ & = [2j_{23} + 2]_q \sqrt{[j_1 + j_2 - j_{12}]_q [j_1 + j_2 + j_{12} + 1]_q} \left\{ \begin{matrix} j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_{12} \\ j_3 & j & j_{23} + \frac{1}{2} \end{matrix} \right\}_q, \end{aligned} \quad (83)$$

and

$$\begin{aligned} & \sqrt{\varsigma(-j_{23} - 1)} \left\{ \begin{matrix} j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_{12} \\ j_3 & j & j_{23} + \frac{1}{2} \end{matrix} \right\}_q - \sqrt{\varsigma(j_{23})} \left\{ \begin{matrix} j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_{12} \\ j_3 & j & j_{23} - \frac{1}{2} \end{matrix} \right\}_q \\ & = [2j_{23} + 1]_q \sqrt{[j_1 + j_2 - j_{12}]_q [j_1 + j_2 + j_{12} + 1]_q} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q, \end{aligned} \quad (84)$$

respectively, where

$$\begin{aligned} \varsigma(j_{23}) &= [j_{23} - j_3 + j_2]_q [j_{23} + j_2 + j_3 + 1]_q [j_{23} - j_1 + j + 1]_q [j + j_1 + j_{23} + 1]_q, \\ \varsigma(-j_{23} - 1) &= [j_{23} + j_3 - j_2 + 1]_q [j_2 + j_3 - j_{23}]_q [j_{23} + j_1 - j + 1]_q [j + j_1 - j_{23}]_q. \end{aligned}$$

The differentiation formulas (61)–(62) give

$$\begin{aligned} & [2j_{12}]_q A_q^- \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} - 1 \end{matrix} \right\}_q - [2j_{23}]_q \tilde{A}_q^- \left\{ \begin{matrix} j_1 & j_2 & j_{12} - 1 \\ j_3 & j & j_{23} \end{matrix} \right\}_q - \\ & \left(\varsigma(j_{23}) [2j_{12}]_q + [j_1 + j_2 + j_{12} + 1]_q [2j_{23}]_q \tilde{\Lambda}(j_{12}, j_{23}, j_1, j_2) \right) \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q = 0 \end{aligned} \quad (85)$$

and

$$\begin{aligned} & [2j_{12}]_q A_q^+ \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} + 1 \end{matrix} \right\}_q + [2j_{23} + 2]_q \tilde{A}_q^- \left\{ \begin{matrix} j_1 & j_2 & j_{12} - 1 \\ j_3 & j & j_{23} \end{matrix} \right\}_q - \\ & \left([2j_{12}]_q \varsigma(-j_{23} - 1) - [2j_{23} + 2]_q [j_1 + j_2 + j_{12} + 1]_q \left(\tilde{\Lambda}(j_{12}, j_{23}, j_1, j_2) + \right. \right. \\ & \left. \left. [j_1 + j_2 - j_{12}]_q [2j_{12}]_q [2j_{23} + 1]_q \right) \right) \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q = 0, \end{aligned} \quad (86)$$

respectively, where A_q^\pm are given by (73), \tilde{A}_q^\pm by (77) and

$$\begin{aligned} \tilde{\Lambda}(j_{12}, j_{23}, j_1, j_2) &= \varsigma \left(\frac{j_{12} - j_1 - j_2}{2} - 1 \right) - \varsigma \left(\frac{j_{12} - j_1 - j_2}{2} \right) - \\ & [2j_{12}]_q \left[j_{23} + \frac{j_1 + j_2 - j_{12}}{2} \right]_q \left[j_{23} + \frac{j_1 + j_2 - j_{12}}{2} + 1 \right]_q. \end{aligned}$$

If we now use the hypergeometric representations (52) and (54) we obtain two new representations of the $6j$ -symbols in terms of the q -hypergeometric function (28)

$$\left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q = (-1)^{j_{12}+j_3+j} \frac{[2j_2]_q! [j_1 + j_2 - j_3 + j]_q!}{[j_3 - j_2 - j_1 + j]_q!} \times$$

$$\sqrt{\frac{[j - j_1 + j_{23}]_q! [j_3 - j_2 + j_{23}]_q!}{[j_1 + j + j_{23} + 1]_q! [j_1 + j - j_{23}]_q! [j_2 - j_3 + j_{23}]_q! [j_2 + j_3 + j_{23} + 1]_q! [j_1 - j + j_{23}]_q!}} \times$$

$$\sqrt{\frac{[j_3 - j_{12} + j]_q! [j_{12} + j_3 - j]_q! [j_3 + j_{12} + j + 1]_q! [j_1 - j_2 + j_{12} + 1]_q!}{[j_3 + j_2 - j_{23}]_q! [j_1 + j_2 - j_{12}]_q! [j_2 - j_1 + j_{12}]_q! [j_{12} - j_3 + j]_q! [j_1 + j_2 + j_{12} + 1]_q!}} \times$$

$${}_4F_3 \left(\begin{matrix} j_{12} - j_1 - j_2, -j_1 - j_2 - j_{12} - 1, j_3 - j_2 - j_{23}, j_{23} + j_3 - j_2 + 1 \\ -2j_2, j_3 - j_1 - j_2 + j + 1, j_3 - j_1 - j_2 - j \end{matrix} \middle| q, 1 \right),$$

and

$$\left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q = (-1)^{j_1+j_{23}+j} \frac{[2j_2]_q! [j_1 + j_2 + j_3 + j]_q! [j_1 + j_2 + j_3 - j]_q!}{\sqrt{[j_3 + j_2 - j_{23}]_q! [j_1 + j_2 - j_{12}]_q!}} \times$$

$$\sqrt{\frac{[j - j_1 + j_{23}]_q! [j_3 - j_2 + j_{23}]_q!}{[j_1 + j + j_{23} + 1]_q! [j_1 + j - j_{23}]_q! [j_2 - j_3 + j_{23}]_q! [j_2 + j_3 + j_{23} + 1]_q! [j_1 - j + j_{23}]_q!}} \times$$

$$\sqrt{\frac{[j_{12} - j_3 + j]_q! [j_{12} + j_1 - j_2 + 1]_q!}{[j_3 - j_{12} + j]_q! [j_{12} + j_3 - j]_q! [j_1 + j_2 + j_{12} + 1]_q! [j_2 - j_1 + j_{12}]_q! [j_3 + j_{12} + j + 1]_q!}} \times$$

$${}_4F_3 \left(\begin{matrix} j_{12} - j_1 - j_2, -j_1 - j_2 - j_{12} - 1, -j_3 - j_2 - j_{23} - 1, j_{23} - j_3 - j_2 \\ -2j_2, -j_1 - j_2 - j_3 - j, j - j_1 - j_2 - j_3 \end{matrix} \middle| q, 1 \right).$$

Notice that from the above representations the values (74) and (75) also follows. Obviously the above formulas give another two alternative explicit formulas for computing the $6j$ -symbols. Finally, from (57)

$$\left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\}_q = \sqrt{\frac{[j_{23} + j_1 + j + 1]_q! [j_1 - j_{23} + j]_q! [j_2 - j_3 + j_{23}]_q! [j_2 + j_3 + j_{23} + 1]_q!}{[j_{23} - j_2 + j_3]_q! [-j_1 + j_{23} + j]_q! [-j_{12} + j_3 + j]_q! [j_1 + j_2 + j_{12} + 1]_q!}} \times$$

$$\sqrt{\frac{[j_1 + j_{23} - j]_q! [j_2 + j_3 - j_{23}]_q! [j_1 + j_2 - j_{12}]_q! [-j_1 + j_2 + j_{12}]_q!}{[-j_3 + j + j_{12}]_q!^{-1} [j_{12} + j_3 + j + 1]_q!^{-1} [j_1 - j_2 + j_{12}]_q!^{-1}}}} \times$$

$$\sum_{l=0}^{j_1+j_2-j_{12}} \frac{(-1)^{l+j_1+j_2+j_3+j} [2j_{23} + 2l - j_1 - j_2 + j_{12} + 1]_q}{[l]_q! [j_1 + j_2 - j_{12} - l]_q! [2j_{23} + l + 1]_q! [j_{23} - j_1 - j_2 + j_{12} + l + j_2 - j_3]_q! [j_2 + j_3 - j_{23} - l]_q!} \times$$

$$\frac{[2j_{23} + l - j_1 - j_2 + j_{12}]_q! [j_{23} + l - j_2 + j_3]_q! [-j_1 + j + j_{23} + l]_q!}{[l + j_{12} - j_2 + j + j_{23} + 1]_q! [j_1 + j - j_{23} - l]_q! [-j_1 + j_{12} + j_3 + j_{23} + l + 1]_q! [-j_2 + j_{12} - j + j_{23} + l]_q!},$$

To conclude this section, let us point out that the orthogonality relations (68) and (69) lead to the orthogonality relations for the alternative Racah polynomials $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ (52) and their duals $\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ (64), respectively, as well as the relation (67) between q -Racah and dual q -Racah corresponds to the symmetry property (70).

3.4 Connection between $\tilde{u}_k^{\alpha,\beta}(x(s), a, b)_q$ and $u_n^{\alpha,\beta}(x(s), a, b)_q$

Let us obtain a formula connecting the two families $\tilde{u}_k^{\alpha,\beta}(x(s), a, b)_q$ and $u_n^{\alpha,\beta}(x(s), a, b)_q$. In fact, Eqs. (71) and (82) suggest the following relation between both Racah polynomials $\tilde{u}_k^{\alpha,\beta}(x(s), a, b)_q$ and $u_n^{\alpha,\beta}(x(s), a, b)_q$

$$\tilde{u}_{b-a-1-n}^{\alpha,\beta}(x(s), a, b)_q = (-1)^{s-a-n} \times$$

$$\frac{\tilde{\Gamma}_q(s-a+\beta+1) \tilde{\Gamma}_q(b+\alpha-s) \tilde{\Gamma}_q(b+\alpha+1+s) \tilde{\Gamma}_q(a+b-\beta-n)}{\tilde{\Gamma}_q(s+a-\beta+1) \tilde{\Gamma}_q(\alpha+1+n) \tilde{\Gamma}_q(\beta+1+n) \tilde{\Gamma}_q(a+b+\alpha+1+n)} u_n^{\alpha,\beta}(x(s), a, b)_q. \quad (87)$$

To prove it is sufficient to substitute the above formula in the difference equation (4) of the $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ polynomials. After some straightforward computations the resulting difference equation becomes into the corresponding difference equation for the polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$.

Notice that from this relation follows that

$$\begin{aligned} & {}_4F_3 \left(\begin{matrix} a-b+n+1, a-b-\alpha-\beta-n, a-s, a+s+1 \\ a-b+1, 2a-\beta+1, a-b-\alpha+1 \end{matrix} \middle| q, 1 \right) \\ &= \frac{(\beta+1|q)_{s-a}(b+\alpha+a+1|q)_{s-a}}{(2a-\beta+1|q)_{s-a}(a-b-\alpha+1)_{s-a}} {}_4F_3 \left(\begin{matrix} -n, \alpha+\beta+n+1, a-s, a+s+1 \\ a-b+1, \beta+1, a+b+\alpha+1 \end{matrix} \middle| q, 1 \right). \end{aligned}$$

This yield to the following identity for terminating ${}_4\phi_3$ basic series, $n, N-n-1, k=0, 1, 2, \dots$,

$$\begin{aligned} & {}_4\varphi_3 \left(\begin{matrix} q^{-N+1}, q^{-n-N+1}A^{-1}B^{-1}, q^{-k}, q^{-k}D \\ q^{1-N}, q^{-2k}DB^{-1}, q^{1-N}A^{-1} \end{matrix} \middle| q, q \right) \\ &= \frac{q^{-kN}}{A^k B^k} \frac{(qB; q)_k (q^{N-2k}DA; q)_k}{(q^{-2k}DB^{-1}; q)_k (q^{1-N}A^{-1}; q)_k} {}_4\varphi_3 \left(\begin{matrix} q^{-n}, ABq^n, q^{-k}, q^{-k}D \\ q^{1-N}, qB, q^{N-2k}DA \end{matrix} \middle| q, q \right). \end{aligned}$$

4 Conclusions

Here we have provided a detailed study of two kind of Racah q -polynomials on the lattice $x(s) = [s]_q[s+1]_q$ and also their comparative analysis with the Racah coefficients or $6j$ -symbols of the quantum algebra $U_q(su(2))$.

To conclude this paper we will briefly discuss the relation of these q -Racah polynomials with the representation theory of the quantum algebra $U_q(su(3))$. In [20, §5.5.3] was shown that the transformation between two different bases (λ, μ) of the irreducible representation of the classical (not q) algebra $su(3)$ corresponding to the reductions $su(3) \supset su(2) \times u(1)$ and $su(3) \supset u(1) \times su(2)$ of the $su(3)$ algebra in two different subalgebras $su(2)$ is given in terms of the Weyl coefficients that are, up to a sign (phase), the Racah coefficients of the algebra $su(2)$. The same statement can be done in the case of the quantum algebra $su_q(3)$ [5, 18]: The Weyl coefficients of the transformation between two bases of the irreducible representation (λ, μ) corresponding to the reductions $su_q(3) \supset su_q(2) \times u_q(1)$ and $su_q(3) \supset u_q(1) \times su_q(2)$ of the quantum algebra $su_q(3)$ in two different quantum subalgebras $su_q(2)$ coincide (up to a sign) with the q -Racah coefficients of the $su_q(2)$.

In fact, the Weyl coefficients satisfy certain difference equations that are equivalent to the differentiation formulas for the q -Racah polynomials $u_n^{\alpha, \beta}(x(s), a, b)_q$ and $\tilde{u}_n^{\alpha, \beta}(x(s), a, b)_q$ so, following the idea in [20, §5.5.3] we can assure that the main properties of the q -Racah polynomials are closely related with the representations of the quantum algebra $U_q(su(3))$. Finally, let us point out that the same assertion can be done but with the non-compact quantum algebra $U_q(su(2, 1))$. This will be carefully done in a forthcoming paper.

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References

- [1] R. Álvarez-Nodarse, *q-analog of the vibrational IBM and the quantum algebra $SU_q(1, 1)$* . Master Thesis in Moscow State University *M. V. Lomonosov*. (November 1992). (In Russian).
- [2] R. Álvarez-Nodarse, *Polinomios generalizados y q-polinomios: propiedades espectrales y aplicaciones*. Tesis Doctoral. Universidad Carlos III de Madrid. Madrid, 1996. (In Spanish)
- [3] R. Álvarez-Nodarse, *Polinomios hipergométricos y q-polinomios*. Monografías del Seminario García Galdeano. Universidad de Zaragoza. Vol. **26**. Pressas Universitarias de Zaragoza, Zaragoza, Spain, 2003. (In Spanish).
- [4] R. Álvarez-Nodarse and Yu. F. Smirnov, *q-Dual Hahn polynomials on the non-uniform lattice $x(s) = [s]_q[s+1]_q$ and the q -algebras $SU_q(1, 1)$ and $SU_q(2)$* . *J. Phys. A: Math. Gen.* **29** (1996), 1435-1451.
- [5] P. M. Asherova, Yu. F. Smirnov, and V. N. Tolstoi, *Weyl q -coefficients for $u_q(3)$ and Racah q -coefficients for $su_q(2)$* , *Sov. J. Nucl. Phys.* **58** (1996), 1859-1872.
- [6] R. Askey and R. Wilson, *A set of orthogonal polynomials that generalize Racah coefficients or $6j$ symbols*. *SIAM J. Math. Anal.* **10** (1979), 1008-1020.

- [7] R. Askey and R. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. *Mem. Amer. Math. Soc.* **319**. Providence, Rhode Island, 1985.
- [8] N. M. Atakishiyev, M. Rahman, and S. K. Suslov, On classical orthogonal polynomials. *Constr. Approx.* **11** (1995), 181-226.
- [9] V.G. Drinfel'd, Quantum Groups, *Proceedings of the Int. Congress of Mathematicians*. Berkeley 1986, 798-820, American Mathematical Society. Providence, R. I. 1987.
- [10] L.D. Faddeev and L. A. Takhtajan, *Lect. Notes Phys.* **246**, (1986) 183
- [11] G. Gasper and M. Rahman, *Basic Hypergeometric Series*. Cambridge University Press, 1990.
- [12] M. Jimbo, Quantum R matrix for the generalized Toda system. *Commun. Math. Phys.* **102** (1986), 537-547.
- [13] A. N. Kirillov and N. Yu. Reshetikhin, *LOMI Preprint E-9-88*, Leningrad, 1988.
- [14] R. Koekoek and R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue. *Reports of the Faculty of Technical Mathematics and Informatics* **No. 98-17**. Delft University of Technology, Delft, 1998.
- [15] H. T. Koelink, Askey-Wilson polynomials and the quantum $SU(2)$ group: Survey and Applications. *Acta Applic. Math.* **44** (1996), 295-352.
- [16] P. P. Kulish and N. Yu. Reshetikhin, *Zapiski Nauchnykh Seminarov LOMI* **101** (1981)
- [17] A. A. Malashin, q -analog of Racah polynomials on the lattice $x(s) = [s]_q[s+1]_q$ and its connections with $6j$ -symbols for the $SU_q(2)$ and $SU_q(1,1)$ quantum algebras. Master Thesis. Moscow State University *M.V. Lomonosov*. (January 1992). (In Russian).
- [18] A. A. Malashin, Yu. F. Smirnov, and Yu.I. Kharitonov, *Sov. J. Nucl.Phys.* **53** (1995), 1105-1119.
- [19] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*. Nauka, Moscow, 1985, Russian Edition (in Russian).
- [20] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable. Springer Series in Computational Physics*. Springer-Verlag, Berlín, 1991. (Russian Edition, Nauka, Moscow, 1985)
- [21] A. F. Nikiforov y V. B. Uvarov, Classical orthogonal polynomials in a discrete variable on nonuniform lattices. *Preprint Inst. Prikl. Mat. Im. M. V. Keldysha Akad. Nauk SSSR*, Moscú, 1983, No. 17, (en ruso).
- [22] A. F. Nikiforov and V. B. Uvarov, Polynomial Solutions of hypergeometric type difference Equations and their classification. *Integral Transform. Spec. Funct.* **1** (1993), 223-249.
- [23] H. Rosengren, An elementary approach to $6j$ -symbols (classical, quantum, rational and elliptic). Preprint 2003, arXiv:math.CA/0312310.
- [24] E. K. Sklyanin, Some algebraic structures connected with the Yang-Baxter equation. *Funct. Anal. Appl.* **16** (1983), 263-270.
- [25] Yu. F. Smirnov, V.N. Tolstoy and Yu.I. Kharitonov, Method of projection operators and the q analog of the quantum theory of angular momentum. Clebsch-Gordan Coefficients and irreducible tensor operators. *Sov. J. Nucl.Phys.* **53** (1991), 593-605.
- [26] Yu. F. Smirnov, V.N. Tolstoy and Yu.I. Kharitonov:, Projection-operator Method and the q analog of the quantum theory of angular momentum. Racah coefficients, $3j$ and $6j$ symbols, and their symmetry properties. *Sov. J. Nucl.Phys.* **53** (1991), 1069-1086.
- [27] Yu. F. Smirnov, V.N. Tolstoy and Yu.I. Kharitonov, Tree technique and irreducible tensor operators for the $SU_q(2)$ quantum algebra. $9j$ symbols. *Sov. J. Nucl.Phys.* **55** (1992), 1599-1604 .
- [28] Yu. F. Smirnov, V. N. Tolstoy y Yu. I. Kharitonov, The tree technique and irreducible tensor operators for the $SU_q(2)$ quantum algebra. The algebra of irreducible tensor operators. *Physics Atom. Nucl.* **56** (1993), 690-700.
- [29] N. Ja. Vilenkin and A. U. Klimyk, *Representations of Lie Groups and Special Functions*. **Vol. II, III**. Kluwer Academic Publishers. Dordrecht, 1992.