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MINIMAL REGULARITY CONDITIONS FOR THE END-POINT ESTIMATE OF BILINEAR CALDERÓN-ZYGMUND OPERATORS

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ABSTRACT. Minimal regularity conditions on the kernels of bilinear operators are identified and shown to be sufficient for the existence of end-point estimates within the context of the bilinear Calderón-Zygmund theory.

1. INTRODUCTION

A crucial property addressed in the linear Calderón-Zygmund theory, going back to the founding article [1], is the fact that operators bounded on L^2 whose kernels possess certain regularity are in fact bounded on every L^p space for $1 < p < \infty$. Moreover, such regularity assumption implies, together with the L^2-boundedness of the operator, a weak-type end-point estimate in L^1. From these continuity properties, the whole range of values of p follows by duality and interpolation. The quest for the minimal amount of regularity needed to guarantee the existence of such end-point estimate has a rich history with several important results that promoted as byproduct numerous developments in harmonic analysis. For classical singular integrals operators with homogeneous kernels, the question was finally settled in the work of Seeger [23] who showed that the kernel of the operators could be quite rough. See also previous work of Christ [5] and Christ and Rubio de Francia [7]. We refer to [23] and its featured review by Hofmann [13] for precise technical details, an account of the history, and relevant references. For more general multiplier operators as well as operators of non-convolution type the regularity of the kernel is very closely related to a Lipschitz-type one. It is convenient for our purposes to recall some well-known facts related to this.

Assume that $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an operator that, at least for $x \notin \operatorname{supp} f$, is given by

$$Tf(x) = \int K(x,y) f(y)\, dy.$$

Here, the kernel K is assumed to be an integrable function on any product $E_1 \times E_2$ of compact sets in $\mathbb{R}^n$, with $E_1 \cap E_2 = \emptyset$. Then, a sufficient condition for the operator to satisfy $T : L^1 \to L^{1,\infty}$ is the gradient conditions on the kernel given by

$$|\nabla_y K(x,y)| \lesssim \frac{1}{|x-y|^{n+1}} \quad for\ \ x \neq y, \tag{1.1}$$

Alternatively one can assume a Lipschitz form of the above,

$$|K(x,y) - K(x,y')| \lesssim \frac{|y-y'|^{\varepsilon}}{|x-y|^{n+\varepsilon}} \quad for\ \ |y-y'| \leq c|x-y|, \tag{1.2}$$

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for some $0 < \varepsilon \leq 1$ and $0 < c < 1$. Moreover, one can also consider a weaker condition which takes the form

$$\int_{|y-y'|\leq c|x-y|} |K(x,y) - K(x,y')|\, dx < \infty. \tag{1.3}$$

This is sometimes referred to as the integral regularity or Hörmander condition. This condition is very convenient when working on more general geometric or measure theoretic contexts where other types of regularity are absent or hard to formalize. Actually, in $\mathbb{R}^n$ it can be written in the more geometric form

$$\sup_{Q} \sup_{y\in Q} \int_{\mathbb{R}^n\setminus Q^*} |K(x,y) - K(x,y_Q)|\, dx < \infty, \tag{1.4}$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the axes, and where Q^* is the cube with same center y_Q as Q and side length an appropriately chosen large multiple c_n of the side length of Q (for example $c_n = 10\sqrt{n}\,\ell(Q)$ will do).

To some extent, the condition (1.3) is barely enough to prove the end-point estimate $L^1 \to L^{1,\infty}$ but, unlike (1.1) or (1.2), it is not enough for other aspects of the Calderón-Zygmund theory. In particular it does not suffice to establish good-λ inequalities between T and the the Hardy-Littlewood maximal function M, and there is no weighted theory for operators satisfying only (1.3); see the work of Martell-Pérez-Trujillo [20].

We also recall that in the case of multiplier operators, that is kernel of the form $K(x-y)$, (1.3) follows from the Hörmander-Mihlin conditions. Let T given by

$$\widehat{Tf}(\xi) = m(\xi)\widehat{f}(\xi).$$

Then, both the Mihlin [21] condition

$$|\partial^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|} \text{ for all } |\alpha| \leq [n/2]+1; \tag{1.5}$$

as well as the weaker Hörmander [14] condtion

$$\sup_{j\in\mathbb{Z}} \|m(2^j\cdot)\varphi\|_{L^2_s(\mathbb{R}^n)} < \infty \text{ for some } s > n/2, \tag{1.6}$$

where φ is a smooth bump supported away from the origin and L^2_s is the usual Sobolev space, imply (1.3).

We now describe the situation of the bilinear Calderón-Zygmund theory. Under regularity assumptions analogous to (1.1) or (1.2) the theory has been by now developed through works of Coifman-Meyer [2], [3], [4], Christ-Journé [6], Kenig-Stein [15] and Grafakos-Torres [10], [11]. Some of the most recent work in the subject was motivated in part by the work of Lacey-Thiele [16], [17] on the bilinear Hilbert transform and a search for the optimal range of exponents where boundedness in Lebesgue spaces can be obtained. Unlike the case of the bilinear Hilbert transform, a more singular operator not covered by the Calderón-Zygmund theory, the boundeness of Calderón-Zygmund operators on the full range of exponents is known. In particular, it was shown in [11] that a bilinear operator bounded from $L^p \times L^q \to L^r$ for some $1/p+1/q = 1/r$, and given by

$$T(f,g)(x) = \int_{\mathbb{R}^{2n}} K(x,y,z) f(y) g(z)\, dydz$$

for $x \notin \operatorname{supp} f \cap \operatorname{supp} g$, also satisfies $T : L^1 \times L^1 \to L^{1/2,\infty}$ provided the Schwartz kernel of the operator has the Lipschitz regularity properties

$$|K(x,y,z) - K(x,y',z)| \lesssim \frac{|y-y'|^\varepsilon}{(|x-y|+|x-z|)^{n+\varepsilon}}, \tag{1.7}$$

whenever $|y-y'| \leq \frac{1}{2}\max\{|x-y|,|x-z|\}$, and

$$|K(x,y,z)-K(x,y,z')| \lesssim \frac{|z-z'|^{\varepsilon}}{(|x-y|+|x-z|)^{n+\varepsilon}}, \tag{1.8}$$

whenever $|z-z'| \leq \frac{1}{2}\max\{|x-y|,|x-z|\}$. As in the linear theory, if T and its two transposes T^{*1} and T^{*2} satisfy the same conditions then interpolation and duality give boundedness for the full range of exponents, $L^p \times L^q \to L^r$ for all $1/p+1/q=1/r$ and $1/2<r<\infty$. Moreover, the regularity assumptions are also good enough to obtain a weighted theory for classical A_p weights, as shown by Grafakos-Torres [12] and Pérez-Torres [22], which was later extended to new optimal multilinear classes of weights by Lerner-Ombrosi-Pérez-Torres-Trujillo [18].

There has been after all thsese works some interest in finding the corresponding analog to the minimal regularity assumption (1.3) in the multilinear setting. Our purpose with this short note is to contribute in this regard.

We note that Maldonado-Naibo [19] have weakened (1.7) and (1.8) to a Diny-type condition that we shall also consider (see Definition 2.2 below) and they simplified the proof of the end-point estimate. However, we are able in this article to identify integral type conditions which are even weaker and further simplify the proof of the $L^1 \times L^1 \to L^{1/2,\infty}$ result. We only focus on the bilinear setting but the interested reader may find analogous conditions in the m-linear case. The reader may also proceed to the next section were we state the conditions, but we want to provide some formal motivation that lead to the conditions we use. To do so, we recall very recent results for bilinear multiplier operators (corresponding to operators with kernels of the form $K(x-y,x-z)$). It is of interest that the parallel with the linear theory in terms of minimal regularity for multipliers appears to break down.

Suppose that T_m is a Coifman-Meyer bilinear multiplier operator. That is,

$$T_m(f,g)(x) = \int m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix(\xi+\eta)}\,d\xi d\eta,$$

where

$$|\partial_\alpha m(\xi,\eta)| \lesssim_\alpha (1+|\xi|+|\eta|)^{-|\alpha|} \text{ for all } \alpha, \tag{1.9}$$

then the kernel of T_m is of the form $K(x-y,x-z)$ and satisfies

$$|\partial_\alpha K((x-y,x-z)|| \lesssim_\alpha \frac{1}{(|x-y|+|x-z|)^{2n+|\alpha|}}.$$

This follows from considering m as a multiplier in $\mathbb{R}^{2n}$ and restricting the corresponding kernel $K(x_1-y,x_2-z)$ to $x_1=x_2$. It also follows then that these operators are bounded from $L^p \times L^q \to L^r$ for the full range of exponents and also from $L^1 \times L^1$ to $L^{1/2,\infty}$.

It is very natural to expect that (1.9) could be relaxed to a Hörmander-Mihlin condition, limiting the number of derivatives of the symbol needed to be controlled. Certainly the arguments used by Coifman-Meyer only need a "sufficiently large" number of derivatives (not easy to track in their computations) but one should expect to require only about "half-the-dimension" number of derivatives. However, the natural dimension in the bilinear setting appears to be $2n$ and so, some interesting situation occurs.

In fact, Tomita [24] recently showed that

$$\sup_{j\in\mathbb{Z}} \|m(2^j\cdot,2^j\cdot)\varphi\|_{L^2_s(\mathbb{R}^{2n})} < \infty \text{ for some } s>n, \tag{1.10}$$

where φ is an appropriate cut-off function in $\mathbb{R}^{2n}$, implies the boundedness

$$T_m : L^p \times L^q \to L^r$$

with $1/p + 1/q = 1/r$, but only for $r > 1$. That is, essentially half-the-dimension" number of derivatives in L^2 to obtain the range $r > 1$. However, to obtain other values of $1/2 < r \leq 1$, which is also natural in the bilinear case, it appears that one needs to impose higher regularity. Grafakos-Si [9] showed after Tomita's work that one can push p, q to $1 + \varepsilon$ (i.e. r to $1/2 + \varepsilon/2$) if

$$\sup_{j \in \mathbb{Z}} \|m(2^j \cdot, 2^j \cdot)\varphi\|_{L_s^{t_\varepsilon}(\mathbb{R}^{2n})} < \infty \tag{1.11}$$

for appropriate $1 < t_\varepsilon \leq 2$ and $s > 2n/t_\varepsilon$. Essentially, one may say that $2n$ derivatives in L^1 may be required to get the full range of exponents. We refer to [9] for the precise technical details. Similar results on the product of Hardy H^p were very recently obtained by Grafakos-Miyachi-Tomita [8] but, as far as we know and unlike the linear case, there are no results of this type that give the end-point estimate $L^1 \times L^1 \to L^{1/2,\infty}$ (except, of course, for the sufficiently large number of derivatives in the arguments of Coifman-Meyer using (1.9) and which give pointwise estimates on the gradient of the kernel).

As already mentioned, we want to find some bilinear analog of the integral condition (1.3) even for non-convolution operators, but it is instructive to see what such a condition could be in the case of multipliers.

If we consider again m as a Fourier multiplier in $\mathbb{R}^{2n}$, then (1.10) is just Hörmander's condition and we have for the kernel of the corresponding linear operator in $\mathbb{R}^{2n}$ for any $y, z \in \mathbb{R}^n$

$$\int_{|y-y'|+|z-z'| \leq c|x_1-y|+|x_2-z|} |K(x_1 - y, x_2 - z) - K(x_1 - y', x_2 - z')| \, dx_1 dx_2 \leq C.$$

When $z = z'$ and performing a simple change of variables, the above can be written as

$$\int_{|y-y'| \leq c|x_1-y|+|x_2-z|} |K(x_1 - y, x_1 - (x_1 - x_2 + z)) - K(x_1 - y', x_1 - (x_1 - x_2 + z))| \, dx_2 dx_1$$
$$= \int_{|y-y'| \leq c|x_1-y|+|x_1-u|} |K(x_1 - y, x_1 - u) - K(x_1 - y', x_1 - u)| \, du dx_1 \leq C.$$

In a more geometric form (and considering a smaller region of integration) we can essentially state the above in the form

$$\sup_Q \sup_{y \in Q} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus Q^*} |K(x - y, x - u) - K(x - y_Q, x - u)| \, dx du \leq C, \tag{1.12}$$

and similarly in the other variable,

$$\sup_Q \sup_{y \in Q} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus Q^*} |K(x - u, x - z) - K(x - u, x - y_Q)| \, dx du \leq C. \tag{1.13}$$

In the non-convolution case of interest to us, the conditions (1.12) and (1.13) would become

$$\sup_Q \sup_{y \in Q} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus Q^*} |K(x, y, u) - K(x, y_Q, u)| \, dx du \leq C \tag{1.14}$$

and

$$\sup_Q \sup_{z \in Q} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus Q^*} |K(x, u, z) - K(x, u, z_Q)| \, dx du \leq C. \tag{1.15}$$

Unfortunately these conditions do not seem to be enough to show the end-point estimate we are looking for. This is not surprising, since as observed, in the convolution case they are implied by Tomita's condition (1.10) which does not seem to be even enough for boundedness for any $r \leq 1$.

We find interesting that a small modification of (1.14) and (1.15) (moving the $\sup_{y\in Q}$ inside the integral) together with a closely related new integral condition do suffice. Moreover we will show that the conditions are implied by a very general Dini-type one, hence showing that they are weaker than all others considered in the literature so far.

2. The new regularity conditions

Definition 2.1. We say that the bilinear operator with kernel K satisfy the bilinear geometric Hörmander conditions (BGHC) if there exits a fixed constant B such that and for any two families of disjoint dyadic cubes D_1 and D_2,

$$\sup_{Q\in D_1}\int_{\mathbb{R}^n}\sup_{y\in Q}\int_{\mathbb{R}^n\setminus(\cup_{R\in D_1}R^*)}|K(x,y,z)-K(x,y_Q,z)|\,dx\,dz\leq B, \tag{2.1}$$

$$\sup_{P\in D_2}\int_{\mathbb{R}^n}\sup_{z\in P}\int_{\mathbb{R}^n\setminus(\cup_{S\in D_2}S^*)}|K(x,y,z)-K(x,y,z_P)|\,dx\,dy\leq B, \tag{2.2}$$

and

$$\sum_{(P,Q)\in D_1\times D_2}|P||Q|\sup_{(y,z)\in P\times Q}\int_{\mathbb{R}^n\setminus(\cup_{R\in D_1}R^*)\cup(\cup_{S\in D_2}S^*)}|K(x,y,z)-K(x,y_P,z_Q)|\,dx \tag{2.3}$$
$$\leq B\left(|\cup_{P\in D_1}P|+|\cup_{Q\in D_2}Q|\right).$$

We want to make some further remarks about these conditions. Write $\Omega_1=\cup_{Q\in D_1}Q$, $\Omega_1^*=\cup_{Q\in D_1}Q^*$, $\Omega_2=\cup_{P\in D_2}P$, and $\Omega_2^*=\cup_{P\in D_1}P^*$. First, note that for any $y',y''\in Q$,

$$\sup_{y\in Q}\int_{\mathbb{R}^n\setminus\Omega_1^*}|K(x,y,z)-K(x,y',z)|\,dx$$
$$\lesssim\sup_{y\in Q}\int_{\mathbb{R}^n\setminus\Omega_1^*}|K(x,y,z)-K(x,y'',z)|\,dx+\int_{\mathbb{R}^n\setminus\Omega_1^*}|K(x,y',z)-K(x,y'',z)|\,dx$$
$$\lesssim\sup_{y\in Q}\int_{\mathbb{R}^n\setminus(\Omega_1^*}|K(x,y,z)-K(x,y'',z)|\,dx,$$

so the inner supremum and integral in (2.1) can be replaced by either

$$\sup_{y'\in Q}\sup_{y\in Q}\int_{\mathbb{R}^n\setminus\Omega_1^*}|K(x,y,z)-K(x,y',z)|\,dx,$$

or

$$\inf_{y'\in Q}\sup_{y\in Q}\int_{\mathbb{R}^n\setminus\Omega_1^*}|K(x,y,z)-K(x,y',z)|\,dx.$$

Similarly with (2.2), while it is also equivalent to replace the supremum and integral in (2.3) by

$$\sup_{(\alpha,\beta)\in P\times Q}\sup_{(y,z)\in P\times Q}\int_{\mathbb{R}^n\setminus(\Omega_1^*\cup\Omega_2^*)}|K(x,y,z)-K(x,\alpha,\beta)|\,dx$$

or

$$\inf_{(\alpha,\beta)\in P\times Q}\sup_{(y,z)\in P\times Q}\int_{\mathbb{R}^n\setminus(\Omega_1^*\cup\Omega_2^*)}|K(x,y,z)-K(x,\alpha,\beta)|\,dx.$$

Next, we note that (2.3) is in some sense the strongest of the three conditions, since it gives a substantial part (though not all) of (2.1) and (2.2). In fact, note that D_1 and D_2 could be the same family and so assuming that (2.3) holds,

$$\sum_{Q\in D_1}|Q|\int_{\Omega_1}\sup_{y\in Q}\int_{\mathbb{R}^n\setminus\Omega_1^*}|K(x,y,z)-K(x,y_Q,z)|\,dx\,dz$$

$$\leq \sum_{Q\in D_1}|Q|\sum_{Q'\in D_1}|Q'|\sup_{z\in Q'}\sup_{y\in Q}\int_{\mathbb{R}^n\setminus\Omega_1^*}|K(x,y,z)-K(x,y_Q,z)|\,dx$$

$$\leq \sum_{(Q,Q')\in D_1\times D_1}|Q|\,|Q'|\sup_{(\alpha,\beta)\in Q\times Q'}\sup_{(y,z)\in Q\times Q'}\int_{\mathbb{R}^n\setminus(\Omega_1^*\cup\Omega_1^*)}|K(x,y,z)-K(x,\alpha,\beta)|\,dx$$

$$\leq B\left(\cup_{Q\in D_1}Q|+|\cup_{Q'\in D_1}Q'|\right)\leq 2B\cup_{Q\in D_1}|Q|.$$

From this it follows that

$$\int_{\Omega_1}\sup_{y\in Q}\int_{\mathbb{R}^n\setminus\Omega_1^*}|K(x,y,z)-K(x,y_Q,z)|\,dx\,dz\leq 2B.$$

So, given (2.3), the condition (2.1) could be replaced with the much weaker one

$$\int_{\mathbb{R}^n\setminus\Omega_1}\inf_{y'\in Q}\sup_{y\in Q}\int_{\mathbb{R}^n\setminus\Omega_1^*}|K(x,y,z)-K(x,y',z)|\,dx\,dz\leq B,$$

and similarly with (2.2).

We now show that the BGHC are implied by a Dini-type one. The following condition is essentially the one considered by Maldonado and Naibo [19], except that we do not require the function Φ involved to be convex.

Definition 2.2. Let Φ increasing and such that

$$\int_0^1\Phi(t)\,\frac{dt}{t}<\infty.$$

We say that K satisfies a bilinear Dini-type condition if

$$|K(x,y,z)-K(x,y',z')|\leq\frac{C}{(|x-y|+|x-z|)^{2n}}\Phi\Big(\frac{|y-y'|+|z-z'|}{|x-y|+|x-z|}\Big)$$

whenever $|y-y'|\leq\frac{1}{2}|x-y|$ and $|z-z'|\leq\frac{1}{2}|x-z|$.

By taking $\Phi(t)=t^{\varepsilon}$ it follows that the bilinear Lipschitz regularity conditions imply the Dini-type one. We now show that the BGHC is actually weaker.

Proposition 2.3. The Dini-type condition implies the BGHC.

Proof. We first show that the Dini-type condition implies

$$\int_{\mathbb{R}^n}\sup_{y\in Q}\int_{\mathbb{R}^n\setminus\Omega_1^*}|K(x,y,z)-K(x,y_Q,z)|\,dx\,dz\leq C$$

for some $C>0$. Indeed,

$$\int_{\mathbb{R}^n}\sup_{y\in Q}\int_{\mathbb{R}^n\setminus\Omega_1^*}|K(x,y,z)-K(x,y_Q,z)|\,dx\,dz$$

$$\lesssim\int_{\mathbb{R}^n}\sup_{y\in Q}\int_{\mathbb{R}^n\setminus Q^*}\frac{1}{(|x-y|+|x-z|)^{2n}}\Phi\Big(\frac{|y-y_Q|}{|x-y|+|x-z|}\Big)\,dx\,dz$$

$$\lesssim\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus Q^*}\frac{1}{(|x-y_Q|+|x-z|)^{2n}}\Phi\Big(\frac{\sqrt{n}\,\ell(Q)}{|x-y_Q|}\Big)\,dx\,dz$$

$$\lesssim \int_{\mathbb{R}^n\setminus Q^*} \frac{1}{|x-y_Q|^n}\Phi\Big(\frac{\sqrt{n}\,\ell(Q)}{|x-y_Q|}\Big)\,dx$$
$$\lesssim \int_{|x|>c_n\ell(Q)} \frac{1}{|x|^n}\Phi\Big(\frac{c_n\ell(Q)}{|x|}\Big)\,dx<\infty.$$

By symmetry, the proof of (2.2) is identical.

We now prove (2.3). Fix P and Q then

$$\sup_{(y,z)\in P\times Q}\int_{\mathbb{R}^n\setminus(\Omega_1^*\cup\Omega_2^*)}|K(x,y,z)-K(x,y_P,z_Q)|\,dx$$
$$\lesssim \sup_{(y,z)\in P\times Q}\left(\int_{\mathbb{R}^n\setminus(P^*\cup Q^*)}\frac{1}{(|x-y|+|x-z|)^{2n}}\Phi\Big(\frac{|y-y_P|+|z-z_Q|}{|x-y|+|x-z|}\Big)\,dx\right)$$
$$\lesssim \sup_{(y,z)\in P\times Q}\left(\int_{\mathbb{R}^n\setminus(P^*\cup Q^*)}\frac{1}{(|x-y|+|x-z|)^{2n}}\Phi\Big(\frac{\sqrt{n}\,(\ell(P)+\ell(Q))}{|x-y_P|+|x-z_Q|}\Big)\,dx\right)$$
$$\lesssim \inf_{(y,z)\in P\times Q}\left(\int_{\mathbb{R}^n\setminus(P^*\cup Q^*)}\frac{1}{(|x-y|+|x-z|)^{2n}}\Phi\Big(\frac{\sqrt{n}\,(\ell(P)+\ell(Q))}{|x-y_P|+|x-z_Q|}\Big)\,dx.\right)$$

Then,

$$\sum_{(P,Q)\in D_1\times D_2}|P||Q|\sup_{(y,z)\in P\times Q}\int_{\mathbb{R}^n\setminus(\Omega_1^*\cup\Omega_2^*)}|K(x,y,z)-K(x,y_P,z_Q)|\,dx$$
$$\lesssim \sum_{(P,Q)\in D_1\times D_2}|P||Q|\inf_{(y,z)\in P\times Q}\left(\int_{\mathbb{R}^n\setminus(P^*\cup Q^*)}\frac{1}{(|x-y|+|x-z|)^{2n}}\Phi\Big(\frac{\sqrt{n}\,(\ell(P)+\ell(Q))}{|x-y_P|+|x-z_Q|}\Big)\,dx\right)$$
$$\lesssim \sum_{(P,Q)\in D_1\times D_2}\int_P\int_Q\int_{\mathbb{R}^n\setminus(P^*\cup Q^*)}\frac{1}{(|x-y|+|x-z|)^{2n}}\Phi\Big(\frac{\sqrt{n}\,(\ell(P)+\ell(Q))}{|x-y_P|+|x-z_Q|}\Big)\,dx\,dy\,dz$$
$$=C\left(\sum_{(P,Q)\in D_1\times D_2:\ell(P)\le\ell(Q)}+\sum_{(P,Q)\in D_1\times D_2:\ell(Q)\le\ell(P)}\right)=I+II$$

We estimate I, the other term is of course similar. Now,

$$I\lesssim \sum_{(P,Q)\in D_1\times D_2}\int_P\int_Q\int_{\mathbb{R}^n\setminus(P^*\cup Q^*)}\frac{1}{(|x-y|+|x-z|)^{2n}}\,\Phi\Big(\frac{2\sqrt{n}\,\ell(Q)}{|x-y_P|+|x-z_Q|}\Big)\,dx\,dy\,dz$$
$$\lesssim \sum_{Q\in D_2}\int_{\mathbb{R}^n\setminus Q^*}\int_Q\sum_{P\in D_1}\int_P\frac{1}{(|x-y|+|x-z|)^{2n}}\,\Phi\Big(\frac{2\sqrt{n}\,\ell(Q)}{|x-z_Q|}\Big)\,dy\,dz\,dx$$
$$\lesssim \sum_{Q\in D_2}\int_{\mathbb{R}^n\setminus Q^*}\int_Q\int_{\mathbb{R}^n}\frac{1}{(|x-y|+|x-z|)^{2n}}\,dy\,dz\,\Phi\Big(\frac{2\sqrt{n}\,\ell(Q)}{|x-z_Q|}\Big)\,dx$$

We can estimate the inner most integral by $\frac{C}{|x-z|^n}$, hence

$$I\lesssim \sum_{Q\in D_2}\int_{\mathbb{R}^n\setminus Q^*}\int_Q\frac{1}{|x-z|^n}\Phi\Big(\frac{2\sqrt{n}\,\ell(Q)}{|x-z_Q|}\Big)\,dx\,dz$$
$$\lesssim \sum_{Q\in D_2}\int_Q\int_{\mathbb{R}^n\setminus Q^*}\frac{1}{|x-z|^n}\Phi\Big(\frac{2\sqrt{n}\,\ell(Q)}{|x-z_Q|}\Big)\,dx\,dz$$
$$\lesssim \sum_{Q\in D_2}\int_Q\int_{\mathbb{R}^n\setminus Q^*}\frac{1}{|x-z_Q|^n}\Phi\Big(\frac{2\sqrt{n}\,\ell(Q)}{|x-z_Q|}\Big)\,dx\,dz$$

$$\lesssim \sum_{Q\in D_2} |Q| \int_{|x|\geq 1} \frac{1}{|x|^n}\Phi\Big(\frac{1}{|x|}\Big)\,dx \lesssim |\cup_{Q\in D_2} Q| \int_0^1 \Phi(t)\,\frac{dt}{t}.$$

Likewise, the term II is bounded by $C|\cup_{P\in D_1} P|$, which completes the proof of (2.3). □

3. THE END-POINT ESTIMATE

Theorem 3.1. *Let T be a bilinear operator satisfying $T : L^{p_1}\times L^{p_2}\to L^{p_3}$ for some $1/p_1+1/p_2=1/p_3$, $1\leq p_1,p_2<\infty$; and the BGHC. Then, $T : L^1\times L^1\to L^{1/2,\infty}$.*

Proof. It is enough to show that

$$|\{x\in\mathbb{R}^n : |T(f_1,f_2)(x)|>\lambda^2\}| \leq C\left(\int_{\mathbb{R}^n}\frac{|f_1(x)|}{\lambda}\,dx\right)^{1/2}\left(\int_{\mathbb{R}^n}\frac{|f_2(x)|}{\lambda}\,dx\right)^{1/2}, \tag{3.1}$$

for all $f_1,f_2\in C_c^\infty(\mathbb{R}^n)$. Moreover, by homogeneity we may assume that $\|f_1\|_1=\|f_2\|_1=1$ and prove that

$$|\{x\in\mathbb{R}^n : |T(f_1,f_2)(x)|>\lambda^2\}| \leq \frac{C}{\lambda}, \tag{3.2}$$

with constant C independent of λ.

Fixed $\lambda>0$. For f_1 we consider the standard Calderón-Zygmund decomposition al level λ and obtain a collection of dyadic non-overlapping cubes $Q_{1,k}$that satisfy

$$\lambda < \frac{1}{|Q_{1,k}|}\int_{Q_{1,k}} |f_1(x)|\,dx \leq 2^n\lambda.$$

If we set $\Omega_1=\bigcup_k Q_{1,k}$, then

$$|\Omega_1|\leq \frac{C}{\lambda}$$

and

$$|f_1(x)|\leq\lambda \qquad a.e.\, x\in\mathbb{R}^n\setminus\Omega_1.$$

As usual, we write $f_1=g_1+b_1$, where g_1 is defined by

$$g_1(x)=\begin{cases} f_1(x), & x\in\mathbb{R}^n\setminus\Omega_1\\ f_{Q_{1,k}}, & x\in Q_{1,k},\end{cases}$$

and where $f_Q=\frac{1}{|Q|}\int_Q f$; the "good" function g_1 satisfies for any $s\geq 1$

$$\|g_1\|_s \leq C\lambda^{1/s'}\|f_1\|_1^{\frac{1}{2s'}+\frac{1}{s}}\|f_2\|_1^{-\frac{1}{2s'}} = C\lambda^{1/s'}; \tag{3.3}$$

and b_1 is written as

$$b_1(x)=\sum_k b_{1,k}(x)=\sum_k (f_1(x)-f_{Q_{1,k}})\chi_{Q_{1,k}}(x).$$

We do the same for f_2. Via the Calderón-Zygmund decomposition at same level λ, we obtain a collection of of dyadic non-overlapping cubes $Q_{2,k}$ with union Ω_2 and analogous decomposition $f_2=g_2+b_2$ with the properties described for f_1. Set

$$\Omega^*=\Omega_1^*\cup\Omega_2^*.$$

We split the distribution set we are trying to estimate in several parts as follows,

$$\begin{aligned}
|\{x\in\mathbb{R}^n : |T(f_1,f_2)(x)|>\lambda^2\}| &\leq |\{x\notin\Omega^* : |T(f_1,f_2)(x)|>\lambda^2\}|+|\Omega^*|\\
&\leq |\{x\in\mathbb{R}^n : |T(g_1,g_2)(x)|>\lambda^2/4\}|\\
&+ |\{x\in\mathbb{R}^n\setminus\Omega^* : |T(g_1,b_2)(x)|>\lambda^2/4\}|\\
&+ |\{x\in\mathbb{R}^n\setminus\Omega^* : |T(b_1,g_2)(x)|>\lambda^2/4\}|
\end{aligned}$$

$$
\begin{aligned}
&+ \quad |\{x \in \mathbb{R}^n \setminus \Omega^* : |T(b_1,b_2)(x)| > \lambda^2/4\}| \\
&+ \quad |\Omega^*| \\
&= \quad |E_1| + |E_2| + |E_3| + |E_4| + |\Omega^*|.
\end{aligned}
$$

Clearly $|\Omega^*|$ is of the right size, so we only need to estimate the other sets. To estimate $|E_1|$, we use the fact that $T : L^{p_1} \times L^{p_2} \to L^{p_3}$ for some $1/p_1 + 1/p_2 = 1/p_3$ to obtain

$$
\begin{aligned}
|E_1| &\le \frac{C}{\lambda^{p_3}} \int_{\mathbb{R}^n} |T(g_1,g_2)(x)|^{p_3} dx \le \frac{C}{\lambda^{2p_3}} \|g_1\|_{p_1}^{p_3} \|g_2\|_{p_2}^{p_3} \\
&\le \frac{C}{\lambda^{2p_3}} \lambda^{p_3(1-1/p_1)} \lambda^{p_3(1-1/p_2)} = \frac{C}{\lambda}.
\end{aligned}
$$

By symmetry, the study of $|E_2|$ and $|E_3|$ are similar, so we only consider $|E_2|$. We have

$$
\begin{aligned}
|E_2| &\le \frac{C}{\lambda^2} \int_{\mathbb{R}^n \setminus \Omega^*} \sum_k |T(g_1, b_{2,k})(x)|\, dx \\
&\le \frac{C}{\lambda^2} \sum_k \int_{\mathbb{R}^n \setminus \Omega^*} \left| \int_{\mathbb{R}^n} \int_{Q_{2,k}} K(x,y,z) g_1(y) b_{2,k}(z)\, dz\, dy \right| dx \\
&\le \frac{C}{\lambda^2} \sum_k \int_{\mathbb{R}^n \setminus \Omega_2^*} \left| \int_{\mathbb{R}^n} g_1(y) \int_{Q_{2,k}} \Big(K(x,y,z) - K(x,y,z_{Q_{2,k}}) \Big) b_{2,k}(z)\, dz\, dy \right| dx \\
&\le \frac{C}{\lambda^2} \sum_k \int_{\mathbb{R}^n} |g_1(y)| \int_{Q_{2,k}} |b_{2,k}(z)| \int_{\mathbb{R}^n \setminus \Omega^*} |K(x,y,z) - K(x,y,z_{Q_{2,k}})|\, dx\, dz\, dy \\
&\le \frac{C}{\lambda^2} \sum_k \int_{\mathbb{R}^n} |g_1(y)| \int_{Q_{2,k}} |b_{2,k}(z)|\, dz \sup_{z \in Q_{2,k}} \int_{\mathbb{R}^n \setminus \Omega^*} |K(x,y,z) - K(x,y,z_{Q_{2,k}})|\, dx\, dy \\
&\le \frac{C}{\lambda} \int_{\mathbb{R}^n} |g_1(y)| \sum_k |Q_{2,k}| \sup_{z \in Q_{2,k}} \int_{\mathbb{R}^n \setminus \Omega^*} |K(x,y,z) - K(x,y,z_{Q_{2,k}})|\, dx\, dy \\
&\le \frac{C}{\lambda} \|g_1\|_\infty \int_{\mathbb{R}^n} \sum_k |Q_{2,k}| \sup_{z \in Q_{2,k}} \int_{\mathbb{R}^n \setminus \Omega^*} |K(x,y,z) - K(x,y,z_{Q_{2,k}})|\, dx\, dy \\
&\le C|\Omega_2| \le \frac{C}{\lambda},
\end{aligned}
$$

where we have used the condition (2.2).

It only remains to estimate

$$
|E_4| = |\{x \in \mathbb{R}^n \setminus \Omega : |T(b_1,b_2)(x)| > \lambda^2/4\}|.
$$

Writing

$$
T(b_1,b_2) = \sum_{l,k} T(b_{1,l}, b_{2,k}),
$$

we have

$$
\begin{aligned}
|E_4| &\le \frac{C}{\lambda^2} \sum_{l,k} \int_{\mathbb{R}^n \setminus \Omega^*} |T(b_{1,l}, b_{2,k})(x)|\, dx \\
&\le \frac{C}{\lambda^2} \sum_{l,k} \int_{\mathbb{R}^n \setminus \Omega^*} \left| \int_{Q_{1,l}} \int_{Q_{2,k}} K(x,y,z) b_{1,l}(y) b_{2,k}(z)\, dz\, dy \right| dx.
\end{aligned}
$$

Fix one of these $Q_{1,l}$ and $Q_{2,k}$

$$\int_{Q_{1,l}}\int_{Q_{2,k}} b_{1,l}(y)b_{2,k}(z)\,dzdy = 0$$

we have

$$\int_{\mathbb{R}^n\setminus\Omega^*}\left|\int_{Q_{1,l}}\int_{Q_{2,k}} K(x,y,z)\,b_{2,k}(z)b_{1,l}(y)dzdy\right|dx$$

$$= \int_{\mathbb{R}^n\setminus\Omega^*}\left|\int_{Q_{1,l}}\int_{Q_{2,k}} (K(x,y,z)-K(x,y_{Q_{1,l}},z_{Q_{2,k}}))\,b_{2,k}(z)b_{1,l}(y)dzdy\right|dx$$

$$\le \int_{\mathbb{R}^n\setminus\Omega^*}\int_{Q_{1,l}}\int_{Q_{2,k}}\left|K(x,y,z)-K(x,y_{Q_{1,l}},z_{Q_{2,k}})\right|dx\,|b_{2,k}(z)||b_{1,l}(y)|\,dzdy$$

$$= \int_{Q_{1,l}}\int_{Q_{2,k}}\int_{\mathbb{R}^n\setminus\Omega^*}\left|K(x,y,z)-K(x,y_{Q_{1,l}},z_{Q_{2,k}})\right|dx\,|b_{2,k}(z)||b_{1,l}(y)|\,dzdy$$

$$\le \int_{Q_{1,l}}\int_{Q_{2,k}}|b_{2,k}(z)||b_{1,l}(y)|dzdy \sup_{(y,z)\in Q_{1,l}\times Q_{2,k}}\int_{\mathbb{R}^n\setminus\Omega^*}\left|K(x,y,z)-K(x,y_{Q_{1,l}},z_{Q_{2,k}})\right|dx$$

$$\le C\lambda^2\,|Q_{1,l}||Q_{2,k}| \sup_{(y,z)\in Q_{1,l}\times Q_{2,k}}\int_{\mathbb{R}^n\setminus\Omega^*}\left|K(x,y,z)-K(x,y_{Q_{1,l}},z_{Q_{2,k}})\right|dx.$$

Finally,

$$|E_4| \le C\sum_{l,k}|Q_{1,l}||Q_{2,k}| \sup_{(y,z)\in Q_{1,l}\times Q}\int_{\mathbb{R}^n\setminus\Omega^*}\left|K(x,y,z)-K(x,y_{Q_{1,l}},z_{Q_{2,k}})\right|dx$$

$$\le C\,(|\cup_l Q_{1,l}| + |\cup_k Q_{2,k}|) = c\,(|\Omega_1|+|\Omega_2|) \le \frac{C}{\lambda}$$

because of the condition (2.3). □

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