MODIFIED CLEBSCH-GORDAN-TYPE EXPANSIONS FOR PRODUCTS OF DISCRETE HYPERGEOMETRIC POLYNOMIALS. [1]

Renato Álvarez-Nodarse

Departamento de Matemáticas, Universidad Carlos III de Madrid
c/ Butarque 15, 28911, Leganés, Madrid, Spain and
Instituto Carlos I de Física Teórica y Computacional,
Universidad de Granada, 18071 Granada, Spain

R. J. Yáñez

Departamento de Matemática Aplicada and Instituto Carlos I
de Física Teórica y Computacional, Universidad de Granada,
18071 Granada, Spain

and

Jesús S. Dehesa

Departamento de Física Moderna and Instituto Carlos I
de Física Teórica y Computacional, Universidad de Granada,
18071 Granada, Spain

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Abstract

Starting from the second-order difference hypergeometric equation satisfied by the set of *discrete* orthogonal polynomials $\{p_n\}$, we find the analytical expressions of the expansion coefficients of any polynomial $r_m(x)$ and of the product $r_m(x)q_j(x)$ in series of the set $\{p_n\}$. These coefficients are given in terms of the polynomial coefficients of the second-order difference equations satisfied by the involved discrete hypergeometric polynomials. Here $q_j(x)$ denotes an arbitrary discrete hypergeometric polynomial of degree j. The particular cases in which $\{r_m\}$ corresponds to the non-orthogonal families $\{x^m\}$, the rising factorials or Pochhammer polynomials $\{(x)_m\}$ and the falling factorial or Stirling polynomials $\{x^{[m]}\}$ are considered in detail. The connection problem between discrete hypergeometric polynomials, which here corresponds to the product case with $m = 0$, is also studied and its complete solution for all the classical discrete orthogonal hypergeometric (CDOH) polynomials is given. Also, the inversion problems of CDOH polynomials associated to the three aforementioned non-orthogonal families are solved.

KEY WORDS AND PHRASES: discrete polynomials, connection and linearization problems, discrete inversion formulas, second-order difference equations.
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[1]E-Mails: `renato@dulcinea.uc3m.es`, `ryanez@ugr.es`, `dehesa@goliat.ugr.es`

1 Introduction.

The hypergeometric polynomials of a discrete variable are objects which not only are important *per se* in the framework of the theory of special functions but also they play a very relevant role in numerous physical and mathematical fields which range from quantum mechanics, theory of group representations and probability theory to numerical analysis, the theory of vibrating strings, the study of random walks with discrete time processes and the theory of Sturm-Liouville difference equations as pointed out by numerous authors [7, 21] and particularly the excellent monograph of A.F. Nikiforov, S.K. Suslov and V.B. Uvarov [41]. This is the case not only for the so-called classical discrete orthogonal polynomials (Hahn, Meixner, Kravchuk and Charlier) but also for discrete orthogonal sets other than the classical ones; see e.g. [12, 24, 39, 49].

The expansion of any arbitrary discrete polynomial $r_m(x)$ in series of a general (albeit fixed) set of discrete hypergeometric polynomial $\{p_n(x)\}$ is a matter of great interest, not yet solved save for some particular classical cases, as briefly summarized by Askey [8] and Gasper [17, 18] up to the middle of seventies and by Ronveaux et al [45, 48], since then up to now. This is particulary true for the deeper problem of linearization of a product of any two discrete polynomials. Usually, the determination of the expansion coefficients in these particular cases required a deep knowledge of special functions and, at times, ingenious induction arguments based in the three-term recurrence relation of the involved orthogonal polynomials [13, 17, 18, 26, 51, 52]. Only recently, general and widely applicable strategies begin to appear [25, 30, 37, 47].

Markett [37] for symmetric orthogonal polynomials, has designed a method which begins with the three-term recurrence relation of the involved orthogonal polynomial system to set up a partial difference equation for the (orthogonal) polynomial, in case of connection problems, or for the product of two (orthogonal) polynomials, in case of linearization problems, to be expanded; then, this equation has to be solved in terms of the initial data.

Ronveaux et al. [47, 48] for classical and semiclassical orthogonal polynomials and Lewanowicz [30, 34] for classical orthogonal polynomials have proposed alternative, simpler techniques of the same type although they require the knowledge of not only the recurrence relation but also the differential-difference relation or/and the second-order difference equation, respectively, satisfied by the polynomials of the orthogonal set of the expansion problem in consideration. See also [5, 11, 23, 31, 45, 46] for further description and applications of this method in the discrete case, and [19, 32] in the continuous case as well as [3, 33] for the q-discrete orthogonality. Koepf and Schmersau [25] has proposed a computer-algebra-based method which, starting from the second order difference hypergeometric equation, produces by symbolic means and in a recurrent way the expansion coefficients of the classical discrete orthogonal hypergeometric polynomials (CDOHP) in terms of the falling factorial polynomials (already obtained analytically by Lesky [28]; see also [41], [42]) as well as the expansion coefficients of its corresponding inverse problem. The combination of these two simple expansion problems allows these authors to solve the connection problems within each specific CDOHP set.

All these four methods provide the expansion coefficients via recursion relationships, what is very useful for the symbolic and/or numerical computation of its values. However, in general, these relationships cannot be analytically solved; so that, in practice, closed

expressions for the expansion coefficients are only obtained, at times, by symbolic means.

The purpose of this paper is to describe a general and constructive approach to solve the expansion formulas of the type

$$r_m(x) = \sum_{n=0}^{m} c_{mn} p_n(x), \qquad (1.1)$$

and

$$r_m(x) q_j(x) = \sum_{n=0}^{m+j} c_{jmn} p_n(x), \qquad (1.2)$$

where $r_m(x)$ and $q_j(x)$ are any mth-degree and jth-degree discrete hypergeometric polynomials, and $\{p_n\}$ denotes an arbitrary set of discrete orthogonal hypergeometric polynomials. Expansions of type (1.1) are usually called as connection or projection formulas while those of type (1.2) are referred to generalized linearization formulas or modified expansions of Clebsch-Gordan type [43]. Here the name Clebsch-Gordan is attached because its structure is similar to the Clebsch-Gordan series for spherical functions [15]. Usually the Clebsch-Gordan expansion [15] or linearization relation involves polynomials which belongs to the same system (i.e., they are all three Hahn polynomials, or all Meixner, etc). When this is not the case, we referred to modified Clebsch-Gordan expansion or generalized linearization [43]. For the sake of completeness and notation, let us also mention that expansions (1.1) are called as inversion formulas when the polynomials $r_m(x)$ belongs to any of the following non-orthogonal families: the power polynomials $\{x^m\}$, the rising factorials or Pochhammer polynomials $\{(x)_m\}$ and the falling factorials or Stirling polynomials $\{x^{[m]}\}$.

The only prerequisite of our approach is the knowledge of the second order difference equation satisfied by the involved hypergeometric polynomials. The resulting expansion coefficients are given in a compact, analytic, closed and formally simple form in terms of the polynomial coefficients of the corresponding second-order difference equation(s). Then, contrary to Market's, Ronveaux et al's and Lewanowicz's methods we do not require information about any kind of recurrence relation about the involved discrete hypergeometric polynomials nor we need to solve any partial difference equation for the polynomial(s) to be expanded, or *"high"* order recurrence relation for the connection coefficients themselves. Let us also underline that, opposite to Koepf and Schmersau's method, we do not use any symbolic means, as well as we directly provide the expansion coefficients in a single step.

The structure of the paper is the following. Firstly, in Section 2, we collect the basic background [41] used in the rest of the work; namely, the second-order hypergeometric difference equation and its polynomial solutions (called as discrete hypergeometric polynomials) as well as the main data of the four classical sets of orthogonal discrete polynomials (Hahn, Meixner, Kravchuk, and Charlier) in the monic form and the principal properties of the aforementioned non-orthogonal families $\{(x)_m\}$ and $\{x^{[m]}\}$. Then, in Section 3, the coefficients of the expansion (1.1) are given explicitly in terms of the polynomial coefficients of the hypergeometric difference equation satisfied by the orthogonal set $\{p_n\}$. Also, as a consequence of the resulting expression, the inversion formulas associated to the Pochhammer, Stirling and the power polynomials are fully solved in the subsection 3.1 and then they are applied to the four classical discrete orthogonal sets.

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In Section 4 the coefficients of the linearization formula (1.2) are found in a fully analytical way in terms of the polynomial coefficients of the second-order difference hypergeometric equations satisfied by the polynomials $p_n(x)$ and $q_j(x)$. Notice that $\{q_j\}$ is not necessarily an orthogonal set, neither $r_m(x)$ is obliged to have a hypergeometric character, what widely extends the linearization formulas considered in the literature [5, 8, 9, 11, 17, 23, 25, 30, 31, 37, 45, 46, 47, 48]. Indeed, most authors study linearizaton formulas between classical discrete polynomials, usually within the same family (see e.g. [10, 11, 14, 53] save some of them, who find a few other formulas which either involve polynomials of diferent classical families [17, 45] or include one of the aforementioned non-orthogonal families together with polynomials of the same classical system [11]. The linearization formulas (1.2) corresponding to the special cases in which $r_m(x) = (x)_m$, $x^{[m]}$ and x^m, are given in subsection 4.1.

In Section 5 the connection problem between discrete hypergeometric polynomials is worked out in detail as the particular case $m = 0$ of the linearization formula fully solved in the previous section. The resulting expressions are used to explicitly obtain the connection formulas between polynomials of each of the four classical families and between all of its possible pairs. This includes also the Hahn system, what we underline because no general results of these type can be encountered in the literature.

Finally, some concluding remarks and a number of references are given.

2 "Discrete" preliminaries.

Here we collect the basic background [40, 41] on hypergeometric discrete polynomials and rising and falling factorials needed in the rest of the work.

2.1 The discrete hypergeometric polynomials.

Let us consider the second-order difference equation of hypergeometric-type [40, 41], i.e., the equation

$$\sigma(x) \bigtriangledown \triangle y(x) + \tau(x) \triangle y(x) + \lambda y(x) = 0 , \tag{2.1}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of degree not greater than 2 and 1, respectively, and λ is a constant. This equation can be written in the self-adjoint form

$$\triangle[\sigma(x)\rho(x) \bigtriangledown y(x)] + \lambda\rho(x)y(x) = 0, \tag{2.2}$$

where the function $\rho(x)$ satisfies the Pearson-type difference equation

$$\triangle[\sigma(x)\rho(x)] = \tau(x)\rho(x) . \tag{2.3}$$

The solutions of Eq. (2.1) with

$$\lambda \equiv \lambda_n = -n \triangle \tau(x) - \frac{1}{2}n(n-1) \triangle^2 \sigma(x) = -n\tau' - \tfrac{1}{2}n(n-1)\sigma'', \tag{2.4}$$

are polynomials of degree n, usually called hypergeometric-type "discrete" polynomials $y = y_n(x) \equiv p_n(x)$. These polynomials [41] are orthogonal in the interval $[a, b-1]$ with respect to the weight function $\rho(x)$, i.e.,

$$\sum_{x_i=a}^{b-1} p_n(x_i)p_m(x_i)\rho(x_i) = \delta_{nm}d_n^2, \qquad x_{i+1} = x_i + 1 , \tag{2.5}$$

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provided that the following condition

$$\sigma(x)\rho(x)x^k \Big|_{x=a,b} = 0\,, \quad \forall k \geq 0,$$ (2.6)

holds. The square of the norm of the polynomial $p_n(x)$ is given [41] by

$$d_n^2 = (-1)^n A_{nn} B_n^2 \sum_{x_i=a}^{b-n-1} \rho_n(x_i) = (-1)^n a_n B_n \sum_{x_i=a}^{b-n-1} \rho_n(x_i)\,,\ ,$$ (2.7)

where a_n is the leading coefficient of the polynomial $p_n(x)$,

$$p_n(x) = a_n x^n + b_n x^{n-1} + \dots\,,$$ (2.8)

and B_n is the normalization constant of the Rodrigues-type formula

$$p_n(x) = \frac{B_n}{\rho(x)} \nabla^n [\rho_n(x)], \quad n = 0, 1, 2, \dots\,, \ .$$ (2.9)

where

$$\rho_n(x) = \rho(x+n) \prod_{m=1}^{n} \sigma(x+m).$$ (2.10)

The use of Eq. (2.9) together with the formula

$$\nabla^n[f(x)] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-k),$$ (2.11)

allows one to obtain [41] an explicit expression for the polynomials. The symbol $\binom{n}{k}$ denotes the binomial coefficients, i.e.,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The constants a_n and B_n are related by

$$a_n = B_n \prod_{k=0}^{n-1} \left[\tau' + \frac{1}{2}(n+k-1)\sigma'' \right]\,, \quad a_0 = b_0\,.$$ (2.12)

For the $k-$difference derivatives of the polynomials $p_n(x)$, it is also fulfilled [41] a Rodrigues-type formula

$$\triangle^k P_n(x) = \frac{A_{nk} B_n}{\rho_k(x)} \nabla^{n-k} [\rho_n(x)],$$ (2.13)

where

$$A_{nk} = \frac{n!}{(n-k)!} \prod_{m=0}^{k-1} [\tau' + \tfrac{1}{2}(n+m-1)\sigma''] \equiv \frac{n!}{(n-k)!} \frac{a_k}{B_k}, \quad A_{n0} = 1.$$ (2.14)

The most general polynomial solution of the hypergemetric difference equation (2.1) corresponds to the case

$$\sigma(x) = A(x-x_1)(x-x_2), \quad \sigma(x) + \tau(x) = A(x-\bar{x}_1)(x-\bar{x}_2).$$

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Without loss of generality we will consider the case $A = -1$ and $x_1 = 0$. In this case, the monic polynomial solutions can be written as follows [2, 42]

$$P_n(x) = \frac{(-\bar{x}_1)_n(-\bar{x}_2)_n}{(x_2 - \bar{x}_1 - \bar{x}_2 + n - 1)_n} \; {}_3\mathrm{F}_2\left(\begin{array}{c} -n, -x \; , \; x_2 - \bar{x}_1 - \bar{x}_2 + n - 1 \\ -\bar{x}_1, \; -\bar{x}_2 \end{array} \middle| 1 \right), \quad (2.15)$$

where the generalized hypergeometric function ${}_p\mathrm{F}_q$ is defined by

$$ {}_p\mathrm{F}_q\left(\begin{array}{c} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{array} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}. \quad (2.16)$$

The four referred families of discrete hypergeometric polynomials are the so-called classical discrete orthogonal polynomials: Hahn $h_n^{\alpha,\beta}(x, N)$, Meixner $M_n^{\gamma,\mu}(x)$, Kravchuk $K_n^p(x, N)$ and Charlier $C_n^\mu(x)$, polynomials [40, 41], whose main data in its monic form are shown in Tables 1-2. They can be expresed in terms of the hypergeometric functions by formulas [41, Section **2.7**,p. 49]:

$$h_n^{\alpha,\beta}(x, N) = \frac{(1 - N)_n(\beta + 1)_n}{(\alpha + \beta + n + 1)_n} \; {}_3\mathrm{F}_2\left(\begin{array}{c} -x, \alpha + \beta + n + 1, -n \\ 1 - N, \beta + 1 \end{array} \middle| 1 \right), \quad (2.17)$$

$$M_n^{\gamma,\mu}(x) = (\gamma)_n \frac{\mu^n}{(\mu - 1)^n} \; {}_2\mathrm{F}_1\left(\begin{array}{c} -n, -x \\ \gamma \end{array} \middle| 1 - \frac{1}{\mu} \right), \quad (2.18)$$

$$K_n^p(x, N) = \frac{(-p)^n N!}{(N - n)!} \; {}_2\mathrm{F}_1\left(\begin{array}{c} -n, -x \\ -N \end{array} \middle| \frac{1}{p} \right), \quad (2.19)$$

$$C_n^\mu(x) = (-\mu)^n \; {}_2\mathrm{F}_0\left(\begin{array}{c} -n, -x \\ - \end{array} \middle| -\frac{1}{\mu} \right). \quad (2.20)$$

These expressions immediately follow from the above representation (2.15) and its different limits (more details can be found in [41, 42]).

2.2 The rising and falling factorials.

The rising factorial polynomials or Pochhammer symbols $(x)_n$ are defined by

$$(x)_n = x(x + 1) \cdots (x + n - 1) \equiv \frac{\Gamma(x + n)}{\Gamma(x)}, \quad (2.21)$$

and they have the properties

$$(-x)_n = \frac{(-1)^n \Gamma(x + 1)}{\Gamma(x - n + 1)}, \qquad \frac{(x)_n}{(x - 1)_{n-1}} = \frac{n + x - 1}{x - 1}, \qquad (x)_{n+k} = (x)_n(x + n)_k, \quad (2.22)$$

as well as the difference equation analogue to the differential equation $(x^n)' = n\,x^{n-1}$,

$$\nabla(x)_n = n(x)_{n-1}. \quad (2.23)$$

The falling factorial polynomials or Stirling polynomials $x^{[n]}$ are polynomials defined by

$$x^{[n]} = x(x - 1) \cdots (x - n + 1) \equiv (-1)^n(-x)_n = \frac{\Gamma(x + 1)}{\Gamma(x - n + 1)} \quad (2.24)$$

They satisfy the equations

$$\triangle x^{[n]} = nx^{[n-1]}, \qquad \nabla x^{[n]} = n(x - 1)^{[n-1]}. \quad (2.25)$$

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Table 1: Main data for monic Hahn and Charlier polynomials.

$P_n(x)$	Hahn $h_n^{\alpha,\beta}(x;N)$	Charlier $C_n^\mu(x)$
(a,b)	$[0, N-1]$	$[0,\infty)$
$\sigma(x)$	$x(N+\alpha-x)$	x
$\tau(x)$	$(\beta+1)(N-1)-(\alpha+\beta+2)x$	$\mu - x$
$\sigma(x)+\tau(x)$	$(x+\beta+1)(N-1-x)$	μ
λ_n	$n(n+\alpha+\beta+1)$	n
$\rho(x)$	$\dfrac{\Gamma(N+\alpha-x)\Gamma(\beta+x+1)}{\Gamma(N-x)\Gamma(x+1)}$ $\alpha,\beta \geq -1\ ,\ n \leq N-1$	$\dfrac{e^{-\mu}\mu^x}{\Gamma(x+1)}$ $\mu > 0$
$\rho_n(x)$	$\dfrac{\Gamma(N+\alpha-x)\Gamma(n+\beta+x+1)}{\Gamma(N-n-x)\Gamma(x+1)}$	$\dfrac{e^{-\mu}\mu^{x+n}}{\Gamma(x+1)}$
B_n	$\dfrac{(-1)^n}{(\alpha+\beta+n+1)_n}$	$(-1)^n$
b_n	$-\dfrac{n}{2}\left(\dfrac{2(\beta+1)(N-1)+(n-1)(\alpha-\beta+2N-2)}{\alpha+\beta+2n}\right)$	$-\dfrac{n}{2}(2\mu+n-1)$
d_n^2	$\dfrac{n!\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+N+n+1)}{(\alpha+\beta+2n+1)(N-n-1)!\Gamma(\alpha+\beta+n+1)(\alpha+\beta+n+1)_n^2}$	$n!\mu^n$

Table 2: Main data for monic Meixner and Kravchuk polynomials.

$P_n(x)$	Meixner $M_n^{\gamma,\mu}(x)$	Kravchuk $K_n^p(x)$
(a,b)	$[0,\infty)$	$[0,N]$
$\sigma(x)$	x	x
$\tau(x)$	$(\mu-1)x+\mu\gamma$	$\dfrac{Np-x}{1-p}$
$\sigma(x)+\tau(x)$	$\mu x+\gamma\mu$	$-\dfrac{p}{1-p}(x-N)$
λ_n	$(1-\mu)n$	$\dfrac{n}{1-p}$
$\rho(x)$	$\dfrac{\mu^x\Gamma(\gamma+x)}{\Gamma(\gamma)\Gamma(x+1)}$ $\gamma>0, 0<\mu<1$	$\dfrac{N!p^x(1-p)^{N-x}}{\Gamma(N+1-x)\Gamma(x+1)}$ $0<p<1,\ n\leq N$
$\rho_n(x)$	$\dfrac{\mu^{x+n}\Gamma(\gamma+x+n)}{\Gamma(\gamma)\Gamma(x+1)}$	$\dfrac{N!p^{x+n}(1-p)^{N-n-x}}{\Gamma(N+1-n-x)\Gamma(x+1)}$
B_n	$\dfrac{1}{(\mu-1)^n}$	$(-1)^n(1-p)^n$
b_n	$n\left(\gamma+\dfrac{n-1}{2}\dfrac{1+\mu}{\mu}\right)\left(\dfrac{\mu}{\mu-1}\right)$	$-n[Np+(n-1)(\tfrac{1}{2}-p)]$
d_n^2	$\dfrac{n!(\gamma)_n\mu^n}{(1-\mu)^{\gamma+2n}}$	$\dfrac{n!N!p^n(1-p)^n}{(N-n)!}$

It is well known that the polynomials $x^{[n]}$ and the x^n are closely related one to another by the formulas

$$x^{[n]} = \sum_{k=0}^{n} s_n^{(k)} x^k, \qquad x^n = \sum_{k=0}^{n} \mathcal{S}_n^{(k)} x^{[k]} \tag{2.26}$$

where $s_n^{(k)}$ and $\mathcal{S}_n^{(k)}$ are the Stirling numbers of the first and second kind respectively [1]. Moreover, they satisfy the relations

$$s_{n+1}^{(k)} = s_n^{(k-1)} - n s_n^{(k)}, \qquad \mathcal{S}_{n+1}^{(k)} = k \mathcal{S}_n^{(k)} + \mathcal{S}_n^{(k-1)} \quad , \quad 1 \le k \le n. \tag{2.27}$$

For the Stirling numbers of the second kind $\mathcal{S}_n^{(k)}$ one has the closed form [1]

$$\mathcal{S}_n^{(k)} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k+j} \left(\begin{array}{c} k \\ j \end{array} \right) j^n.$$

In particular $\mathcal{S}_n^{(1)} = \mathcal{S}_n^{(n)} = 1$ and $\mathcal{S}_n^{(n-1)} = \frac{n}{2}(n-1)$.

3 Expansions of a polynomial $r_m(x)$.

Here we find the explicit expression of the coefficients c_{mn} in the expansion of an arbitrary polynomial $r_m(x)$ in series of the orthogonal discrete hypergeometric set $\{p_n\}$, i.e.

$$r_m(x) = \sum_{n=0}^{m} c_{mn} p_n(x) \ , \tag{3.1}$$

The expansion coefficients will be given in terms of the polynomial coefficients $\sigma(x)$ and $\tau(x)$ of the difference equation (2.1) satisfied by the polynomials $p_n(x)$.

Theorem 3.1 *The explicit expression of the coefficients c_{mn} in the expansion (3.1) is*

$$c_{mn} = \frac{(-1)^n B_n}{d_n^2} \sum_{x=a}^{b-1} \nabla^n r_m(x) \rho(x) \prod_{k=0}^{n-1} \sigma(x-k). \tag{3.2}$$

Proof: Multiplying both sides of Eq. (3.1) by $p_k(x)\rho(x)$, and summing between a and $b-1$, the orthogonality relation (2.5) immediately gives

$$c_{mn} = \frac{1}{d_n^2} \sum_{x=a}^{b-1} r_m(x) p_n(x) \rho(x) dx \ . \tag{3.3}$$

Use the Rodrigues formula (2.9) for $p_n(x)$ gives

$$c_{mn} = \frac{B_n}{d_n^2} \sum_{x=a}^{b-1} r_m(x) \nabla^n \left[\rho_n(x) \right] dx \ . \tag{3.4}$$

Using $n-$times the following formula of summation by parts

$$\sum_{x=a}^{b-1} f(x) \nabla g(x) = f(x) g(x) \Big|_{a-1}^{b-1} - \sum_{x=a}^{b-1} g(x-1) \nabla f(x), \tag{3.5}$$

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and taking into account the orthogonality condition (2.6) as well as Eqs. (2.6) and (2.23), one obtains

$$
c_{mn} = \frac{(-1)^n B_n}{d_n^2} \sum_{x=a}^{b-1} \triangledown^n r_m(x) \rho_n(x-n) = \frac{1}{n! a_n} \frac{\displaystyle\sum_{x=a}^{b-1} \triangledown^n r_m(x) \rho_n(x-n)}{\displaystyle\sum_{x=a}^{b-n-1} \rho_n(x)} \ . \tag{3.6}
$$

Finally, using the expression of $\rho_n(x)$ as given by (2.10), Eq.(3.6) transforms into the searched Eq. (3.2). ∎

Keeping in mind Eqs. (2.7) and (2.10), one observes that Eq. (3.2) allows us to determine the expansion formula (3.1) directly from the expression $r_m(x)$ and the polynomial coefficients which characterize the difference equation verified by the polynomials $\{p_n\}$.

3.1 Expansion of the polynomials $(x)_m$, $x^{[m]}$ and x^m.

Let us to apply the above equations (3.1) and (3.2) to the special cases $r_m(x) = (x)_m$ and $r_m(x) = x^{[m]}$. Since

$$
\triangledown^n (x)_m = \frac{m!}{(m-n)!}(x)_{m-n}, \qquad \text{and} \qquad \triangledown^n \, x^{[m]} = \frac{m!}{(m-n)!}(x-n)^{[m-n]},
$$

then,

$$
(x)_m = \sum_{n=0}^m a_{mn} p_n(x), \qquad a_{mn} = \frac{(-1)^n m!}{(m-n)!} \frac{B_n}{d_n^2} \sum_{x=a}^{b-1} (x)_{m-n} \rho_n(x-n) \ , \tag{3.7}
$$

$$
x^{[m]} = \sum_{n=0}^m d_{mn} p_n(x), \qquad d_{mn} = \frac{(-1)^n m!}{(m-n)!} \frac{B_n}{d_n^2} \sum_{x=a}^{b-1} (x-n)^{[m-n]} \rho_n(x-n) \ . \tag{3.8}
$$

To solve the problem

$$
x^m = \sum_{n=0}^m e_{mn} p_n(x), \tag{3.9}
$$

we notice that

$$
x^m = \sum_{k=0}^m \mathcal{S}_m^{(k)} x^{[k]} = \sum_{k=0}^m \mathcal{S}_m^{(k)} \sum_{n=0}^k d_{kn} p_n(x) = \sum_{n=0}^m \left(\sum_{k=n}^m d_{kn} \mathcal{S}_m^{(k)} \right) p_n(x).
$$

Then,

$$
e_{mn} = \sum_{k=n}^m d_{kn} \mathcal{S}_m^{(k)}. \tag{3.10}
$$

The expressions (3.7)-(3.10) complement and extend similar inversion formulas of classical discrete polynomials previously and differently found by various authors for the Stirling polynomials $x^{[m]}$ in a purely analitical way [17] or recurrently [11, 25, 48, 55]. We should also mention here that the inversion problems of type (3.7) and (3.8) can be easily solved in the classical case by use of the hypergeometric-function representation of these polynomials [53].

3.1.1 Application: Inversion problems of classical polynomials.

Here we will give the explicit closed expressions for the coefficients of the inversion formulas (3.7) and (3.8) of the classical discrete polynomials associated to the polynomials $(x)_m$ and $x^{[m]}$, respectively. From then and together with Eq. (3.10), the corresponding inversion formulas associated to the polynomials x^m follow in a straightforward manner.

Charlier Polynomials $C_n^\mu(x)$.

The use of the inversion formula (3.7) related to $(x)_m$ and the main data of the monic Charlier polynomials (see Table 1), as well as formula (A.6), allows us to find the corresponding expansion coefficients

$$
a_{mn} = \begin{cases} 1 & m = n = 0 \\[2ex] \mu m! \,_1\mathrm{F}_1 \left(\begin{matrix} 1 - m \\ 2 \end{matrix} \,\middle|\, -\mu \right) & m \neq 0, n = 0 \\[3ex] \left(\begin{matrix} m \\ n \end{matrix} \right) \dfrac{\Gamma(m)}{\Gamma(n)} \,_1\mathrm{F}_1 \left(\begin{matrix} n - m \\ n \end{matrix} \,\middle|\, -\mu \right) & m \neq 0, n \neq 0 \end{cases} .
$$

For the expansion of $x^{[m]}$, we use Eq. (3.8) and Eq. (A.5), to obtain that

$$
d_{mn} = \left(\begin{matrix} m \\ n \end{matrix} \right) \mu^{m-n}.
$$

Meixner polynomials $M_n^{\gamma,\mu}(x)$.

Analogously, for the monic Meixner polynomials we find

$$
a_{mn} = \begin{cases} 1 & m = n = 0 \\[2ex] \dfrac{\mu \gamma m!}{1 - \mu} \,_2\mathrm{F}_1 \left(\begin{matrix} 1 - m \,,\, 1 + \gamma \\ 2 \end{matrix} \,\middle|\, \dfrac{\mu}{\mu - 1} \right) & m \neq 0, n = 0 \\[3ex] \left(\begin{matrix} m \\ n \end{matrix} \right) \dfrac{\Gamma(m)}{\Gamma(n)} \,_2\mathrm{F}_1 \left(\begin{matrix} n - m \,,\, n + \gamma \\ n \end{matrix} \,\middle|\, \dfrac{\mu}{\mu - 1} \right) & m \neq 0, n \neq 0 \end{cases} ,
$$

and

$$
d_{mn} = \left(\begin{matrix} m \\ n \end{matrix} \right) (\gamma + n)_{m-n} \left(\dfrac{\mu}{1 - \mu} \right)^{m-n} .
$$

Kravchuk polynomials $K_n^p(x, N)$.

For the monic Kravchuk polynomials, we obtain

$$
a_{mn} = \begin{cases} 1 & m = n = 0 \\[2mm] Npm! \; {}_2F_1\left(\begin{array}{c} 1-m \; , \; 1-N \\ 2 \end{array} \middle| p \right) & m \neq 0, n = 0 \\[3mm] \left(\begin{array}{c} m \\ n \end{array} \right) \dfrac{\Gamma(m)}{\Gamma(n)} {}_2F_1\left(\begin{array}{c} n-m \; , \; n-N \\ n \end{array} \middle| p \right) & m \neq 0, n \neq 0 \end{cases} ,
$$

and

$$
d_{mn} = \left(\begin{array}{c} m \\ n \end{array} \right) p^{m-n}(N - m + 1)_{m-n}.
$$

Hahn polynomials $h_n^{\alpha,\beta}(x, N)$.

Finally, for the monic Hahn polynomials, one has

$$
a_{mn} = \begin{cases} 1 & m = n = 0 \\[2mm] \dfrac{m!(\beta+1)(N-1)}{\alpha+\beta+2} \; {}_3F_2\left(\begin{array}{c} m+1, 2-N, 2+\beta \\ 2, 2-N-\alpha \end{array} \middle| 1 \right) & m \neq 0, n = 0 \\[3mm] \left(\begin{array}{c} m \\ n \end{array} \right) \dfrac{\Gamma(m)}{\Gamma(n)} \; {}_3F_2\left(\begin{array}{c} n-m, 1+n-N, n+\beta+1 \\ n, 2n+\alpha+\beta+2 \end{array} \middle| 1 \right) & m \neq 0, n \neq 0 \end{cases} ,
$$

and

$$
d_{mn} = \left(\begin{array}{c} m \\ n \end{array} \right) \frac{(N-m)_{m-n}(n+\beta+1)_{m-n}}{(2n+\alpha+\beta+2)_{m-n}}.
$$

Some of the above formulas have been found by different authors using different approaches. This is so for the Stirling inversion problems of the Charlier [11, 25, 48, 55], Meixner [25, 48, 55], Kravchuk [25, 48, 55] and Hahn [17] polynomials.

4 Expansion of the product of a polynomial $r_m(x)$ and a hypergeometric polynomial.

Here, we face and solve the modified Clebsch-Gordan linearization problem, which consists of finding the expansion coefficients c_{jmn} of the relation

$$
r_m(x) q_j(x) = \sum_{n=0}^{m+j} c_{jmn} p_n(x) , \tag{4.1}
$$

where $\{p_n\}$ is a discrete orthogonal set of hypergeometric polynomials which satisfy the difference equation (2.1) and $r_m(x)$ and $q_j(x)$ are arbitrary polynomials.

Theorem 4.1 *The explicit expression of the coefficients c_{jmn} in the expansion (4.1) is given by*

$$c_{jmn} = \frac{(-1)^n B_n}{d_n^2} \sum_{x=a}^{b-1} \rho_n(x-n) \, \nabla^n \left[r_m(x) q_j(x) \right] \, . \tag{4.2}$$

Proof: In order to find explicit formulas for the coefficients c_{jmn} we can multiply Eq. (4.1) by $\rho(x) p_k(x)$ and summing on x. Then, by using the orthogonality properties of the polynomials $p_k(x)$ we get

$$c_{jmn} = \sum_{x=a}^{b-1} \frac{q_j(x) r_m(x) p_n(x) \rho(x)}{d_n^2} = \frac{B_n}{d_n^2} \sum_{x=a}^{b-1} q_j(x) r_m(x) \, \nabla^n \, \rho_n(x) \, . \tag{4.3}$$

Now, the same method that leads to Eq. (3.7) gives us (4.2). ∎

In the special case when $q_j(x)$ is the jth-degree hypergeometric polynomial satisfying the following second-order difference equation

$$\tilde{\sigma}(x) \, \triangle \, \nabla y + \tilde{\tau}(x) \, \triangle \, y + \tilde{\lambda} y = 0, \quad y \equiv q_j(x),$$

the following theorem follows

Theorem 4.2 *The explicit expression of the coefficients c_{jmn} in the expansion (4.1) is given by*

$$c_{jmn} = \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \times$$

$$\times \sum_{k=k_-}^{k_+} \binom{n}{k} \tilde{A}_{jk} \sum_{x=a}^{b-1} \sum_{l=0}^{j-k} (-1)^l \binom{j-k}{l} \frac{\rho_n(x-n)}{\tilde{\rho}_k(x-k)} \tilde{\rho}_j(x-k-l) [\nabla^{n-k} r_m(x-k)] \, , \tag{4.4}$$

where $k_- = \max(0, n-m)$ and $k_+ = \min(n, j)$.

Proof: We will start from Eq. (4.2)

$$c_{jmn} = \frac{(-1)^n B_n}{d_n^2} \sum_{x=a}^{b-1} \rho_n(x-n) \, \nabla^n \left[r_m(x) q_j(x) \right] \, . \tag{4.5}$$

Applying the Leibniz's rule for the n-th difference derivative of a product,

$$\nabla^n \left[f(x) g(x) \right] = \sum_{k=0}^{n} \binom{n}{k} [\nabla^k f(x)][\nabla^{n-k} g(x-k)] \, ,$$

to $r_m(x) q_j(x)$,

$$\nabla^n \left[r_m(x) q_j(x) \right] = \sum_{k=0}^{n} \binom{n}{k} [\nabla^k q_j(x)][\nabla^{n-k} r_m(x-k)] \, , \tag{4.6}$$

together with Eq. (2.9) for the orthogonal polynomial $q_j(x)$,

$$\nabla^k q_j(x) \equiv \triangle^k q_j(x-k) = \frac{\tilde{A}_{jk} \tilde{B}_j}{\tilde{\rho}_k(x-k)} \, \nabla^{j-k} \left[\tilde{\rho}_j(x-k) \right] \, , \tag{4.7}$$

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where the parameters $\tilde{B}_j$ and $\tilde{A}_{kj}$ are defined in terms of $\tilde{\sigma}(x)$ and $\tilde{\tau}(x)$ as in Eqs. (2.13) and (2.14) respectively and using Eq. (2.23) we find the expression

$$c_{jmn} = \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=0}^{n} \binom{n}{k} \tilde{A}_{jk} \sum_{x=a}^{b-1} [\triangledown^{n-k} r_m(x-k)] \frac{\rho_n(x-n)}{\tilde{\rho}_k(x-k)} \triangledown^{j-k} [\tilde{\rho}_j(x-k)] . \quad (4.8)$$

Furthermore, the Rodrigues parameter $\tilde{B}_j$ is directly related to the leading coefficient $\tilde{a}_j$ of the polynomial $q_j(x)$ as in Eq. (2.12). Eq. (4.8) can be written as

$$c_{jmn} = \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=k_-}^{k_+} \binom{n}{k} \tilde{A}_{jk} \sum_{x=a}^{b-1} [\triangledown^{n-k} r_m(x-k)] \frac{\rho_n(x-n)}{\tilde{\rho}_k(x-k)} \triangledown^{j-k} [\tilde{\rho}_j(x-k)] . \quad (4.9)$$

where

$$k_- = \max(0, n-m) , \quad k_+ = \min(n, j) , \quad (4.10)$$

The use of the explicit expression of $\triangledown^{j-k} [\tilde{\rho}_j(x-k)]$ allows us the find the wanted equation (4.4). ∎

The expansions considered in the previous Section can be considered as the particular case $j = 0$ of the present one. Notice that B_n/d_n^2 and $\tilde{B}_j \tilde{A}_{jk}$ can be expressed in terms of the leading coefficients a_n and $\tilde{a}_j$, respectively, of the polynomials $p_n(x)$ and $q_j(x)$ by means of the Eqs. (2.7), (2.12) and (2.14), respectively.

Up to now, to the best of our information, there only exists a formal, recurrent way [11] to evaluate linearization coefficients. Its application to the simplest case, i.e., to the expansion of the products of two Charlier polynomials in series of Charlier polynomials of the same type, leads to a six-term recursive relation for the corresponding linearization coefficients which has not yet been possible to be solved, even not at a hypergeometric level by symbolic means (Petrovšek algorithm [44]). Also , Dunkl [14] for Hahn polynomials and Askey and Gasper [10] for Kravchuk polynomials have been able to calculate explicitly the expansion coefficients of the Clebsch-Gordan-type or conventional linearization problems (i.e., those problems which involve polynomials of the same system). They are collected in [53].

All these results are generalized by means of Eqs. (4.9) or (4.4). In particular, these expressions allow us to explicitly solve not only all Clebsch-Gordan-type expansion problems of the classical discrete polynomials but also the modified ones which involve polynomials of any classical system. There are 64 linearization formulas corresponding to the expansions of all possible products of pairs of classical discrete polynomials in terms of each classical discrete set, which can be described in full detail by using the Eq. (4.4).

4.1 Some special cases.

Let us apply the above formula for the cases $r_m(x) = (x)_m$, $r_m(x) = x^{[m]}$ and x^m.

Corollary 4.1.1 *The coefficients of the expansion*

$$(x)_m q_j(x) = \sum_{n=0}^{m+j} c_{jmn} p_n(x),$$ (4.11)

are given by

$$c_{jmn} = \frac{(-1)^n B_n \tilde{B}_j m!}{d_n^2} \sum_{k=k_-}^{k_+} \left(\begin{array}{c} n \\ k \end{array} \right) \frac{\tilde{A}_{jk}}{(m-n+k)!} \times$$

$$\times \sum_{l=0}^{j-k} (-1)^l \left(\begin{array}{c} j-k \\ l \end{array} \right) \sum_{x=a}^{b-1} \frac{\rho_n(x-n)\tilde{\rho}_j(x-k-l)}{\tilde{\rho}_k(x-k)} (x-k)_{m-n+k} .$$

Corollary 4.1.2 *The coefficient of the expansion*

$$x^{[m]} q_j(x) = \sum_{n=0}^{m+j} d_{jmn} p_n(x),$$ (4.12)

are given by

$$d_{jmn} = \frac{(-1)^n B_n \tilde{B}_j m!}{d_n^2} \sum_{k=k_-}^{k_+} \left(\begin{array}{c} n \\ k \end{array} \right) \frac{\tilde{A}_{jk}}{(m-n+k)!} \times$$

$$\times \sum_{l=0}^{j-k} (-1)^l \left(\begin{array}{c} j-k \\ l \end{array} \right) \sum_{x=a}^{b-1} \frac{\rho_n(x-n)\tilde{\rho}_j(x-k-l)}{\tilde{\rho}_k(x-k)} (x-n)^{[m-n+k]} .$$

Corollary 4.1.3 *The coefficients of the expansion*

$$x^m q_j(x) = \sum_{n=0}^{m+j} e_{mn} p_n(x)$$ (4.13)

are given by

$$e_{jmn} = \sum_{k=n}^{m+j} d_{jkn} \mathcal{S}_m^{(k)}.$$

The expressions (4.11)-(4.13) complement and considerably generalize some linearization formulas of similar type recently found by Belmehdi et al. [11]. Indeed, these authors found recurrently the coefficients of the expansion (4.12) in the cases for which both polynomials $q_j(x)$ and $p_n(x)$ are of the same Charlier, Meixner or Kravchuk character.

To conclude this section we will show two simple examples of the linearization problem, which require the use of Theorems 4.1 and 4.2, respectively.

4.2   Examples.

Firstly, the linearization of a product of two Stirling polynomials $x^{[m]} x^{[j]}$ in terms of the Charlier polynomials

$$x^{[m]} x^{[j]} = \sum_{n=0}^{m+j} c_{m,j,n} C_n^\mu(x),$$ (4.14)

is solved by the use of Th. 4.1 to give that

$$c_{m,j,n} = \begin{pmatrix} m \\ p-j \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix} \frac{j!}{(p-m)!} \, \mu^{p-n} \times$$

$${}_3\mathrm{F}_3 \left(\begin{array}{c} p-m-j, \ p+1, \ 1 \\ p-j+1, p-m+1, p-n+1 \end{array} \middle| -\mu \right),$$

$$(4.15)$$

where $p = \max(n, m, j)$.

Next, we apply Th. 4.2 to find the solution of the following linearization problem

$$x^{[m]} C_j^\gamma(x) = \sum_{n=0}^{m+j} c_{m,j,n} C_n^\mu(x), \qquad (4.16)$$

obtaining

$$c_{m,j,n} = \sum_{k=\max(0,n-m)}^{j} \begin{pmatrix} j \\ k \end{pmatrix} \begin{pmatrix} m \\ p-k \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix} \frac{k! \, (-\gamma)^{j-k} \mu^{p-n}}{(p-m)!} \times$$

$${}_3\mathrm{F}_3 \left(\begin{array}{c} p-m-k, \ p+1, \ 1 \\ p-k+1, p-m+1, p-n+1 \end{array} \middle| -\mu \right), \qquad p = \max(n, m, k).$$

$$(4.17)$$

This result can be alternatively found by means of Eqs. (4.14) and (4.15) together with Eqs. (2.20) and (2.24). Notice the finiteness of the k-summation and the terminating character of the involved hypergeometric function ${}_3\mathrm{F}_3$.

Expressions similar to Eq. (4.16) referred to the rest of classical discrete hypergeometric polynomials with the non-orthogonal polynomials $x^{[m]}$ and $(x)_m$ may be equally found.

5 The connection problem between discrete hypergeometric polynomials.

A very important particular case of the expansion (4.1) is that corresponding to $m = 0$, i.e., the connection problem

$$q_j(x) = \sum_{n=0}^{j} c_{j0n} p_n(x), \qquad (5.1)$$

which has received a lot of attention in the literature [5, 8, 17, 20, 25, 30, 47, 48] but still not fully solved for discrete hypergeometric polynomials. Here, this solution immediatly follows from the general linearization formulas (4.9) or (4.4), what enable us to find easily connection coefficients in terms of a terminating hypergeometric function. The latter is illustrated in Subsection 5.1 for all posible pairs of classical discrete orthogonal hypergeometric polynomials.

Indeed, one has from Eq. (4.9) with $m = 0$ that the connection coefficients are

$$c_{j0n} = \frac{(-1)^n B_n \tilde{B}_j \tilde{A}_{jn}}{d_n^2} \sum_{x=a}^{b-1} \frac{\rho_n(x-n)}{\tilde{\rho}_n(x-n)} \, \nabla^{j-n} \left[\tilde{\rho}_j(x-n) \right] , \qquad (5.2)$$

which using (2.11) and (2.14) becomes into

$$c_{j0n} = \frac{(-1)^n j!}{(j-n)!} \frac{B_n \tilde{B}_j}{\tilde{B}_n d_n^2} \sum_{x=a}^{b-1} \sum_{k=0}^{j-n} \frac{\rho_n(x-n)}{\tilde{\rho}_n(x-n)} \left(\begin{array}{c} j-n \\ k \end{array} \right) (-1)^k \tilde{\rho}_j(x-n-k) . \qquad (5.3)$$

The problem (5.1) is also a particular case of the expansion (3.1); namely when the discrete polynomial $r_m(x)$ posesses the hypergeometric character. So, Eq. (5.3) may be alternatively obtained by the inclusion of that fact in Eq. (3.2). In the case when $\tilde{\sigma}(x) = \sigma(x)$ and $(\tilde{a}, \tilde{b}) = (a, b)$, the corresponding connection problem (5.1) involves discrete hypergeometric polynomials orthogonal in the same interval, and the expansion coeffiecients given by (3.2) reduce as

$$c_{j0n} = \frac{(-1)^n B_n \tilde{B}_j \tilde{A}_{jn}}{d_n^2} \sum_{x=a}^{b-1} \sum_{k=0}^{j-n} \frac{\rho(x)}{\tilde{\rho}(x)} \left(\begin{array}{c} j-n \\ k \end{array} \right) (-1)^k \tilde{\rho}_j(x-n-k) . \qquad (5.4)$$

For completeness, let us point out that there is another equivalent expression for the connection coefficients c_{j0n} which sometimes is very useful. The general polynomial solution of the equation (2.1) is given by (2.15). Then, the solution for the *direct* connection problem

$$q_j(x) = \sum_{k=0}^{j} a_{jk} x^{[k]}, \qquad (5.5)$$

is given by

$$a_{jk} = \frac{(-1)^k (-\bar{x}_1)_j (-\bar{x}_2)_j (x_2 - \bar{x}_1 - \bar{x}_2 + j - 1)_k (-n)_k}{(-\bar{x}_1)_k (-\bar{x}_2)_k (x_2 - \bar{x}_1 - \bar{x}_2 + j - 1)_j k!}. \qquad (5.6)$$

This formula inmediately follows from the identity $x^{[k]} = (-1)^k (x)_k$ and the definition of the generalized hypergeometric function (2.16). Let us also remark that sometimes it is better to use the combination of the above formula with formula (3.8) wo obtain the searched expansion coefficients. Notice that

$$q_j(x) = \sum_{k=0}^{j} a_{jk} x^{[k]} = \sum_{k=0}^{j} a_{jk} \sum_{n=0}^{k} d_{kn} p_n(x) = \sum_{n=0}^{j} \underbrace{\left(\sum_{k=0}^{j-n} a_{j\,k+n} d_{k+n\,n} \right)}_{c_{j0n}} p_n(x), \qquad (5.7)$$

where $a_{j\,k+n}$ and $d_{k+n\,n}$ are given by (5.6) and (3.8), respectively. Again here, the coefficients c_{j0n} depend only on the coefficients of the second order difference equation of hypergeometric type (2.1).

Finally, let us mention that from Eqs. (5.1)-(5.7) one obtains, as a byproduct, the solution for the conventional connection problem; i.e., that associated with the four classical orthogonal discrete hypergeometric polynomials (Hahn, Meixner, Kravchuk and Charlier).

5.1 Application to all possible pairs of classical polynomials.

In this section we will provide the formulas connecting the different families of classical hypergeometric discrete polynomials, which generalize results already obtained by different authors using different approaches, e.g. [5, 17, 25, 30, 48], in particular, the most general case involving two Hahn polynomials is given (see formula (5.17) from below) from where, the most general connection formula given by Gasper [17, Eq. (4.1), pag. 188] is

obtained as a particular case.

The first eight cases can be computed by using (5.3) and the other ones with the help of (5.7). Notice that if we equate both expressions (3.8) and (5.7) one can obtain different summation formulas involving terminating hypergeometric series of the type given in the Appendix.

5.1.1 Charlier-Charlier

From formula (5.4) and using the main data of the Charlier polynomials (see Table 1) we find for the connection coefficients between the families

$$C_j^\mu(x) = \sum_{n=0}^{j} c_{j0n} C_n^\gamma(x),$$

the expression

$$c_{j0n} = \left(\begin{array}{c} j \\ n \end{array} \right) (\gamma - \mu)^{j-n}. \tag{5.8}$$

5.1.2 Meixner-Meixner

For the Meixner-Meixner problem we have

$$M_j^{\gamma,\mu}(x) = \sum_{n=0}^{j} c_{j0n} M_n^{\alpha,\beta}(x),$$

where

$$c_{j0n} = \left(\begin{array}{c} j \\ n \end{array} \right) \frac{(1-\beta)^{n+\alpha}\mu^{j-n}\Gamma(j+\gamma)}{\Gamma(\alpha+n)(\mu-1)^{j-n}} \times$$

$$\times \sum_{k=0}^{j-n} (-1)^k \left(\begin{array}{c} j-n \\ k \end{array} \right) \left(\frac{\beta}{\mu} \right)^k \frac{\Gamma(n+k+\alpha)}{\Gamma(n+k+\gamma)} {}_2\mathrm{F}_1 \left(\begin{array}{c} n+k+\alpha \ , \ j+\gamma \\ n+k+\gamma \end{array} \middle| \beta \right).$$

Using the transformation formula (A.3), the identity $\left(\begin{array}{c} j-n \\ k \end{array} \right) = (-1)^k \frac{(n-j)_k}{k!}$ as well as formula (A.4) we finally obtain

$$c_{j0n} = \left(\begin{array}{c} j \\ n \end{array} \right) \left(\frac{\mu}{\mu-1} \right)^{j-n} (\gamma+n)_{j-n} \ {}_2\mathrm{F}_1 \left(\begin{array}{c} n-j \ , \ n+\alpha \\ n+\gamma \end{array} \middle| \frac{\beta(1-\mu)}{\mu(1-\beta)} \right). \tag{5.9}$$

In particular, for the special case $\alpha = \gamma$, Eq. (5.9) becomes

$$c_{j0n} = \left(\begin{array}{c} j \\ n \end{array} \right) (\gamma+n)_{j-n} \left(\frac{\beta-\mu}{(\beta-1)(\mu-1)} \right)^{j-n},$$

The second case corresponds to $\beta = \mu$, then (5.9) becomes

$$c_{j0n} = \left(\begin{array}{c} j \\ n \end{array} \right) \left(\frac{\mu}{\mu-1} \right)^{j-n} (\gamma-\alpha)_{j-n}.$$

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5.1.3 Kravchuk-Kravchuk.

For the Kravchuk-Kravchuk expansion,

$$K_j^p(x, N) = \sum_{n=0}^{j} c_{j0n} \, K_n^q(x, M), \qquad j \leq \min\{N, M\},$$

the same procedure used in the Meixner-Meixner case gives us

$$c_{j0n} = \begin{pmatrix} j \\ n \end{pmatrix} (M - j + 1)_{j-n} (-p)^{j-n} \, {}_2\mathrm{F}_1 \begin{pmatrix} n - j \ , \ n - N \\ n - M \end{pmatrix} \begin{vmatrix} q \\ p \end{pmatrix}. \qquad (5.10)$$

In the particular case $p = q$ its reduces to

$$c_{j0n} = \begin{pmatrix} j \\ n \end{pmatrix} p^{j-n} (N - M)_{j-n},$$

and for the case $M = N$

$$c_{j0n} = \begin{pmatrix} j \\ n \end{pmatrix} \left(\frac{p}{q}\right)^{j-n} (q - p)^{j-n} (N - j + 1)_{j-n}.$$

5.1.4 Meixner-Charlier.

In this case we have the expansion

$$M_j^{\gamma,\mu}(x) = \sum_{n=0}^{j} c_{j0n} \, C_n^\alpha(x),$$

with

$$c_{j0n} = \begin{pmatrix} j \\ n \end{pmatrix} \frac{e^{-\alpha} \mu^{j-n} \Gamma(j + \gamma)}{(\mu - 1)^{j-n}} \sum_{k=0}^{j-n} \frac{(-1)^k}{\Gamma(\gamma + n + k)} \begin{pmatrix} j - n \\ k \end{pmatrix} \left(\frac{\alpha}{\mu}\right)^k {}_1\mathrm{F}_1 \begin{pmatrix} j + \gamma \\ m + k + \gamma \end{pmatrix} \alpha \end{pmatrix}.$$

If we use the transformation formula (A.6) and the sumation formula (A.7) we find

$$c_{j0n} = \begin{pmatrix} j \\ n \end{pmatrix} \left(\frac{\mu}{\mu - 1}\right)^{j-n} (\gamma + n)_{j-n} \, {}_1\mathrm{F}_1 \begin{pmatrix} n - j \\ n + \gamma \end{pmatrix} \frac{\alpha(1 - \mu)}{\mu} \end{pmatrix}. \qquad (5.11)$$

5.1.5 Charlier-Meixner.

For the Charlier-Meixner expansion

$$C_j^\alpha(x) = \sum_{n=0}^{j} c_{j0n} \, M_n^{\gamma,\mu}(x),$$

one finds from Eq. (5.4) that

$$c_{j0n} = \begin{pmatrix} j \\ n \end{pmatrix} (-\alpha)^{j-n} \, {}_2\mathrm{F}_0 \begin{pmatrix} n - j \ , \ \gamma + n \\ - \end{pmatrix} \frac{\mu}{\alpha(1 - \mu)} \end{pmatrix}. \qquad (5.12)$$

5.1.6 Meixner-Kravchuk.

In the Meixner-Kravchuk case,

$$M_j^{\gamma,\mu}(x) = \sum_{n=0}^{j} c_{j0n} \, K_n^p(x,N), \qquad j \le N$$

we find

$$c_{j0n} = \begin{pmatrix} j \\ n \end{pmatrix} (n+\gamma)_{j-n} \left(\frac{\mu}{\mu-1} \right)^{j-n} {}_2\mathrm{F}_1 \left(\begin{array}{c} n-j \ , \ n-N \\ n+\gamma \end{array} \middle| \frac{p(\mu-1)}{\mu} \right). \qquad (5.13)$$

5.1.7 Kravchuk-Meixner.

For the Kravchuk-Meixner connection problem,

$$K_j^p(x,N) = \sum_{n=0}^{j} c_{j0n} \, M_n^{\alpha,\beta}(x), \qquad j \le N$$

we have

$$c_{j0n} = \begin{pmatrix} j \\ n \end{pmatrix} (N+1-j)_{j-n}(-p)^{j-n} \, {}_2\mathrm{F}_1 \left(\begin{array}{c} n-j \ , \ n+\alpha \\ n-N \end{array} \middle| \frac{\beta}{(\beta-1)p} \right). \qquad (5.14)$$

5.1.8 Kravchuk-Charlier.

For the Kravchuk-Charlier connection problem,

$$K_j^p(x,N) = \sum_{n=0}^{j} c_{j0n} \, C_n^\mu(x), \qquad j \le N,$$

we have

$$c_{j0n} = \begin{pmatrix} j \\ n \end{pmatrix} (N+1-j)_{j-n}(-p)^{j-n} \, {}_1\mathrm{F}_1 \left(\begin{array}{c} n-j \\ n-N \end{array} \middle| -\frac{\mu}{p} \right). \qquad (5.15)$$

5.1.9 Charlier-Kravchuk.

For the Charlier-Kravchuk problem,

$$C_j^\mu(x) = \sum_{n=0}^{j} c_{j0n} \, K_n^p(x,N), \qquad j \le N,$$

we have

$$c_{j0n} = \begin{pmatrix} j \\ n \end{pmatrix} (-\mu)^{j-n} \, {}_2\mathrm{F}_0 \left(\begin{array}{c} n-j \ , \ n-N \\ - \end{array} \middle| -\frac{p}{\mu} \right). \qquad (5.16)$$

5.1.10 Hahn-Hahn

For the Hahn-Hahn problem, we use Eq. (5.7). A straightforward study of the problem

$$h_j^{\gamma,\mu}(x,M) = \sum_{n=0}^{j} c_{j0n} \, h_n^{\alpha,\beta}(x,N), \qquad j \le \min\{N-1, M-1\},$$

allows us to find

$$c_{j0n} = \left(\begin{array}{c} j \\ n \end{array} \right) \frac{(1+n-M)_{j-n}(1+n+\mu)_{j-n}}{(1+n+j+\gamma+\mu)_{j-n}} \times$$

$$\times \; {}_4F_3 \left(\begin{array}{c} n-j \, , \, 1+n-N \, , \, n+\beta+1 \, , \, 1+j+n+\gamma+\mu \\ 1+n-M \, , \, n+\mu+1 \, , \, 2n+\alpha+\beta+2 \end{array} \middle| 1 \right).$$

(5.17)

In the particular case $N = M$ (5.17) reduces to

$$c_{j0n} = \left(\begin{array}{c} j \\ n \end{array} \right) \frac{(1+n-N)_{j-n}(1+n+\mu)_{j-n}}{(1+n+j+\gamma+\mu)_{j-n}} \; {}_3F_2 \left(\begin{array}{c} n-j \, , \, n+\beta+1 \, , \, 1+j+n+\gamma+\mu \\ n+\mu+1 \, , \, 2n+\alpha+\beta+2 \end{array} \middle| 1 \right).$$

5.1.11   Hahn-Charlier

For the Hahn-Charlier problem,

$$h_j^{\alpha+\beta}(x, N) = \sum_{n=0}^{j} c_{j0n} \, C_n^\alpha(x), \qquad j \le M - 1,$$

we find that

$$c_{j0n} = \left(\begin{array}{c} j \\ n \end{array} \right) \frac{(1+n-N)_{j-n}(1+n+\beta)_{j-n}}{(1+n+j+\alpha+\beta)_{j-n}} \; {}_2F_2 \left(\begin{array}{c} n-j \, , \, 1+j+n+\alpha+\beta \\ 1+n-N \; n+\beta+1 \end{array} \middle| -\alpha \right). \quad (5.18)$$

5.1.12   Charlier-Hahn

For the Charlier-Hahn problem,

$$C_j^\mu(x) = \sum_{n=0}^{j} c_{j0n} \, h_n^{\alpha,\beta}(x, N), \qquad j \le N - 1,$$

we find that

$$c_{j0n} = \left(\begin{array}{c} j \\ n \end{array} \right) (-\mu)^{j-n} \; {}_3F_1 \left(\begin{array}{c} n-j \, , \, 1+n-N \, , \, n+\beta+1 \\ 2n+\alpha+\beta+2 \end{array} \middle| -\frac{1}{\mu} \right). \qquad (5.19)$$

5.1.13   Hahn-Meixner

For the Hahn-Meixner problem,

$$h_j^{\alpha,\beta}(x, N) = \sum_{n=0}^{j} c_{j0n} \, M_n^{\gamma,\mu}(x), \qquad j \le N - 1,$$

we find that

$$c_{j0n} = \left(\begin{array}{c} j \\ n \end{array} \right) \frac{(1+n-N)_{j-n}(1+n+\mu)_{j-n}}{(1+n+j+\gamma+\mu)_{j-n}} \times$$

$$\times \; {}_3F_2 \left(\begin{array}{c} n-j \, , \, \alpha+\beta+j+n+1 \, , \, \gamma+n \\ 1+n-N \, , \, n+\beta+1 \end{array} \middle| \frac{\mu}{\mu-1} \right).$$

(5.20)

5.1.14 Meixner-Hahn

In the Meixner-Hahn case,

$$M_j^{\gamma,\mu}(x) = \sum_{n=0}^{j} c_{j0n}\, h_n^{\alpha,\beta}(x, N), \qquad j \le N - 1,$$

we find that

$$c_{j0n} = \binom{j}{n} \left(\frac{\mu}{\mu - 1}\right)^{j-n} (\gamma + n)_{j-n}\, {}_3\mathrm{F}_2\left(\begin{array}{c} n - j\,,\, 1 + n - N\,,\, n + \beta + 1 \\ \alpha + \beta + 2n + 2\,,\, \gamma + n \end{array} \middle| \frac{\mu - 1}{\mu} \right). \qquad (5.21)$$

5.1.15 Hahn-Kravchuk

For the Hahn-Kravchuk problem,

$$h_j^{\alpha,\beta}(x, N) = \sum_{n=0}^{j} c_{j0n}\, K_n^p(x, M), \qquad j \le \min\{M - 1, N\},$$

we find that

$$c_{j0n} = \binom{j}{n} \frac{(1 + n - N)_{j-n}(1 + n + \mu)_{j-n}}{(1 + n + j + \gamma + \mu)_{j-n}} \times$$

$$\times\ {}_3\mathrm{F}_2\left(\begin{array}{c} n - j\,,\, \alpha + \beta + j + n + 1\,,\, n - M \\ 1 + n - N\,,\, n + \beta + 1 \end{array} \middle| p \right). \qquad (5.22)$$

5.1.16 Kravchuk-Hahn

In the Kravchuk-Hahn case,

$$K_j^p(x, M) = \sum_{n=0}^{j} c_{j0n}\, h_n^{\alpha,\beta}(x, N), \qquad j \le \min\{N - 1, M\},$$

we find that

$$c_{j0n} = \binom{j}{n} p^{j-n}(n - M)_{j-n}\, {}_3\mathrm{F}_2\left(\begin{array}{c} n - j\,,\, 1 + n - N\,,\, n + \beta + 1 \\ \alpha + \beta + 2n + 2\,,\, n - M \end{array} \middle| \frac{1}{p} \right). \qquad (5.23)$$

Some of the above connection formulas have been previously found in a different manner either analytically [17] or recurrently [5, 20, 25, 30, 48]; at times, the recurrence relation for the expansion coefficients may be solved by symbolic means. Gasper [17] gave the explicit solution of the Charlier-Charlier problem as well as the hypergeometric representation of the expansion coefficients of the Meixner-Meixner, Kravchuk-Kravchuk, Hahn-Hahn (with the same interval of orthogonality), Kravchuk-Charlier, Kravchuk-Meixner and Kravchuk-Hahn problems; that is, he only considers the seven connection problems with positive coefficients. Lewanowicz [30] found recurrently the expansion coefficients for all possible pairs of the Charlier, Meixner and Kravchuk families; however, he is only able to lead to an explicit or hypergeometric-function solution in the Charlier-Charlier, Meixner-Charlier and Kravchuk-Charlier cases. Ronveaux et al. [48] are able to alternatively solve in a recurrent way the Charlier-Charlier, Charlier-Kravchuk, Charlier-Meixner, Meixner-Meixner, Meixner-Charlier and Kravchuk-Kravchuk cases; however, they are only

able to find an explicit solution in the Charlier-Charlier case, and the hypergeometric-function solution in the Kravchuk-Kravchuk case by symbolic means. To this respect see also [20]. Finally, Koepf and Schmarsau [25] have found with their computer-algebra-based method the explicit expression for the coefficients of the connection problem between classical discrete polynomials of the same type for some particular choice of the parameters (for example, they consider polynomials in the same interval of orthogonality or polynomials with equal parameters) save in the Charlier case, of course, where they obtained the complete solution.

It is worth to mention here that the connection coefficients of the aforementioned sixteen cases may be alternatively obtained by use of general theorems on expansion of generalized hypergeometric functions in series of functions of the same kind [29, 30], such us those described in [36, §9.1], [16]. Moreover, theorems of similar kind [36, §12.4], [54] may be potentially used to produce recursion formulas for the coefficients of the above expressions. This hypergeometric approach is being developed by S. Lewanowicz [35].

Finally, notice that, since the coefficients (5.9)-(5.16) are terminating hypergeometric series of the type $_2F_1$, $_1F_1$ and $_2F_0$, they can be identified with some classical hypergeometric polynomials, e.g. the Jacobi, Meixner or Kravchuk ($_2F_1$), Laguerre ($_1F_1$) and Charlier ($_2F_0$) polynomials.

Summary and Conclusions

We have studied

- The expansion of a general discrete polynomial $r_m(x)$ in series of an arbitrary (albeit fixed) orthogonal set of discrete hypergeometric polynomial $\{p_n\}$, and

- The expansion of the product $r_m(x)q_j(x)$ in series of the orthogonal set $\{p_n\}$, where $q_j(x)$ is any discrete hypergeometric polynomial.

The corresponding expansion coefficients are given in a compact and closed form by means of the coefficients which characterize the second-order difference equations satisfied by the involved polynomial(s) as well as the leading coefficient of their explicit expression. The resulting expressions, which are the main contributions of this work, are given by Eqs. (3.2), (4.9) or (4.4) and (5.3) or (5.7). They allow us to calculate both analytically and symbolically the expansion coefficients what is greatly useful to solve very involved mathematical problems, such us, e.g. some of queuing theory, birth and death processes and coding theory, and to deeply understand some physical phenomena which often require to obtain the matrix representation of quantum-mechanical observables; the determination of the corresponding matrix elements makes often use of connection and/or linearization formulas of the type here considered. Let us point out that our results, specially Eq. (4.9) or (4.4), opens a research avenue to determine practical connection and linearization formulas for arbitrary discrete hypergeometric polynomials. Its extension to generalized linearization expression for products of any number of discrete hypergeometric polynomials of great actual interest [23, 27, 31, 38] may be easily carried out.

The mathematical usefulness of these general expressions is illustrated by the explicit of the expansion coefficients of the non-orthogonal families $(x)_m$, $x^{[m]}$ and x^m in terms

of each classical discrete family (Hahn, Meixner, Kravchuk and Charlier). Also the coefficients of the expansion of the products $(x)_m q_j(x)$, $x^{[m]} q_j(x)$ and $x^m q_j(x)$ are found for an arbitrary discrete hypergeometric polynomial. Furthermore, as a nice byproduct, the complete solution of the conventional connection problem for all possible pairs of classical polynomials (Hahn, Meixner, Kravchuk and Charlier) is given from the same root in a unifying way. Some specific cases of this problem can be encountered dispersely in the literature.

Finally, we believe that the constructive approach here presented is a complementary, very useful alternative to the methods of Markett, Ronveaux et al and Lewanowicz to attack succesfully the long-standing connection and linearization problems of discrete hypergeometric polynomials. The same approach has been used for hypergeometric polynomials of a continuous variable in both connection [50] and linearization [6] problems, as well as for q-polynomials [4].

A   Some formulas involving hypergeometric functions.

In this section we will enumerate some relations involving hypergeometric functions which were useful in order to obtain the results of the paper.

Formulas involving $_2\mathrm{F}_1$.

Special values [1, Chapter 15]

$$_2\mathrm{F}_1\left(\begin{array}{cc} a & b \\ & c \end{array}\middle|1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad c \neq 0, -1, -2, ..., \Re(c-a-b) > 0. \quad (\mathrm{A}.1)$$

$$_2\mathrm{F}_1\left(\begin{array}{cc} a & b \\ & b \end{array}\middle|x\right) = (1-x)^{-a} \qquad \forall b \in \mathbb{R}. \quad (\mathrm{A}.2)$$

Linear transformation formulas [22, p. 425]

$$_2\mathrm{F}_1\left(\begin{array}{cc} a & b \\ & c \end{array}\middle|x\right) = (1-x)^{-a} {}_2\mathrm{F}_1\left(\begin{array}{cc} a & c-b \\ & c \end{array}\middle|\frac{x}{x-1}\right) =$$

$$= (1-x)^{c-a-b} {}_2\mathrm{F}_1\left(\begin{array}{cc} c-a & c-b \\ & c \end{array}\middle|x\right). \quad (\mathrm{A}.3)$$

A summation formula [22, Eq. 65.2.2, p. 426]

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} y^k {}_2\mathrm{F}_1\left(\begin{array}{cc} c-a & c-b \\ & c+k \end{array}\middle|x\right) = (1-x)^{a+b-c} {}_2\mathrm{F}_1\left(\begin{array}{cc} a & b \\ & c \end{array}\middle|x+y-xy\right). \quad (\mathrm{A}.4)$$

Formulas involving $_1\mathrm{F}_1$.

Special values [1, Chapter 13]

$$_1\mathrm{F}_1\left(\begin{array}{c} a \\ a \end{array}\middle|x\right) = e^x, \qquad \forall a \in \mathbb{R}. \quad (\mathrm{A}.5)$$

Linear transformation formula [22, p. 431]

$$_1F_1\left(\begin{array}{c} a \\ c \end{array}\middle| x\right) = e^x {}_1F_1\left(\begin{array}{c} c-a \\ c \end{array}\middle| -x\right).\tag{A.6}$$

A summation formula [22, Eq. (66.2.5), p. 431]

$$\sum_{k=0}^{\infty}\frac{(c-a)_k}{k!(c)_k}y^k {}_1F_1\left(\begin{array}{c} a \\ c+k \end{array}\middle| x\right) = e^y {}_1F_1\left(\begin{array}{c} a \\ c \end{array}\middle| x-y\right).\tag{A.7}$$

Finally, we want to remind, as we already pointed out in Section **5.1**, that equating the expressions (3.8) and (5.7) one can obtain different summation formulas involving terminating hypergeometric series of higher order.

Akcnowledgmets

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