Existence of positive solution of a nonlocal logistic population model

Claudianor O. Alves, Manuel Delgado, Marco A. S. Souto and Antonio Suárez

Abstract. In this paper, we study the existence of positive solutions for a class of nonlocal problem arising in population dynamic. Basically, we prove our results via bifurcation theory.

Mathematics Subject Classification (2010). 35J15, 35J60, 92B05 .

Keywords. Nonlocal logistic equations, a priori bounds, positive solutions.

1. Introduction

In this paper, we study the existence of positive solution for the following class of nonlocal problem

$$\begin{cases} -\Delta u = u\left(\lambda - \displaystyle\int_{\Omega} K(x,y)u^p(y)dy\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N, N \geq 1$, is a smooth bounded domain, $p > 0$ and $K : \Omega \times \Omega \to \mathbb{R}$, is a non-negative function with $K \in L^\infty(\Omega \times \Omega)$ and verifying other hypotheses that will be detailed below.

Our motivation to study the above problem begins with the most used equation to model the behaviour of a species inhabiting in a domain Ω, that is the classical logistic equation

$$\begin{cases} -\Delta u = u(\lambda - b(x)u^p) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, $u(x)$ is the population density at location $x \in \Omega$, $\lambda \in \mathbb{R}$ is the growth rate of the species, and b is a positive function denoting the carrying capacity, that is, $b(x)$ describes the limiting effect of crowding of the population. In (1), we are assuming that Ω is surrounded by inhospitable areas, due to the homogeneous Dirichlet boundary conditions. Equation (1) is a local equation, and so the crowding effect of the population u at x only depends on the value of the population in the same point x. It seems more realistic (see for instance [3]) to consider that this crowding effect depends also on the value of the population around of x, that is, the crowding effect depends on the value of u in a neighborhood of x, $B_r(x)$, the centered ball at x of radius $r > 0$. So, we consider the equation

$$\begin{cases} -\Delta u = u\left(\lambda - \displaystyle\int_{\Omega \cap B_r(x)} b(y)u^p(y)dy\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where b is a nonnegative and nontrivial continuous function. In fact, we are going to study a more general problem, that is, problem (P).

We would like to mention that the nonlocal term has been also used to model the selection process of a population structured by phenotypical trait, see [7].

Before to statement our main results, we will recall some known results with respect to (P), involving different conditions on K. Hereafter, λ_1 denotes the first eigenvalue of the Laplacian under homogeneous Dirichlet boundary conditions.

When K is separable variable, i.e.,

$$K(x,y)=g(x)h(y), h\geq 0, h\neq 0 \text{ and } g(x)>0 \text{ in } \Omega, \tag{K_1}$$

it is proved in [4] that (P) possesses a unique positive solution for $\lambda>\lambda_1$. Moreover, in [5] and [6], assuming $g\equiv 1$, $p>1$ and under homogeneous Neumann boundary conditions, it is proved that the positive solution of (P) attracts all the possible solutions of the corresponding parabolic associated to (P). When $g\geq 0$, $g\neq 0$, $g\equiv 0$ in $\Omega_0\subset\Omega$, then (P) possesses a unique positive solution for $\lambda\in(\lambda_1,\lambda_0)$ where λ_0 is the principal eigenvalue of the Laplacian in Ω_0, see [4].

In [1], a similar result is proved when $K(x,y)=K_\delta(|x-y|)$ is a mollifier in $\mathbb{R}^N$, i. e., $K_\delta(|x-y|)\in C_0^\infty$, $\int_{\mathbb{R}^N}K_\delta(|x-y|)dy=1$ for any x with

$$K_\delta(|x-y|)=0 \text{ if } |x-y|\geq\delta \tag{K_2}$$

and

$$K_\delta(|x-y|) \text{ bounded away from zero is } |x-y|<\mu<\delta. \tag{K_3}$$

Observe that in this case, K vanishes away from the diagonal of $\Omega\times\Omega$.

For kernel functions $K(x,y)$ verifying that

$$K(x,y)\geq K_0>0 \text{ for all } (x,y)\in\Omega\times\Omega, \tag{K_4}$$

in [6] it was proved that there exists a positive solution of (P) if, and only if, $\lambda>\lambda_1$, see also [5].

In [2], when $p=1$ and $K\in C(\overline{\Omega}\times\overline{\Omega})$ is a nonnegative function such that for all $\phi>0$ it holds that

$$\int_\Omega K(x,y)\phi(y)dy>0 \tag{K_5}$$

then it is shown the existence of $\lambda^*>\lambda_1$ such that (P) possesses at least a positive solution for $\lambda\in(\lambda_1,\lambda^*]$; for that the authors use the implicit function theorem.

Finally, (P) has been studied also for the case $N=1$. Indeed, in [6] the existence of positive solution it proved if

$$K(x,x)\geq K_0>0 \text{ for all } x\in\Omega \tag{K_6}$$

and also in [9] for $K(x,y)=K_1(|x-y|)$ and $\Omega=(-1,1)$, where $K_1:[0,2]\mapsto(0,\infty)$ is a nondecreasing and piecewise continuous map with

$$\int_0^2 K_1(y)dy>0. \tag{K_7}$$

In this paper we are interested in giving new conditions on K to assure the existence of positive solution for all $\lambda>\lambda_1$ or the non-existence of positive solution for λ large. To this end, we introduce the class $\mathcal{K}$, which is formed by functions $K:\Omega\times\Omega\to\mathbb{R}$ verifying:

i) $K\in L^\infty(\Omega\times\Omega)$ *and* $K(x,y)\geq 0$ *for all* $x,y\in\Omega$.

ii) If w *is measurable and* $\displaystyle\int_{\Omega\times\Omega}K(x,y)|w(y)|^p w(x)^2dxdy=0$, *then* $w=0$ *a.e in* Ω.

Theorem 1. *Suppose that* $K\in\mathcal{K}$. *Then problem* (P) *has a positive solution if, and only if,* $\lambda>\lambda_1$.

Here, we would like to point out that Theorem 1 implies that (2) has a positive solution if, and only if, $\lambda>\lambda_1$, because the function $K(x,y)=b(y)\chi_{B_r(x)}(y)$ belongs to $\mathcal{K}$, once that b is a positive function on Ω. Moreover, we observe that Theorem 1 also improves the above results, allowing that K vanishes in some part of $\Omega\times\Omega$ in a general way, not only in a symmetric as in $(K_2)-(K_3)$.

If K does not belong to $(\mathcal{K})$, then K vanishes in some neighborhood of the diagonal of $\Omega\times\Omega$, see Section 3. Here, we are also able to prove results of existence and non-existence of positive solution for some values of λ if K belongs to a class $\mathcal{K}'$, which is formed by functions $K:\Omega\times\Omega\to\mathbb{R}$ verifying the following condition:

There are $r > 0$ and m connected open sets $\Omega_1, \Omega_2, \Omega_3, ..., \Omega_m \subset \Omega$ such that $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$, $i \neq j$, and

$$K(x,y) > 0, \text{ for all } (x,y) \in \Omega \times \Omega \text{ such that } x \notin U := \bigcup_{j=1}^{m} \Omega_j \text{ and } |x-y| < r.$$

Our main results involving this class of function is

Theorem 2. *Suppose that $K \in \mathcal{K}'$. Then, for any $\lambda_1 < \lambda < \min\{\lambda_1(\Omega_1), , ..., \lambda_1(\Omega_m)\}$, problem (P) has a positive solution.*

In the above result, $\lambda_1(\Omega_i)$ denotes the principal eigenvalue of the Laplacian in Ω_i under homogeneous Dirichlet boundary conditions. As a by product, we have the following corollary

Corollary 3. *Suppose that $K \in \mathcal{K}'$ with $m = 1$ and $K(x,y) = 0$ in U for any $y \in \Omega$. Suppose that ∂U is C^1. For any $\lambda_1 < \lambda < \lambda_1(U)$, there exists a positive solution u for the problem (P). Moreover, (P) does not have any positive solution for $\lambda \geq \lambda_1(U)$.*

An outline of the paper is as follows: in Section 2 we show, using bifurcation arguments, the existence of positive solution under class $\mathcal{K}$. Section 3 is devoted to the case when K belongs to $\mathcal{K}'$.

2. Proof of Theorem 1

In whole this section, we are assuming that $K \in \mathcal{K}$. Moreover, for any $w \in L^\infty(\Omega)$, we will consider the function $\phi_w : \Omega \to \mathbb{R}$ given by

$$\phi_w(x) := \int_\Omega K(x,y)|w(y)|^p dy.$$

Once that K and w are bounded, we have that ϕ_w is well defined. Furthermore, the ensuing properties will be useful along the paper:

$$t^p \phi_w = \phi_{tw}; \text{ for all } w \in L^\infty(\Omega), t > 0; \tag{ϕ_1}$$

$$||\phi_w||_\infty \leq ||K||_\infty |\Omega| \, ||w||_\infty^p, \text{ for all } w \in L^\infty(\Omega); \tag{ϕ_2}$$

$$||\phi_w - \phi_v||_\infty \leq ||K||_\infty |\Omega| \, || \, |w|^p - |v|^p||_\infty, \text{ for all } w, v \in L^\infty(\Omega); \tag{ϕ_3}$$

and

$$\phi : L^\infty(\Omega) \to L^\infty(\Omega), \ \phi(u) = \phi_u \text{ is uniformly continuous in } L^\infty(\Omega). \tag{ϕ_4}$$

Using the above notation, it is easy to observe (P) can be rewritten by

$$\begin{cases} -\Delta u + \phi_u u = \lambda u & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_1}$$

Here, we recall that u satisfies the above problem in weak sense, if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega \phi_u u v dx = \lambda \int_\Omega u v dx \ \forall v \in H_0^1(\Omega). \tag{3}$$

Hereafter, we intend to solve problem (P_1) by using the classical bifurcation result of Rabinowitz, see [8]. To this end, we recall that there exists $c_\infty = c_\infty(\Omega) > 0$ such that: for each $f \in L^\infty(\Omega)$, there exists a unique $\omega \in C^1(\overline{\Omega})$ satisfying

$$\begin{cases} -\Delta \omega = f(x) & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$||\omega||_{C^1(\overline{\Omega})} \leq c_\infty ||f||_\infty.$$

Hence, the solution operator $S : C^0(\overline{\Omega}) \to C^1(\overline{\Omega})$ given by

$$Sv = \omega_1 \iff \begin{cases} -\Delta\omega_1 = v & \text{in } \Omega, \\ \omega_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

is well-defined, it is linear and verifies

$$||Sv||_{C^1(\overline{\Omega})} \leq c_\infty ||v||_{C^0(\overline{\Omega})}, \ \forall v \in C^0(\overline{\Omega}).$$

Moreover, using the Schuart imbeeding, $S : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is a compact operator. Related to spectrum of S, it easy to see that

$$\sigma(S) = \{\lambda_j^{-1} \ : \ \lambda_j \text{ is a eigenvalue of the Laplacian}\},$$

On the other hand, define the nonlinear operator $G : C^0(\overline{\Omega}) \to C^1(\overline{\Omega})$ given by

$$G(v) = \omega_2 \iff \begin{cases} -\Delta\omega_2 + \phi_v v = 0 & \text{in } \Omega, \\ \omega_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

which is continuous and satisfies

$$||G(v)||_{C^1(\overline{\Omega})} \leq c_\infty ||\phi_v||_\infty ||v||_{C^0(\overline{\Omega})}, \forall v \in C^0(\overline{\Omega}).$$

Using again the Schuart imbedding, we have that $G : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is compact. Furthermore, since

$$||G(v)||_{C^0(\overline{\Omega})} \leq ||G(v)||_{C^1(\overline{\Omega})},$$

we have

$$\left\| \frac{G(v)}{||v||_{C^0(\overline{\Omega})}} \right\|_{C^0(\overline{\Omega})} \leq \frac{||G(v)||_{C^1(\overline{\Omega})}}{||v||_{C^0(\overline{\Omega})}} \leq c_\infty ||\phi_v||_\infty,$$

from where it follows that

$$\lim_{v \to 0} \frac{G(v)}{||v||_{C^0(\overline{\Omega})}} = 0, \qquad (\mathcal{G})$$

i.e.,

$$G(v) = o(||v||_{C^0(\overline{\Omega})}).$$

Of course, under these new notations: (λ, u) solves (P) if, and only if,

$$u = F(\lambda, u) := \lambda S u + G(u).$$

Now, as a direct consequence of [8], we have the following result

Theorem 4. *(***Global bifurcation***) Let E be a Banach space. Suppose that S is a compact linear operator and $\lambda^{-1} \in \sigma(S)$ and its multiplicity is odd. If G satisfies condition $(\mathcal{G})$, then set*

$$\Sigma = \overline{\{(\lambda, u) \in \mathbb{R} \times E : u = \lambda S u + G(u), u \neq 0\}}$$

has a closed connected component $\mathcal{C} = \mathcal{C}_\lambda$ such that $(\lambda, 0) \in \mathcal{C}$ and

(i) *$\mathcal{C}$ is unbounded in $\mathbb{R} \times E$, or*
(ii) *there exists $\hat{\lambda} \neq \lambda$ such that $(\hat{\lambda}, 0) \in \mathcal{C}$ and $\hat{\lambda}^{-1} \in \sigma(S)$.*

It is known that the first eigenfunction φ_1 associated to λ_1 can be chosen positive. Moreover, λ_1^{-1} is an eigenvalue with odd multiplicity for S.

From global bifurcation theorem, there exists a closed connected component $\mathcal{C} = \mathcal{C}_{\lambda_1}$ of solutions for (P), which satisfies (i) or (ii).

Lemma 5. *There exists $\delta > 0$ such that if $(\lambda, u) \in \mathcal{C}$ with $|\lambda - \lambda_1| + ||u||_{C^0(\overline{\Omega})} < \delta$ and $u \neq 0$, then u has defined signal, i.e.,*

$$u(x) > 0 \ \forall x \in \Omega \quad \textit{or} \quad u(x) < 0 \ \forall x \in \Omega.$$

Proof: Take (u_n) in $C^0(\overline{\Omega})$ and $\lambda_n \to \lambda_1$ such that,

$$u_n \neq 0, \ ||u_n||_{C^0(\overline{\Omega})} \to 0 \text{ and } u_n = F(\lambda_n, u_n).$$

Consider $w_n = u_n/||u_n||_{C^0(\overline{\Omega})}$ and observe that

$$\left\{ \begin{array}{ll} -\Delta w_n + \phi_{u_n} w_n = \lambda_n w_n & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{4}$$

It is easy to check that

$$||w_n||_{C^1(\overline{\Omega})} \leq c_\infty(\lambda_n + ||\phi_{u_n}||_\infty)||w_n||_{C^0(\overline{\Omega})} \leq c_\infty(\lambda_n + ||\phi_{u_n}||_\infty) \ \forall n \in \mathbb{N}$$

Once that (u_n) is bounded in $C^0(\overline{\Omega})$, it follows from (ϕ_2) that $(||\phi_{u_n}||_\infty)$ is bounded, therefore (w_n) is bounded in $C^1(\overline{\Omega})$. By using Arzelá-Áscoli theorem, (w_n) converges to some $w \in C^1(\overline{\Omega})$, uniformly in $\overline{\Omega}$, under a convenient subsequence. Of course $||w||_{C^0(\overline{\Omega})} = 1$, showing that $w \neq 0$ in Ω.

Now, by (ϕ_3), we know that (ϕ_{u_n}) is a Cauchy sequence in $C^0(\overline{\Omega})$. Then, this fact combined with the below inequality

$$\begin{aligned} ||w_n - w_m||_{C^1(\overline{\Omega})} \leq c_\infty[||\lambda_n u_n - \lambda_m u_m||_{C^0(\overline{\Omega})} + ||\phi_{u_n} - \phi_{u_m}||_\infty \\ + ||\phi_{u_n}||_\infty ||w_n - w_m||_{C^0(\overline{\Omega})}], \end{aligned}$$

give that (w_n) converges to w in $C^1(\overline{\Omega})$, and so, passing to the limit in

$$\int_\Omega \nabla w_n \cdot \nabla v dx + \int_\Omega \phi_{u_n} w_n v dx = \lambda_n \int_\Omega w_n v dx,$$

and recalling that by (ϕ_2), $\phi_{u_n} w_n \to 0$ in $C^0(\overline{\Omega})$, we get

$$\left\{ \begin{array}{ll} -\Delta w = \lambda_1 w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{array} \right.$$

Since $w \neq 0$, by spectral theory, we must have

$$w(x) > 0 \ \forall x \in \Omega \text{ or } w(x) < 0 \ \forall x \in \Omega.$$

Without loss of generality, we can suppose that $w(x) > 0$ for all $x \in \Omega$. As w is the $C^1(\overline{\Omega})$-limit of (w_n), we must have $w_n(x) > 0$ for all $x \in \Omega$ for n large enough. Thereby, the sign of u_n is the same of w_n for n large enough finishing the proof.

□

It is easy to check that: if $(\lambda, u) \in \Sigma$, the pair $(\lambda, -u)$ also is in Σ. In what follows, we decompose $\mathcal{C}$ into $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$ where

$$\mathcal{C}^+ = \{(\lambda, u) \in \mathcal{C} : u(x) \geq 0, \forall x \in \Omega\}$$

and

$$\mathcal{C}^- = \{(\lambda, u) \in \mathcal{C} : u(x) \leq 0, \forall x \in \Omega\}.$$

A simple computation gives that $\mathcal{C}^- = \{(\lambda, u) \in \mathcal{C} : (\lambda, -u) \in \mathcal{C}^+\}$, $\mathcal{C}^+ \cap \mathcal{C}^- = \{(\lambda_1, 0)\}$ and $\mathcal{C}^+$ is unbounded if, and only if, $\mathcal{C}^-$ is also unbounded.

Lemma 6. *$\mathcal{C}^+$ is unbounded.*

Proof: Suppose that $\mathcal{C}^+$ is bounded. Then $\mathcal{C}$ is also bounded. From global bifurcation theorem, $\mathcal{C}$ contains $(\hat{\lambda}, 0)$, where $\hat{\lambda} \neq \lambda_1$ and $\hat{\lambda}^{-1} \in \sigma(S)$.

In this way, we can take (u_n) in $C^0(\overline{\Omega})$ and $\lambda_n \to \hat{\lambda}$ such that,

$$u_n \neq 0, \ ||u_n||_{C^0(\overline{\Omega})} \to 0 \text{ and } u_n = F(\lambda_n, u_n).$$

Considering $w_n = u_n/||u_n||_{C^0(\overline{\Omega})}$, we know that it satisfies problem (4). Moreover, as in the proof of previous lemma, under an adequate subsequence, (w_n) converges to w in $C^1(\overline{\Omega})$, which is a nonzero solution of the eigenvalue problem

$$\left\{ \begin{array}{ll} -\Delta w = \hat{\lambda} w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{array} \right.$$

showing that w is a eigenfunction related to $\hat{\lambda}$. Since $\hat{\lambda} \neq \lambda_1$, w must change sign. Then, for n large, each w_n must change sign, and the same should be hold for $u_n = ||u_n||_{C^0(\overline{\Omega})} w_n$. But this is not possible, because $(\lambda_n, u_n) \in \mathcal{C}^+$ or $(\lambda_n, u_n) \in \mathcal{C}^-$.

□

2.1. A priori estimate

From Lemma 6, the connected component $\mathcal{C}^+$ is unbounded. Now, our goal it is to show that this component intersects any set of the form $\{\lambda\} \times H_0^1(\Omega)$, for $\lambda > \lambda_1$.

Lemma 7. *Suppose that $K \in \mathcal{K}$. For any $\Lambda > 0$, there exists $r > 0$ such that: if $(\lambda, u) \in \mathcal{C}^+$ and $\lambda \leq \Lambda$, we must have $||u||_{C^0(\overline{\Omega})} \leq r$.*

Proof: From now on, we denote by $||\ ||$ the usual norm in $H_0^1(\Omega)$, i.e.,

$$||u||^2 = ||u||^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2 dx.$$

We start with the following claim:

Claim. *For any $\Lambda > 0$, there exists $r > 0$ such that: if $(\lambda, u) \in \mathcal{C}^+$ and $\lambda \leq \Lambda$, we should have $||u|| \leq r$.*

Indeed, arguing by contradiction, if it is not true, there are $(u_n) \subset H_0^1(\Omega)$ and $(\lambda_n) \subset [0, \Lambda]$ such that,

$$||u_n|| \to \infty \text{ and } u_n = F(\lambda_n, u_n).$$

Considering $w_n = u_n/||u_n||$, it follows that

$$\int_\Omega \nabla w_n \cdot \nabla v dx + \int_\Omega \phi_{u_n} w_n v dx = \lambda_n \int_\Omega w_n v dx, \quad \forall v \in H_0^1(\Omega).$$

Once that (w_n) is bounded in $H_0^1(\Omega)$, without loss of generality, we can suppose that there is $w \in H_0^1(\Omega)$ verifying

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega), w_n \to w \text{ in } L^2(\Omega) \text{ and } w_n(x) \to w(x) \text{ a.e. in } \Omega.$$

Taking $v = \dfrac{u_n}{||u_n||^{p+1}}$ as a test function, and recalling that $t^p \phi_{u_n} = \phi_{tu_n}$ for all $t > 0$, we obtain

$$\frac{1}{||u_n||^p} + \int_\Omega \phi_{w_n} w_n^2 dx = \frac{\lambda_n}{||u_n||^p} \int_\Omega w_n^2 dx, \ \forall n.$$

Passing to the limit in the above equality, we derive

$$\lim_n \int_\Omega \phi_{w_n} w_n^2 dx = 0.$$

From Fatou Lemma

$$\int_\Omega \phi_w w^2 dx \leq \lim_n \int_\Omega \phi_{w_n} w_n^2 dx = 0,$$

and so,

$$\int_{\Omega\times\Omega} K(x,y)|w(y)|^p |w(x)|^2 \ dxdy = 0.$$

Since $K \in \mathcal{K}$, we should have $w \equiv 0$. Thereby, (w_n) converges to 0 in $L^2(\Omega)$. Taking $v = w_n$ as test function, we see that

$$\int_\Omega |\nabla w_n|^2 dx + \int_\Omega \phi_{u_n} w_n^2 dx = \lambda_n \int_\Omega w_n^2 dx.$$

Since (λ_n) is bounded from above by Λ and $\int_\Omega \phi_{u_n} w_n^2 dx \geq 0$, we have

$$\int_\Omega |\nabla w_n|^2 dx \leq \Lambda \int_\Omega w_n^2 dx.$$

Taking the limit, we conclude that $||w_n|| \to 0$, which is an absurd, because $||w_n|| = 1$ for all n, proving the claim.

Since (u_n) is bounded in $H_0^1(\Omega)$, iteration arguments imply that (u_n) is bounded in $L^\infty(\Omega)$, and the proof is done.

□

Next, we will show the non-existence of solution for $\lambda \le \lambda_1$, proving that $\mathcal{C}^+$ does not intersect $[0, \lambda_1] \times H_0^1(\Omega)$. In fact, suppose that

$$(\lambda, u) \in \mathcal{C}^+ \cap ([0, \lambda_1] \times H_0^1(\Omega)).$$

Using $v = \varphi_1$ as the test function in (3), we get

$$\lambda_1 \int_\Omega u\varphi_1 dx < \lambda_1 \int_\Omega u\varphi_1 dx + \int_\Omega \phi_u u\varphi_1 dx = \lambda \int_\Omega u\varphi_1 dx.$$

Since $\int_\Omega u\varphi_1 dx > 0$, the above inequality leads to $\lambda_1 < \lambda$.

Corollary 8. *Consider problem (2). There exists at least a positive solution of (2) if and only if $\lambda > \lambda_1$.*

Proof: It is clear that

$$K(x, y) = \chi_{\Omega \cap B_r(x)}(y) b(y) = \begin{cases} b(y) & y \in \Omega \cap B_r(x), \\ 0 & y \notin \Omega \cap B_r(x), \end{cases}$$

belongs to class $\mathcal{K}$. □

3. Proof of Theorem 2

We begin this section observing that if kernel K does not belong to $\mathcal{K}$, then there exists a measurable function $w : \Omega \to \mathbb{R}$ such that

$$\int_{\Omega \times \Omega} K(x, y)|w(y)|^p w(x)^2 dxdy = 0 \quad \text{but} \quad w \neq 0.$$

Thus, there exists $a > 0$ such that $A = \{x \in \Omega : |w(x)| \ge a\}$ has positive measure.

Observe that

$$a^{p+2} \int_{A \times A} K(x, y) dxdy \le \int_{\Omega \times \Omega} K(x, y)|w(y)|^p w(x)^2 dxdy = 0,$$

i.e, $K = 0$ a.e. in $A \times A$.

Lemma 9. *If $K \in \mathcal{K}'$ and*

$$\int_\Omega \phi_w w^2 dx = 0,$$

then $w = 0$, a.e. in $\Omega \setminus U$.

Proof: It is easy to verify that $\phi_w(x) w(x)^2 = 0$, for all $x \in \Omega$. The, fixing $\varepsilon > 0$ and $A_\varepsilon = \{x \in \Omega \setminus U : |w(x)| \ge \varepsilon\}$, it follows that $\phi_w(x) = 0$ for all $x \in A_\varepsilon$ and

$$\begin{aligned} 0 = \phi_w(x) = \int_\Omega K(x, y)|w(y)|^p dy &\ge \int_{B_r(x) \cap A_\varepsilon} K(x, y)|w(y)|^p dy \\ &\ge \varepsilon^p \int_{B_r(x) \cap A_\varepsilon} K(x, y) dy. \end{aligned}$$

Once that $K \in \mathcal{K}'$, we can deduce that $|A_\varepsilon \cap B_r(x)| = 0$, for all $x \in A_\varepsilon$. Hence, $|A_\varepsilon| = 0$ for all $\varepsilon > 0$, from where it follows that $w = 0$ a.e. in $\Omega \setminus U$.

□

In what follows, for $D \subset \Omega$, $\lambda_1(D)$ denotes the first eigenvalue for the problem

$$\begin{cases} -\Delta w = \mu w & \text{in } D, \\ w = 0 & \text{on } \partial D, \end{cases}$$

which has a positive associated eigenfunction.

Now, we are ready to prove Theorem 2:

Proof of Theorem 2: It is enough to obtain an a priori estimate for $(\lambda, u) \in \mathcal{C}^+$ such that $\lambda \in [\lambda_1, \lambda^*]$ with $\lambda^* < \Lambda = \min\{\lambda_1(\Omega_1), \lambda_1(\Omega_2), ..., \lambda_1(\Omega_m)\}$.

Hereafter, we proceed as in the proof of Lemma 7. Let $(\lambda_n, u_n) \in \mathcal{C}^+$ such that

$$\lambda_n \in [\lambda_1, \lambda^*],\ u_n \neq 0,\ ||u_n|| \to \infty \text{ and } F(\lambda_n, u_n) = 0.$$

Then, $w_n = u_n/||u_n||$ satisfies

$$\int_\Omega \nabla w_n \cdot \nabla v dx + \int_\Omega \phi_{u_n} w_n v dx = \lambda_n \int_\Omega w_n v dx, \quad \forall v \in H_0^1(\Omega)$$

and under a subsequence, there exists $w \in H_0^1(\Omega)$ verifying

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega), w_n \to w \text{ in } L^2(\Omega), w_n(x) \to w(x) \text{ a.e. in } \Omega$$
$$\text{and } \int_\Omega \phi_w |w|^2 dx = 0.$$

We claim that $w \neq 0$. In fact, if (w_n) converges to 0 in $L^2(\Omega)$, we will get a contradiction as in the proof of Lemma 7.

Since $w \neq 0$, from Lemma 9, there exists some $j \in \{1, 2, ..., m\}$ such that $w \neq 0$, a. e. in Ω_j. Of course $w = w|_{\Omega_j} \in H_0^1(\Omega_j)$.

For any $v \in H_0^1(\Omega_j)$ with $v \geq 0$, we have

$$\int_\Omega \nabla w_n \cdot \nabla v dx \leq \int_\Omega \nabla w_n \cdot \nabla v dx + \int_\Omega \phi_{u_n} w_n v dx = \lambda_n \int_\Omega w_n v dx \leq \lambda^* \int_\Omega w_n v dx.$$

Passing to the limit in this last inequality, we derive

$$\int_{\Omega_j} \nabla w \cdot \nabla v dx \leq \lambda^* \int_{\Omega_j} w v dx, \text{ for any } v \in H_0^1(\Omega_j), v \geq 0.$$

Taking $v = \varphi_0$, where φ_0 is a positive eigenfunction associated to eigenvalue $\lambda_1(\Omega_j)$, we find

$$\lambda_1(\Omega_j) \int_{\Omega_j} w \varphi_0 dx = \int_{\Omega_j} \nabla w \cdot \nabla \varphi_0 dx \leq \lambda^* \int_{\Omega_j} w \varphi_0 dx,$$

which is a contradiction, because $\int_{\Omega_j} w\varphi_0 dx > 0$. □

In the following result, we prove the non-existence of positive solution for λ large when K vanishes in a sub-domain.

Lemma 10. *Assume that $D \subset \Omega$ is a sub-domain and suppose that $K(x, y) = 0$ in $D \subset \Omega$ for any $y \in \Omega$. Then, (P) does not possess positive solution for $\lambda \geq \lambda_1(D)$.*

Proof. Take φ_0, a positive eigenfunction associated to eigenvalue $\lambda_1(D)$, and v the prolongation by zero of φ_0 as test function in (P). Then,

$$\int_D \nabla u \cdot \nabla \varphi_0 dx + \int_\Omega \phi_u u \varphi_0 dx = \lambda \int_D u \varphi_0 dx.$$

We can check that

$$\int_\Omega \phi_u u \varphi_0 dx = \int_D \left(\int_\Omega K(x, y) |u(y)|^p dy \right) u \varphi_0 dx$$
$$= \int_\Omega \left(\int_D K(x, y) \varphi_0 dx \right) |u(y)|^p u dy = 0,$$

because $K(x, y) = 0$ in D for any $y \in \Omega$.

Using the Green identity,

$$\int_D \nabla u \cdot \nabla \varphi_0 dx = -\int_D u\Delta\varphi_0 dx + \int_{\partial D} u\frac{\partial \varphi_0}{\partial \nu} d\sigma < \lambda_1(D)\int_D u\varphi_0 dx,$$

because $\dfrac{\partial \varphi_0}{\partial \nu} < 0$ on ∂D, where ν is the outward unit normal vector.

Combining everything we have

$$\lambda \int_D u\varphi_0 dx = \int_D \nabla u \cdot \nabla \varphi_0 dx < \lambda_1(D)\int_D u\varphi_0 dx$$

which implies $\lambda < \lambda_1(D)$. □

As a direct consequence of Lemma 10 and Theorem 2 we have the Corollary 3.

Remark 11. *In the hypothesis of the Corollary* 3 *we see that* $(\lambda_1(U), \infty)$ *is a bifurcation at infinity for equation* $u = F(\lambda, u)$.

We finish with two examples in $N = 1$, $\Omega = (0, \pi)$ and so $\lambda_1((0, \pi)) = 1$. In the first one, K vanishes in the diagonal and there exists positive solution for all $\lambda > 1$. Consider

$$K(x, y) = \begin{cases} 1 & \text{if } (x, y) \in (0, \pi/2) \times (\pi/2, \pi) \cup (\pi/2, \pi) \times (0, \pi/2), \\ 0 & \text{in other cases.} \end{cases}$$

Observe that

$$A = \int_{\pi/2}^{\pi} \sin^p(y)dy = \int_0^{\pi/2} \sin^p(y)dy,$$

and then a solution for (P) is

$$u(x) = \left(\frac{\lambda - 1}{A}\right)^{1/p} \sin(x).$$

In the second example, K is positive only in a horizontal band and there exists positive solution for all $\lambda > 1$. Consider $a_0 \in (\pi/2, \pi)$ and

$$K(x, y) = \begin{cases} 1 & \text{if } y > a_0, \\ 0 & \text{in other cases.} \end{cases}$$

Denote

$$A_0 = \int_{a_0}^{\pi} \sin^p(y)dy,$$

then a solution for (P) is

$$u(x) = \left(\frac{\lambda - 1}{A_0}\right)^{1/p} \sin(x).$$

Acknowledgements. C.O. Alves was partially supported by CNPq - Grant 304036/2013-7. M.D. and A.S. are supported by by MINECO (Spain) Project MTM2012-31304. M. Souto is supported by CAPES, INCT-MAT, casadinho/PROCAD-CNPq/Brazil 552.464/2011-2 and CNPq/Brazil 304.652/2011-3. This paper was written while C. O. Alves and M. A. S. Souto were visiting Dpto. EDAN of the University of Seville. They want to thank the warm hospitality of all members of the Department.

References

[1] W. Allegretto and P. Nistri, On a class of nonlocal problems with applications to mathematical biology. Differential equations with applications to biology (Halifax, NS, 1997), 114, Fields Inst. Commun., 21, Amer. Math. Soc., Providence, RI, 1999.

[2] S. Chen and J. Shi, Stability and Hopf bifurcation in a diffusive logistic population model with nonlocal delay effect, J. Differential Equations, **253** (2012), 3440-3470.

[3] M. Chipot, Remarks on some class of nonlocal elliptic problems, Recent Advances on Elliptic and Parabolic Issues, World Scientific (2006), 79-102.

[4] F. J. S. A. Corrêa, M. Delgado and A. Suárez, Some nonlinear heterogeneous problems with nonlocal reaction term, Advances in Differential Equations, **16**, (2011), 623-641.

[5] J. Coville, Convergence to equilibrium for positive solutions of some mutation-selection model, arXiv:1308.647 (2013).

[6] H. Leman, S. Méléard and S. Mirrahimi, Influence of a spatial structure on the long time behavior of a competitive Lotka-Volterra type system, arXiv:1401.1182 (2014).

[7] B. Perthame, Transport equations in biology, Transport Equations in Biology, Frontiers in Mathematics, vol. 12, Birkhauser Basel, 2007, pp. 126.

[8] P. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Anal., **7** (1971), 487-513.

[9] L. Sun, J. Shi and Y. Wang, Existence and uniqueness of steady state solutions of a nonlocal diffusive logistic equation, Z. Angew. Math. Phys., **64** (2013), 1267-1278.

Claudianor O. Alves

Manuel Delgado

Marco A. S. Souto

Antonio Suárez

(C. O. Alves) Universidade Federal de Campina Grande, Unidade Acadêmica de Matemática, 58429-900, Campina Grande-PB, Brazil. coalves@dme.ufcg.edu.br

(M. Delgado) Departamento de Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Calle Tarfia s/n, 41012-Sevilla, Spain. madelgado@us.es

(M. A. S. Souto) Universidade Federal de Campina Grande, Acadêmica de Matemática, 58429-900, Campina Grande-PB, Brazil. marco@dme.ufcg.edu.br

(A. Suárez) Departamento de Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Calle Tarfia s/n, 41012-Sevilla, Spain. suarez@us.es