BORDERLINE WEIGHTED ESTIMATES FOR COMMUTATORS OF SINGULAR INTEGRALS

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ABSTRACT. In this paper we establish the following estimate

$$w\left(\{x \in \mathbb{R}^n : |[b,T]f(x)| > \lambda\}\right) \le \frac{c_T}{\varepsilon^2} \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{\lambda}\right) M_{L(\log L)^{1+\varepsilon}}w(x)dx$$

where $w \ge 0$, $0 < \varepsilon < 1$ and $\Phi(t) = t(1 + \log^+(t))$. This inequality relies upon the following sharp L^p estimate

$$\|[b,T]f\|_{L^p(w)} \le c_T (p')^2 p^2 \left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}} \|b\|_{BMO} \|f\|_{L^p(M_{L(\log L)^{2p-1+\delta}}w)}$$

where $1 < p < \infty$, $w \ge 0$ and $0 < \delta < 1$. As a consequence we recover the following estimate essentially contained in [18]:

$$w\left(\{x \in \mathbb{R}^n : |[b,T]f(x)| > \lambda\}\right) \le c_T [w]_{A_\infty} \left(1 + \log^+[w]_{A_\infty}\right)^2 \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{\lambda}\right) Mw(x)dx$$

We also obtain the analogue estimates for symbol-multilinear commutators for a wider class of symbols.

1. INTRODUCTION

Motivated by a classical inequality due to C. Fefferman and E. Stein for the Hardy-Littlewood maximal function, namely

$$\|Mf\|_{L^{1,\infty}(w)} \le c \int_{\mathbb{R}^n} |f|Mwdx$$

where M denotes the Hardy-Littlewood maximal operator and w is a weight, i.e. a locally non negative integrable function, B. Muckenhoupt and R. Wheeden conjectured that

$$\|Hf\|_{L^{1,\infty}(w)} \le c \int_{\mathbb{R}^n} |f|Mwdx$$

where H is the Hilbert transform. This conjecture was recently disproved by M. C. Reguera and C. Thiele [26] (see also [4] for the result in higher dimensions). The failure of this conjecture was suggested by the first author in [20] where the following positive result was obtained

$$(1) \qquad \|Tf\|_{L^{1,\infty}(w)} \le c_{\varepsilon,T} \int_{\mathbb{R}^n} |f|M_{L(\log L)^\varepsilon}(w)dx \qquad w \ge 0,$$

where T is a Calderón-Zygmund operator (CZO). In the recent work [10], T. Hytönen and the first author improved the control on the $c_{\varepsilon,T}$ constant and were able to consider the

maximal singular operator T^* obtaining the following estimate (see [6] for an improvement of this result)

$$(2) \qquad \|T^*f\|_{L^{1,\infty}(w)} \leq \frac{c_T}{\varepsilon} \int_{\mathbb{R}^n} |f(x)| M_{L(\log L)^\varepsilon}(w)(x)dx \qquad w \geq 0$$

which implies

$$(3) \qquad \|T^*f\|_{L^{1,\infty}(w)} \leq c_T \log\left(e + [w]_{A_\infty}\right) \int_{\mathbb{R}^n} |f| M w dx,$$

when $w \in A_\infty$. This result improves the main theorem from [15], namely

$$(4) \qquad \|T^*\|_{L^1(w) \to L^{1,\infty}(w)} \leq c_T [w]_{A_1} \log\left(e + [w]_{A_\infty}\right).$$

It seemed that the logarithmic factor was superfluous and that it could be removed. However, this is not the case by a very impressive negative result obtained by F. Nazarov, A. Reznikov, V. Vasyunin and A. Volberg in [17]. In this work the authors disprove the so called A_1 conjecture, namely they prove

$$\sup_{w \in A_1} \frac{\|H\|_{L^1(w) \to L^{1,\infty}(w)}}{[w]_{A_1}} = \infty$$

where H is the classical Hilbert transform. Furthermore, the same conclusion holds if the linear constant $[w]_{A_1}$ is replaced by $[w]_{A_1} \log(e + [w]_{A_1})^\alpha$ for a positive $\alpha < \frac{1}{5}$. This is indicating that most probably (4) is fully optimal.

The main purpose of this paper is to prove estimates similar to (2) for commutators of CZOs T with BMO functions b, usually called the symbol. These operators are defined formally by the expression

$$[b, T]f = bT(f) - T(bf),$$

These commutators were introduced by Coifman, Rochberg and Weiss in [3] in connection with the classical factorization theorem for Hardy spaces. However, many other applications were found much later, specially in the theory of elliptic operators [13], [2]. Another interesting aspect of the theory is its connection with the following nonlinear commutators introduced by R. Rochberg and G. Weiss in [27]:

$$f \to Nf = T(f \log |f|) - Tf \log |Tf|.$$

This operator is interesting due to its relationship with the Jacobian mapping and with nonlinear P.D.E. as shown in [12] and [8].

The main result from [3] states that $[b, T]$ is a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, when b is a BMO function and T is a singular integral operator. In fact, the BMO condition of b is also a necessary condition for the L^p-boundedness of the commutator when T is the Hilbert transform.

From the theoretical point of view, these commutators are of interest because they are more singular than CZOs. For instance, the first author proved in [21] that these commutators are not of weak type $(1, 1)$ obtaining a suitable replacement, namely the following $L \log L$ endpoint estimate:

$$w\left(\{x \in \mathbb{R}^n \,:\, |[b, T]\, f(x)| > \lambda\}\right) \leq c \int_{\mathbb{R}^n} \Phi\left(\frac{|f|}{\lambda} \|b\|_{BMO}\right) w dx$$

where $\Phi(t) = t\left(1 + \log^+(t)\right)$, $w \in A_1$ and the constant c depends upon the A_1 constant of the weight. The approach to prove this result was based on an appropriate non-standard good-λ inequality using the Fefferman-Stein "sharp" maximal function. However, this method

does not produce good results for further developments and in particular when considering non-A_∞ weights.

Later on, the first author together with G. Pradolini ([23]) established the following estimate for arbitrary weights $w \geq 0$,

$$w\left(\{x \in \mathbb{R}^n \,:\, |[b,T]\,f(x)| > \lambda\}\right) \leq c_{T,\varepsilon} \int_{\mathbb{R}^n} \Phi\left(\frac{|f|}{\lambda}\|b\|_{BMO}\right) M_{L(\log L)^{1+\varepsilon}} w \, dx$$

for any $\varepsilon > 0$. The aim of this paper is to give a quantitative version of this estimate with a good control on the constant $C_{T,\varepsilon}$ in terms of ε. In the simplest situation our estimate can be stated as follows

$$(5) \qquad w\left(\{x \in \mathbb{R}^n \,:\, |[b,T]\,f(x)| > \lambda\}\right) \leq \frac{c_T}{\varepsilon^2} \int_{\mathbb{R}^n} \Phi\left(\frac{|f|}{\lambda}\|b\|_{BMO}\right) M_{L(\log L)^{1+\varepsilon}} w \, dx.$$

See Theorem 2 for the general situation. In fact, we will obtain a wider class of results since we will be considering symbol-multilinear commutators with symbols in $Osc_{\exp L^s}$ classes which are subspaces of the BMO space (cf. Section 3.2). We will see that this choice of symbols will be reflected in the maximal operator on the right hand side of the inequality. As a consequence of these type of estimates we can recover, among other results, the following endpoint A_1 result from C. Ortiz [18] (see Corollary 1):

$$w\left(\{x \in \mathbb{R}^n \,:\, |[b,T]\,f(x)| > \lambda\}\right) \leq c_T\, \Phi\left([w]_{A_1}\right)^2 \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{\lambda}\right) w(x)\, dx.$$

Estimate (5) should be compared with the case of CZOs (2). It is not clear whether is possible or not to establish (5) using techniques based on sparse operators as in [6]. Another open question is the analogue of the Muckenhoupt-Wheeden conjecture for the commutator, namely whether

$$w\left(\{x \in \mathbb{R}^n \,:\, |[b,T]\,f(x)| > \lambda\}\right) \leq c_w \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{\lambda}\right) M^2(w)(x)\, dx.$$

holds for every weight w or not. Techniques used in [26] and [4] rely upon an endpoint extrapolation result, firstly established in [5], or upon variations of it as in [6]. It is not clear how to perform a similar extrapolation from the $L \log L$ estimate that commutators satisfy.

This paper is organized as follows: Section 2 contains the statements of our main results and the proof of Corollary 1. Section 3 contains precise definitions and facts which will be used throughout the paper. In section 4 we give the proof of the strong type theorem, namely Theorem 1, and all the needed technical results. The proof of the endpoint estimate (Theorem 2) is presented in Section 5.

Acknowledgments

2. Main results

To state our main results we need to introduce some notation. Let $b_i \in Osc_{\exp L^{s_i}}$ $s_i \geq 1$, $i = 1, \cdots, k$ (cf. section 3.2 after Lemma 1) and T a CZO with associated kernel K. We

define the symbol-multilinear commutator with respect to the symbol $\vec{b} = (b_1, \ldots, b_k)$ as follows

$$T_{\vec{b}}f(x) = \int_{\mathbb{R}^n} \prod_{i=1}^{k} (b_i(x) - b_i(y)) \, K(x,y) f(y) dy$$

We also denote

$$\frac{1}{s} = \sum_{i=1}^{k} \frac{1}{s_i}$$

and

$$\|\vec{b}\| = \prod_{i=1}^{k} \|b_i\|_{Osc_{\exp L^{s_i}}}.$$

Our main results are the following.

Theorem 1. *Let be $T_{\vec{b}}$ defined as above and let w be a weight. Then*

$$\|T_{\vec{b}}f\|_{L^p(w)} \leq c_T \, (p')^{k+1} \, p^{1+\frac{1}{s}} \left(\frac{p-1}{\delta} \right)^{\frac{1}{p'}} \|\vec{b}\| \, \|f\|_{L^p(M_{L(\log L)^{(1+\frac{1}{s})p-1+\delta}}w)}$$

for every $\delta \in (0,1)$ and $p \in (1, \infty)$.

This result can be applied to derive the following endpoint estimate.

Theorem 2. *Let be $T_{\vec{b}}$ and w as above. Then*

$$w\left(\left\{x \in \mathbb{R}^n : |T_{\vec{b}}f| > \lambda\right\}\right) \leq \frac{c_T}{\varepsilon^{k+1}} \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}}\left(\|\vec{b}\| \frac{|f(x)|}{\lambda}\right) M_{L(\log L)^{\frac{1}{s}+\varepsilon}}w(x)dx$$

for every $\varepsilon \in (0,1)$ where $\Phi_\rho(t) = t(1 + \log^+(t))^\rho$, $\rho > 0$.

There is an interesting application of Theorem 2 from which we can recover one of the main results of [18].

Corollary 1. *Let T be a CZO and let $b \in BMO$.*

(1) *If $w \in A_\infty$ then*

$$w\left(\left\{x \in \mathbb{R}^n : |[b,T]f(x)| > \lambda\right\}\right) \leq c[w]_{A_\infty} \left(1 + \log^+[w]_{A_\infty}\right)^2 \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO} \frac{|f(x)|}{\lambda}\right) Mw(x)dx,$$

(2) *If $w \in A_1$ then*

$$w\left(\left\{x \in \mathbb{R}^n : |[b,T]f(x)| > \lambda\right\}\right)$$

$$\leq c[w]_{A_1}[w]_{A_\infty} \left(1 + \log^+[w]_{A_\infty}\right)^2 \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO} \frac{|f(x)|}{\lambda}\right) w(x)dx$$

$$\leq c\Phi\left([w]_{A_1}\right)^2 \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO} \frac{|f(x)|}{\lambda}\right) w(x)dx$$

where $\Phi(t) = t(1 + \log^+ t)$.

Proof. For the proof of the corollary we follow the arguments in [10]. First observe that for every $\alpha > 0$ we have that $\log t \leq \frac{t^\alpha}{\alpha}$. Then we can write

$$\log(t)^{1+\varepsilon} \leq \frac{t^{\alpha(1+\varepsilon)}}{\alpha^{1+\varepsilon}},$$

and hence

$$M_{L(\log L)^{1+\varepsilon}}w \leq \frac{1}{\alpha^{1+\varepsilon}} M_{L^{1+\alpha(1+\varepsilon)}}.$$

Let us take $\alpha = \frac{1}{c_n[w]_{A_\infty}(1+\varepsilon)}$. Then, using the reverse Hölder inequality (Theorem 3),

$$\frac{1}{\varepsilon^2} M_{L(\log L)^{1+\varepsilon}} \leq \frac{1}{\varepsilon^2} \left[c_n[w]_{A_\infty}(1+\varepsilon)\right]^{1+\varepsilon} M_{L^{1+\alpha(1+\varepsilon)}} w$$

$$\leq \frac{2}{\varepsilon^2} \left[c_n[w]_{A_\infty}(1+\varepsilon)\right]^{1+\varepsilon} Mw.$$

If we choose $\varepsilon = \frac{1}{1+\log^+([w]_{A_\infty})}$ we obtain the desired results just recalling that $[w]_{A_\infty} \leq [w]_{A_1}$.

$\square$

3. Preliminaries and notation

In this section we gather some definitions and properties which will be used throughout the paper.

3.1. A_p weights.

We recall that a weight w belongs to the class A_p, $1 < p < \infty$, if

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w\right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}}\right)^{p-1} < \infty.$$

A weight w belongs to the class A_1 if there is a finite constant C such that

$$\frac{1}{|Q|} \int_Q w(y)\, dy \leq C \inf_Q w,$$

and the infimum of these constants C is called the A_1 constant of w denoted by $[w]_{A_1}$. Since the A_p classes are increasing with respect to p, the A_∞ class of weights is defined in a natural way by $A_\infty = \cup_{p>1} A_p$. These classes of weights were introduced by B. Muckenhoupt in [16] where it was shown that for $1 < p < \infty$

$$w \in A_p \iff M : L^p(w) \longrightarrow L^p(w)$$

and also

$$w \in A_1 \iff M : L^1(w) \longrightarrow L^{1,\infty}(w).$$

From the definition of A_∞ it is not clear how to define an appropriate constant. However, Fujii proved essentially in [7] another characterization:

$$w \in A_\infty \iff [w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w) dx < \infty$$

which was also rediscovered later on by Wilson in [29]. Recently, this quantity was defined as the A_∞ constant in [9] since it was proved to be the most suitable one. In particular, the following optimal reverse Hölder's inequality obtained in [9] (see also [11] for a better proof and [28] for some other related results) was used in the proof of Corollary 1.

Theorem 3. *Let $w \in A_\infty$, then there exists a dimensional constant τ_n such that*

$$\left(\frac{1}{|Q|} \int_Q w^{r_w}\right)^{\frac{1}{r_w}} \leq \frac{2}{|Q|} \int_Q w.$$

where

$$r_w = 1 + \frac{1}{\tau_n[w]_{A_\infty}}$$

3.2. **Orlicz maximal functions.** We recall that Φ is a Young function if it is a continuous, nonnegative, strictly increasing and convex function defined on $[0, \infty)$ such that $\Phi(0) = 0$ and $\lim_{t \to \infty} \Phi(t) = \infty$. We define the localized Luxembourg norm of a function f with respect to a Young function Φ as follows

$$\|f\|_{\Phi,Q} = \|f\|_{\Phi(L),Q} = \inf\left\{ \lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}$$

which is equivalent to the following

$$\|f\|'_{\Phi,Q} = \inf_{\mu > 0}\left\{ \mu + \frac{\mu}{|Q_j|}\int_{Q_j} \Phi\left(\frac{|f(x)|}{\mu}\right) dx \right\}.$$

This result is due to Krasnosel'skiĭ, M. A. and Rutickiĭ, Ja. B. [14, p. 92] (see also [25, p. 69]). In fact,

$$\|f\|_{\Phi,Q} \leq \|f\|'_{\Phi,Q} \leq 2\|f\|_{\Phi,Q}$$

which will be quite useful for our purposes. Observe that the case $\Phi(t) = t$ corresponds to the usual average and we can see these localized norms as a "different" way of taking averages. We can also define the maximal function associated to Φ as

$$M_\Phi f(x) = \sup_{x \in Q} \|f\|_{\Phi,Q}.$$

Some useful examples that will be quite useful in the sequel are $L \log L$ functions

$$\Phi_\rho(t) = t(1 + \log^+(t))^\rho \quad \text{with} \quad t \geq 0$$

where $\log^+(t) = \chi_{(1,\infty)}(t)\log(t)$ and $\rho > 0$. For such Φ we shall denote

$$\|f\|_{\Phi,Q} = \|f\|_{L(\log L)^\rho,Q}.$$

Another useful property that makes interesting these "non-standard averages" is the following generalized Hölder inequality.

Lemma 1. *Let* $\Phi_0, \Phi_1, \Phi_2, \ldots, \Phi_k$ *be Young functions. If*

(6) $$\Phi_1^{-1}(t)\Phi_2^{-1}(t)\ldots\Phi_k^{-1}(t) \leq \kappa\Phi_0^{-1}(t).$$

then for all functions $f_1, \ldots, f_m$ *and all cubes* Q *we have that*

$$\|f_1 f_2 \ldots f_k\|_{\Phi_0,Q} \leq k\kappa\|f_1\|_{\Phi_1,Q}\|f_2\|_{\Phi_2,Q}\ldots\|f_k\|_{\Phi_k,Q}.$$

A particular case of interest, an especially in this paper, are the spaces defined by

$$\|f\|_{Osc_{expL^s}} = \sup_Q \|f - f_Q\|_{\Psi_s,Q}$$

where

$$\Psi_s(t) = e^{t^s} - 1 \qquad t \geq 0,$$

with $s > 0$, is a Young function. Then the space $Osc_{\exp L^s}$ is defined as

$$Osc_{\exp L^s} = \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{Osc_{expL^s}} < \infty \right\}.$$

We observe that John-Nirenberg's theorem yields $BMO = Osc_{\exp L}$. It's also clear that for every $s > 1$

$$Osc_{\exp L^s} \subsetneq BMO.$$

Now we state a result borrowed from [24] that will be used in the proof of Theorem 2.

Lemma 2. *Let $\Phi_0, \ldots, \Phi_k$ be continuous, nonnegative, strictly increasing functions on $[0, \infty)$ with $\Phi(0) = 0$ and $\lim_{t \to \infty} \Phi(t) = \infty$ such that*

$$\Phi_1^{-1}(t)\Phi_2^{-1}(t)\ldots\Phi_k^{-1}(t) \leq \Phi_0^{-1}(t) \qquad t \geq 0,$$

then for all $0 \leq x_1, x_2, \ldots, x_k < \infty$

$$\Phi_0(x_1 x_2 \ldots x_k) \leq \Phi_1(x_1) + \Phi_2(x_2) + \cdots + \Phi_k(x_k).$$

To close this section we provide a proof of Lemma 1 and also a corollary of it that will be quite useful in the proof of Theorem 2.

Proof. Fix $(x_1, \ldots, x_k)$ and consider $t_0 = \Phi_1(x_1) + \Phi_2(x_2) + \cdots + \Phi_k(x_k)$. Combining (6) and the fact that each Φ_i is increasing it readily follows that

$$\Phi_0\left(\frac{\Phi_1^{-1}(t_0)\Phi_2^{-1}(t_0)\ldots\Phi_k^{-1}(t_0)}{\kappa}\right) \leq t_0$$

and also that

$$\Phi_i^{-1}(t_0) \geq \Phi_i^{-1}(\Phi_i(x_i)) = x_i.$$

Then we have that

$$(7) \qquad \Phi_0\left(\frac{x_1 x_2 \ldots x_k}{\kappa}\right) \leq \Phi_1(x_1) + \Phi_2(x_2) + \cdots + \Phi_k(x_k)$$

We observe that this argument gives us a proof of Lemma 2. Coming back to our proof, let us consider now $t_i > \|f_i\|_{\Phi_i, Q}$. We have that using 7,

$$\frac{1}{m}\frac{1}{|Q|}\int_Q \Phi_0\left(\frac{|f_1 \ldots f_k|}{\kappa t_1 \ldots t_k}\right)$$
$$\leq \frac{1}{m}\left(\frac{1}{|Q|}\int_Q \Phi_1\left(\frac{|f_1|}{t_1}\right) + \cdots + \frac{1}{|Q|}\int_Q \Phi_k\left(\frac{|f_k|}{t_k}\right)\right)$$
$$< 1$$

Consequently

$$\|f_1 \ldots f_k\|_{\Phi_0, Q} \leq \kappa t_1 \ldots t_k$$

and it is enough to take the infimum on each t_i to finish the proof of the lemma. $\qquad \square$

As a particular case, the following corollary holds which will be used several times in this paper.

Corollary 2. *Let $s_1, \ldots, s_k \geq 1$ and denote $\sum_{i=1}^{k}\frac{1}{s_i}$. Then*

$$\frac{1}{|Q|}\int_Q |f_1 \ldots f_k g| \leq c_s \|f_1\|_{\exp L^{s_1}, Q} \cdots \|f_k\|_{\exp L_k^s, Q}\|g\|_{L(\log L)^{\frac{1}{s}}, Q}$$

Proof. We denote $\varphi_\eta(t) = e^{t^\eta} - 1$. Then $\varphi_\eta^{-1}(t) = \log(x+1)^{\frac{1}{\eta}}$ and we have that

$$\varphi_{s_1}^{-1}(t)\ldots\varphi_{s_k}^{-1}(t)\Phi_{\frac{1}{s}}^{-1}(t) \simeq \varphi_{s_1}^{-1}(t)\ldots\varphi_{s_k}^{-1}(t)\frac{x}{\log(x+1)^{\frac{1}{s}}} \leq x$$

and Lemma 1 gives the desired result. $\qquad \square$

3.3. **Symbol-multilinear commutators.** We recall that an operator T initially defined on the Schwartz spaces and taking values into the space of tempered distributions $T : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ is a CZO if,

(1) T is bounded on $L^2(\mathbb{R}^n)$.
(2) For each smooth and compactly supported function f, Tf admits the following representation

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \qquad x \notin \operatorname{supp} f$$

where K is a standard kernel. Recall that a kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \longrightarrow \mathbb{R}$, where Δ is the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$, is a locally integrable function such that for some constants $C_1, C_2, \gamma > 0$ the following conditions hold:

(a) Size condition

$$|K(x,y)| \le C_1 \frac{1}{|x-y|^n} \qquad \text{if } x \ne y.$$

(b) Regularity condition

$$|K(x,y) - K(x',y)| + |K(y,x') - K(y,x)| \le C_2 \frac{|x-x'|^\gamma}{|x-y|^{n+\gamma}}$$

provided that $|x - x'| \le \frac{1}{2}|x-y|$.

The symbol-multilinear commutator $T_{\vec{b}}$ with vector symbol $\vec{b} = (b_1, \cdots, b_k)$, $b_i \in Osc_{\exp L^{s_i}}$, $i = 1, \cdots, k$, and CZO T with kernel K is defined for smooth functions f as follows

$$T_{\vec{b}}f(x) = \int_{\mathbb{R}^n} \prod_{i=1}^k (b_i(x) - b_i(y))\, K(x,y)f(y)dy \qquad x \notin \operatorname{supp} f.$$

Let $b = \{b_1, b_2, \ldots, b_k\}$ be a set of symbols with $b_i \in Osc_{\exp L^{s_i}}$, $i = 1, \cdots, k$. Also, let $b = \sigma \cup \sigma'$ where σ and σ' are pairwise disjoint sets be a splitting of b. If we identify i and b_i we can introduce the following notation

$$(b(x) - \lambda)_\sigma = \prod_{i \in \sigma} (b_i(x) - \lambda_i)$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and also to write $\sum_{i \in \sigma} \frac{1}{s_i}$.

By $C_j(b)$ we refer the family of all the subsets σ of b such that $\#\sigma = j$. We shall also omit the set of symbols and write just C_j^k. Finally if σ is a subset of b we write

$$T_{\vec{\sigma}}f(x) = \int_{\mathbb{R}^n} \prod_{i \in \sigma} (b_i(x) - b_i(y))\, K(x,y)f(y)dy = \int_{\mathbb{R}^n} (b(x) - b(y))_\sigma\, K(x,y)f(y)dy \qquad x \notin \operatorname{supp} f.$$

We end this section with some further notation. We write

$$\|\vec{b}\| = \prod_{b_i \in b} \|b_i\|_{Osc_{\exp L^{s_i}}}$$

and similarly

$$\|\vec{\sigma}\| = \prod_{b_i \in \sigma} \|b_i\|_{Osc_{\exp L^{s_i}}}.$$

We will denote by $\#\sigma$ the cardinal of the set of symbols σ.

3.4. **Some estimates involving the sharp function.** In this paper we will use two classical operators and some of their variations. The first one is the Hardy-Littlewood maximal operator,

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where each Q is a cube with sides parallel to the axis. Also, M^d will denote its dyadic version, where the supremum is taken over dyadic cubes. We will also use the following variants, $M_\varepsilon(f) = M(|f|^\varepsilon)^{\frac{1}{\varepsilon}}$, and similarly for M_ε^d where $\varepsilon \in (0, \infty)$. The second operator is the Fefferman-Stein sharp maximal function, namely

$$M^\sharp f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy,$$

and its dyadic counterpart $M^{\sharp,d}$. Similarly as above we define the following useful variation

$$M_\delta^\sharp(f) = M^\sharp(|f|^\delta)^{\frac{1}{\delta}}$$

with $\delta \in (0, \infty)$.

The first result that we state in this section is borrowed from [19].

Lemma 3. *Let $0 < p < \infty$, $0 < \delta < 1$ and let $w \in A_\infty$. Then*

$$\|f\|_{L^p(w)} \leq cp[w]_{A_\infty} \left\| M_\delta^{\sharp,d} f \right\|_{L^p(w)}$$

for any function f such that $|\{x : |f(x)| > t\}| < \infty$ for all $t > 0$.

Using the preceding lemma and following the proof of Lemma 3.1 in [18] we can derive the following improvement.

Lemma 4. *Let $0 < p < \infty$, $0 < \varepsilon \leq 1$ and $w \in A_\infty$. Suppose that*

$$|\{x : |f(x)| > t\}| < \infty$$

for all $t > 0$. Then there is a constant $c = c_{n,\varepsilon}$ such that

$$\left\| M_\varepsilon^d f \right\|_{L^p(w)} \leq cp[w]_{A_\infty} \left\| M_\varepsilon^{\sharp,d} f \right\|_{L^p(w)}$$

Proof. Applying previous lemma with $\delta = \varepsilon_0$ and $0 < \varepsilon_0 < \varepsilon < 1$

$$\left\| M_\varepsilon^d f \right\|_{L^p(w)} \leq cp[w]_{A_\infty} \left\| M_{\varepsilon_0}^{\sharp,d} \left(M_\varepsilon^d f \right) \right\|_{L^p(w)}.$$

Now it suffices to prove that

$$M_{\varepsilon_0}^{\sharp,d} \left(M_\varepsilon^d f \right)(x) \leq c M_\varepsilon^{\sharp,d} f(x).$$

But this was done in Lemma 3.1 of [18]. $\square$

The reason why it is important to deal with $M_\varepsilon^\sharp$ for small ε will be clear after the following pointwise estimate proved in [24].

Lemma 5. *Let $T_{\vec{b}}$ be the symbol-multilinear commutator defined above and let $0 < \delta < \varepsilon < 1$. Then there exists a constant $c > 0$, depending only on δ and ε such that*

$$M_\delta^\sharp \left(T_{\vec{b}} f \right)(x) \leq c_{\delta,\varepsilon} \left(\|\vec{b}\| M_{L(\log L)^{\frac{1}{s}}}(f) + \sum_{j=1}^k \sum_{\sigma \in C_j^k} \|\vec{\sigma}\| M_\varepsilon \left(T_{b_{\vec{\sigma'}}} f \right)(x) \right)$$

for any bounded function f with compact support.

The first result, corresponding to the case $k = 0$, namely

$$(8) \qquad\qquad M_\delta^\sharp(Tf)(x) \le c_\delta \, Mf(x) \qquad 0 < \delta < 1$$

can be found in [1] and the case $k = 1$ was established in [21].

4. Proof of Theorem 1

In this section we prove Theorem 1. The first two subsections will be devoted to the core of the proof for all the cases, namely $k = 1$ and $k > 1$. Both cases rely upon a two weight inequality that will be established in the third subsection. A careful control of some Young's functions inverses will be required to obtain that two weight inequality.

4.1. Case $k = 1$.

Proof. In this proof we follow techniques in [15] and [18]. Let us call $v = M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}}w$. If $\kappa = c_T \, (p')^2 \, p^{1+\frac{1}{s}} \left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}$, by duality, it suffices to show that

$$\left\| \frac{[b,T]^t f}{v} \right\|_{L^{p'}(v)} \le \kappa \left\| \frac{f}{w} \right\|_{L^{p'}(w)}$$

Where $[b,T]^t$ is the adjoint of $[b,T]$. Calculating the norm by duality allows us to find a non-negative function $h \in L^p(v)$ with $\|h\|_{L^p(v)} = 1$ such that

$$\left\| \frac{[b,T]^t f}{v} \right\|_{L^{p'}(v)} = \int_{\mathbb{R}^n} \frac{|[b,T]^t f|}{v} \, hv \, dx = \int_{\mathbb{R}^n} |[b,T]^t| f \, h \, dx = I.$$

Let us consider the operator

$$S(h) = \frac{M\left(hv^{\frac{1}{p}}\right)}{v^{\frac{1}{p}}}.$$

We build the Rubio de Francia algorithm R using the operator S.

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{\|S\|_{L^p(v)}^k}.$$

R satisfies the following properties:

(1) $0 \le h \le R(h)$
(2) $\|R(h)\|_{L^p(v)} \le 2\|h\|_{L^p(v)}$
(3) $R(h)v^{\frac{1}{p}} \in A_1$ and furthermore $\left[R(h)v^{\frac{1}{p}}\right]_{A_1} \le cp'$.

Using Lemma 4.2 in [10]

$$\left[v^{\frac{1}{2p}}\right]_{A_1}^2 \le c_n$$

Taking that into account

$$[Rh]_{A_3} = \left[R(h)v^{\frac{1}{p}}\left(v^{-\frac{1}{p(1-q)}}\right)^{1-q}\right]_{A_q} \le \left[R(h)v^{\frac{1}{p}}\right]_{A_1}\left[v^{\frac{1}{2p}}\right]_{A_1}^2 \le c_n p'$$

Applying Lemma 3, with $p = 1$, $w = Rh$ and $\gamma \in (0,1)$, together with $[Rh]_{A_\infty} \le [Rh]_{A_3} \le c_n p'$, we have

$$I \le \int_{\mathbb{R}^n} \left| [b,T]^t f \right| Rh dx$$

$$\le c_n [Rh]_{A_\infty} \int_{\mathbb{R}^n} M_\gamma^\sharp ([b,T]^t f) Rh(x) dx \le c_n p' \int_{\mathbb{R}^n} M_\gamma^\sharp ([b,T]^t f) Rh(x) dx$$

Now we observe that $[b,T]^t = -[b,T^t]$. Consequently, $[b,T]^t$ is a commutator and if we choose $\varepsilon \in (\delta, 1)$ in Lemma 5 we can continue with

$$\le c_n p' \|b\|_{Osc_{expL^s}} \left[\int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}}} f(x) Rh(x) dx + \int_{\mathbb{R}^n} M_\varepsilon \left(T^t f \right) (x) Rh(x) dx \right]$$

$$= c_n p' \|b\|_{Osc_{expL^s}} (I_1 + I_2)$$

To estimate I_1 we use Hölder's inequality and the second property of the operator R

$$I_1 = \int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}}} f(x) Rh(x) dx \le \left(\int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}}} f(x)^{p'} v(x)^{1-p'} dx \right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^n} Rh(x)^p v(x) dx \right)^{\frac{1}{p}}$$

$$\le 2 \left\| \frac{M_{L(\log L)^{\frac{1}{s}}} f}{v} \right\|_{L^{p'}(v)}$$

To bound I_2 we apply lemma 4 with $w = Rh$ and $p = 1$

$$I_2 \le c_n [Rh]_{A_\infty} \int_{\mathbb{R}^n} M_\varepsilon^\sharp (T^t f)(x) Rh(x) dx \le c_n p' \int_{\mathbb{R}^n} M_\varepsilon^\sharp (T^t f)(x) Rh(x) dx.$$

Using now (8) for the adjoint of T, namely, $M_\varepsilon^\sharp (T^t f) \le c_\varepsilon M f$, we have

$$I_2 \le c_{n,\varepsilon} p' \int_{\mathbb{R}^n} M f Rh.$$

Proceeding now as we did for I_1 we derive to

$$I_2 \le c_n p' \left\| \frac{Mf}{v} \right\|_{L^{p'}(v)}.$$

Consequently

$$\left\| \frac{[b,T]^t f}{v} \right\|_{L^{p'}(v)} \le c_n (p')^2 \|b\|_{Osc_{expL^s}} \left\| \frac{M_{L(\log L)^{\frac{1}{s}}} f}{v} \right\|_{L^{p'}(v)},$$

and recalling that $v = M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}} w$, everything is reduced to establish the following inequality,

$$(9) \qquad \left\| \frac{M_{L(\log L)^{\frac{1}{s}}} f}{v} \right\|_{L^{p'}(v)} \le c_n p^{1+\frac{1}{s}} \left(\frac{p-1}{\delta} \right)^{\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}$$

which will be proved in Lemma 6 below. This concludes the proof of the Theorem in the case $k = 1$.

$\square$

4.2. **Case $k > 1$.**

Proof. Due to the homogeneity of the operator we may assume that

$$\|b_1\|_{Osc_{expL^{s_1}}} = \|b_2\|_{Osc_{expL^{s_2}}} = \cdots = \|b_k\|_{Osc_{expL^{s_k}}} = 1$$

Let us denote $v = M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}}w$. If $\kappa = c_T \left(p'\right)^{k+1} p^{1+\frac{1}{s}} \left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}$, by duality, it suffices to show that

$$\left\|\frac{T_{\vec{b}}^t f}{v}\right\|_{L^{p'}(v)} \leq \kappa \left\|\frac{f}{w}\right\|_{L^{p'}(w)},$$

where $T_{\vec{b}}^t$ is the adjoint of $T_{\vec{b}}$. Using duality we can find a non-negative function $h \in L^p(v)$ with $\|h\|_{L^p(v)} = 1$ such that

$$\left\|\frac{T_{\vec{b}}^t f}{v}\right\|_{L^{p'}(v)} = \int_{\mathbb{R}^n} \frac{|T_{\vec{b}}^t f|}{v} hv dx = \int_{\mathbb{R}^n} |T_{\vec{b}}^t f| \, h dx = I.$$

As in the case $k = 1$ we use again Lemma 3 with $p = 1$, $w = Rh$ and $\gamma \in (0,1)$. Hence, since $[Rh]_{A_\infty} \leq [Rh]_{A_3} \leq c_n p'$, we have

$$I \leq \int_{\mathbb{R}^n} |T_{\vec{b}}^t f| \, Rh dx$$

$$\leq c_n [Rh]_{A_\infty} \int_{\mathbb{R}^n} M_\gamma^\sharp (T_{\vec{b}}^t f) \, Rh dx \leq c_n p' \int_{\mathbb{R}^n} M_\gamma^\sharp (T_{\vec{b}}^t f) \, Rh dx$$

$T_{\vec{b}}^t f$ is a commutator so if we take $\varepsilon \in (\gamma, 1)$ Lemma 5 yields

$$c_n p' \int_{\mathbb{R}^n} M_\gamma^\sharp \left(T_{\vec{b}}^t f\right) \, Rh(x) dx$$

$$\leq c_n p' \left[\int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}}} f(x) \, Rh dx + \sum_{j=1}^k \sum_{\sigma \in C_j^k} \int_{\mathbb{R}^n} M_\varepsilon \left(T_{\vec{\sigma}'}^t f\right) \, Rh dx\right]$$

$$= c_n p' (I_1 + I_2)$$

Now we have to estimate I_1 and I_2. For I_1 we proceed as in the case $k = 1$ obtaining

$$I_1 \leq 2 \left\|\frac{M_{L(\log L)^{\frac{1}{s}}} f}{v}\right\|_{L^{p'}(v)}$$

To estimate I_2 we need to control each term of the sum. To accomplish this we claim that for every $\varepsilon \in (0,1)$:

$$(10) \qquad \int_{\mathbb{R}^n} M_\varepsilon \left(T_{\vec{\sigma}'}^t f\right)(x) Rh(x) dx \leq c\Gamma(\#\sigma')(p')^{\#\sigma'+1} \int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i \in \sigma'} \frac{1}{s_i}}} f(x) Rh(x) dx$$

where

$$\Gamma(j) = \begin{cases} 1 & j = 0 \\ 2 & j = 1 \\ 2 + \sum_{i=1}^{j-1} \binom{j}{i} \Gamma(i) & j > 1 \end{cases}$$

which will be proved by induction on the number of symbols of $T_{\vec{\sigma}'}^t$, i.e., $\#\sigma'$. For simplicity with the notation, we prove the claim for $T_{\vec{\sigma}'}$ instead of $T_{\vec{\sigma}'}^t$. We can do this since both them

are commutators with the same number of symbols. Let us call $m = \#\sigma'$. If the number of symbols is zero $T_{\vec{\sigma}'} = T$ and then combining Lemma 4 and (8) we obtain

$$\int_{\mathbb{R}^n} M_\varepsilon\left(Tf\right)(x) Rh(x) dx \le cp' \int_{\mathbb{R}^n} Mf(x) Rh(x) dx$$

since we assume $\sum_{i \in \emptyset} \frac{1}{s_i} = 0$. If $m = 1$, then $T_{\vec{\sigma}'} = [b_1, T]$. Applying Lemma 4 with $w = Rh$, $p = 1$ we have that

$$\int_{\mathbb{R}^n} M_\varepsilon\left([b_1, T]f\right)(x) Rhdx \le c_n [Rh]_{A_3} \int_{\mathbb{R}^n} M_\varepsilon^\sharp([b_1, T]f) Rhdx$$

$$\le c_n p' \int_{\mathbb{R}^n} M_\varepsilon^\sharp([b_1, T]f) Rhdx$$

Now, if we take $0 < \varepsilon < \tilde{\varepsilon} < 1$, Lemma 5 produces the following bound of the last expression

$$\le cp' \int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s_1}}} f(x) Rh(x) dx + cp' \int_{\mathbb{R}^n} M_{\tilde{\varepsilon}}(Tf(x)) Rh(x) dx$$

$$\le cp' \int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s_1}}} f(x) Rh(x) dx + c(p')^2 \int_{\mathbb{R}^n} Mf(x) Rh(x) dx$$

$$\le 2c(p')^2 \int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s_1}}} f(x) Rh(x) dx.$$

This proves claim (10) in the case $m = 1$. Let us suppose now that the result holds for $0 \le l < m$ symbols, namely, if $0 \le \#\tau < m$, then for every $\varepsilon \in (0,1)$

$$\int_{\mathbb{R}^n} M_\varepsilon\left(T_{\vec{\tau}} f\right)(x) Rhdx \le c\Gamma(\#\tau)(p')^{\#\tau+1} \int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i \in \tau} \frac{1}{s_i}}} f(x) Rhdx$$

Combining Lemma 4, with $w = Rh$ and $p = 1$, and Lemma 5 we have for $\varepsilon < \tilde{\varepsilon} < 1$,

$$\int_{\mathbb{R}^n} M_\varepsilon\left(T_{\vec{\sigma}'} f\right) Rhdx$$

$$\le cp' \int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i \in \sigma'} \frac{1}{s_i}}} f\, Rhdx + \sum_{j=1}^{m} \sum_{\tau \in C_j^m} cp' \int_{\mathbb{R}^n} M_{\tilde{\varepsilon}}\left(T_{b_{\tau'}^{\vec{}}} f\right) Rhdx$$

$$\le cp' \int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i \in \sigma'} \frac{1}{s_i}}} f\, Rhdx + \sum_{j=1}^{k} \sum_{\tau \in C_j^k} cp' \int_{\mathbb{R}^n} M_{\tilde{\varepsilon}}\left(T_{b_{\tau'}^{\vec{}}} f\right) Rhdx.$$

Using now the induction hypothesis we continue with

$$\le cp' \int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i \in \sigma'} \frac{1}{s_i}}} f(x) Rhdx + \sum_{j=1}^{m} \sum_{\tau \in C_j^m} \Gamma(\#\tau') c(p')^{\#\tau'+1} \int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i \in \tau'} \frac{1}{s_i}}} f(x) Rhdx$$

$$\le cp' \int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i \in \sigma'} \frac{1}{s_i}}} f(x) Rhdx + \left(\sum_{j=1}^{m} \sum_{\tau \in C_j^m} \Gamma(\#\tau')\right) (p')^{m+1} \int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i \in \sigma'} \frac{1}{s_i}}} f(x) Rhdx$$

$$\le \left(1 + \sum_{j=1}^{m} \sum_{\tau \in C_j^m} \Gamma(\#\tau')\right) c(p')^{k+1} \int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i \in \sigma'} \frac{1}{s_i}}} f(x) Rhdx.$$

It's easy to check that

$$1 + \sum_{j=1}^{m} \sum_{\tau \in C_j^m} \Gamma(\#\tau') = \Gamma(m)$$

Then the main claim (10) is proved in full generality. This yields, combining estimates

$$I_2 \le c_{n,k,\delta,\varepsilon} \, (p')^{k+1} \int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}}} f(x) \, Rh dx.$$

Proceeding as we did for I_1 we obtain the following estimate

$$I_2 \le c_{n,\delta,\varepsilon} \, (p')^{k+1} \left\| \frac{M_{L(\log L)^{\frac{1}{s}}} f}{v} \right\|_{L^{p'}(v)}.$$

Consequently

$$\left\| \frac{T_{\vec{b}}^t f}{v} \right\|_{L^{p'}(v)} \le c_n \, (p')^{k+1} \left\| \frac{M_{L(\log L)^{\frac{1}{s}}} f}{v} \right\|_{L^{p'}(v)}.$$

This concludes the proof since by Lemma 6

$$\left\| \frac{M_{L(\log L)^{\frac{1}{s}}} f}{v} \right\|_{L^{p'}(v)} \le c p^{1+\frac{1}{s}} \left(\frac{p-1}{\delta} \right)^{\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}$$

that $v = M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}} w$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.3. A key two weight inequality. As we already mentioned, we prove the following lemma that was used several times during the proof of Theorem 1.

Lemma 6. *Let $w \ge 0$ be a weight. Let $s \ge 1$ and $0 < \delta < 1$. Then for every $p \in (1,\infty)$ we have that*

$$(11) \qquad \left\| \frac{M_{L(\log L)^{\frac{1}{s}}} f}{v} \right\|_{L^{p'}(v)} \le c p^{1+\frac{1}{s}} \left(\frac{p-1}{\delta} \right)^{\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}.$$

where $v = M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}} w$.

The proof of this lemma will follow ideas from [22]. In particular, we are going to obtain a precise version of the two weight inequality that appears in the proof of Theorem 2 of that work. To do that we need precise estimates of the following inverse functions.

Lemma 7. *Let $\rho > 0$, $A_\rho(t) = t \left(1 + \log^+ (t)\right)^\rho$ and $X_\rho(t) = \frac{t}{\left(1+\log^+(t)\right)^\rho}$. Then*

$$\left(\frac{1}{1+\rho} \right)^\rho t \le X_\rho(A_\rho(t)) \le t.$$

Proof. Observe that

$$X_\rho(A_\rho(t)) = \frac{t \left(1 + \log^+ (t)\right)^\rho}{\left(1 + \log^+ \left(t \left(1 + \log^+ (t)\right)^\rho\right)\right)^\rho}$$

The upper bound is straightforward since

$$\left(1 + \log^+ (t)\right)^\rho \le \left(1 + \log^+ \left(t \left(1 + \log^+ (t)\right)^\rho\right)\right)^\rho.$$

Now we prove the lower bound. It suffices to prove that

$$\frac{1 + \log^+ (t)}{1 + \log^+ \left(t \left(1 + \log^+ (t)\right)^\rho\right)} \geq \frac{1}{1 + \rho}.$$

If $0 < t \leq 1$ there's nothing to prove since $\log^+ (t) = \log^+ \left(t \left(1 + \log^+ (t)\right)^\rho\right) = 0$. Suppose now that $t > 1$. Then we have that

$$\frac{1 + \log^+ (t)}{1 + \log^+ \left(t \left(1 + \log^+ (t)\right)^\rho\right)} = \frac{1 + \log (t)}{1 + \log \left(t \left(1 + \log (t)\right)^\rho\right)}$$

$$= \frac{1 + \log (t)}{1 + \log (t) + \rho \log \left(1 + \log (t)\right)} \geq \frac{1 + \log (t)}{1 + \log (t) + \rho \left(1 + \log (t)\right)}$$

$$= \frac{1}{1 + \rho}.$$

$\square$

Lemma 8. *Let* $\rho > 1$, $A_\rho(t) = t \left(1 + \log^+ (t)\right)^\rho$ *and* $\tilde{X}_\rho(t) = \frac{t}{\left(1 + \log^+ \left(\frac{t}{t_\rho}\right)\right)^\rho}$ *with* $t_\rho = \rho^\rho$.
Then

$$\left(1 - \frac{1}{e}\right)^\rho t \leq A_\rho(\tilde{X}_\rho(t)) \leq t \left(1 + \rho \log (\rho)\right)^\rho.$$

Proof. Observe first that

$$A_\rho(\tilde{X}_\rho(t)) = t \left(\frac{1 + \log^+ \left(\frac{t}{\left(1 + \log^+ \left(\frac{t}{t_\rho}\right)\right)^\rho}\right)}{1 + \log^+ \left(\frac{t}{t_\rho}\right)}\right)^\rho = t \Phi(t)^\rho$$

We begin studying the lower bound.
If $t \in (0, 1)$ then

$$A_\rho(\tilde{X}_\rho(t)) = t \Phi(t)^\rho = t$$

and there's nothing to prove.
If $t \in [1, t_\rho]$ then

$$A_\rho(\tilde{X}_\rho(t)) = t \Phi(t)^\rho = t \left(1 + \log^+ (t)\right)^\rho \geq t$$

Now if $t > t_\rho$, it's easy to check that $\frac{t}{\left(1 + \log \left(\frac{t}{t_\rho}\right)\right)^\rho} \geq 1$. Then

$$A_\rho(\tilde{X}_\rho(t)) = t \left(\frac{1 + \log \left(\frac{t}{\left(1 + \log \left(\frac{t}{t_\rho}\right)\right)^\rho}\right)}{1 + \log \left(\frac{t}{t_\rho}\right)}\right)^\rho$$

Now we observe that

$$\frac{1 + \log \left(\frac{t}{\left(1 + \log \left(\frac{t}{t_\rho}\right)\right)^\rho}\right)}{1 + \log \left(\frac{t}{t_\rho}\right)} = \frac{1 + \log (t) - \rho \log \left(1 + \log \left(\frac{t}{t_\rho}\right)\right)}{1 + \log \left(\frac{t}{t_\rho}\right)}.$$

Let us choose $t = e^\lambda$ and $t_\rho = e^{\lambda_\rho}$. Then

$$\frac{1 + \lambda - \rho \log \left(1 + \log \left(\frac{e^\lambda}{e^{\lambda_\rho}} \right) \right)}{1 + \log \left(\frac{e^\lambda}{e^{\lambda_\rho}} \right)} = 1 + \frac{\lambda_\rho - \rho \log (1 + \lambda - \lambda_\rho)}{1 + \lambda - \lambda_\rho} = 1 + g_\rho(\lambda)$$

Now we minimize $g_\rho(\lambda)$. It's easy to check that g_ρ reaches its minimum when $\lambda = e^{1 + \frac{\lambda_\rho}{\rho}} + \lambda_\rho - 1$. We observe that

$$g_\rho \left(e^{1 + \frac{\lambda_\rho}{\rho}} + \lambda_\rho - 1 \right) = \frac{-\rho}{e^{1 + \frac{\lambda_\rho}{\rho}}}$$

and since $t_\rho = \rho^\rho$

$$\frac{-\rho}{e^{1 + \frac{\lambda_\rho}{\rho}}} = -\frac{1}{e}$$

and we obtain the desired lower bound. To finish the proof we focus on the bound. If $t \in (0, 1)$, then $A_\rho(\tilde{X}_\rho(t)) = t$ and there's nothing to prove. If $t \in [1, t_\rho]$ then we have that

$$A_\rho(\tilde{X}_\rho(t)) = t(1 + \log t)^\rho \le t(1 + \log t_\rho)^\rho = t(1 + \rho \log \rho)^\rho.$$

Finally if $t \in (t_\rho, \infty)$ then it's easy to check that

$$A_\rho(\tilde{X}_\rho(t)) \le t \left(1 + \log (t_\rho) \right)^\rho.$$

$\square$

Finally, with the precise control of the inverses at our disposal we are ready to give the proof of lemma 6.

Proof of Lemma 6. Proving (11) is equivalent to prove that

$$\int_{\mathbb{R}^n} M_{L \log L^{\frac{1}{s}}} \left(f w^{\frac{1}{p}} \right)^{p'} \left(M_{L(\log L)^{\left(1 + \frac{1}{s} \right) p - 1 + \delta}} w \right)^{1 - p'} \le c_n^{p'} \left(p^{1 + \frac{1}{s}} \right)^{p'} \left(\frac{p - 1}{\delta} \right) \int_{\mathbb{R}^n} |f|^{p'}$$

Using now the notation of Lemma 7, we can write $A_{\frac{1}{s}}(t) = t(1 + \log^+ t)^{\frac{1}{s}}$ and $X_{\frac{1}{s}}(t) = \frac{t}{\left(1 + \log^+ t \right)^{\frac{1}{s}}}$ and we have that

$$A_{\frac{1}{s}}^{-1}(t) \ge X_{\frac{1}{s}}(t)$$

We observe now that

$$X_{\frac{1}{s}}(t) = \frac{t}{\left(1 + \log^+ t \right)^{\frac{1}{s}}} = \frac{t^{\frac{1}{p}}}{\left(1 + \log^+ t \right)^{\frac{1}{s} + \frac{p - 1 + \delta}{p}}} \cdot t^{\frac{1}{p'}} \left(1 + \log^+ t \right)^{\frac{p - 1 + \delta}{p}}$$

$$= \left(\frac{t}{\left(1 + \log^+ t \right)^{\left(1 + \frac{1}{s} \right) p - 1 + \delta}} \right)^{\frac{1}{p}} \left(t \left(1 + \log^+ t \right)^{1 + \delta(p' - 1)} \right)^{\frac{1}{p'}} = F_1(t)^{\frac{1}{p}} \cdot F_2(t)^{\frac{1}{p'}}$$

Using again the notation of Lemma 7,

$$F_1(t) = X_{\left(1 + \frac{1}{s} \right) p - 1 + \delta}(t) = \frac{t}{\left(1 + \log^+ t \right)^{\left(1 + \frac{1}{s} \right) p - 1 + \delta}}.$$

From that lemma it readily follows that

$$F_1(t)^{\frac{1}{p}} \ge \left(\frac{1}{\left(1 + \frac{1}{s} \right) p + \delta} \right)^{\frac{\left(1 + \frac{1}{s} \right) p - 1 + \delta}{p}} A_{\left(1 + \frac{1}{s} \right) p - 1 + \delta}^{-1}(t)^{\frac{1}{p}}.$$

Analogously, following the notation of Lemma 8

$$F_2(t) = A_{1+\delta(p'-1)}(t) = t\left(1 + \log^+ t\right)^{1+\delta(p'-1)}$$

From that lemma it follows that

$$F_2(t)^{\frac{1}{p'}} \geq \left(\frac{e-1}{e}\right)^{\frac{1+\delta(p'-1)}{p'}} \tilde{X}^{-1}_{1+\delta(p'-1)}(t)^{\frac{1}{p'}}.$$

Taking into account (4.3) and (4.3) we obtain the following estimate

$$A_{\frac{1}{s}}^{-1}(t)\,(e')^{\frac{1+\delta(p'-1)}{p'}}\left(\left(1+\frac{1}{s}\right)p+\delta\right)^{\frac{\left(1+\frac{1}{s}\right)p-1+\delta}{p}} \geq A^{-1}_{\left(1+\frac{1}{s}\right)p-1+\delta}(t)^{\frac{1}{p}}\tilde{X}^{-1}_{1+\delta(p'-1)}(t)^{\frac{1}{p'}} \qquad t > 0.$$

Using now generalized Hölder inequality (Lemma 1) and taking into account that, since $\delta \in (0,1)$,

$$(e')^{\frac{1+\delta(p'-1)}{p'}}(2p+\delta)^{\frac{\left(1+\frac{1}{s}\right)p-1+\delta}{p}} \leq cp^{1+\frac{1}{s}}$$

and also that $\|w\|_{\Psi(L)} = \|w^p\|_{\Psi\left(L^{\frac{1}{p}}\right)}^{\frac{1}{p}}$ if Ψ is a Young function, we have that

$$\left\|fw^{\frac{1}{p}}\right\|_{L(\log L)^{\frac{1}{s}},Q} \leq cp^{1+\frac{1}{s}}\|f\|_{\tilde{X}_{1+\delta(p'-1)}(L^{p'}),Q}\|w\|_{A_{\left(1+\frac{1}{s}\right)p-1+\delta}(L),Q}^{\frac{1}{p}}$$

and consequently

$$M_{L(\log L)^{\frac{1}{s}}}\left(fw^{\frac{1}{p}}\right) \leq cp^{1+\frac{1}{s}}M_{\tilde{X}_{1+\delta(p'-1)}(L^{p'})}(f)M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}}(w)^{\frac{1}{p}}.$$

Using this estimate we have that

$$\int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}}}\left(fw^{\frac{1}{p}}\right)^{p'}\left(M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}}w\right)^{1-p'} dx$$

$$\leq \int_{\mathbb{R}^n}\left(cp^{\left(1+\frac{1}{s}\right)}M_{\tilde{X}_{1+\delta(p'-1)}(L^{p'})}(f)M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}}(w)^{\frac{1}{p}}\right)^{p'}\left(M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}}w\right)^{1-p'} dx$$

$$= \left(cp^{1+\frac{1}{s}}\right)^{p'}\int_{\mathbb{R}^n} M_{\tilde{X}_{1+\delta(p'-1)}(L^{p'})}(f)^{p'} dx$$

Lemma 2.1 of [10] yields

$$\left(\int_{\mathbb{R}^n} M_{\tilde{X}_{1+\delta(p'-1)}(L^{p'})}f(x)^{p'} dx\right)^{\frac{1}{p'}} \leq c\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^n}|f|^{p'}(x)dx\right)^{\frac{1}{p'}},$$

since

$$\left(\int_1^\infty \frac{\tilde{X}_{1+\delta(p'-1)}(t^{p'})}{t^{p'}}\frac{dt}{t}\right)^{\frac{1}{p'}} = \left(\frac{(1+(p'-1)\delta)}{p'}\log\left(1+(p'-1)\delta\right) + \frac{1}{(p'-1)\delta}\right)^{\frac{1}{p'}}$$

and $0 < \delta < 1$ allows us to write

$$\left(\frac{(1+(p'-1)\delta)}{p'}\log\left(1+(p'-1)\delta\right) + \frac{1}{(p'-1)\delta}\right)^{\frac{1}{p'}} \leq c\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}.$$

Consequently we have that

$$\left\|M_{L(\log L)^{\frac{1}{s}}}\left(fw^{\frac{1}{p}}\right)\right\|_{L^{p'}(v^{1-p'})} \leq cp^{1+\frac{1}{s}}\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}\|f\|_{L^{p'}(\mathbb{R}^n)}.$$

This concludes the proof of (4.3).   □

5. Proof of Theorem 2

5.1. Case $k = 1$.

Proof. By homogeneity we shall suppose that $\|b\|_{Osc_{expL^s}} = 1$. We consider the Calderón-Zygmund decomposition of f at height λ. That decomposition allows us to obtain a family of dyadic cubes $\{Q_j\}$ which are pairwise disjoint such that

$$\lambda \leq \frac{1}{|Q_j|} \int_{Q_j} |f| \leq 2^n \lambda.$$

Let us denote

$$\Omega = \bigcup_j Q_j$$

As usual, we write $f = g + h$ where g, the "good" part of f, is defined as

$$g(x) = \begin{cases} f(x) & x \in \Omega^c \\ f_{Q_j} & x \in Q_j \end{cases}$$

and verifies that $|g(x)| \leq 2^n \lambda$ a.e. and $h = \sum h_j$ where $h_j = (f - f_{Q_j}) \chi_{Q_j}$ and $f_{Q_j} = \frac{1}{|Q_j|} \int_{Q_j} f(x)dx$. We denote $w^*(x) = w(x)\chi_{\mathbb{R}^n \setminus \tilde{\Omega}}(x)$ and $w_j(x) = w(x)\chi_{\mathbb{R}^n \setminus \tilde{Q}_j}$ where $\tilde{Q}_j = 5\sqrt{n}Q_j$ and $\tilde{\Omega} = \bigcup_j \tilde{Q}_j$. Using that decomposition we can write

$$w\left(\{x \in \mathbb{R}^n : |[b,T]f(x)| > \lambda\}\right) \leq w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]g(x)| > \frac{\lambda}{2}\right\}\right) + w(\tilde{\Omega})$$

$$+ w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]h(x)| > \frac{\lambda}{2}\right\}\right)$$

$$= I + II + III$$

To end the proof we have to estimate I, II and III. Let us begin with I. If $p > 0$, Chebyschev's inequality gives

$$w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]g(x)| > \frac{\lambda}{2}\right\}\right) \leq \frac{2^p}{\lambda^p} \int_{\mathbb{R}^n} |[b,T]g(x)|^p w^*(x)dx.$$

Let us choose $1 + \frac{\varepsilon}{3(1+\frac{1}{s})} < p < 1 + \frac{\varepsilon}{2(1+\frac{1}{s})}$ y $\delta = \varepsilon - (1 + \frac{1}{s})(p-1)$. For that choice of p and δ, is easy to check that

$$(p')^{2p} p^{(1+\frac{1}{s})p} \left(\frac{p-1}{\delta}\right)^{\frac{p}{p'}} \leq c_s \frac{1}{\varepsilon^2} \quad \text{and} \quad \left(1 + \frac{1}{s}\right)p - 1 + \delta = \frac{1}{s} + \varepsilon.$$

Using now Theorem 1, we have that

$$\frac{2^p}{\lambda^p} \int_{\mathbb{R}^n} |[b,T]g(x)|^p w^*(x)dx$$

$$\leq c (p')^{2p} p^{(1+\frac{1}{s})p} \left(\frac{p-1}{\delta}\right)^{\frac{p}{p'}} \int_{\mathbb{R}^n} |g(x)|^p M_{L(\log L)^{(1+\frac{1}{s})p-1+\delta}} w^*(x)dx$$

$$\leq c \frac{1}{\varepsilon^2} \frac{2^p}{\lambda^p} \int_{\mathbb{R}^n} |g(x)|^p M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w^*(x)dx \leq c \frac{1}{\varepsilon^2} \frac{1}{\lambda} \int_{\mathbb{R}^n} |g(x)| M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w^*(x)dx$$

$$\leq c \frac{1}{\varepsilon^2} \frac{1}{\lambda} \left(\int_{\mathbb{R}^n \setminus \Omega} |f(x)| M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w(x)dx + \int_{\Omega} |g(x)| M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w^*(x)dx \right)$$

and it suffices to estimate last integral. Indeed,

$$\int_{\Omega} |g(x)| M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w^*(x) dx \le \sum_j |f|_{Q_j} \int_{Q_j} M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w_j(x) dx$$

$$\le c \sum_j |Q_j| \frac{1}{|Q_j|} \int_{Q_j} |f(y)| dy \inf_{z \in Q_j} M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w_j(z)$$

$$= c \sum_j \int_{Q_j} |f(y)| \inf_{z \in Q_j} M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w_j(z) dy \le c \sum_j \int_{Q_j} |f(y)| M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w_j(y) dy$$

$$\le c \int_{\Omega} |f(y)| M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w(y) dy.$$

Summarizing, we obtain that

$$I \le c \frac{1}{\varepsilon^2} \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w(y) dy.$$

For II we have the following standard estimate

$$II = w(\tilde{\Omega}) \le \sum_j \int_{5\sqrt{n}Q_j} w(x) dx = \sum_j |5\sqrt{n}Q_j| \frac{1}{|5\sqrt{n}Q_j|} \int_{5\sqrt{n}Q_j} w(x) dx$$

$$\le \sum_j (5\sqrt{n})^n |Q_j| \inf_{z \in Q_j} Mw(z) \le (5\sqrt{n})^n \sum_j \frac{1}{\lambda} \int_{Q_j} f(y) dy \inf_{z \in Q_j} Mw(z)$$

$$\le (5\sqrt{n})^n \sum_j \frac{1}{\lambda} \int_{Q_j} Mw(y) f(y) dy \le (5\sqrt{n})^n \int_{\mathbb{R}^n} \frac{f(y)}{\lambda} Mw(y) dy$$

To estimate III we split the operator as follows

$$[b,T]h = \sum_j [b,T]h_j = \sum_j (bT(h_j) - T(bh_j)) = \sum_j (b - b_{Q_j}) T(h_j) - \sum_j T\left((b - b_{Q_j}) h_j\right).$$

Then we continue with

$$III \le w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : \left|\sum_j (b(x) - b_{Q_j}) Th_j(x)\right| > \frac{\lambda}{4}\right\}\right)$$

$$+ w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : \left|\sum_j T\left([b - b_{Q_j}] h_j\right)(x)\right| > \frac{\lambda}{4}\right\}\right)$$

$$= A + B$$

To estimate A we use standard computations based on the smoothness property of the kernel K and the cancellation of each h_j,

$$
\begin{aligned}
A &\le \frac{c}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} \sum_j |b(x) - b_{Q_j}| \, |Th_j(x)| \, w(x) dx \\
&\le \frac{c}{\lambda} \sum_j \int_{\mathbb{R}^n \setminus \tilde{Q}_j} |b(x) - b| w(x) \int_{Q_j} |h_j(y)| \, \left| K(x,y) - K(x, x_{Q_j}) \right| dy dx \\
&\le \frac{c}{\lambda} \sum_j \int_{Q_j} |h_j(y)| \int_{\mathbb{R}^n \setminus \tilde{Q}_j} |K(x,y) - K(x, x_{Q_j})| |b(x) - b_{Q_j}| w_j(x) dx dy \\
&\le \frac{c}{\lambda} \sum_j \int_{Q_j} |h_j(y)| \int_{\mathbb{R}^n \setminus \tilde{Q}_j} \frac{|y - x_{Q_j}|^\gamma}{|x - x_{Q_j}|^{n+\gamma}} |b(x) - b_{Q_j}| w_j(x) dx dy \\
&\le \frac{c}{\lambda} \sum_j \int_{Q_j} |h_j(y)| \sum_{k=1}^\infty \int_{2^k l(Q_j) \le |x - x_{Q_j}| < 2^{k+1} l(Q_j)} \frac{|y - x_{Q_j}|^\gamma}{|x - x_{Q_j}|^{n+\gamma}} |b(x) - b_{Q_j}| w_j(x) dx dy \\
&\le \frac{c}{\lambda} \sum_j \left(\int_{Q_j} |h_j(y)| dy \right) \sum_{k=1}^\infty \frac{2^{-\gamma k}}{|2^{k+1} Q_j|} \int_{2^{k+1} Q_j} |b(x) - b_{Q_j}| w_j(x) dx
\end{aligned}
$$

We now fix one term of the sum. Using generalized Hölder inequality, Lemma 1, we have

$$
\begin{aligned}
&\sum_{k=0}^\infty \frac{2^{-\gamma k}}{|2^{k+1} Q_j|} \int_{2^{k+1} Q_j} |b(x) - b_{Q_j}| w_j(x) dx \\
&\le \sum_{k=0}^\infty \frac{2^{-\gamma k}}{|2^{k+1} Q_j|} \int_{2^{k+1} Q_j} |b(x) - b_{2^{k+1} Q_j}| w_j(x) dx \\
&\quad + \sum_{k=0}^\infty \frac{2^{-\gamma k}}{|2^{k+1} Q_j|} \int_{2^{k+1} Q_j} |b_{2^{k+1} Q_j} - b_{Q_j}| w_j(x) dx \\
&\le \sum_{k=1}^\infty 2^{-\gamma k} \|b - b_{2^{k+1} Q_j}\|_{\exp L^s, 2^{k+1} Q_j} \|w_j\|_{L \log L^{\frac{1}{s}}, 2^{k+1} Q_j} \\
&\quad + \sum_{k=1}^\infty 2^{-\gamma k} (k+1) \|b\|_{Osc_{expL^s}} \inf_{z \in Q_j} M w_j(z) \\
&\le \sum_{k=1}^\infty 2^{-\gamma k} \|b\|_{Osc_{expL^s}} \inf_{z \in Q_j} M_{L \log L^{\frac{1}{s}}} w_j(z) \\
&\quad + \sum_{k=1}^\infty 2^{-\gamma k} (k+1) \|b\|_{Osc_{expL^s}} \inf_{z \in Q_j} M w_j(z) \\
&\le c \left(\inf_{z \in Q_j} M_{L \log L^{\frac{1}{s}}} w_j(z) \sum_{k=1}^\infty 2^{-\gamma k} + \inf_{z \in Q_j} M w_j(z) \sum_{k=1}^\infty 2^{-\gamma k} (k+1) \right) \\
&\le c \inf_{z \in Q_j} M_{L \log L^{\frac{1}{s}}} w_j(z)
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
A &\leq \frac{c}{\lambda} \sum_j \int_{Q_j} |h_j(y)| dy \inf_{y \in Q_j} M_{L \log L^{\frac{1}{s}}}(w_j)(y) \\
&\leq \frac{c}{\lambda} \sum_j \int_{Q_j} M_{L \log L^{\frac{1}{s}}}(w_j)(y) |h_j(y)| dy \\
&\leq \frac{c}{\lambda} \left(\int_{\mathbb{R}^n} |f(y)| M_{L \log L^{\frac{1}{s}}}(w_j)(y) dy + \sum_j \int_{Q_j} M_{L \log L^{\frac{1}{s}}}(w_j)(y) |f_{Q_j}| dy \right) \\
&\leq \frac{c}{\lambda} \left(\int_{\mathbb{R}^n} |f(y)| M_{L \log L^{\frac{1}{s}}}(w_j)(y) dy + \sum_j \int_{Q_j} f(y) dy \inf_{z \in Q_j} M_{L \log L^{\frac{1}{s}}}(w_j)(z) dy \right) \\
&\leq \frac{c}{\lambda} \int_{\mathbb{R}^n} |f(y)| M_{L \log L^{\frac{1}{s}}}(w_j)(y) dy
\end{aligned}
$$

To end the proof we estimate B. Theorem 1.1 from [10] gives

$$
\begin{aligned}
B &= w^* \left(\left\{ x \in \mathbb{R}^n : \left| \sum_j T \left([b - b_{Q_j}] h_j \right)(x) \right| > \frac{\lambda}{4} \right\} \right) \\
&\leq c \frac{1}{\varepsilon} \frac{1}{\lambda} \int_{\mathbb{R}^n} \left| \sum_j \left(b(x) - b_{Q_j} \right) h_j \right|(x) M_{L(\log L)^\varepsilon}(w^*)(x) dx \\
&\leq c \frac{1}{\varepsilon} \frac{1}{\lambda} \sum_j \int_{Q_j} |b(x) - b_{Q_j}| |f(x) - f_{Q_j}| M_{L(\log L)^\varepsilon}(w_j)(x) dx \\
&\leq c \frac{1}{\varepsilon} \frac{1}{\lambda} \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon}(w_j)(z) \left(\int_{Q_j} |b(x) - b_{Q_j}| |f(x)| dx + \int_{Q_j} |b(x) - b_{Q_j}| |f_{Q_j}| dx \right) \\
&= \frac{1}{\varepsilon} (B_1 + B_2)
\end{aligned}
$$

For B_2

$$
\begin{aligned}
B_2 &= \frac{c}{\lambda} \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon}(w_j)(z) \int_{Q_j} |b(x) - b_{Q_j}| |f_{Q_j}| dx \\
&\leq \frac{c}{\lambda} \sum_j \frac{1}{|Q_j|} \int_{Q_j} |b(x) - b_{Q_j}| dx \int_{Q_j} |f(y)| M_{L(\log L)^\varepsilon}(w_j)(y) dx \\
&\leq \frac{c}{\lambda} \sum_j \|b\|_{Osc_{expL^s}} \int_{Q_j} |f(y)| M_{L(\log L)^\varepsilon}(w_j)(y) dx \\
&\leq c \sum_j \int_{Q_j} \frac{|f(y)|}{\lambda} M_{L(\log L)^\varepsilon}(w_j)(y) dy \\
&\leq c \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} M_{L(\log L)^\varepsilon} w(x) dx.
\end{aligned}
$$

For B_1 we use the generalized Hölder inequality Lemma 1 and we obtain

$$B_1 = \frac{c}{\lambda} \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon}(w_j)(z) \int_{Q_j} |b(x) - b_{Q_j}| \, |f(x)| \, dx$$

(12)
$$\leq c \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon}(w_j)(z) \frac{1}{\lambda} |Q_j| \, \|b\|_{Osc_{expL^s}} \|f\|_{L(\log L)^{\frac{1}{s}} L, Q_j}$$

$$= c \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon}(w_j)(z) \frac{1}{\lambda} |Q_j| \, \|f\|_{L(\log L)^{\frac{1}{s}}, Q_j}.$$

Now we see that

(13)
$$\frac{1}{\lambda} |Q_j| \, \|f\|_{L(\log L)^{\frac{1}{s}}, Q_j} \leq \frac{1}{\lambda} |Q_j| \inf_{\mu > 0} \left\{ \mu + \frac{\mu}{|Q_j|} \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\mu} \right) dx \right\}$$

$$\leq \frac{1}{\lambda} |Q_j| \left(\lambda + \frac{\lambda}{|Q_j|} \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx \right) = |Q_j| + \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx$$

$$\leq \frac{1}{\lambda} \int_{Q_j} |f(x)| dx + \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx \leq 2 \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx.$$

Consequently

$$B_1 \leq c \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon}(w_j)(z) \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx$$

$$\leq c \sum_j \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) M_{L(\log L)^\varepsilon}(w_j)(x) dx$$

$$\leq c \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) M_{L(\log L)^\varepsilon}(w)(x) dx.$$

$\square$

5.2. Case $k > 1$.

Proof. Let us suppose that the desired inequality holds for $l \leq k-1$ symbols. By homogeneity we may assume that $\|b\|_{Osc_{expL^{s_1}}} = \cdots = \|b\|_{Osc_{expL^{s_k}}} = 1$. Using the Calderón-Zygmund decomposition with the same notation used in the case $k = 1$ we can write

$$w\left(\left\{ x \in \mathbb{R}^n : |T_{\vec{b}}f(x)| > \lambda \right\}\right) \leq w\left(\left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : |T_{\vec{b}}g(x)| > \frac{\lambda}{2} \right\}\right) + w(\tilde{\Omega})$$

$$+ w\left(\left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : |T_{\vec{b}}h(x)| > \frac{\lambda}{2} \right\}\right)$$

$$= I + II + III$$

We consider now each term separately. To estimate I we use Chebyschev's inequality for $p > 1$ that will be chosen appropriately,

$$w\left(\left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : |T_{\vec{b}}g(x)| > \frac{\lambda}{2} \right\}\right) \leq \frac{2^p}{\lambda^p} \int_{\mathbb{R}} |T_{\vec{b}}g(x)|^p w^*(x) dx.$$

Let us choose, as we did in the case $k = 1$, p such that $1 + \frac{\varepsilon}{3(1+\frac{1}{s})} < p < 1 + \frac{\varepsilon}{(1+\frac{1}{s})2}$ and $\delta = \varepsilon - \left(1 + \frac{1}{s}\right) p$. For this choice of p and δ we have that

$$(p')^{(k+1)p} p^{\left(1+\frac{1}{s}\right)p} \left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}} \leq c_s \frac{1}{\varepsilon^{k+1}} \quad \text{and} \quad \left(1 + \frac{1}{s}\right)p - 1 + \delta = \frac{1}{s} + \varepsilon$$

Using now theorem 1 and the choice of δ and p we have that

$$\frac{2^p}{\lambda^p} \int_{\mathbb{R}} |T_{\vec{b}} g(x)|^p w^*(x) dx \leq c_n \, (p')^{(k+1)p} \, p^{\left(1+\frac{1}{s}\right)p} \left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}} \int_{\mathbb{R}} |g(x)|^p M_{L(\log L)^{\left(1+\frac{1}{s}\right)p - 1 + \delta}} w^*(x) dx$$

$$\leq c \frac{1}{\varepsilon^{k+1}} \frac{2^p}{\lambda^p} \int_{\mathbb{R}} |g(x)|^p M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w^*(x) dx$$

Arguing as in the case $k = 1$ we obtain that

$$I \leq c_n \frac{1}{\varepsilon^{k+1}} \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w(y) dy.$$

For II, as in the case $k = 1$, we have the following estimate

$$II \leq 3^n \int_{\mathbb{R}^n} \frac{f(y)}{\lambda} M w(y) dy$$

It remains to estimate III. Following the computations of page 684 of [24] we can write

$$\begin{aligned}
T_{\vec{b}} f(x) &= (b_1(x) - \lambda_1) \ldots (b_k(x) - \lambda_k) T f(x) \\
&\quad + (-1)^k T \left((b_1 - \lambda_1) \ldots (b_k - \lambda_k) f\right)(x) \\
&\quad + \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} (-1)^{k-i} \left(b(x) - \vec{\lambda}\right)_\sigma \int_{\mathbb{R}^n} \left(b(y) - \vec{\lambda}\right)_{\sigma'} K(x,y) f(y) dy.
\end{aligned}$$
(14)

Now we work on the last double summation. We observe that for each term we can write

$$\left(b(x) - \vec{\lambda}\right)_\sigma \int_{\mathbb{R}^n} \left(b(y) - \vec{\lambda}\right)_{\sigma'} K(x,y) f(y) dy$$

$$= \int_{\mathbb{R}^n} \left(b(y) - \vec{\lambda}\right)_{\sigma'} \left([b(x) - b(y)] + \left[b(y) - \vec{\lambda}\right]\right)_\sigma K(x,y) f(y) dy$$

$$\overset{\tau \cup \tau' = \sigma}{=} \int_{\mathbb{R}^n} \left(b(y) - \vec{\lambda}\right)_{\sigma'} \sum_{j=0}^{\#\sigma} \sum_{\tau \in C_j(\sigma)} (b(x) - b(y))_\tau \left(b(y) - \vec{\lambda}\right)_{\tau'} K(x,y) f(y) dy$$

$$= \sum_{j=0}^{\#\sigma} \sum_{\tau \in C_j(\sigma)} \int_{\mathbb{R}^n} (b(x) - b(y))_\tau \left(b(y) - \vec{\lambda}\right)_{\sigma' \cup \tau'} K(x,y) f(y) dy$$

$$= \sum_{j=0}^{\#\sigma} \sum_{\tau \in C_j(\sigma)} T_{\vec{\tau}} \left(\left(b - \vec{\lambda}\right)_{\sigma' \cup \tau'} f\right)$$

$$= T \left((b_1 - \lambda_1) \ldots (b_k - \lambda_k) f\right)(x) + \sum_{j=1}^{\#\sigma} \sum_{\tau \in C_j(\sigma)} T_{\vec{\tau}} \left(\left(b - \vec{\lambda}\right)_{\sigma' \cup \tau'} f\right).$$

Plugging this into the double summation of (14), since $\tau \cup \tau' \cup \sigma' = b$ we can write,

$$\sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} (-1)^{k-i} \left(b(x) - \vec{\lambda}\right)_\sigma \int_{\mathbb{R}^n} \left(b(y) - \vec{\lambda}\right)_{\sigma'} K(x,y) f(y) dy$$

$$= c_k T\left((b_1 - \lambda_1) \ldots (b_k - \lambda_k) f\right)(x) + \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} c_\sigma T_{\vec{\sigma}} \left(\left(b - \vec{\lambda}\right)_{\sigma'} f\right)$$

where c_σ is a constant that counts the number of repetitions of each $T_{\vec{\sigma}}$. Summarizing

$$T_{\vec{b}} f(x) = (b_1(x) - \lambda_1) \ldots (b_k(x) - \lambda_k) T f(x)$$
$$+ c_k T\left((b_1 - \lambda_1) \ldots (b_k - \lambda_k) f\right)(x)$$
$$+ \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} c_\sigma T_{\vec{\sigma}} \left(\left(b(y) - \vec{\lambda}\right)_{\sigma'} f\right)(x)$$

Using this for each h_j and summing on j,

$$\sum_j T_{\vec{b}} h_j(x) = \sum_j (b_1(x) - \lambda_1) \ldots (b_k(x) - \lambda_k) T h_j(x)$$
$$+ \sum_j c_k T\left((b_1 - \lambda_1) \ldots (b_k - \lambda_k) h_j\right)(x)$$
$$+ \sum_j \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} c_\sigma T_{\vec{\sigma}} \left(\left(b - \vec{\lambda}\right)_{\sigma'} h_j\right)(x)$$

Then we can estimate III as follows

$$III \le w\left(\left\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : \left|\sum_j \left(b_1(x) - (b_1)_{Q_j}\right) \ldots \left(b_k(x) - (b_k)_{Q_j}\right) T h_j(x)\right| > \frac{\lambda}{6}\right\}\right)$$

$$+ w\left(\left\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : \left|\sum_j c_k T\left(\left(b_1 - (b_1)_{Q_j}\right) \ldots \left(b_k - (b_k)_{Q_j}\right) h_j\right)(x)\right| > \frac{\lambda}{6}\right\}\right)$$

$$+ w\left(\left\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : \left|\sum_j \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^k} c_\sigma T_{\vec{\sigma}} \left(\left(b - \vec{b_{Q_j}}\right)_{\sigma'} h_j\right)(x)\right| > \frac{\lambda}{6}\right\}\right)$$

$$= L_1 + L_2 + L_3$$

To estimate L_1 we denote $w_j = \chi_{\mathbb{R}^n \setminus 5\sqrt{n}Q_j} w$ and $B(x) = \prod_{i=1}^{k} \left| b_i(x) - (b_i)_{Q_j} \right|$. Then

$$
\begin{aligned}
L_1 &\leq \frac{c}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} \left| \sum_j \left(b_1(x) - (b_1)_{Q_j} \right) \dots \left(b_k(x) - (b_k)_{Q_j} \right) Th_j(x) \right| w(x) dx \\
&\leq \sum_j \frac{c}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} B(x) |Th_j(x)| w(x) dx = \\
&\leq \sum_j \frac{c}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} B(x) w(x) \left(\int_{Q_j} |h_j(y)| |K(x,y) - K(x, x_{Q_j})| dy \right) dx \\
&\leq \sum_j \frac{c}{\lambda} \int_{Q_j} |h_j(y)| \int_{\mathbb{R}^n \setminus 5\sqrt{n}Q_j} B(x) w_j(x) |K(x,y) - K(x, x_{Q_j})| dx dy
\end{aligned}
$$

A standard computation using the smoothness condition of K yields that the latter is bounded by

$$
\begin{aligned}
(15) \qquad & \sum_j \frac{c}{\lambda} \int_{Q_j} |h_j(y)| \sum_m \int_{2^m l(Q_j) \leq |x - x_{Q_j}| \leq 2^{m+1} l(Q_j)} B(x) w_j(x) \frac{|y - x_{Q_j}|^\gamma}{|x - x_{Q_j}|^{n+\gamma}} dx dy \\
& \leq \sum_j \frac{c}{\lambda} \int_{Q_j} |h_j(y)| \sum_m \frac{2^{-m\gamma}}{(2^{m+1} l(Q_j))^n} \int_{|x - x_{Q_j}| \leq 2^{m+1} l(Q_j)} B(x) w_j(x) dx dy =
\end{aligned}
$$

Let us estimate the inner sum. We have that

$$
\sum_m \frac{2^{-m\gamma}}{(2^{m+1} l(Q_j))^n} \int_{|x - x_{Q_j}| \leq 2^{m+1} l(Q_j)} B(x) w_j(x) dx
$$

$$
\leq \sum_m \frac{2^{-m\gamma}}{(2^{m+1} l(Q_j))^n} \int_{2^{m+1} Q_j} \prod_{i=1}^k \left| b_i(x) - (b_i)_{Q_j} \right| w_j(x) dx
$$

$$
= \sum_m \frac{2^{-m\gamma}}{(2^{m+1} l(Q_j))^n} \int_{2^{m+1} Q_j} \prod_{i=1}^k \left(\left| b_i(x) - (b_i)_{2^{m+1} Q_j} \right| + \left| (b_i)_{2^{m+1} Q_j} - (b_i)_{Q_j} \right| \right) w_j(x) dx
$$

$$
= \sum_m \frac{2^{-m\gamma}}{(2^{m+1} l(Q_j))^n} \int_{2^{m+1} Q_j} \sum_{l=0}^k \sum_{\sigma \in C_l^k} \left(\prod_{i \in \sigma} \left| b_i(x) - (b_i)_{2^{m+1} Q_j} \right| \right) \left(\prod_{i \in \sigma'} \left| (b_i)_{2^{m+1} Q_j} - (b_i)_{Q_j} \right| \right) w_j(x) dx
$$

$$
= \sum_m \sum_{l=0}^k \sum_{\sigma \in C_l(b)} \left(\prod_{i \in \sigma'} \left| (b_i)_{2^{m+1} Q_j} - (b_i)_{Q_j} \right| \right) \frac{2^{-m\gamma}}{(2^{m+1} l(Q_j))^n} \int_{2^{m+1} Q_j} \left(\prod_{i \in \sigma} \left| b_i(x) - (b_i)_{2^{m+1} Q_j} \right| \right) w_j(x) dx
$$

$$
\leq \sum_m \sum_{l=0}^k \sum_{\sigma \in C_l(b)} \left(\prod_{i \in \sigma'} \| b_i \|_{Osc_{expL^{s_i}}} \right) \frac{2^{-m\gamma}}{(2^{m+1} l(Q_j))^n} \int_{2^{m+1} Q_j} \left(\prod_{i \in \sigma} \left| b_i(x) - (b_i)_{2^{m+1} Q_j} \right| \right) w_j(x) dx
$$

Applying Corollary 2 we have that

$$
\frac{1}{(2^{m+1} l(Q_j))^n} \int_{2^{m+1} Q_j} \left(\prod_{i \in \sigma} \left| b_i(x) - (b_i)_{2^{m+1} Q_j} \right| \right) w_j(x) dx
$$

$$
\leq c \left(\prod_{i \in \sigma} \| b_i \|_{Osc_{expL^{s_i}}} \right) \inf_{z \in 2^{m+1} Q_j} M_{L(\log L)^{\sum_{i \in \sigma} \frac{1}{s_i}}}(w_j)(x)
$$

Then for each $y \in Q_j$

$$\sum_m \sum_{l=0}^{k} \sum_{\sigma \in C_l(b)} \left(\prod_{i \in \sigma'} \|b_i\|_{Osc_{expL^{s_i}}} \right) \frac{2^{-m\gamma}}{(2^{m+1}l(Q_j))^n} \int_{2^{m+1}Q_j} \left(\prod_{i \in \sigma} \left| b_i(x) - (b_i)_{2^{m+1}Q_j} \right| \right) w_j(x) dx$$

$$\leq c \sum_m \frac{1}{2^{m\gamma}} \sum_{l=0}^{k} \sum_{\sigma \in C_l(b)} \left(\prod_{s \in \sigma'} \|b_s\|_{Osc_{expL^{\frac{1}{r_s}}}} \right) \left(\prod_{i \in \sigma} \|b_i\|_{Osc_{expL^{\frac{1}{r_i}}}} \right) \inf_{z \in 2^{m+1}Q_j} M_{L(\log L)^{\sum_{i \in \sigma} \frac{1}{s_i}}}(w_j)(x)$$

$$\leq c_k M_{L(\log L)^{\frac{1}{s}}}(w_j)(y) \sum_m \frac{1}{2^{\gamma m}} = c_k M_{L(\log L)^{\frac{1}{s}}}(w_j)(y).$$

Continuing the computation in (15) we have that by standard estimates,

$$\sum_j \frac{c}{\lambda} \int_{Q_j} |h_j(y)| \sum_m \frac{2^{-m\varepsilon}}{(2^{m+1}l(Q_j))^n} \int_{|x-x_{Q_j}| \leq 2^{m+1}l(Q_j)} B(x) w_j(x) dx dy$$

$$\leq \frac{c_k}{\lambda} \sum_j \int_{Q_j} |h_j(y)| M_{L(\log L)^{\frac{1}{s}}}(w_j)(y) dy$$

$$\leq \frac{c_k}{\lambda} \int_{\mathbb{R}^n} |f(y)| M_{L(\log L)^{\frac{1}{s}}}(w)(y) dy.$$

Summarizing

$$L_1 \leq \frac{c_k}{\lambda} \int_{Q_j} |f(y)| M_{L(\log L)^{\frac{1}{s}}}(w)(y) dy.$$

We shall work now on L_2. Theorem 1.1 from [10] gives

$$L_2 = \tilde{w} \left(\left\{ y \in \mathbb{R}^n : \left| c_k T \left(\sum_j \left(b_1 - (b_1)_{Q_j} \right) \dots \left(b_k - (b_k)_{Q_j} \right) h_j \right)(x) \right| > \frac{\lambda}{6} \right\} \right)$$

$$\leq \frac{c}{\lambda} \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \left| \sum_j \left[\left(b_1(x) - (b_1)_{Q_j} \right) \dots \left(b_k(x) - (b_k)_{Q_j} \right) h_j \right] \right| M_{L(\log L)^\varepsilon} \tilde{w}(x) dx$$

$$\leq \frac{c}{\lambda} \frac{1}{\varepsilon} \sum_j \int_{Q_j} B(x) \left| f(x) - f_{Q_j} \right| M_{L(\log L)^\varepsilon} w_j(x) dx$$

$$\leq \frac{c}{\lambda} \frac{1}{\varepsilon} \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon} w_j(z) \left(\int_{Q_j} B(x)|f(x)|dx + \int_{Q_j} B(x)|f_{Q_j}|dx \right)$$

$$= \frac{c}{\lambda} \frac{1}{\varepsilon} \left(\sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon} w_j(z) \int_{Q_j} B(x)|f(x)|dx + \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon} w_j(z) \int_{Q_j} B(x)|f_{Q_j}|dx \right)$$

$$= \frac{1}{\varepsilon} (L_{21} + L_{22})$$

We estimate first L_{22} as follows

$$\frac{c}{\lambda} \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon} w_j(z) \int_{Q_j} B(x)|f_{Q_j}|dx$$

$$= \frac{c}{\lambda} \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon} w_j(z) \left(\frac{1}{|Q_j|} \int_{Q_j} B(x)dx \right) \left(\int_{Q_j} |f(x)|dx \right)$$

Using Corrollary 2 with $g = 1$ and $f_i = \left|b_i - (b_i)_{Q_j}\right|$, we obtain the following estimate

$$(16) \qquad \frac{1}{|Q_j|} \int_{Q_j} B(x) dx \leq c \prod_{i=1}^{m} \left\| b_i - (b_i)_{Q_j} \right\|_{\exp L^{s_i}, Q_j} \leq c \|\vec{b}\| = c.$$

Then

$$\frac{c}{\lambda} \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon} w_j(z) \left(\frac{1}{|Q_j|} \int_{Q_j} B(x) dx \right) \left(\int_{Q_j} |f(x)| dx \right)$$

$$\leq \frac{c}{\lambda} \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon} w_j(z) \left(\int_{Q_j} |f(x)| dx \right)$$

$$\leq \frac{c}{\lambda} \sum_j \int_{Q_j} |f(x)| M_{L(\log L)^\varepsilon} w_j(x) dx$$

$$\leq \frac{c}{\lambda} \int_{\mathbb{R}^n} |f(x)| M_{L(\log L)^\varepsilon} w_j(x) dx.$$

Let us estimate now L_{21}. Using generalized Hölder inequality (Lemma 1) similarly as we did in (12)

$$L_{21} = \frac{c}{\lambda} \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon} w_j(z) \int_{Q_j} B(x) |f(x)| dx$$

$$\leq \frac{c}{\lambda} \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon} w_j(z) |Q_j| \|f\|_{L(\log L)^{\frac{1}{s}}, Q_j}$$

since $\|\vec{b}\| = 1$. Also the same computation used in (13) based on properties of the Calderón-Zygmund cubes Q_j yields

$$\frac{1}{\lambda} |Q_j| \|f\|_{L(\log L)^{\frac{1}{s}}, Q_j} \leq 2 \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx$$

Hence

$$L_{21} \leq \frac{c}{\lambda} \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon} w_j(z) |Q_j| \|f\|_{L(\log L)^{\frac{1}{s}}, Q_j}$$

$$\leq c \sum_j \inf_{z \in Q_j} M_{L(\log L)^\varepsilon} w_j(z) 2 \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx$$

$$\leq c \sum_j \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) M_{L(\log L)^\varepsilon} w_j(x) dx$$

$$\leq c \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) M_{L(\log L)^\varepsilon} w(x) dx.$$

Putting L_{21} and L_{22} together we have that

$$L_2 \leq c \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) M_{L(\log L)^\varepsilon} w(x) dx$$

To conclude the proof we are left with estimating L_3 as follows

$$L_3 = w \left(\left\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : \left| \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} c_\sigma T_{\vec{\sigma}} \left(\sum_j \left(b - \vec{\lambda} \right)_{\sigma'} h_j \right) (x) \right| > \frac{\lambda}{6} \right\} \right)$$

$$\leq w \left(\left\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : \left| \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} c_\sigma T_{\vec{\sigma}} \left(\sum_j \left(b - \vec{\lambda} \right)_{\sigma'} f \chi_{Q_j} \right) (x) \right| > \frac{\lambda}{12} \right\} \right)$$

$$+ w \left(\left\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : \left| \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} c_\sigma T_{\vec{\sigma}} \left(\sum_j \left(b - \vec{\lambda} \right)_{\sigma'} f_{Q_j} \chi_{Q_j} \right) (x) \right| > \frac{\lambda}{12} \right\} \right)$$

$$= L_{31} + L_{32}$$

To estimate L_{31} we use the inductive hypothesis.

L_{31}

$$= w \left(\left\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : \left| \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} c_\sigma T_{\vec{\sigma}} \left(\sum_j \left(b - \vec{\lambda} \right)_{\sigma'} f \chi_{Q_j} \right) (x) \right| > \frac{\lambda}{12} \right\} \right)$$

$$\leq c \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} w \left(\left\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : \left| T_{\vec{\sigma}} \left(\sum_j \left(b - \vec{\lambda} \right)_{\sigma'} f \chi_{Q_j} \right) (x) \right| > \frac{\lambda}{c_k} \right\} \right)$$

$$\leq c \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} \sum_j \frac{1}{\varepsilon^{\#\sigma+1}} \int_{Q_j} \Phi_{\sum_{i \in \sigma} \frac{1}{s_i}} \left(\|\vec{\sigma}\| \frac{|f(x)|}{\lambda} \left(b(x) - b_{Q_j} \right)_{\sigma'} \right) M_{L(\log L)^{\sum_{i \in \sigma} \frac{1}{s_i} + \varepsilon}} (w_j)(x) dx$$

Since we are assuming that $\|b_1\|_{Osc_{expL^{s_1}}} = \|b_2\|_{Osc_{expL^{s_2}}} = \cdots = \|b_k\|_{Osc_{expL^{s_k}}} = 1$, for each $\sigma \subseteq b$ we have that $\|\vec{\sigma}\| = 1$. Then,

$$c \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} \sum_j \frac{1}{\varepsilon^{\#\sigma+1}} \int_{Q_j} \Phi_{\sum_{i \in \sigma} \frac{1}{s_i}} \left(\frac{|f(x)|}{\lambda} \left(b(x) - b_{Q_j} \right)_{\sigma'} \right) M_{L(\log L)^{\sum_{i \in \sigma} \frac{1}{s_i} + \varepsilon}} (w_j)(x) dx$$

$$\leq c \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} \sum_j \frac{1}{\varepsilon^{\#\sigma+1}} \inf_{z \in Q_j} M_{L(\log L)^{\sum_{i \in \sigma} \frac{1}{s_i} + \varepsilon}} (w_j)(z) \int_{Q_j} \Phi_{\sum_{i \in \sigma} \frac{1}{s_i}} \left(\frac{|f(x)|}{\lambda} \left(b(x) - b_{Q_j} \right)_{\sigma'} \right) dx$$

Let us consider now

$$\Phi_u^{-1}(t) = \frac{t}{\log(e + t)^u} \qquad \varphi_v^{-1}(t) = \log(1 + t)^{\frac{1}{v}}.$$

Then

$$\Phi_{\frac{1}{s}}^{-1}(t) \prod_{i \in \sigma} \varphi s_i^{-1}(t) = \frac{t}{\log(e + t)^{\sum \frac{1}{s_i}}} \prod_{i \in \sigma} \log(1 + t)^{\frac{1}{s_i}} \leq \frac{t}{\log(e + t)^{\sum_{i \in \sigma'} \frac{1}{s_i}}} = \Phi_{\sum_{i \in \sigma'} \frac{1}{s_i}}^{-1}(t)$$

and also we know that

$$\Phi_u(t) \simeq t \left(1 + \log^+ t \right)^u, \varphi_v(t) = e^{t^v} - 1.$$

Taking that into account, Lemma [2] gives

$$\int_{Q_j} \Phi_{\sum_{i\in\sigma} \frac{1}{s_i}} \left(\frac{|f(x)|}{\lambda} \left(b(x) - b_{Q_j}\right)_{\sigma'} \right) dx$$

$$\leq \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx + \sum_{i\in\sigma} \int_{Q_j} \left(\exp\left(\left| b_i(x) - (b_i)_{Q_j} \right|^{s_i} \right) - 1 \right) dx$$

$$(17) \qquad \leq \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx + c \sum_{i\in\sigma} |Q_j| \|b_i\|_{expL^{s_i}, Q_j}$$

$$\leq \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx + c \sum_{i\in\sigma} |Q_j| \|b_i\|_{Osc_{expL^{s_i}}} \qquad \left[\|b_i\|_{Osc_{expL^{s_i}}} = 1 \right]$$

$$\leq \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx + ck|Q_j|$$

$$\leq c_k \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx.$$

In the last step we used properties of the Calderón-Zygmund cubes.

Plugging now that estimate,

$$c \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^k} \sum_j \frac{1}{\varepsilon^{\#\sigma+1}} \inf_{z\in Q_j} M_{L(\log L)^{\sum_{i\in\sigma'} \frac{1}{s_i}+\varepsilon}}(w_j)(z) \int_{Q_j} \Phi_{\sum_{i\in\sigma'} \frac{1}{s_i}} \left(\frac{|f(x)|}{\lambda} \left(b(x) - b_{Q_j}\right)_\sigma \right) dx$$

$$\leq c_k \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^m} \sum_j \frac{1}{\varepsilon^{\#\sigma+1}} \inf_{z\in Q_j} M_{L(\log L)^{\sum_{i\in\sigma'} \frac{1}{s_i}+\varepsilon}}(w_j)(z) \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx$$

$$\leq c_k \frac{1}{\varepsilon^k} \sum_j \inf_{z\in Q_j} M_{L(\log L)^{\frac{1}{s}+\varepsilon}}(w_j)(z) \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx$$

$$\leq c_k \frac{1}{\varepsilon^k} \sum_j \int_{Q_j} M_{L(\log L)^{\frac{1}{s}+\varepsilon}}(w_j)(x) \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx$$

$$\leq c_k \frac{1}{\varepsilon^k} \int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}+\varepsilon}}(w_j)(x) \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx$$

For L_{32} arguing in the same way we have that

$$E_2 \leq c_k \frac{1}{\varepsilon^k} \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^k} \sum_j \inf_{z\in Q_j} M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w_j(z) \int_{Q_j} \Phi_{\sum_{i\in\sigma'} \frac{1}{s_i}} \left(\frac{|f_{Q_j}|}{\lambda} \left(b(x) - b_{Q_j}\right)_\sigma \right) dx$$

The same computation used to obtain (17) yields

$$\int_{Q_j} \Phi_{\sum_{i\in\sigma'} \frac{1}{s_i}} \left(\frac{|f_{Q_j}|}{\lambda} \left(b(x) - b_{Q_j}\right)_\sigma \right) dx \leq \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f_{Q_j}|}{\lambda} \right) dx + ck|Q_j|.$$

Now we see that using Jensen's inequality,

$$\int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f_{Q_j}|}{\lambda} \right) dx \leq |Q_j| \Phi_{\frac{1}{s}} \left(\frac{|f|_{Q_j}}{\lambda} \right)$$

$$\leq |Q_j| \frac{1}{|Q_j|} \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx = \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx.$$

Hence

$$\int_{Q_j} \Phi_{\sum_{i \in \sigma'} \frac{1}{s_i}} \left(\frac{|f_{Q_j}|}{\lambda} \left(b(x) - b_{Q_j} \right)_\sigma \right) dx \leq \int_{Q_j} \Phi_{\frac{1}{s}} \left(\frac{|f(x)|}{\lambda} \right) dx + ck|Q_j|$$

and we finish the estimate arguing as we did for L_{31}. □

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