Publicacions Matemàtiques, Vol 35 (1991), 169–186.

WEIGHTED NORM INEQUALITIES FOR GENERAL MAXIMAL OPERATORS

C. Pérez

1. Introduction

In [13] Muckenhoupt proved the fundamental result characterizing all the weights for which the Hardy–Littlewood maximal operator is bounded; the surprisingly simple necessary and sufficient condition is the so called A_p–condition (see below). A different approach to this characterization was found by Jawerth (cf. [9]). An advantage with this approach is that it generalizes to more general situations; for instance, to Hardy–Littlewood type maximal operators, obtained by replacing the cubes by any collection of sets in $\mathbf{R}^n$, and to spaces of homogeneous type. For a general introduction, and historical comments we refer to [7].

The main purpose of this paper is to use some of the results and techniques in [9] to further investigate weighted norm inequalities for Hardy–Littlewood type maximal operators. We start by introducing some notation. By a **basis** $\mathcal{B}$ in $\mathbf{R}^n$ we mean a collection of open sets in $\mathbf{R}^n$. We say that w is a **weight** associated to the basis $\mathcal{B}$ if w is a non-negative measurable function in $\mathbf{R}^n$ such that $w(B) = \int_B w(y)\,dy < \infty$ for each B in $\mathcal{B}$. $M_{\mathcal{B},w}$ is the corresponding maximal operator defined by

$$M_{\mathcal{B},w}f(x) = \sup_{x\in B} \frac{1}{w(B)} \int_B |f(y)|\, w(y)dy$$

if $x \in \cup_{B\in\mathcal{B}}$ and $M_{\mathcal{B},w}f(x) = 0$ otherwise. If $w \equiv 1$, we just write $M_{\mathcal{B}}f(x)$.

We say that the weight w belongs to the class $A_{p,\mathcal{B}}$, $1 < p < \infty$, if there is a constant c such that

$$\left(\frac{1}{|B|}\int_B w(y)\,dy\right)\left(\frac{1}{|B|}\int_B w(y)^{1-p'}\,dy\right)^{p-1} \leq c$$

for all $B \in \mathcal{B}$. p' will always denote the dual of p, that is

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

In the limit case $p = 1$ we have that w belongs to the class $A_{1,\mathcal{B}}$ if

$$\left(\frac{1}{|B|}\int_B w(y)\,dy\right) ess.sup_B(w^{-1}) \le c$$

for all $B \in \mathcal{B}$; this is equivalent to saying

$$M_{\mathcal{B}}w(x) \le c\,w(x)$$

almost everywhere $x \in \mathbf{R}^n$. For the other limit case, $p = \infty$, we set

$$A_{\infty,\mathcal{B}} = \cup_{p>1} A_{p,\mathcal{B}}.$$

It follows from these definitions and Hölder's inequality that

$$A_{p,\mathcal{B}} \subset A_{q,\mathcal{B}}$$

if $1 \le p \le q \le \infty$.

In section (5) we shall use also the following notation. The weight w belongs to the class $A_{p,\mathcal{B}}(d\mu)$, $1 < p < \infty$, if there is a constant c such that

$$\left(\frac{1}{\mu(B)}\int_B w(y)\,d\mu(y)\right)\left(\frac{1}{\mu(B)}\int_B w(y)^{1-p'}\,d\mu(y)\right)^{p-1} \le c$$

for all $B \in \mathcal{B}$. We also denote by $M_{\mathcal{B},wd\mu}$ the maximal operator defined by

$$M_{\mathcal{B},wd\mu}f(x) = \sup_{x\in B}\frac{1}{(w\mu)(B)}\int_B |f(y)|\,w(y)d\mu(y)$$

if $x \in \cup_{B\in\mathcal{B}}$ and $M_{\mathcal{B},wd\mu}f(x) = 0$ otherwise, with $(wd\mu)(B) = \int_B w(y)\,d\mu(y)$.

One of the main results in [9] is the following Theorem.

Theorem 1.1 (Jawerth). *Let $1 < p < \infty$. Suppose that $\mathcal{B}$ is a basis and that w is a weight, and set $\sigma = w^{1-p'}$. Then*

$$\begin{cases} M_{\mathcal{B}} : L^p(w) \to L^p(w) \\ M_{\mathcal{B}} : L^{p'}(\sigma) \to L^{p'}(\sigma) \end{cases}$$

if and only if

$$\begin{cases} w \in A_{p,\mathcal{B}} \\ M_{\mathcal{B},w} : L^{p'}(w) \to L^{p'}(w) \\ M_{\mathcal{B},\sigma} : L^p(\sigma) \to L^p(\sigma). \end{cases}$$

Theorem 1.1 includes Muckenhoupt's result, mentioned above, that for a fixed $1 < p < \infty$

$$M_Q : L^p(d\mu) \to L^p(d\mu)$$

if and only if $d\mu = w(y)dy$, with $w \in A_{p,Q}$. Here Q is the basis of all open cubes in $\mathbf{R}^n$.

A key fact concerning Theorem 1.1 is that the proof completely avoids the (difficult) "Reverse Hölder inequality."

Acknowledgements. The content of this paper is part of my Washington University Ph. D. Thesis. I would like to express my deepest gratitude to my teacher Björn Jawerth for his guidance and all his teaching. I also would like to thank R. Howard and A. R. Schep from the University of South Carolina, for several conversations concerning their work in [8]. The referee has made several useful observations for which I am grateful. Finally, it is a great honour for me to dedicate this work to the memory of José Luis Rubio de Francia. He introduced me to the field of Harmonic Analysis, and, later, always kindly supported and encouraged me.

2. One-weight theory

It is a fundamental fact that $M_{Q,w}$ is bounded in $L^p(w)$ for each $1 < p < \infty$, if the weight w is doubling (cf. [7] p.144). In particular $M_{Q,w}$ is bounded if w is a $A_{\infty,Q}$ weight. R. Fefferman in [3] and B. Jawerth and A. Torchinsky in [11] (also cf. [7] p.463) proved that the weighted strong maximal operator $M_{\mathcal{R},w}$, that is the weighted maximal operator associated with the basis $\mathcal{B} = \mathcal{R}$ of all rectangles in $\mathbf{R}^n$ with sides parallel to the coordinate axes, is also bounded in $L^p(w)$ whenever the weight w belongs to the class $A_{\infty,\mathcal{R}}$. The proof of this result is based on a geometric covering lemma which goes back to the work of Córdoba (cf. [1]). In this section we show that these results are particular phenomena of a general fact.

Theorem 2.1. *Let $\mathcal{B}$ be a basis. The following statements are equivalent.*

i) For each $1 < p < \infty$, and whenever $w \in A_{p,\mathcal{B}}$

$$M_{\mathcal{B}} : L^p(w) \to L^p(w); \tag{1}$$

ii) for each $1 < p < \infty$, and whenever $w \in A_{\infty,\mathcal{B}}$

$$M_{\mathcal{B},w} : L^p(w) \to L^p(w). \tag{2}$$

Proof: Assume that the basis $\mathcal{B}$ satisfies ii). Fix $1 < p < \infty$, and let $w \in A_{p,\mathcal{B}}$. By denoting $\sigma = w^{1-p'}$, we have that $\sigma \in A_{p',\mathcal{B}} \subset A_{\infty,\mathcal{B}}$, and thus

$$\begin{cases} M_{\mathcal{B},w} : L^{p'}(w) \to L^{p'}(w) \\ M_{\mathcal{B},\sigma} : L^p(\sigma) \to L^p(\sigma). \end{cases}$$

Applying now Theorem 1.1 we get

$$\begin{cases} M_{\mathcal{B}} : L^p(w) \to L^p(w) \\ M_{\mathcal{B}} : L^{p'}(\sigma) \to L^{p'}(\sigma), \end{cases}$$

which in particular gives us i) .

Assuming now i), we fix $1 < p < \infty$, and we take $w \in A_\infty$. Suppose that $w \in A_q$, $1 < q < \infty$. There are two cases.

a) $q \leq p'$;

b) $q > p'$.

In the first case we have that $w \in A_{p'}$, which means that $w^{1-p} \in A_p$. Hence, by hypothesis,

$$\begin{cases} M_{\mathcal{B}} : L^{p'}(w) \to L^{p'}(w) \\ M_{\mathcal{B}} : L^p(w^{1-p}) \to L^p(w^{1-p}). \end{cases}$$

And applying again Theorem 1.1 we obtain

$$M_{\mathcal{B},w} : L^p(w) \to L^p(w).$$

In the second case we have that

$$\begin{cases} M_{\mathcal{B}} : L^q(w) \to L^q(w) \\ M_{\mathcal{B}} : L^{q'}(w^{1-q'}) \to L^{q'}(w^{1-q'}). \end{cases}$$

since $w^{1-q'} \in A_{q'}$. Now by using Theorem 1.1 we get

$$M_{\mathcal{B},w} : L^{q'}(w) \to L^{q'}(w).$$

Since we always have that

$$M_{\mathcal{B},w} : L^\infty(w) \to L^\infty(w),$$

we can interpolate to get

$$M_{\mathcal{B},w} : L^p(w) \to L^p(w),$$

for every $q' < p < \infty$, and hence $p' < q$. This concludes the proof. ∎

Example 2.2. Let $\mathbf{M}_{\Re}$ be the Córdoba–Zygmund maximal operator. The basis $\Re$ defining this maximal operator is formed by those rectangles in $\mathbf{R}^3$ with sides parallel to the coordinate axes whose sidelengths are of the form $\{s, t, st\}$. It has been shown by R. Fefferman (see for instance **[4]**) that $M_{\Re}$ is bounded in $L^p(w)$, for each $1 < p < \infty$, if and only if $w \in A_{p,\Re}$. Hence, by Theorem 2.1

$$M_{\Re,w} : L^p(w) \to L^p(w).$$

for each $1 < p < \infty$, whenever $w \in A_{\infty,\Re}$

In view of all these important examples we make the following definition.

Definition 2.3. We say that the basis $\mathcal{B}$ is a *Muckenhoupt* basis if for each $1 < p < \infty$, and every $w \in A_{p,\mathcal{B}}$

$$M_{\mathcal{B}} : L^p(w) \to L^p(w).$$

With this definition, Theorem 2.1 can be stated as follows

$\mathcal{B}$ is a Muckenhoupt basis

if and only if

$$(3) \qquad M_{\mathcal{B},w} : L^p(w) \to L^p(w),$$

for each $1 < p < \infty$, and whenever $w \in A_{\infty,\mathcal{B}}$

Next result is an extension of Lin's result (cf. [**12**]) for the strong maximal operator to any maximal operator whose basis is a Muckenhoupt basis.

Corollary 2.4. *Suppose that $\mathcal{B}$ is a Muckenhoupt basis. Let $1 < p < \infty$, and suppose that $w \in A_{\infty,\mathcal{B}}$, then the following Fefferman–Stein inequality holds*

$$(4) \qquad \int_{\mathbf{R}^n} M_{\mathcal{B}}f(y)^p\, w(y)dy \le c \int_{\mathbf{R}^n} f(y)^p M_{\mathcal{B}}\, w(y)dy.$$

Proof: Suppose that $w \in A_{r,\mathcal{B}}$, for some $1 < r < \infty$. The following pointwise inequality then follows easily from Hölder's inequality and from the $A_{r,\mathcal{B}}$ – condition

$$(5) \qquad M_{\mathcal{B}}(\chi_E)(x) \le c\,(M_{\mathcal{B},w}(\chi_E)(x))^{1/r}.$$

Since $\mathcal{B}$ is a Muckenhoupt basis, Theorem 2.1 yields

$$(6) \qquad M_{\mathcal{B},w} : L^p(w) \to L^p(w),$$

and together with (5) we easily conclude that for each measurable E and each $0 < \lambda < \infty$ the following inequality holds:

$$(7) \qquad w\,(M_{\mathcal{B}}(\chi_E) > \lambda) \le c(\lambda)w(E).$$

Since also $M_{\mathcal{B}} : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$, we are now in a position where we can proceed as in the proof of Lemma 7.1 in [**9**], to conclude the proof of the Corollary. ∎

Remark 2.5. We point out that for (4) to hold, we do not need to assume that $w \in A_{\infty,\mathcal{B}}$; (7) for some $0 < \lambda < 1$ would be sufficient.

For the sake of completness we observe that Muckenhoupt bases satisfy Jones' factorization theorem. We just state the result and refer the reader to [**9**], Corollary 6.1, for the proof under a weaker condition on the basis.

Proposition 2.6. *Suppose that $\mathcal{B}$ is a Muckenhoupt basis, and let $1 < p < \infty$. Suppose that $w \in A_{p,\mathcal{B}}$. Then there are weights $w_1, w_2 \in A_{1,\mathcal{B}}$ such that $w = w_1 w_2^{1-p}$.*

3. Vector–valued inequalities

It is well known by now that there is an intimate connection between vector valued inequalities and weigthed norm inequalities (cf. [7] Chapter 5). In this section we shall use the results from the previous one to obtain an extension of the classical Fefferman–Stein vector-valued inequality (cf. [2])

$$\left\| \left(\sum_{i=0}^{\infty} |M_Q f_i|^q \right)^{1/q} \right\|_{L^p} \le c_{p,q} \left\| \left(\sum_{i=0}^{\infty} |f_i|^q \right)^{1/q} \right\|_{L^p}, \tag{8}$$

where $1 < p < \infty$ and $1 < q \le \infty$, to any maximal operator $M_{\mathcal{B}}$ whose basis is a Muckenhoupt basis (cf. the definition above). We would like to point out that (8) has played a fundamental role in the analysis made in the recent works [5] and [10]. We shall assume throughout the section that $\mathcal{B}$ is a Muckenhoupt basis.

For a fix $1 < p < \infty$, and motivated by the method introduced by J. L. Rubio de Francia (cf. section 5.5 in [7] and section 6 in [9]), we denote by $R_{\mathcal{B}}$ the operator

$$R_{\mathcal{B}} : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$$
$$f \to \sum_{i=0}^{\infty} \frac{M_{\mathcal{B}}^i f}{(2K)^i},$$

where $M_{\mathcal{B}}^0 = I_d$ and $M_{\mathcal{B}}^i$ is the ith iterate of the operator $M_{\mathcal{B}}$. K is the norm of $M_{\mathcal{B}}$, as an operator on $L^p(\mathbf{R}^n)$. Although $R_{\mathcal{B}}$ is pointwise larger than $M_{\mathcal{B}}$, it preserves most of its properties, namely

i)
$$u \le R_{\mathcal{B}}\, u$$

ii)
$$\|R_{\mathcal{B}}\, u\|_{L^p(\mathbf{R}^n)} \le 2\|u\|_{L^p(\mathbf{R}^n)}.$$

Furthermore, $R_{\mathcal{B}}$ has the property that

iii)
$$R_{\mathcal{B}}\, f \in A_{1,\mathcal{B}}$$

for each $f \in L^p(\mathbf{R}^n)$. The last statement is not true for $M_{\mathcal{B}} f$ as the following argument shows. Consider the Hardy–Littlewood maximal operator $M = M_Q$, and take any positive function $\varphi \in L^1(\mathbf{R}^n)$. We shall show that $M\varphi$ is not even an A_∞ weight. Indeed, suppose that $M\varphi \in A_p$, for some $1 < p < \infty$. Then $(M\varphi)^{1-p'} \in A_{p'}$ and, by Muckenhoupt's theorem,

$$\int_{\mathbf{R}^n} Mf(y)^{p'}\, M\varphi(y)^{1-p'} dy \le c \int_{\mathbf{R}^n} f(y)^{p'}\, M\varphi(y)^{1-p'} dy.$$

By taking $f = \varphi$ we see by Lebesgue's differentiation theorem, that the right hand side of the inequality is finite, while the left hand side is infinite[1].

Lemma 3.1. *Let $1 < p, t < \infty$, and let $\mathcal{B}$ be a Muckenhoupt basis. Then for each nonnegative function $u \in \cup_{s>1} L^s(\mathbf{R}^n)$ we have*

$$\int_{\mathbf{R}^n} M_{\mathcal{B}} f(y)^p \, u(y) dy \le c \int_{\mathbf{R}^n} f(y)^p \, R_{\mathcal{B}} u(y) dy,$$

and

$$\int_{\mathbf{R}^n} M_{\mathcal{B}} f(y)^p \, \frac{1}{(R_{\mathcal{B}}(u^t))(y)^{1/t}} \, dy \le c \int_{\mathbf{R}^n} f(y)^p \, \frac{1}{u(y)} dy, \tag{9}$$

for each $t > \frac{1}{p-1}$.

Proof: The first inequality follows from above remarks and (4). For the second we observe that $(R_{\mathcal{B}}\, u)^{-1/t} = \left[(R_{\mathcal{B}}\, u)^{1/t(p-1)} \right]^{1-p} \in A_{p,\mathcal{B}}$, and this, in turn, follows from the fact that if $w \in A_{1,\mathcal{B}}$ then $w^\delta \in A_{1,\mathcal{B}}$, if $0 \le \delta < 1$, and from the easy part of (2.6). Finally, (9) follows from the definition of Muckenhoupt basis and from above remarks about $R_{\mathcal{B}}$. ■

Remark 3.2. We point out that for the case $\mathcal{B} = \mathcal{Q}$ the following inequality holds:

$$\int_{\mathbf{R}^n} M f(y)^p \, \frac{dy}{M(u^t)(y)^{1/t}} \le c \int_{\mathbf{R}^n} f(y)^p \, \frac{dy}{u}, \tag{10}$$

for $1 < p < \infty$ and $t > \frac{1}{p-1}$. The result is false if $t = \frac{1}{p-1}$, (cf. [14]).

We may think of $R_{\mathcal{B}}$ as being the right substitute in the general case.

Once we have this lemma then the vector-valued inequality for $M_{\mathcal{B}}$ follows from Theorem 5.2 Chapter 5 in [7].

Theorem 3.3. *Let $1 < p < \infty$, and $1 < q \le \infty$. If $\mathcal{B}$ is a Muckenhoupt basis, then*

$$\left\| \left(\sum_{i=0}^{\infty} |M_{\mathcal{B}} f_i|^q \right)^{1/q} \right\|_{L^p} \le c_{p,q} \left\| \left(\sum_{i=0}^{\infty} |f_i|^q \right)^{1/q} \right\|_{L^p}. \tag{11}$$

Remark 3.4. Although we have mentioned that the theorem follows from Lemma 3.1, it is to be mentioned that we just need the first half of it. Indeed, this and a standar procedure would give the case $q < p$. Now, since the theorem is obvious for $q = p$, and also for $q = \infty$, the case $1 < p < q < \infty$ is obtained by interpolation.

In the same spirit we make the following observation about the class $A_{1,\mathcal{B}}$.

[1]Following the method in [16] it is possible to prove that $M\varphi$ may not be even doubling. We are indebted to F. Soria for showing this to us.

Lemma 3.5. *Assume that $\mathcal{B}$ is a Muckenhoupt basis. Let $0 < q \leq 1$, and let $w \in \cup_{p>1} L^p(\mathbf{R}^n)$. Then*

$$w \in A_{1,\mathcal{B}}$$

if and only if there is a positive function $g \in \cup_{p>1} L^p(\mathbf{R}^n)$, such that for some large enough constant A

$$w \approx \left[\sum_{i=0}^{\infty} \left(\frac{M_{\mathcal{B}}^i g}{A^i}\right)^q\right]^{1/q}. \tag{12}$$

Proof: By iteration $M_{\mathcal{B}}^i w \leq c^i w$, for some constant c. Then putting $A = c2^{1/q}$ and $g = w$ we get

$$\left[\sum_{i=0}^{\infty} \left(\frac{M_{\mathcal{B}}^i g}{A^i}\right)^q\right]^{1/q} \leq 2w.$$

Since obviously

$$w \leq \left(\sum_{i=0}^{\infty} \left(\frac{M_{\mathcal{B}}^i g}{A^i}\right)^q\right)^{1/q},$$

(12) follows.

To prove the converse, let G denote the right hand side of (12). Since $w \approx G$, it is enough to deal with G. We first show that G is in $\cup_{p>1} L^p(\mathbf{R}^n)$. Indeed, suppose that $g \in L^p(\mathbf{R}^n)$, $1 < p < \infty$. Then

$$\|G\|_{L^p}^q = \left\|\sum_{i=0}^{\infty} \left(\frac{M_{\mathcal{B}}^i g}{A^i}\right)^q\right\|_{L^{p/q}} =$$

$$= \sum_{i=0}^{\infty} \int_{\mathbf{R}^n} \left(\frac{M_{\mathcal{B}}^i g(y)}{A^i}\right)^q u(y)dy,$$

for some $u \in L^{(p/q)'}(\mathbf{R}^n)$ with unit norm. Therefore, by the above remarks and by iterating (4), the last expression is dominated by

$$\sum_{i=0}^{\infty} \int_{\mathbf{R}^n} \left(\frac{M_{\mathcal{B}}^i g(y)}{A^i}\right)^q R_{\mathcal{B}} u(y)dy \leq \sum_{i=0}^{\infty} \left(\frac{F}{A}\right)^{qi} \int_{\mathbf{R}^n} g(y)^q R_{\mathcal{B}} u(y)dy.$$

Here F denotes the smallest constant for which (4) holds. Finally, by taking $A = 2^{1/q}F$, using Hölder's inequality, and ii) above obtain that $\|G\|_{L^p} \leq 2^{1/q}\|g\|_{L^p}$.

We shall prove now that $G \in A_{1,\mathcal{B}}$. For each $B \in \mathcal{B}$ we shall see that

$$\frac{1}{|B|}\int_B G(y)\,dy \leq A\,G(x),$$

a. e. $x \in B$. Indeed, by Minkowski's integral inequality with $r = \frac{1}{q} > 1$

$$\left(\frac{1}{|B|}\int_B G(y)\,dy\right)^{1/r} = \left[\frac{1}{|B|}\int_B \left(\sum_{i=0}^{\infty}\left(\frac{M_{\mathcal{B}}^i g(y)}{A^i}\right)^q\right)^r dy\right]^{1/r} \leq$$

$$\leq \sum_{i=0}^{\infty}\left(\frac{1}{|B|}\int_B \frac{M_{\mathcal{B}}^i g(y)}{A^i}\,dy\right)^{1/r} \leq A^q \sum_{i=0}^{\infty}\left(\frac{M_{\mathcal{B}}^{i+1} g(x)}{A^{i+1}}\right)^q = A^q G(x)^q. \quad \blacksquare$$

4. An alternative formulation of Muckenhoupt's theorem

In this section we give a different criterion to decide whether the operator $M_{\mathcal{B}}$ is bounded on $L^p(w)$, assuming that the basis is a Muckenhoupt basis. In particular this result applies to the case $\mathcal{B} = \mathcal{Q}$, providing a different characterization of Muckenhoupt's theorem. This approach is inspired by the results in [**8**].

Theorem 4.1. *Let $1 < p < \infty$, and suppose that $\mathcal{B}$ is a Muckenhoupt basis. Let w be a weight for $\mathcal{B}$. Then*

$$M_{\mathcal{B}} : L^p(w) \to L^p(w) \tag{13}$$

if and only if

$$M_{\mathcal{B}}\left(w\,(M_{\mathcal{B}} g_0)^{p-1}\right) \leq c\, w\, g_0^{\,p-1}, \tag{14}$$

for some nonnegative measurable function g_0, with $\int_B g_0(y)\,dy < \infty$ for each B in $\mathcal{B}$, and for some positive constant c.

Proof: Assuming (13) it is standard to see that $w \in A_{p,\mathcal{B}}$, and hence $\sigma = w^{1-p'} \in A_{p',\mathcal{B}}$. Since $\mathcal{B}$ is a Muckenhoupt basis $M_{\mathcal{B}} : L^p(w) \to L^p(w)$, and $M_{\mathcal{B}} : L^{p'}(\sigma) \to L^{p'}(\sigma)$. Hence

$$S_1 : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$$

$$f \to w^{1/p}\, M_{\mathcal{B}}(w^{-1/p} f)$$

and

$$S_2 : L^{p'}(\mathbf{R}^n) \to L^{p'}(\mathbf{R}^n)$$

$$f \to w^{-1/p}\, M_{\mathcal{B}}(w^{1/p} f),$$

which implies

$$T : L^{p'}(\mathbf{R}^n) \to L^{p'}(\mathbf{R}^n)$$

$$f \to S_2\left((S_1 f^{p'/p})^{p-1}\right).$$

Hence, by using the iteration technique of Rubio de Francia (cf.[7] p. 434, [9] p.392, or the previous work of Gagliardo in [6]), there exists a nonnegative measurable function $h \in L^{p'}(\mathbf{R}^n)$ for which

$$S_2\left((S_1 h^{p'/p})^{p-1}\right) \le c\, h,$$

or, equivalently,

$$M_{\mathcal{B}}\left(w\,\left(M_{\mathcal{B}}(w^{-1/p}h^{p'/p})\right)^{p-1}\right) \le c\, w^{1/p}h.$$

Taking $g_0 = w^{-1/p}h^{p'/p}$ we obtain (14). Note that $\int_B g_0(y)\,dy < \infty$ for each B in $\mathcal{B}$ since

$$\int_B g_0(y)\,dy \le \left(\int_B w(y)^{1-p'}\,dy\right)^{1/p'} \left(\int_B h(y)^{p'}\,dy \le\right)^{1/p} < \infty,$$

by Hölder's inequality, and because $w^{1-p'} \in A_{p',\mathcal{B}}$ and $h \in L^{p'}(\mathbf{R}^n)$.

To prove the converse, we note that it is enough to show that w satisfies the $A_{p,\mathcal{B}}$–condition since $\mathcal{B}$ is a Muckenhoupt basis. For each $B \in \mathcal{B}$ we define the constant A by

$$A = \frac{1}{|B|}\int_B w(y)\,dy\left(\frac{1}{|B|}\int_B w(y)^{1-p'}\,dy\right)^{p-1} =$$

$$= \frac{1}{|B|}\int_B w(y)\,dy\left(\frac{1}{|B|}\int_B g_0^{1/p'}g_0^{-1/p'}w(y)^{1-p'}\,dy\right)^{p-1}.$$

Applying Hölder's inequality and the hypothesis about g_0 we estimate A by

$$\frac{1}{|B|}\int_B w(y)\,dy\left(\frac{1}{|B|}\int_B g_0(y)\,dy\right)^{(p-1)/p'}\left(\frac{1}{|B|}\int_B g_0(y)^{1-p}w(y)^{-p'}\,dy\right)^{(p-1)/p} =$$

$$= \frac{1}{|B|}\int_B w(y)\,dy\left(\frac{1}{|B|}\int_B g_0(y)\,dy\right)^{p-1}$$

$$\times\left(\frac{1}{|B|}\int_B g_0(y)\,dy\right)^{-1/p'}\left(\frac{1}{|B|}\int_B g_0(y)^{1-p}w(y)^{-p'}\,dy\right)^{1/p'} \le$$

$$\le c\,\inf_B w\,g_0^{p-1}\left(\frac{1}{|B|}\int_B g_0(y)\,dy\right)^{-1/p'}\left(\frac{1}{|B|}\int_B g_0(y)^{1-p}w(y)^{-p'}\,dy\right)^{1/p'} \le$$

$$\le c\left(\frac{1}{|B|}\int_B g_0(y)\,dy\right)^{-1/p'}\left(\frac{1}{|B|}\int_B g_0(y)\,dy\right)^{1/p'} = c,$$

concluding the proof of the Theorem. ■

5. A characterization of the reverse Hölder inequality classes for general bases

We denote by $RH_{q,\mathcal{B}}$, $1 < q < \infty$, the class of weights satisfying a reverse Hölder inequality of order q uniformly on each $B \in \mathcal{B}$. That is

$$RH_{q,\mathcal{B}} = \{w : (\frac{1}{|B|} \int_B w(y)^q \, dy)^{1/q} \leq \frac{c}{|B|} \int_B w(y) \, dy, \ B \in \mathcal{B}\} \tag{15}$$

for some positive constant c, independent of $B \in \mathcal{B}$.

To illustrate the interest of the reverse Hölder classes, consider the simpler operator

$$m_{w,B} f(x) = \frac{1}{w(B)} \int_B f(y) \, w(y) dy \, \chi_B(x),$$

where B is an arbitrary fixed set in $\mathcal{B}$. It readily follows from the next (easy) lemma that for $1 < p < \infty$

$$m_{w,B} : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$$

if and only if

$$w \in RH_{p',\mathcal{B}}.$$

Lemma 5.1. *Let $1 < p < \infty$. The following statements are equivalent.*

i) There is a positive constant c, independent of $B \in \mathcal{B}$, such that for every nonnegative locally integrable function f

$$\frac{1}{w(B)} \int_B f(y) \, w(y) dy \leq c \left(\frac{1}{|B|} \int_B f(y)^p \, dy \right)^{1/p} ;$$

ii)

$$w \in RH_{p',\mathcal{B}}.$$

Proof: ii) follows from i) by taking $f = w^{p'/p}$, and i) follows from ii) by Hölder's inequality:

$$\frac{1}{w(B)} \int_B f(y) \, w(y) dy \leq$$

$$\leq \frac{1}{w(B)} \left(\int_B f(y)^p \, dy \right)^{1/p} \left(\int_B w(y)^{p'} \, dy \right)^{1/p'} =$$

$$= \frac{|B|}{w(B)} \left(\frac{1}{|B|} \int_B f(y)^p \, dy \right)^{1/p} \left(\frac{1}{|B|} \int_B w(y)^{p'} \, dy \right)^{1/p'} \leq$$

$$\leq c \left(\frac{1}{|B|} \int_B f(y)^p \, dy \right)^{1/p} . \quad \blacksquare$$

To obtain a result for $M_{\mathcal{B},w}$, we need a way of measuring how the different $m_{w,\mathcal{B}}$'s interfere. This is intimately connected with the geometry of the particular basis $\mathcal{B}$, and hence to covering properties of families of sets belonging to $\mathcal{B}$. This, in turn, is essentially equivalent to mapping properties of the maximal operators.

Our characterization is the following

Theorem 5.2. *Let $1 < p < \infty$. Suppose that $\mathcal{B}$ is a basis and that w is a weight. Then*

$$\begin{cases} M_{\mathcal{B},w} : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n) \\ M_{\mathcal{B},w} : L^{p'}(w^{p'}) \to L^{p'}(w^{p'}) \end{cases}$$

if and only if

$$\begin{cases} w \in RH_{p',\mathcal{B}} \\ M_{\mathcal{B}} : \ L^{p'}(\mathbf{R}^n) \to L^{p'}(\mathbf{R}^n) \\ M_{\mathcal{B},w^{p'}} : L^p(w^{p'}) \to L^p(w^{p'}). \end{cases}$$

Proof: Let dw denote the measure $dw = wdx$. Noticing that $w \in RH_{p',\mathcal{B}}$ is equivalent to $w^{-1} \in A_{p,\mathcal{B}}(dw)$, and writing $dx = w^{-1}dw$, and $w^{p'} = \sigma^{-1}dw$, where $\sigma = w^{1-p'}$, the following equivalence follows

$$\begin{cases} w \in RH_{p',\mathcal{B}} \\ M_{\mathcal{B}} : \ L^{p'}(\mathbf{R}^n) \to L^{p'}(\mathbf{R}^n) \\ M_{\mathcal{B},w^{p'}} : L^p(w^{p'}) \to L^p(w^{p'}) \end{cases}$$

if and only if

$$\begin{cases} w^{-1} \in A_{p,\mathcal{B}}(dw) \\ M_{\mathcal{B},w^{-1}dw} : L^{p'}(w^{-1}dw) \to L^{p'}(w^{-1}dw) \\ M_{\mathcal{B},\sigma^{-1}dw} : L^p(\sigma^{-1}dw) \to L^p(\sigma^{-1}dw) \end{cases}$$

If we now apply Theorem refA with the Lebesgue measure replaced by the measure dw, this is equivalent to

$$\begin{cases} M_{\mathcal{B},dw} : L^p(w^{-1}dw) \to L^p(w^{-1}dw) \\ M_{\mathcal{B},dw} : L^{p'}(\sigma^{-1}dw) \to L^{p'}(\sigma^{-1}dw) \end{cases}$$

that is

$$\begin{cases} M_{\mathcal{B},w} : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n) \\ M_{\mathcal{B},w} : L^{p'}(w^{p'}) \to L^{p'}(w^{p'}) \end{cases}$$

And this concludes the proof of the Theorem. ■

In particular we have the following corollary,

Corollary 5.3. *Let $1 < p < \infty$. Suppose that $\mathcal{B}$ is a basis such that $M_{\mathcal{B}} : L^{p'}(\mathbf{R}^n) \to L^{p'}(\mathbf{R}^n)$, and that $w \in RH_{p',\mathcal{B}}$ implies $M_{\mathcal{B},w^{p'}} : L^p(w^{p'}) \to L^p(w^{p'})$. Then*

$$M_{\mathcal{B},w} : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n) \tag{16}$$

if and only if

$$w \in RH_{p',\mathcal{B}}. \tag{17}$$

Proof: (17) follows from (16) by taking $f = w^{p'/p}\chi_B$, and using the definition of $M_{\mathcal{B},w}$. The converse is immediate from Theorem 5.2. ∎

Corollary 5.4. *Let $1 < p < \infty$. Then*

$$M_{Q,w} : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n) \tag{18}$$

if and only if

$$w \in RH_{p',Q}. \tag{19}$$

Proof: $M_Q : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ is the Hardy-Littlewood maximal Theorem. For $M_{Q,w^{p'}} : L^p(w^{p'}) \to L^p(w^{p'})$, it would be enough to prove that $w^{p'}$ is doubling, but in fact something better holds (cf. [17])

$$w \in RH_{p',Q} \Leftrightarrow w^{p'} \in A_{\infty,Q}. \quad ∎ \tag{20}$$

Remark 5.5. There is an $L^p - L^q$ version of this characterization. Let $0 \le \beta < n$, and assume that $1 < p \le q < \infty$, satisfy the "Sobolev relation",

$$\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}.$$

Then, the following statements are equivalent.

i)

$$M_{\beta,w} : L^p(\mathbf{R}^n) \to L^q(\mathbf{R}^n);$$

ii)

$$w \in RH_{p',Q}.$$

Here $M_{\beta,w}$ is the following weighted fractional maximal operator

$$M_{\beta,w}f(x) = \sup_{x\in Q} \frac{|Q|^{\beta/n}}{w(Q)} \int_Q f(y)\, w(y)dy,$$

where the supremum is taken over all cubes.

Remark 5.6. There is another proof of the nontrivial part of Corollary 5.4, closer in spirit to the classical proof of Muckenhoupt's theorem. Suppose $w \in RH_{p',Q}$. It was discovered by Gehring that $w \in RH_{(p-\epsilon)',Q}$ for some tiny $\epsilon > 0$. Then by Lemma 5.1 we have

$$\frac{1}{w(Q)} \int_Q f(y)\, w(y) dy \leq \left(\frac{1}{|Q|} \int_Q f(y)^{p-\epsilon}\, dy \right)^{1/(p-\epsilon)},$$

and by picking $\epsilon > 0$ smaller if necessary, we have

$$\int_{\mathbf{R}^n} M_{Q,w} f(y)^p\, dy \leq \int_{\mathbf{R}^n} \left(M_Q(f^{p-\epsilon})(y) \right)^{p/(p-\epsilon)} dy \leq c \int_{\mathbf{R}^n} f(y)^p\, dy,$$

by the Hardy-Littlewood maximal Theorem.

Corollary 5.7. *Let* $1 < p < \infty$. *Then*

$$M_{\mathcal{R},w} : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n) \tag{21}$$

if and only if

$$w \in RH_{p',\mathcal{R}}. \tag{22}$$

Proof: As in the proof of Corollary 5.4 we just need to check both $M_{\mathcal{R}} : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ and $M_{\mathcal{R},w^{p'}} : L^p(w^{p'}) \to L^p(w^{p'})$. The first one is the classical Theorem of Jessen, Marcinkiewicz and Zygmund. Now, since $\mathcal{R}$ is a Muckenhoupt basis, (3) yields the boundedness of $M_{\mathcal{R},w^{p'}}$ if we show that

$$w^{p'} \in A_{\infty,\mathcal{R}} \Leftrightarrow w \in RH_{p',\mathcal{R}}.$$

To prove this, we use Theorem (6.7) p. 458 in [7], and the proof of the case $\mathcal{B} = Q$ in [17] applies mutatis mutandis to the case $\mathcal{B} = \mathcal{R}$. ■

6. Two weight theory

In this section we discuss two weighted norm inequalities for $M_{\mathcal{B}}$. We shall extend the two weights results in [9] to the Lorentz spaces.

We recall that a function f belongs to the Lorentz space $L(r,s)$ if

$$\|f\|_{L(r,s)(\mu)} = \left[\int_0^\infty \left(t\, \mu\{x \in \mathbf{R}^n : |f(x)| > t\}^{1/r} \right)^s \frac{dt}{t} \right]^{1/s} < \infty$$

Theorem 6.1. *Let $1 < p, q < \infty$, and let (v, u) be a couple of weights. Call $\sigma = u^{1-p'}$. Assume that the basis $\mathcal{B}$ satisfies the condition that for every set G which is a union of sets in $\mathcal{B}$ the following holds*

$$\left[\int_G M_{\mathcal{B}}(\sigma\chi_G)(y)^q\, v(y)dy\right]^{1/q} \leq c\sigma(G)^{1/p}.$$

Then for each smooth f

$$\|M_{\mathcal{B}} f\|_{L^q(v)} \leq c\|M_{\mathcal{B},\sigma}(f/\sigma)\|_{L^{p,q}(\sigma)} \tag{23}$$

for some constant c independent of f, v, and σ.

Proof: For each integer k consider the set

$$E_k = \{y \in \mathbf{R}^n : 2^k < M_{\mathcal{B}} f(y) \leq 2^{k+1}\}.$$

From the definition of $M_{\mathcal{B}}$, $E_k \subset \cup_j B_{k,j}$, where $B_{k,j} \in \mathcal{B}$ satisfies

$$2^k < \frac{1}{|B_{k,j}|}\int_{B_{k,j}} f(y)\, dy.$$

Define

$$E_{k,1} = B_{k,1} \cap E_k,$$

and for $j > 1$

$$E_{k,j} = (B_{k,j} \setminus \cup_{s<j} B_{k,s}) \cap E_k.$$

Each of the sets E_k is the disjoint union of the sets $E_{k,j}$.

We now can write

$$\int_{\mathbf{R}^n} M_{\mathcal{B}} f(y)^q\, v(y)dy = \sum_k \int_{E_k} M_{\mathcal{B}} f(y)^q\, v(y)dy =$$

$$= \sum_{k,j} \int_{E_{k,j}} M_{\mathcal{B}} f(y)^q\, v(y)dy \leq 2^q \sum_{k,j} 2^{kq} v(E_{k,j}) \leq$$

$$\leq C \sum_{k,j} v(E_{k,j}) \left(\frac{1}{|B_{k,j}|}\int_{B_{k,j}} f(y)\, dy\right)^q =$$

$$= c\sum_{k,j} v(E_{k,j}) \left(\frac{1}{|B_{k,j}|}\int_{B_{k,j}} \sigma(y)\, dy\right)^q [(\frac{1}{\sigma(B_{k,j})}\int_{B_{k,j}} \frac{f(y)}{\sigma(y)}\, \sigma(y)dy\,)^p]^{q/p} =$$

$$= c\sum_{k,j} \mu_{k,j} g_{k,j}^{q/p},$$

where

$$\mu_{k,j} = v(E_{k,j}) \left(\frac{1}{|B_{k,j}|} \int_{B_{k,j}} \sigma(y)\, dy \right)^q ,$$

and

$$g_{k,j} = \left(\frac{1}{\sigma(B_{k,j})} \int_{B_{k,j}} \frac{f(y)}{\sigma(y)} \sigma(y) dy \right)^p .$$

We view the sum $\sum_{k,j} \mu_{k,j} g_{k,j}^{q/p}$, as an integral on a measure space (X, μ) built over the set $X = \{k, j\}$, assigning to each (k, j) the measure $\mu_{k,j}$. For $\lambda > 0$, set

$$\Gamma(\lambda) = \{(k,j) : g_{k,j} > \lambda\},$$

$$G(\lambda) = \cup_{(k,j)\in\Gamma(\lambda)} B_{k,j}.$$

Then

$$\sum_{k,j} g_{k,j}^{q/p}\, \mu_{k,j} = \int_0^\infty \lambda^{q/p} \mu(\Gamma(\lambda))\, \frac{d\lambda}{\lambda}.$$

We can estimate $\mu(\Gamma(\lambda))$ as follows

$$\mu(\Gamma(\lambda)) = \sum_{(k,j)\in\Gamma(\lambda)} \mu_{k,j} \leq$$

$$\sum_{(k,j)\in\Gamma(\lambda)} \int_{E_{k,j}} M_{\mathcal{B}}\,(\sigma \chi_{B_{k,j}})(y)^q\, v(y) dy \leq$$

$$\leq c\sigma(G(\lambda))^{q/p} \leq c\sigma\left(\{y \in \mathbf{R}^n : M_{\mathcal{B},\sigma}(f/\sigma)(y)^p > \lambda\}\right)^{q/p} .$$

Here we have used the hypothesis on $M_{\mathcal{B},\sigma}$ in the third inequality. Finally by making a change of variables we obtain

$$\int_{\mathbf{R}^n} M_{\mathcal{B}} f(y)^q\, v(y) dy \leq$$

$$\leq c \int_0^\infty \left(\lambda \sigma\left(\{y \in \mathbf{R}^n : M_{\mathcal{B},\sigma}(f/\sigma)(y) > \lambda\}\right)^{1/p} \right)^q \frac{d\lambda}{\lambda} =$$

$$= c\|M_{\mathcal{B},\sigma}(f/\sigma)\|^q_{L(p,q)},$$

concluding the proof. ■

As a consequence of this theorem we can deduce the following characterization.

Corollary 6.2. *Let $1 < p \leq q < \infty$, and let (v,u) be a couple of weights. Suppose that $M_{\mathcal{B},\sigma} : L^p(\sigma) \to L^p(\sigma)$, where $\sigma = u^{1-p'}$. Then*

$$M_{\mathcal{B}} : L^p(u) \to L^q(v) \tag{24}$$

if and only if

$$\left(\int_G M_{\mathcal{B}}(\sigma\chi_{G(\lambda)})(y)^q\, v(y)dy\right)^{1/q} \leq c\sigma(G(\lambda))^{1/p}, \tag{25}$$

for every set G which is a union of sets in $\mathcal{B}$.

Proof: By setting $f = \sigma\chi_{G(\lambda)}$ in (24) we readily get (25). To prove the converse we use Theorem 6.1, that $p \leq q$, and our hypothesis on $M_{\mathcal{B},\sigma}$

$$\|M_{\mathcal{B}} f\|_{L^q(v)} \leq c\|M_{\mathcal{B},\sigma}(f/\sigma)\|_{L^{p,q}(\sigma)} \leq$$

$$\leq c\,\|M_{\mathcal{B},\sigma}(f/\sigma)\|_{L^p(\sigma)} \leq c\,\|f/\sigma\|_{L^p(\sigma)} = c\,\|f\|_{L^p(u)} \quad \blacksquare$$

As a consequence of this result we can obtain Sawyer's characterization of those couple of weights (v,u) for which the Hardy–Littlewood is bounded from $L^p(u)$ to $L^q(v)$. We just state the result since the proof is like the given for the case $p = q$ in [7] p.432, with some obvious modifications.

Corollary 6.3. *Let $1 < p \leq q < \infty$, and let (v,u) be a couple of weights, and $\sigma = u^{1-p'}$. Then*

$$M : L^p(u) \to L^q(v) \tag{26}$$

if and only if

$$\left(\int_Q M(\sigma\chi_Q)(y)^q\, v(y)dy\right)^{1/q} \leq c\sigma(Q)^{1/p}, \tag{27}$$

for every cube Q .

References

1. A. CÓRDOBA, On the Vitali covering properties of a differentiation basis, *Studia Math.* **57** (1976), 91–95.
2. C. FEFFERMAN AND E. M. STEIN, Some maximal inequalities, *Amer. J. Math.* **93** (1971), 107–115.
3. R. FEFFERMAN, Strong differentiation with respect to measures, *Amer. J. Math.* **103** (1981), 33-40.

4. R. FEFFERMAN, "*Multiparameter Fourier Analysis*," Annals of Math. Studies **112**, Beijing Lectures in Harmonic Analysis, Princeton University Press, Princeton, 1986, pp. 47–130.
5. M. FRAZIER AND B. JAWERTH, A discrete transform and decompositions of distribution spaces, to appear in *J. Func. Anal.*, also in *MSRI reports* **00321-89** and **00421-89** (1988).
6. E. GAGLIARDO, On integral tranformations with positive kernels, *Proc. Amer. Math. Soc.* **16** (1965), 429–434.
7. J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA, "*Weighted norm inequalities and related topics*," North Holland Math. Studies **116**, North Holland, Amsterdam, 1985.
8. R. HOWARD AND A. R. SCHEP, Norms of positive operators on L^p–spaces, *Proc. Amer. Math. Soc.* **109** (1990), 135–146.
9. B. JAWERTH, Weighted inequalities for maximal operators: linearization, localization, and factorization, *Amer. J. Math.* **108** (1986), 361–414.
10. B. JAWERTH, C. PÉREZ AND G. WELLAND, "*The positive cone in Triebel-Lizorkin spaces and the relation among potential and maximal operators*," Contemporary Mathematics, Harmonic Analysis and Partial Differential Equations (M. Milman and T. Schonbek editors) **107**, Amer. Math. Soc., Providence, 1990.
11. B. JAWERTH AND A. TORCHINSKY, The strong maximal function with respect to measures, *Studia Math.* **80** (1984), 261–285.
12. K. C. LIN, Harmonic Analysis on the bidisc, Thesis, U. C. L. A. (1984).
13. B. MUCKENHOUPT, Weighted norm inequalities for the Hardy–Littlewood maximal function, *Trans. Amer. Math. Soc.* **165** (1972), 207–226.
14. C. PÉREZ, Weighted norm inequalities for potential and maximal operators, Thesis, Washington University (1989).
15. E. T. SAWYER, A characterization of a two weight norm weight inequality for maximal operators, *Studia Math.* **75** (1982), 1–11.
16. F. SORIA, A remark on A_1–weights theory for the strong maximal function, *Proc. Amer. Math. Soc.* **100** (1987), 46–48.
17. J.-O. STRÖMBERG AND R.L. WHEEDEN, Fractional integrals on weighted H^p and L^p spaces, *Trans. Amer. Math. Soc.* **287** (1983), 293–321.

Department of Mathematics
Campus Box 1146
Washington University
Saint Louis, Missouri 63130
U.S.A.

current address : Department of Mathematical Sciences
New Mexico State University
Las Cruces, NEW MEXICO 88003